The Formal Foundations of Mathematical Analysis

-- The text which follows is a mixture of formulae and comments, acceptable to the Referee proof verifier and verified thereby, which gives a sequence of formal definitions and proofs covering the foundations of mathematical analysis from its set theoretic roots. These definitions and proofs are expected to extend in time to culminate in a proof of the Cauchy integral theorem of complex analysis. The verifier uses an extended form of natural deduction, in which each inference within a proof consists of a 'hint' part defining the inference primitive used to derive a statement, followed (after an occurrence of the separator "==>") by the statement itself. Just 15 inference primitives, are used, namely

ELEM	extended mlss inference
Suppose	opens natural deduction context
Discharge	closes natural deduction context if current context is contradictory
Citation	statement citation followed by extended mlss inference
Tcitation	theorem citation followed by extended mlss inference
EQUAL	use of equalities during blobbing, followed by extended mlss inference
ALGEBRA	use of algebraic identities during blobbing, followed by extended mlss inference
SIMPLF	simplify nested setformers, generating identities for use in extended mlss inference
Monot	handle quantifiers and exploit set-theoretic monotonicity relationships
Def	make a (possibly recursive) definition
Use_def	cite and use a definition
Loc_def	make a local definition of some auxiliary constant
ENTER_THEORY	enter the context defined by a theory
Assump	cite a theory assumption
APPLY	apply a theory

The formal material presented falls into various sections, each roughly corresponding to some recognized area of mathematics.

1 Basic Operations of set theory and the theory of ordinals

-- Our first step is to recast the axioms of choice and of infinity, which are built-in assumptions of set theory, as the two following small utility theorems.

-- Axiom of Choice

Theorem 1 (0) $(\mathsf{S} = \emptyset \;\&\; \mathbf{arb}(\mathsf{S}) = \emptyset) \vee (\mathbf{arb}(\mathsf{S}) \in \mathsf{S} \;\&\; \mathbf{arb}(\mathsf{S}) \cap \mathsf{S} = \emptyset)$. PROOF:

Suppose_not(s) $\Rightarrow$ $\neg\big((\mathsf{s} = \emptyset \;\&\; \mathbf{arb}(\mathsf{s}) = \emptyset) \vee (\mathbf{arb}(\mathsf{s}) \in \mathsf{s} \;\&\; \mathbf{arb}(\mathsf{s}) \cap \mathsf{s} = \emptyset)\big)$
Assump $\Rightarrow$ *Stat1* : $\langle \forall \mathsf{s} \mid (\mathsf{s} = \emptyset \;\&\; \mathbf{arb}(\mathsf{s}) = \emptyset) \vee (\mathbf{arb}(\mathsf{s}) \in \mathsf{s} \;\&\; \mathbf{arb}(\mathsf{s}) \cap \mathsf{s} = \emptyset)\rangle$
$\langle \mathsf{s} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem simply restates the axiom of infinity.

-- Axiom of Infinity

Theorem 2 (00) $\mathsf{s\_inf} \neq \emptyset \;\&\; \langle \forall \mathsf{x} \in \mathsf{s\_inf} \mid \{\mathsf{x}\} \in \mathsf{s\_inf}\rangle$. PROOF:

Suppose_not $\Rightarrow$ $\neg(\mathsf{s\_inf} \neq \emptyset \;\&\; \langle \forall \mathsf{x} \in \mathsf{s\_inf} \mid \{\mathsf{x}\} \in \mathsf{s\_inf}\rangle)$
Assump $\Rightarrow$ $\mathsf{s\_inf} \neq \emptyset \;\&\; \langle \forall \mathsf{x} \in \mathsf{s\_inf} \mid \{\mathsf{x}\} \in \mathsf{s\_inf}\rangle$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- After this, we begin by making a trick, purely set-theoretic definition of the notion of ordered pair, and proving a few simple lemmas which tell us how to extract the first and second components of a pair.

-- Ordered pair

DEF 1. $[\mathsf{X}, \mathsf{Y}] \;=_{\mathrm{Def}}\; \{\{\mathsf{X}\}, \{\{\mathsf{X}\}, \{\{\mathsf{Y}\}, \mathsf{Y}\}\}\}$

-- Our first result is a lemma stating a basic property of 'arb'. Its proof is elementary.

Theorem 3 (1) $\mathbf{arb}(\{\mathsf{X}\}) = \mathsf{X}$. PROOF:

Suppose_not(c) $\Rightarrow$ $\mathbf{arb}(\{\mathsf{c}\}) \neq \mathsf{c}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we give a lemma which prepares for definition of the first-component extractor from an ordered pair. Its proof is elementary.

Theorem 4 (2) $\mathsf{X} \in \mathsf{Y} \rightarrow \mathbf{arb}(\{\mathsf{Y}, \mathsf{X}\}) = \mathsf{X}$. PROOF:

Suppose_not(c, d) $\Rightarrow$ $\mathsf{c} \in \mathsf{d} \;\&\; \mathbf{arb}(\{\mathsf{d}, \mathsf{c}\}) \neq \mathsf{c}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma also prepares for definition of the first-component extractor from an ordered pair. Its proof is elementary.

Theorem 5 (3) $\mathbf{arb}([\mathsf{X},\mathsf{Y}]) = \{\mathsf{X}\}$. PROOF:

Suppose_not(c, d) ⇒ $\mathbf{arb}([\mathsf{c},\mathsf{d}]) \neq \{\mathsf{c}\}$
Use_def([·, ·]) ⇒ $\mathbf{arb}(\{\{\mathsf{c}\},\{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\}\}) \neq \{\mathsf{c}\}$
ELEM ⇒ false; Discharge ⇒ QED

-- The following is a third lemma which prepares for definition of the first-component extractor from an ordered pair. Its proof is elementary.

Theorem 6 (4) $\mathbf{arb}(\{\mathsf{X},\{\mathsf{X},\mathsf{Y}\}\}) = \mathsf{X}$. PROOF:

Suppose_not(x, y) ⇒ $\mathbf{arb}(\{\mathsf{x},\{\mathsf{x},\mathsf{y}\}\}) \neq \mathsf{x}$
ELEM ⇒ false; Discharge ⇒ QED

-- Finally, we give the formula for the first-component extractor from an ordered pair, along with its proof, which remains elementary.

Theorem 7 (5) $\mathbf{arb}(\mathbf{arb}([\mathsf{X},\mathsf{Y}])) = \mathsf{X}$. PROOF:

Suppose_not(c, d) ⇒ $\mathbf{arb}(\mathbf{arb}([\mathsf{c},\mathsf{d}])) \neq \mathsf{c}$
Use_def([·, ·]) ⇒ *Stat1* : $\mathbf{arb}(\mathbf{arb}(\{\{\mathsf{c}\},\{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\}\})) \neq \mathsf{c}$
$\langle\ \rangle$ ELEM ⇒ $\mathbf{arb}(\{\{\mathsf{c}\},\{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\}\}) = \{\mathsf{c}\}$ & $\mathbf{arb}(\{\mathsf{c}\}) = \mathsf{c}$
EQUAL $\langle$*Stat1*$\rangle$ ⇒ false; Discharge ⇒ QED

-- Next we give a lemma which prepares for definition of the second-component extractor from an ordered pair. Its proof is elementary.

Theorem 8 (6) $\mathbf{arb}(\mathbf{arb}(\mathbf{arb}([\mathsf{X},\mathsf{Y}] \setminus \{\mathbf{arb}([\mathsf{X},\mathsf{Y}])\}) \setminus \{\mathbf{arb}([\mathsf{X},\mathsf{Y}])\})) = \mathsf{Y}$. PROOF:

Suppose_not(c, d) ⇒ $\mathbf{arb}(\mathbf{arb}(\mathbf{arb}([\mathsf{c},\mathsf{d}] \setminus \{\mathbf{arb}([\mathsf{c},\mathsf{d}])\}) \setminus \{\mathbf{arb}([\mathsf{c},\mathsf{d}])\})) \neq \mathsf{d}$
$\langle$c, d$\rangle \hookrightarrow$ *T3* ⇒ $\mathbf{arb}([\mathsf{c},\mathsf{d}]) = \{\mathsf{c}\}$
EQUAL ⇒ $\mathbf{arb}(\mathbf{arb}(\mathbf{arb}([\mathsf{c},\mathsf{d}] \setminus \{\{\mathsf{c}\}\}) \setminus \{\{\mathsf{c}\}\})) \neq \mathsf{d}$
Use_def([·, ·]) ⇒ $\mathbf{arb}(\mathbf{arb}(\mathbf{arb}(\{\{\mathsf{c}\},\{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\}\} \setminus \{\{\mathsf{c}\}\}) \setminus \{\{\mathsf{c}\}\})) \neq \mathsf{d}$
TELEM ⇒ $\mathbf{arb}(\{\{\mathsf{c}\},\{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\}\} \setminus \{\{\mathsf{c}\}\}) = \{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\}$
EQUAL ⇒ *Stat1* : $\mathbf{arb}(\mathbf{arb}(\{\{\mathsf{c}\},\{\{\mathsf{d}\},\mathsf{d}\}\} \setminus \{\{\mathsf{c}\}\})) \neq \mathsf{d}$
$\langle$*Stat1*$\rangle$ ELEM ⇒ false; Discharge ⇒ QED

-- Now we can give formal definitions of both ordered-pair component extractor functions.

DEF 2. $X^{[1]} =_{\text{Def}} \mathbf{arb}(\mathbf{arb}(X))$

DEF 3. $X^{[2]} =_{\text{Def}} \mathbf{arb}(\mathbf{arb}(\mathbf{arb}(X \setminus \{\mathbf{arb}(X)\}) \setminus \{\mathbf{arb}(X)\}))$

-- The following recursive definitions of a lexicographic order for hereditarily finite sets play a role in the 'mirroring lemma' discussion preparatory to the proof of Goedel's theorem in the book associated with this collection of formal proofs. Def xxx: [Lexicographic discriminant] discr (p, q) := {v in p |q incs {w in p |discr (v + w, v * w) * w = 0}}-q Def yyy: [Lexicographic ordering] Smaller (x, y) := discr (x + y, x * y) * y = 0 The following basic property of the first-component extractor function is an elementary consequence of the preceding lemmas.

Theorem 9 (7) $[X, Y]^{[1]} = X$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $[x, y]^{[1]} \neq x$

Use_def$([1, \cdot]) \Rightarrow$ $\mathbf{arb}(\mathbf{arb}([x, y])) \neq x$

$\langle x, y \rangle \hookrightarrow T5 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Similarly, the basic property of the first-component extractor function is an elementary consequence of the preceding lemmas.

Theorem 10 (8) $[X, Y]^{[2]} = Y$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $[x, y]^{[2]} \neq y$

Use_def$([\cdot, 2]) \Rightarrow$ $\mathbf{arb}(\mathbf{arb}(\mathbf{arb}([x, y] \setminus \{\mathbf{arb}([x, y])\}) \setminus \{\mathbf{arb}([x, y])\})) \neq y$

$\langle x, y \rangle \hookrightarrow T6 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following basic relationship between ordered pair formation and component extraction is also an elementary consequence of the preceding lemmas.

-- Ordered pair Property

Theorem 11 (9) $[X, Y] = \left[[X, Y]^{[1]}, [X, Y]^{[2]}\right]$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $[x, y] \neq \left[[x, y]^{[1]}, [x, y]^{[2]}\right]$

$\langle x, y \rangle \hookrightarrow T7 \Rightarrow$ $[x, y]^{[1]} = x$

$\langle x, y \rangle \hookrightarrow T8 \Rightarrow$ $[x, y]^{[2]} = y$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The fact that any pair is the pair of its first and second components is equally elementary.

Theorem 12 (10) $\mathsf{U} = [\mathsf{A},\mathsf{B}] \rightarrow \mathsf{U} = \left[\mathsf{U}^{[1]},\mathsf{U}^{[2]}\right]$. PROOF:

Suppose_not$(\mathsf{u},\mathsf{a},\mathsf{b}) \Rightarrow$ $\mathsf{u} = [\mathsf{a},\mathsf{b}] \;\&\; \mathsf{u} \neq \left[\mathsf{u}^{[1]},\mathsf{u}^{[2]}\right]$
$\langle \mathsf{a},\mathsf{b}\rangle \hookrightarrow T7 \Rightarrow$ $[\mathsf{a},\mathsf{b}]^{[1]} = \mathsf{a}$
$\langle \mathsf{a},\mathsf{b}\rangle \hookrightarrow T8 \Rightarrow$ $[\mathsf{a},\mathsf{b}]^{[2]} = \mathsf{b}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEORY setformer$\left(\mathsf{e}(\mathsf{x}),\mathsf{ep}_1(\mathsf{x}),\mathsf{s},\mathsf{p}(\mathsf{x}),\mathsf{pp}_1(\mathsf{x})\right)$
-- Elementary properties of setformers
END setformer

-- The following small utility theory encapsulates the fact that the value of a set former is defined uniquely by the expression and predicate it contains. The consequence of this theory is available within our mechanism of variable substitution, making most explicit uses of the theory unnecessary.

ENTER_THEORY setformer

-- The following theorem results easily by use of our mechanism of variable substitution into setformers known to be different.

DEF setformer · 0. $\mathsf{x}_\Theta \;=_{\text{Def}}\; \mathbf{arb}\left(\{\mathsf{x} \in \mathsf{s} \,|\, \mathsf{e}(\mathsf{x}) \neq \mathsf{e}'(\mathsf{x}) \vee \neg\left(\mathsf{P}(\mathsf{x}) \leftrightarrow \mathsf{PP}(\mathsf{x})\right)\}\right)$

Theorem 13 (setformer · 1) $\mathsf{x}_\Theta \notin \mathsf{s} \vee \left(\mathsf{e}(\mathsf{x}_\Theta) = \mathsf{e}'(\mathsf{x}_\Theta) \;\&\; \left(\mathsf{P}(\mathsf{x}_\Theta) \leftrightarrow \mathsf{PP}(\mathsf{x}_\Theta)\right)\right) \rightarrow \{\mathsf{e}(\mathsf{x}) : \mathsf{x} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{x})\} = \{\mathsf{e}'(\mathsf{x}) : \mathsf{x} \in \mathsf{s} \,|\, \mathsf{PP}(\mathsf{x})\}$. PROOF:

Suppose_not$(\mathsf{s}) \Rightarrow$ $\mathsf{x}_\Theta \notin \mathsf{s} \vee \left(\mathsf{e}(\mathsf{x}_\Theta) = \mathsf{e}'(\mathsf{x}_\Theta) \;\&\; \left(\mathsf{P}(\mathsf{x}_\Theta) \leftrightarrow \mathsf{PP}(\mathsf{x}_\Theta)\right)\right) \;\&\; Stat1: \{\mathsf{e}(\mathsf{x}) : \mathsf{x} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{x})\} \neq \{\mathsf{e}'(\mathsf{x}) : \mathsf{x} \in \mathsf{s} \,|\, \mathsf{PP}(\mathsf{x})\}$

-- For let s be a counterexample to our assertion, and let c be some element of one of the sets $\{\mathsf{e}(\mathsf{x}) : \mathsf{x} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{x})\}$ and $\{\mathsf{e}'(\mathsf{x}) : \mathsf{x} \in \mathsf{s} \,|\, \mathsf{PP}(\mathsf{x})\}$ but not the other. Supposing that c belongs to the first of these sets but not the second, a contradiction results immediately from the axiom of choice, and similarly in the symmetric case. So the negative of our assertion leads to a contradiction in every case, proving the present theorem.

$\langle \mathsf{c}\rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{c} \in \mathsf{s} \;\&\; \mathsf{e}(\mathsf{c}) \neq \mathsf{e}'(\mathsf{c}) \vee \left(\mathsf{P}(\mathsf{c}) \;\&\; \neg\mathsf{PP}(\mathsf{c})\right) \vee \left(\neg\mathsf{P}(\mathsf{c}) \;\&\; \mathsf{PP}(\mathsf{c})\right)$
Suppose $\Rightarrow$ $Stat2: \mathsf{c} \notin \left\{\mathsf{x} \in \mathsf{s} \,|\, \mathsf{e}(\mathsf{x}) \neq \mathsf{e}'(\mathsf{x}) \vee \neg\left(\mathsf{P}(\mathsf{x}) \leftrightarrow \mathsf{PP}(\mathsf{x})\right)\right\}$
$\langle \mathsf{c}\rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ $\left\{\mathsf{x} \in \mathsf{s} \,|\, \mathsf{e}(\mathsf{x}) \neq \mathsf{e}'(\mathsf{x}) \vee \neg\left(\mathsf{P}(\mathsf{x}) \leftrightarrow \mathsf{PP}(\mathsf{x})\right)\right\} \neq \emptyset$

$\langle\, \{x \in s \,|\, e(x) \neq e'(x) \vee \neg(P(x) \leftrightarrow PP(x))\}\,\rangle \hookrightarrow T0 \Rightarrow$ $\mathbf{arb}(\{x \in s \,|\, e(x) \neq e'(x) \vee \neg(P(x) \leftrightarrow PP(x))\}) \in \{x \in s \,|\, e(x) \neq e'(x) \vee \neg(P(x) \leftrightarrow PP(x))\}$

Use_def $(x_\Theta) \Rightarrow$ *Stat3* : $x_\Theta \in \{x \in s \,|\, e(x) \neq e'(x) \vee \neg(P(x) \leftrightarrow PP(x))\}$

$\langle\ \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The following small utility theory describes a few elementary cases in which we can be sure that a set is non-null

THEORY setformer$_0(e(x), s, P(x))$

-- More elementary properties of setformers

END setformer$_0$

ENTER_THEORY setformer$_0$

-- The following theorem results easily by use of our mechanism of variable substitution into a setformer known to be non-null.

Theorem 14 (setformer$_{01}$) $S \neq \emptyset \to \{e(x) : x \in S\} \neq \emptyset$. PROOF:

Suppose_not$(s) \Rightarrow$ *Stat1* : $s \neq \emptyset$ & *Stat2* : $\{e(x) : x \in s\} = \emptyset$

$\langle c \rangle \hookrightarrow Stat1 \Rightarrow$ $c \in s$

$\langle c \rangle \hookrightarrow Stat2 \Rightarrow$ $\neg(e(c) = e(c) \;\&\; c \in s)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem also results easily by use of our mechanism of variable substitution into a setformer known to be non-null.

Theorem 15 (setformer$_{02}$) $\{x \in S \,|\, P(x)\} \neq \emptyset \to \{e(x) : x \in S \,|\, P(x)\} \neq \emptyset$. PROOF:

Suppose_not$(s) \Rightarrow$ *Stat1* : $\{x \in s \,|\, P(x)\} \neq \emptyset$ & $\{e(x) : x \in s \,|\, P(x)\} = \emptyset$

$\langle c \rangle \hookrightarrow Stat1 \Rightarrow$ $c \in s \;\&\; P(c)$

ELEM $\Rightarrow$ *Stat2* : $e(c) \notin \{e(x) : x \in s \,|\, P(x)\}$

$\langle c \rangle \hookrightarrow Stat2 \Rightarrow$ $\neg(e(c) = e(c) \;\&\; c \in s \;\&\; P(c))$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The following utility theory is the two-variable analog of the 'setformer' theory given above. It encapsulates the fact that the value of a two-variable set former is defined uniquely by the expression and predicate it contains. The consequence of this theory is available within our mechanism of variable substitution, making most explicit uses of the theory unnecessary.

THEORY $\mathsf{setformer_2}\big(\mathsf{e(x,y), e'(x,y), f(x), f'(x), s, p(x,y), p'(x,y)}\big)$

-- More elementary properties of setformers

END $\mathsf{setformer_2}$

ENTER_THEORY $\mathsf{setformer_2}$

DEF $\mathsf{setformer_2} \cdot 0a.$ $\quad \mathsf{xy_\Theta} \quad =_{\text{Def}} \quad \mathbf{arb}\big(\{[\mathsf{x,y}] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \cup \mathsf{f'(x)} \mid \mathsf{f(x)} \neq \mathsf{f'(x)} \vee \mathsf{e(x,y)} \neq \mathsf{e'(x,y)} \vee \neg\big(\mathsf{P(x,y)} \leftrightarrow \mathsf{PP(x,y)}\big)\}\big)$

DEF $\mathsf{setformer_2} \cdot 0b.$ $\quad \mathsf{x_\Theta} \quad =_{\text{Def}} \quad \mathsf{xy_\Theta}^{[1]}$

DEF $\mathsf{setformer_2} \cdot 0c.$ $\quad \mathsf{y_\Theta} \quad =_{\text{Def}} \quad \mathsf{xy_\Theta}^{[2]}$

-- The following theorem results easily along the same line of proof used in proving Theorem setformer. 1.

Theorem 16 ($\mathsf{setformer_{21}}$) $\big(\mathsf{x_\Theta} \notin \mathsf{s} \vee \mathsf{y_\Theta} \notin \mathsf{f(x_\Theta)} \cup \mathsf{f'(x_\Theta)}\big) \vee \Big(\mathsf{f(x_\Theta)} = \mathsf{f'(x_\Theta)} \,\&\, \mathsf{e(x_\Theta, y_\Theta)} = \mathsf{e'(x_\Theta, y_\Theta)} \,\&\, \big(\mathsf{P(x_\Theta, y_\Theta)} \leftrightarrow \mathsf{PP(x_\Theta, y_\Theta)}\big)\Big) \rightarrow$ $\{\mathsf{e(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \mid \mathsf{P(x,y)}\} = \{\mathsf{e'(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f'(x)} \mid \mathsf{PP(x,y)}\}$. PROOF:

-- For let s be a counterexample to our assertion, and let c be some element of one of the sets $\{\mathsf{e(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \mid \mathsf{P(x,y)}\}$ and $\{\mathsf{e'(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f'(x)} \mid \mathsf{PP(x,y)}\}$ but not the other. Supposing that c belongs to the first of these sets but not the second, a contradiction results easily from the axiom of choice.

Suppose_not(s) $\Rightarrow$ $\big(\mathsf{x_\Theta} \notin \mathsf{s} \vee \mathsf{y_\Theta} \notin \mathsf{f(x_\Theta)} \cup \mathsf{f'(x_\Theta)}\big) \vee \Big(\mathsf{f(x_\Theta)} = \mathsf{f'(x_\Theta)} \,\&\, \mathsf{e(x_\Theta, y_\Theta)} = \mathsf{e'(x_\Theta, y_\Theta)} \,\&\, \big(\mathsf{P(x_\Theta, y_\Theta)} \leftrightarrow \mathsf{PP(x_\Theta, y_\Theta)}\big)\Big) \,\&\, \mathit{Stat1}:$
$\{\mathsf{e(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \mid \mathsf{P(x,y)}\} \neq$
$\{\mathsf{e'(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f'(x)} \mid \mathsf{PP(x,y)}\}$

$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathsf{c} \in \{\mathsf{e(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \mid \mathsf{P(x,y)}\} \leftrightarrow \mathsf{c} \notin \{\mathsf{e'(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f'(x)} \mid \mathsf{PP(x,y)}\}$

Suppose $\Rightarrow$ $\mathit{Stat2}:$ $\mathsf{c} \in \{\mathsf{e(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \mid \mathsf{P(x,y)}\} \,\&\, \mathsf{c} \notin \{\mathsf{e'(x,y)} : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f'(x)} \mid \mathsf{PP(x,y)}\}$

-- Indeed, there must exist x and y in s and f(x) respectively for which $\mathsf{c} = \mathsf{e(x,y)}$ but for which one of the clauses of $\mathsf{y} \in \mathsf{f'(x)} \,\&\, \mathsf{PP(x,y)} \,\&\, \mathsf{c} = \mathsf{e'(x,y)}$ is false.

$\langle \mathsf{x,y,x,y} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{y} \in \mathsf{f(x)} \,\&\, \mathsf{P(x,y)} \,\&\, \mathsf{c} = \mathsf{e(x,y)} \,\&$
$\neg\big(\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{y} \in \mathsf{f'(x)} \,\&\, \mathsf{PP(x,y)} \,\&\, \mathsf{c} = \mathsf{e'(x,y)}\big)$

Suppose $\Rightarrow$ $\mathit{Stat3}:$ $[\mathsf{x,y}] \notin \big\{[\mathsf{x,y}] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{f(x)} \cup \mathsf{f'(x)} \mid \mathsf{f(x)} \neq \mathsf{f'(x)} \vee \mathsf{e(x,y)} \neq \mathsf{e'(x,y)} \vee \neg\big(\mathsf{P(x,y)} \leftrightarrow \mathsf{PP(x,y)}\big)\big\}$

$\langle x, y\rangle \hookrightarrow \textit{Stat3} \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\} \neq \emptyset$

-- This is easily seen to imply that the set

$$\{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}$$

is non-null, so that by the axiom of choice xy_{Θ} is a member of it. But then the components x_{Θ} and y_{Θ} of xy_{Θ} plainly violate the hypotheses of the present theorem, and so rule out our initial supposition.

$\langle \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\} \rangle \hookrightarrow \textit{T0} \Rightarrow$
$\quad \mathbf{arb}(\{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}) \in$
$\qquad \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}$
Use_def $(xy_{\Theta}) \Rightarrow$ $\quad \textit{Stat4}: xy_{\Theta} \in \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}$
$\langle x_2, y_2\rangle \hookrightarrow \textit{Stat4} \Rightarrow$ $\quad xy_{\Theta} = [x_2, y_2] \;\&\; x_2 \in s \;\&\; y_2 \in f(x_2) \cup f'(x_2) \;\&\; f(x_2) \neq f'(x_2) \vee e(x_2, y_2) \neq e'(x_2, y_2) \vee \neg(P(x_2, y_2) \leftrightarrow PP(x_2, y_2))$
ELEM $\Rightarrow$ $\quad x_2 = xy_{\Theta}{}^{[1]} \;\&\; y_2 = xy_{\Theta}{}^{[2]}$
Use_def $(x_{\Theta}) \Rightarrow$ $\quad x_{\Theta} = x_2$
Use_def $(y_{\Theta}) \Rightarrow$ $\quad y_{\Theta} = y_2$
EQUAL $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \textit{Stat5}: c \in \{e'(x, y) : x \in s, y \in f'(x) \mid PP(x, y)\} \;\&\; c \notin \{e(x, y) : x \in s, y \in f(x) \mid P(x, y)\}$

-- The same argument can be given in the symmetric case, and so the negative of our assertion leads to a contradiction in every case, proving the present theorem.

$\langle xx, yy, xx, yy\rangle \hookrightarrow \textit{Stat5} \Rightarrow$ $\quad xx \in s \;\&\; yy \in f'(xx) \;\&\; PP(xx, yy) \;\&\; c = e'(xx, yy) \;\&$
$\quad \neg(xx \in s \;\&\; yy \in f(xx) \;\&\; P(xx, yy) \;\&\; c = e(xx, yy))$
Suppose $\Rightarrow$ $\quad \textit{Stat6}: [xx, yy] \notin \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}$
$\langle xx, yy\rangle \hookrightarrow \textit{Stat6} \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\} \neq \emptyset$
$\langle \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\} \rangle \hookrightarrow \textit{T0} \Rightarrow$
$\quad \mathbf{arb}(\{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}) \in$
$\qquad \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}$
Use_def $(xy_{\Theta}) \Rightarrow$ $\quad \textit{Stat7}: xy_{\Theta} \in \{[x, y] : x \in s, y \in f(x) \cup f'(x) \mid f(x) \neq f'(x) \vee e(x, y) \neq e'(x, y) \vee \neg(P(x, y) \leftrightarrow PP(x, y))\}$
$\langle x_3, y_3\rangle \hookrightarrow \textit{Stat4} \Rightarrow$ $\quad xy_{\Theta} = [x_3, y_3] \;\&\; x_3 \in s \;\&\; y_3 \in f(x_3) \cup f'(x_3) \;\&\; f(x_3) \neq f'(x_3) \vee e(x_3, y_3) \neq e'(x_3, y_3) \vee \neg(P(x_3, y_3) \leftrightarrow PP(x_3, y_3))$
ELEM $\Rightarrow$ $\quad x_3 = xy_{\Theta}{}^{[1]} \;\&\; y_3 = xy_{\Theta}{}^{[2]}$
Use_def $(x_{\Theta}) \Rightarrow$ $\quad x_{\Theta} = x_3$
Use_def $(y_{\Theta}) \Rightarrow$ $\quad y_{\Theta} = y_3$
EQUAL $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

ENTER_THEORY Set_theory

-- The following utility theory encapsulates the fact that given any set and predicate $\mathsf{P}(\mathsf{x})$, one can always obtain a (generally smaller) set consisting of precisely those entities which satisfy P.

THEORY $\mathsf{comprehension}\big(\mathsf{s}, \mathsf{P}(\mathsf{x})\big)$
END comprehension

ENTER_THEORY comprehension

-- We begin by defining the element considered in the proof which follows.

DEF $00g$. $\quad \mathsf{tt}_\Theta \quad =_{\text{Def}} \quad \{\mathsf{x} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{x})\}$

Theorem 17 (comprehension$_1$) $\mathsf{X} \in \mathsf{tt}_\Theta \leftrightarrow \mathsf{X} \in \mathsf{s} \,\&\, \mathsf{P}(\mathsf{X})$. PROOF:

Suppose_not$(\mathsf{x}, \mathsf{tt}_\Theta) \Rightarrow \quad \neg\big(\mathsf{x} \in \mathsf{S} \,\&\, \mathsf{P}(\mathsf{x}) \leftrightarrow \mathsf{x} \in \mathsf{tt}_\Theta\big)$

-- For the contrary assumption would lead to the following contradiction:

Use_def$(\mathsf{tt}_\Theta) \Rightarrow \quad \neg\big(\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{P}(\mathsf{x}) \leftrightarrow \mathsf{x} \in \{\mathsf{u} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{u})\}\big)$
Suppose $\Rightarrow \quad \mathsf{x} \in \mathsf{S} \,\&\, \mathsf{P}(\mathsf{x}) \,\&\, \mathit{Stat1}: \; \mathsf{x} \notin \{\mathsf{u} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{u})\}$
$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \neg\big(\mathsf{x} \in \mathsf{S} \,\&\, \mathsf{P}(\mathsf{x})\big) \,\&\, \mathit{Stat2}: \; \mathsf{x} \in \{\mathsf{u} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{u})\}$
$\langle \, \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

ENTER_THEORY Set_theory

-- The following small THEORY summarizes what has just been proved.

DISPLAY comprehension

THEORY $\mathsf{comprehension}(\mathsf{s}, \mathsf{P})$
$\quad \big\langle \forall \mathsf{x} \in \mathsf{s} \,|\, \mathsf{P}(\mathsf{x}) \longrightarrow \mathsf{x} \in \mathsf{s} \big\rangle$
$\Rightarrow (\mathsf{tt})$
$\quad \mathsf{X} \in \mathsf{tt} \leftrightarrow \mathsf{P}(\mathsf{X})$
END comprehension

-- Our more serious work begins now as we start to prove the basic properties of ordinals. We take a first small step in this direction by giving von Neumann's definition of ordinals: an ordinal is a set which is transitively closed and totally ordered under membership.

DEF 10. $\mathcal{O}(\mathsf{X}) \quad \leftrightarrow_{\text{Def}} \quad \langle \forall \mathsf{x} \in \mathsf{X} \mid \mathsf{x} \subseteq \mathsf{X} \rangle \;\&\; \langle \forall \mathsf{x} \in \mathsf{X}, \mathsf{y} \in \mathsf{X} \mid \mathsf{x} \in \mathsf{y} \vee \mathsf{y} \in \mathsf{x} \vee \mathsf{x} = \mathsf{y} \rangle$

-- Next we prove a first basic property of ordinals: any member of an ordinal is an ordinal.

Theorem 18 (11) $\mathcal{O}(\mathsf{S}) \;\&\; T \in \mathsf{S} \rightarrow \mathcal{O}(T)$. PROOF:

Suppose_not(s, t) $\Rightarrow \quad \mathcal{O}(\mathsf{s}) \;\&\; \mathsf{t} \in \mathsf{s} \;\&\; \neg\mathcal{O}(\mathsf{t})$

-- We proceed by contradiction. If our theorem is false, there is an ordinal s having a member t which is not an ordinal.

Use_def ($\mathcal{O}$) $\Rightarrow \quad \mathit{Stat1}: \; \neg(\langle \forall \mathsf{x} \in \mathsf{t} \mid \mathsf{x} \subseteq \mathsf{t} \rangle \;\&\; \langle \forall \mathsf{x} \in \mathsf{t}, \mathsf{y} \in \mathsf{t} \mid \mathsf{x} \in \mathsf{y} \vee \mathsf{y} \in \mathsf{x} \vee \mathsf{x} = \mathsf{y} \rangle)$

-- Hence, by definition of ordinal, t must either have a member a not included in t, or a pair b, c of distinct members not related by membership.

$\langle \mathsf{a}, \mathsf{b}, \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad (\mathsf{a} \in \mathsf{t} \;\&\; \mathsf{a} \not\subseteq \mathsf{t}) \vee \big(\mathsf{b}, \mathsf{c} \in \mathsf{t} \;\&\; \neg(\mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{b} = \mathsf{c})\big)$

-- But since s is an ordinal, it must include its member t, so that the second case is impossible.

Use_def ($\mathcal{O}$) $\Rightarrow \quad \mathit{Stat2}: \; \langle \forall \mathsf{x} \in \mathsf{s} \mid \mathsf{x} \subseteq \mathsf{s} \rangle \;\&\; \mathit{Stat3}: \; \langle \forall \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{s} \mid \mathsf{x} \in \mathsf{y} \vee \mathsf{y} \in \mathsf{x} \vee \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{t} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{t} \subseteq \mathsf{s}$
Suppose $\Rightarrow \quad \mathsf{b}, \mathsf{c} \in \mathsf{t} \;\&\; \neg(\mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{b} = \mathsf{c})$
$\langle \mathsf{b}, \mathsf{c} \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathsf{b}, \mathsf{c} \in \mathsf{s} \rightarrow \mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{b} = \mathsf{c}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad \mathit{Stat4}: \; \mathsf{a} \not\subseteq \mathsf{t} \;\&\; \mathsf{a} \in \mathsf{t}$

-- Thus we need only consider the first case, in which a is a member but not a subset of t. In this case there plainly exists a d in a but not in t. Plainly a is a member of s, and thus a subset of s; so d is also a member of s.

$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad \mathsf{d} \in \mathsf{a} \;\&\; \mathsf{d} \notin \mathsf{t}$
$\langle \mathsf{a} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{a} \subseteq \mathsf{s}$
ELEM $\Rightarrow \quad \mathsf{d} \in \mathsf{s}$

-- By definition of ordinal, it follows that d either equals t, is a member of t, or that t is a member of d. But all three of these cases are impossible, since any would imply the existence of a membership cycle. This contradiction proves our theorem.

$\langle \mathsf{d}, \mathsf{t} \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathsf{d} \in \mathsf{t} \vee \mathsf{t} \in \mathsf{d} \vee \mathsf{t} = \mathsf{d}$
$\langle \mathit{Stat4} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We continue by considering and proving various useful forms of the principle of transfinite induction, formulating these as a succession of utility theories. We begin with an induction principle for ordinals. This tells us that if an ordinal has a certain property, it must include some subset which is an ordinal with this same property, but which contains no member that has the property in question. In fact, this subset can be defined as $\mathbf{arb}(\mathsf{aux\_set})$, where $\mathsf{aux\_set} =_{\mathrm{Def}} \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x}) \,\&\, \mathsf{P}(\mathsf{x})\}$ The following property of ordinals follows directly from their definition.

Theorem 19 (12) $\mathcal{O}(\mathsf{S}) \,\&\, T \in \mathsf{S} \rightarrow T \subseteq \mathsf{S}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s},\mathsf{t}) \Rightarrow \quad \mathcal{O}(\mathsf{s}) \,\&\, \mathsf{t} \in \mathsf{s} \,\&\, \mathsf{t} \nsubseteq \mathsf{s}$
$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad \mathit{Stat1}: \ \langle \forall \mathsf{x} \in \mathsf{s} \mid \mathsf{x} \subseteq \mathsf{s} \rangle$
$\langle \mathsf{t} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

THEORY ordinal_induction$(\mathsf{o}, \mathsf{P}(\mathsf{x}))$
$\mathcal{O}(\mathsf{o}) \,\&\, \mathsf{P}(\mathsf{o})$
END ordinal_induction

ENTER_THEORY ordinal_induction

-- We begin by defining the element considered in the proof which follows.

DEF ordinal_induction · 0. $\quad \mathsf{t}_{\Theta} \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x}) \,\&\, \mathsf{P}(\mathsf{x})\})$

Theorem 20 ($\mathsf{ord\_ind}_1$) $\mathcal{O}(\mathsf{t}_{\Theta}) \,\&\, \mathsf{P}(\mathsf{t}_{\Theta}) \,\&\, \mathsf{t}_{\Theta} \subseteq \mathsf{o} \,\&\, \langle \forall \mathsf{x} \in \mathsf{t}_{\Theta} \mid \neg \mathsf{P}(\mathsf{x}) \rangle$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{t}_{\Theta},\mathsf{o}) \Rightarrow \quad \mathit{Stat1}: \ \neg\big(\langle \forall \mathsf{x} \in \mathsf{t}_{\Theta} \mid \neg \mathsf{P}(\mathsf{x}) \rangle \,\&\, \mathcal{O}(\mathsf{t}_{\Theta}) \,\&\, \mathsf{P}(\mathsf{t}_{\Theta}) \,\&\, \mathsf{t}_{\Theta} \subseteq \mathsf{o}\big)$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \neg\big(\mathcal{O}(\mathsf{t}_{\Theta}) \,\&\, \mathsf{P}(\mathsf{t}_{\Theta}) \,\&\, \mathsf{t}_{\Theta} \subseteq \mathsf{o}\big) \vee \big(\mathsf{c} \in \mathsf{t}_{\Theta} \,\&\, \mathsf{P}(\mathsf{c})\big)$

-- We proceed by contradiction, and begin by noting that the set aux_set displayed above is not empty. Indeed, o is obviously a member of it.

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat2}: \ \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x}) \,\&\, \mathsf{P}(\mathsf{x})\} = \emptyset$
$\mathsf{ELEM} \Rightarrow \quad \mathit{Stat3}: \ \mathsf{o} \notin \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x}) \,\&\, \mathsf{P}(\mathsf{x})\}$
$\langle \mathsf{o} \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \neg\big(\mathcal{O}(\mathsf{o}) \,\&\, \mathsf{P}(\mathsf{o})\big)$
$\mathsf{Assump} \Rightarrow \quad \mathcal{O}(\mathsf{o}) \,\&\, \mathsf{P}(\mathsf{o})$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x}) \,\&\, \mathsf{P}(\mathsf{x})\} \neq \emptyset$

-- The axiom of choice now tells us that t_Θ as defined above must be a minimal element of aux_set, and so must clearly satisfy $\mathcal{O}(\mathsf{t}_\Theta)$ & $\mathsf{P}(\mathsf{t}_\Theta)$ & $\mathsf{t}_\Theta \subseteq \mathsf{o}$. This rules out the first of the two cases listed above, leaving only the second.

Use_def $(\mathsf{t}_\Theta) \Rightarrow \quad \mathsf{t}_\Theta = \mathbf{arb}(\{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{P}(\mathsf{x})\})$
$\langle\, \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{P}(\mathsf{x})\}\,\rangle \hookrightarrow T0 \Rightarrow \quad Stat4:$
$\quad \mathsf{t}_\Theta \in \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{P}(\mathsf{x})\}\ \&\ \mathsf{t}_\Theta \cap \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{P}(\mathsf{x})\} = \emptyset$
$\langle \mathsf{t}_\Theta\rangle \hookrightarrow Stat4 \Rightarrow \quad \mathcal{O}(\mathsf{t}_\Theta)\ \&\ \mathsf{P}(\mathsf{t}_\Theta)\ \&\ \mathsf{t}_\Theta \subseteq \mathsf{o}$
ELEM $\Rightarrow \quad \mathsf{c} \in \mathsf{t}_\Theta\ \&\ \mathsf{P}(\mathsf{c})$

-- Since t_Θ is an ordinal, it must by definition include its member c, which must therefore also be a subset of o and an ordinal.

$\langle \mathsf{t}_\Theta, \mathsf{c}\rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{c} \subseteq \mathsf{t}_\Theta$
ELEM $\Rightarrow \quad \mathsf{c} \subseteq \mathsf{o}$
$\langle \mathsf{t}_\Theta, \mathsf{c}\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{c})$

-- Thus c has the properties required to make it an element of aux_set. But this contradicts the minimality of t_Θ, and so proves our theorem.

Suppose $\Rightarrow \quad Stat5:\ \mathsf{c} \notin \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{P}(\mathsf{x})\}$
$\langle \mathsf{c}\rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathsf{c} \in \{\mathsf{x} \subseteq \mathsf{o} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{P}(\mathsf{x})\}$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad$ QED

ENTER_THEORY Set_theory

-- The following small THEORY summarizes what has just been proved.

DISPLAY ordinal_induction

THEORY ordinal_induction(o, P)
$\quad \mathcal{O}(\mathsf{o})\ \&\ \mathsf{P}(\mathsf{o})$
$\Rightarrow (\mathsf{t})$
$\quad \mathcal{O}(\mathsf{t})\ \&\ \mathsf{P}(\mathsf{t})\ \&\ \mathsf{t} \subseteq \mathsf{o}\ \&\ \langle \forall \mathsf{x} \in \mathsf{t} \mid \neg \mathsf{P}(\mathsf{x})\rangle$
END ordinal_induction

2 Other versions of the principle of transfinite induction

-- Next we consider and prove several other forms of the principle of transfinite induction which are sometimes easier to use. We aim to show that if a set has a specified property, it contains a subset having this property, none of whose members have the property. The following definition, of the set of all elements which are linked to a set s by a chain of memberships, and so are its 'ultimate members', starts to prepare for this.

-- Transitive membership closure of S

DEF 35*a*. $\mathsf{Ult\_membs}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathsf{X} \cup \{\mathsf{y} : \mathsf{u} \in \{\mathsf{Ult\_membs}(\mathsf{x}) : \mathsf{x} \in \mathsf{X}\}, \mathsf{y} \in \mathsf{u}\}$

-- Our goal is to prove that $\mathsf{Ult\_membs}(\mathsf{S})$ includes S, and that $\mathsf{Ult\_membs}(\mathsf{S})$ is transitively closed under membership. The proof of this second fact, which is given somewhat below, will involve use of the axiom of infinity. The first of these facts, captured in the following lemma, is an immediate consequence of the definition of Ult_membs.

Theorem 21 (13) $\mathsf{S} \subseteq \mathsf{Ult\_membs}(\mathsf{S})$. PROOF:

Suppose_not(s) ⇒ $\mathsf{s} \not\subseteq \mathsf{Ult\_membs}(\mathsf{s})$
Use_def(Ult_membs) ⇒ $\mathsf{Ult\_membs}(\mathsf{s}) = \mathsf{s} \cup \{\mathsf{y} : \mathsf{u} \in \{\mathsf{Ult\_membs}(\mathsf{x}) : \mathsf{x} \in \mathsf{s}\}, \mathsf{y} \in \mathsf{u}\}$
ELEM ⇒ false; Discharge ⇒ QED

-- Our next small lemma simply reformulates the definition of Ult_membs as an identity.

Theorem 22 (14) $\mathsf{Ult\_membs}(\mathsf{S}) = \mathsf{S} \cup \{\mathsf{v} : \mathsf{x} \in \mathsf{S}, \mathsf{v} \in \mathsf{Ult\_membs}(\mathsf{x})\}$. PROOF:

Suppose_not(s) ⇒ $\mathsf{Ult\_membs}(\mathsf{s}) \neq \mathsf{s} \cup \{\mathsf{v} : \mathsf{x} \in \mathsf{s}, \mathsf{v} \in \mathsf{Ult\_membs}(\mathsf{x})\}$
Use_def(Ult_membs) ⇒ $\mathsf{Ult\_membs}(\mathsf{s}) = \mathsf{s} \cup \{\mathsf{v} : \mathsf{u} \in \{\mathsf{Ult\_membs}(\mathsf{x}) : \mathsf{x} \in \mathsf{s}\}, \mathsf{v} \in \mathsf{u}\}$
SIMPLF ⇒ false; Discharge ⇒ QED

-- It follows immediately from the definition of $\mathsf{Ult\_membs}$ that $\mathsf{Ult\_membs}$ includes all the members of its members.

Theorem 23 (15) $\mathsf{X} \in \mathsf{S} \,\&\, \mathsf{Y} \in \mathsf{X} \rightarrow \mathsf{Y} \in \mathsf{Ult\_membs}(\mathsf{S})$. PROOF:

Suppose_not(x, s, y) ⇒ $\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{y} \in \mathsf{x} \,\&\, \mathsf{y} \notin \mathsf{Ult\_membs}(\mathsf{s})$
$\langle \mathsf{s} \rangle \hookrightarrow T14$ ⇒ *Stat1* : $\mathsf{y} \notin \{\mathsf{v} : \mathsf{x} \in \mathsf{s}, \mathsf{v} \in \mathsf{Ult\_membs}(\mathsf{x})\}$
$\langle \mathsf{x}, \mathsf{y} \rangle \hookrightarrow Stat1$ ⇒ $\mathsf{y} \notin \mathsf{Ult\_membs}(\mathsf{x})$
$\langle \mathsf{x} \rangle \hookrightarrow T14$ ⇒ $\mathsf{y} \notin \mathsf{x}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- For the special case of an ordinal s we can show, using the principle of ordinal induction established above, that $\mathsf{Ult\_membs}(\mathsf{s}) = \mathsf{s}$.

Theorem 24 (16) $\mathcal{O}(\mathsf{S}) \rightarrow \mathsf{Ult\_membs}(\mathsf{S}) = \mathsf{S}$. PROOF:

-- We proceed by contradiction. If our theorem is false, there is an ordinal s which is not identical to $\mathsf{Ult\_membs}(\mathsf{s})$, and so, by Theorem 14, is not included in s.

Suppose_not(s) $\Rightarrow$ $\mathcal{O}(\mathsf{s})\ \&\ \mathsf{Ult\_membs}(\mathsf{s}) \neq \mathsf{s}$
$\langle \mathsf{s} \rangle \hookrightarrow T13 \Rightarrow$ $\mathcal{O}(\mathsf{s})\ \&\ \mathsf{Ult\_membs}(\mathsf{s}) \not\subseteq \mathsf{s}$

-- Thus the principle of ordinal induction tells us that s contains a minimal ordinal t with this property.

APPLY $\langle \mathsf{t}_\Theta : \mathsf{t} \rangle$ ordinal_induction$(\mathsf{o} \mapsto \mathsf{s}, \mathsf{P}(\mathsf{x}) \mapsto \mathsf{Ult_membs}(\mathsf{x}) \not\subseteq \mathsf{x}) \Rightarrow$
$Stat2 : \mathcal{O}(\mathsf{t})\ \&\ \mathsf{Ult\_membs}(\mathsf{t}) \not\subseteq \mathsf{t}\ \&\ Stat2a : \langle \forall \mathsf{x} \in \mathsf{t} \mid \neg \mathsf{Ult\_membs}(\mathsf{x}) \not\subseteq \mathsf{x} \rangle$
$\langle \mathsf{a}_0 \rangle \hookrightarrow Stat2 \Rightarrow$ $\mathcal{O}(\mathsf{t})\ \&\ Stat1 : \mathsf{Ult\_membs}(\mathsf{t}) \not\subseteq \mathsf{t}$

-- It follows that $\mathsf{Ult\_membs}(\mathsf{t})$ has an element c which does not belong to t. By Theorem 14, c must belong to $\{\mathsf{v} : \mathsf{x} \in \mathsf{t}, \mathsf{v} \in \mathsf{Ult\_membs}(\mathsf{x})\}$, and so there must exist an $\mathsf{x} \in \mathsf{t}$ such that $\mathsf{c} \in \mathsf{Ult\_membs}(\mathsf{x})$.

$\langle \mathsf{c} \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{c} \in \mathsf{Ult\_membs}(\mathsf{t})\ \&\ \mathsf{c} \notin \mathsf{t}$
$\langle \mathsf{t} \rangle \hookrightarrow T14 \Rightarrow$ $Stat3 : \mathsf{c} \in \{\mathsf{v} : \mathsf{x} \in \mathsf{t}, \mathsf{v} \in \mathsf{Ult\_membs}(\mathsf{x})\}$
$\langle \mathsf{x}, \mathsf{v} \rangle \hookrightarrow Stat3 \Rightarrow$ $\mathsf{x} \in \mathsf{t}\ \&\ \mathsf{c} \in \mathsf{Ult\_membs}(\mathsf{x})$

-- By Stat4 4 above, x must satisfy $\mathsf{Ult\_membs}(\mathsf{x}) \subseteq \mathsf{x}$, so c must belong to x.

$\langle \mathsf{x} \rangle \hookrightarrow Stat2a \Rightarrow$ $\mathsf{Ult\_membs}(\mathsf{x}) \subseteq \mathsf{x}\ \&\ \mathsf{c} \in \mathsf{x}$

-- Since t is an ordinal, its member x must be included in it; so c must be a member of t, contrary to what has been proved above. This contradiction proves our theorem.

$\langle \mathsf{t}, \mathsf{x} \rangle \hookrightarrow T12 \Rightarrow$ $\mathsf{x} \subseteq \mathsf{t}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next small lemma expresses $\mathsf{Ult\_membs}(\{\mathsf{s}\})$ in terms of $\mathsf{Ult\_membs}(\mathsf{s})$.

Theorem 25 (17) $\mathsf{Ult\_membs}(\{S\}) = \{S\} \cup \mathsf{Ult\_membs}(S)$. PROOF:

Suppose_not(s) ⇒ $\mathsf{Ult\_membs}(\{s\}) \neq \{s\} \cup \mathsf{Ult\_membs}(s)$
Use_def(Ult_membs) ⇒ $\mathsf{Ult\_membs}(\{s\}) = \{s\} \cup \{y : u \in \{\mathsf{Ult\_membs}(x) : x \in \{s\}\}, y \in u\}$
ELEM ⇒ $\{s\} \cup \{y : u \in \{\mathsf{Ult\_membs}(x) : x \in \{s\}\}, y \in u\} \neq \{s\} \cup \mathsf{Ult\_membs}(s)$

-- If our theorem were false, we could use the definition of Ult_membs to obtain the set inequality displayed above, which simplifies to an impossibility:

SIMPLF ⇒ $\{y : u \in \{\mathsf{Ult\_membs}(x) : x \in \{s\}\}, y \in u\} = \{y : x \in \{s\}, y \in \mathsf{Ult\_membs}(x)\}$
SIMPLF ⇒ $\{y : x \in \{s\}, y \in \mathsf{Ult\_membs}(x)\} = \{y : y \in \mathsf{Ult\_membs}(s)\}$
SIMPLF ⇒ $\{y : y \in \mathsf{Ult\_membs}(s)\} = \mathsf{Ult\_membs}(s)$
ELEM ⇒ false; Discharge ⇒ QED

-- We also note the following elementary consequence of Theorems T16 and 7, which tells us that the set of ultimate members of a singleton consisting of just one ordinal is its successor ordinal.

Theorem 26 (18) $\mathcal{O}(S) \rightarrow \mathsf{Ult\_membs}(\{S\}) = S \cup \{S\}$. PROOF:

Suppose_not(s) ⇒ $\mathcal{O}(s)\ \&\ \mathsf{Ult\_membs}(\{s\}) \neq s \cup \{s\}$
$\langle s \rangle \hookrightarrow T17$ ⇒ $\mathsf{Ult\_membs}(\{s\}) = \{s\} \cup \mathsf{Ult\_membs}(s)$
$\langle s \rangle \hookrightarrow T16$ ⇒ false; Discharge ⇒ QED

-- The 'union set' of a set s is the union of all its members, or, equivalently, the set of all members of members of s. For sets of ordinals, this is the least upper bound.

-- Union Set

DEF 25. $\bigcup X \quad =_{\text{Def}} \quad \{x : y \in X, x \in y\}$

-- Now we start to prepare more closely for the proof of a preliminary version of the principle of transfinite induction by making a few auxiliary definitions. First we introduce 'the set of all x which are either members of s or members of members of s:'

DEF 7g. $\mathsf{membs}_2(X) \quad =_{\text{Def}} \quad X \cup \bigcup X$

-- Next, using the axiom of infinity and the set s_inf which it provides, we extend this definition recursively.

DEF 7h. $\mathsf{membs\_x}(X, Y) \quad =_{\text{Def}} \quad \textbf{if } Y = \mathbf{arb}(\mathsf{s\_inf}) \textbf{ then } X \textbf{ else } \mathsf{membs}_2\left(\bigcup \{\mathsf{membs\_x}(X, y) : y \in Y\}\right) \textbf{ fi}$

-- This lets us define a set including s which we will show to be transitively closed under membership.

Def 7*i*. $\mathsf{Ult\_memb}_1(\mathsf{X}) \;=_{\mathrm{Def}}\; \bigcup\{\mathsf{membs\_x}(\mathsf{X},\mathsf{x}) : \mathsf{x} \in \mathsf{s\_inf}\}$

-- First we need the following simple lemma:

Theorem 27 (19) $\mathsf{X} \in \mathsf{s\_inf} \rightarrow \mathsf{membs\_x}(\mathsf{S},\{\mathsf{X}\}) = \mathsf{membs\_x}(\mathsf{S},\mathsf{X}) \cup \bigcup \mathsf{membs\_x}(\mathsf{S},\mathsf{X})$. Proof:

Suppose_not$(\mathsf{x},\mathsf{s}) \Rightarrow$ $\mathsf{x} \in \mathsf{s\_inf}\ \&\ \mathsf{membs\_x}(\mathsf{s},\{\mathsf{x}\}) \neq \mathsf{membs\_x}(\mathsf{s},\mathsf{x}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$

-- Since $\mathsf{x} \in \mathsf{s\_inf}$, $\{\mathsf{x}\} \neq \mathbf{arb}(\mathsf{s\_inf})$, and so $\mathsf{membs\_x}(\mathsf{s},\{\mathsf{x}\}) = \mathsf{membs}_2\left(\bigcup\{\mathsf{membs\_x}(\mathsf{s},\mathsf{y}) : \mathsf{y} \in \{\mathsf{x}\}\}\right)$ by definition

ELEM $\Rightarrow$ $\{\mathsf{x}\} \neq \mathbf{arb}(\mathsf{s\_inf})$

Use_def$(\mathsf{membs_x}) \Rightarrow$ $\mathsf{membs\_x}(\mathsf{s},\{\mathsf{x}\}) = \textbf{if } \{\mathsf{x}\} = \mathbf{arb}(\mathsf{s\_inf}) \textbf{ then } \mathsf{s} \textbf{ else } \mathsf{membs}_2\left(\bigcup\{\mathsf{membs\_x}(\mathsf{s},\mathsf{y}) : \mathsf{y} \in \{\mathsf{x}\}\}\right) \textbf{ fi}$

ELEM $\Rightarrow$ $\mathsf{membs}_2\left(\bigcup\{\mathsf{membs\_x}(\mathsf{s},\mathsf{y}) : \mathsf{y} \in \{\mathsf{x}\}\}\right) \neq \mathsf{membs\_x}(\mathsf{s},\mathsf{x}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$

-- This inequality simplifies to $\mathsf{membs}_2\big(\mathsf{membs\_x}(\mathsf{s},\mathsf{x})\big) \neq \mathsf{membs\_x}(\mathsf{s},\mathsf{x}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$, which contradicts the definition of $\mathsf{membs}_2\big(\mathsf{membs\_x}(\mathsf{s},\mathsf{x})\big)$, and so proves our lemma.

SIMPLF $\Rightarrow$ $\mathsf{membs}_2\big(\bigcup\{\mathsf{membs\_x}\}(\mathsf{s},\mathsf{x})\big) \neq \mathsf{membs\_x}(\mathsf{s},\mathsf{x}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$

Use_def$(\bigcup) \Rightarrow$ $\mathsf{membs}_2\left(\{\mathsf{u} : \mathsf{y} \in \{\mathsf{membs\_x}\}(\mathsf{s},\mathsf{x}), \mathsf{u} \in \mathsf{y}\}\right) \neq \mathsf{membs\_x}(\mathsf{s},\mathsf{x}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$

SIMPLF $\Rightarrow$ $\{\mathsf{u} : \mathsf{y} \in \{\mathsf{membs\_x}\}(\mathsf{s},\mathsf{x}), \mathsf{u} \in \mathsf{y}\} = \{\mathsf{u} : \mathsf{u} \in \mathsf{membs\_x}(\mathsf{s},\mathsf{x})\}$

SIMPLF $\Rightarrow$ $\{\mathsf{u} : \mathsf{u} \in \mathsf{membs\_x}(\mathsf{s},\mathsf{x})\} = \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$

EQUAL $\Rightarrow$ $\mathsf{membs}_2\big(\mathsf{membs\_x}(\mathsf{s},\mathsf{x})\big) \neq \mathsf{membs\_x}(\mathsf{s},\mathsf{x}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s},\mathsf{x})$

Use_def$(\mathsf{membs}_2) \Rightarrow$ false; Discharge $\Rightarrow$ Qed

-- Now we can prove that, for any set s, $\mathsf{Ult\_memb}_1(\mathsf{s})$ includes s and is membership-transitive.

Theorem 28 (20) $\mathsf{S} \subseteq \mathsf{Ult\_memb}_1(\mathsf{S})\ \&\ \big(\mathsf{X} \in \mathsf{Ult\_memb}_1(\mathsf{S})\ \&\ \mathsf{Y} \in \mathsf{X} \rightarrow \mathsf{Y} \in \mathsf{Ult\_memb}_1(\mathsf{S})\big)$. Proof:

-- We proceed by contradiction. Suppose that our theorem is false, and let s, x, and y be a counterexample.

Suppose_not$(\mathsf{s},\mathsf{x},\mathsf{y}) \Rightarrow$ $\mathsf{s} \not\subseteq \mathsf{Ult\_memb}_1(\mathsf{s}) \vee \big(\mathsf{x} \in \mathsf{Ult\_memb}_1(\mathsf{s})\ \&\ \mathsf{y} \in \mathsf{x}\ \&\ \mathsf{y} \notin \mathsf{Ult\_memb}_1(\mathsf{s})\big)$

-- The first of these cases is impossible, since an $\mathsf{xx} \in \mathsf{s}$ but not in $\mathsf{Ult\_memb}_1(\mathsf{s})$ could not be in any of the sets $\mathsf{membs\_x}(\mathsf{s}, \mathsf{v})$ where v belongs to $\mathsf{s\_inf}$, contradicting the fact that $\mathbf{arb}(\mathsf{s\_inf})$ and $\{\mathbf{arb}(\mathsf{s\_inf})\}$ both belong to $\mathsf{s\_inf}$, while $\mathsf{membs\_x}(\mathsf{s}, \{\mathbf{arb}(\mathsf{s\_inf})\}) \supseteq \mathsf{membs\_x}(\mathsf{s}, \mathbf{arb}(\mathsf{s\_inf})) = \mathsf{s}$. Hence we need only consider the second case.

Suppose $\Rightarrow$ $\quad$ *Stat1* : $\mathsf{s} \not\subseteq \mathsf{Ult\_memb}_1(\mathsf{s})$
$\langle \mathsf{xx} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad$ $\mathsf{xx} \in \mathsf{s} \ \& \ \mathsf{xx} \notin \mathsf{Ult\_memb}_1(\mathsf{s})$
Use_def $(\mathsf{Ult\_memb}_1)$ $\Rightarrow$ $\quad$ $\mathsf{xx} \notin \bigcup \{\mathsf{membs\_x}(\mathsf{s}, \mathsf{v}) : \mathsf{v} \in \mathsf{s\_inf}\}$
Use_def $(\bigcup)$ $\Rightarrow$ $\quad$ $\mathsf{xx} \notin \{\mathsf{y} : \mathsf{u} \in \{\mathsf{membs\_x}(\mathsf{s}, \mathsf{v}) : \mathsf{v} \in \mathsf{s\_inf}\}, \mathsf{y} \in \mathsf{u}\}$
SIMPLF $\Rightarrow$ $\quad$ *Stat2* : $\mathsf{xx} \notin \{\mathsf{y} : \mathsf{v} \in \mathsf{s\_inf}, \mathsf{y} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{v})\}$
T00 $\Rightarrow$ $\quad$ $\mathsf{s\_inf} \neq \emptyset \ \&$ *Stat3* : $\langle \forall \mathsf{v} \in \mathsf{s\_inf} \mid \{\mathsf{v}\} \in \mathsf{s\_inf} \rangle$
$\langle \mathsf{s\_inf} \rangle \hookrightarrow$ *T0* $\Rightarrow$ $\quad$ $\mathbf{arb}(\mathsf{s\_inf}) \in \mathsf{s\_inf}$
$\langle \mathbf{arb}(\mathsf{s\_inf}) \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad$ $\{\mathbf{arb}(\mathsf{s\_inf})\} \in \mathsf{s\_inf}$
$\langle \{\mathbf{arb}(\mathsf{s\_inf})\}, \mathsf{xx} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\quad$ $\mathsf{xx} \notin \mathsf{membs\_x}(\mathsf{s}, \{\mathbf{arb}(\mathsf{s\_inf})\})$
$\langle \mathbf{arb}(\mathsf{s\_inf}), \mathsf{s} \rangle \hookrightarrow$ *T19* $\Rightarrow$ $\quad$ $\mathsf{xx} \notin \mathsf{membs\_x}(\mathsf{s}, \mathbf{arb}(\mathsf{s\_inf}))$
Use_def $(\mathsf{membs\_x})$ $\Rightarrow$ $\quad$ $\mathsf{membs\_x}(\mathsf{s}, \mathbf{arb}(\mathsf{s\_inf})) = \mathsf{s}$
ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ *Stat4* : $\mathsf{x} \in \mathsf{Ult\_memb}_1(\mathsf{s}) \ \& \ \mathsf{y} \in \mathsf{x} \ \& \ \mathsf{y} \notin \mathsf{Ult\_memb}_1(\mathsf{s})$

-- But in this case there must exist some d in $\mathsf{s\_inf}$ such that x in $\mathsf{membs\_x}(\mathsf{s}, \mathsf{d})$, and then $\mathsf{membs\_x}(\mathsf{s}, \{\mathsf{d}\}) = \mathsf{membs\_x}(\mathsf{s}, \mathsf{d}) \cup \{\mathsf{w} : \mathsf{v} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{d}), \mathsf{w} \in \mathsf{v}\}$ must have y as a member. Since $\{\mathsf{d}\}$ is a member of $\mathsf{s\_inf}$, this contradicts the fact that $\mathsf{y} \notin \mathsf{Ult\_memb}_1(\mathsf{s})$, and so proves our theorem.

Use_def $(\mathsf{Ult\_memb}_1)$ $\Rightarrow$ $\quad$ $\mathsf{x} \in \bigcup \{\mathsf{membs\_x}(\mathsf{s}, \mathsf{v}) : \mathsf{v} \in \mathsf{s\_inf}\}$
Use_def $(\bigcup)$ $\Rightarrow$ $\quad$ $\mathsf{x} \in \{\mathsf{w} : \mathsf{u} \in \{\mathsf{membs\_x}(\mathsf{s}, \mathsf{v}) : \mathsf{v} \in \mathsf{s\_inf}\}, \mathsf{w} \in \mathsf{u}\}$
SIMPLF $\Rightarrow$ $\quad$ *Stat5* : $\mathsf{x} \in \{\mathsf{w} : \mathsf{v} \in \mathsf{s\_inf}, \mathsf{w} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{v})\}$
$\langle \mathsf{d}, \mathsf{w} \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\quad$ $\mathsf{d} \in \mathsf{s\_inf} \ \& \ \mathsf{x} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{d})$
$\langle \mathsf{d} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad$ $\{\mathsf{d}\} \in \mathsf{s\_inf}$
Use_def $(\mathsf{Ult\_memb}_1)$ $\Rightarrow$ $\quad$ $\mathsf{y} \notin \bigcup \{\mathsf{membs\_x}(\mathsf{s}, \mathsf{v}) : \mathsf{v} \in \mathsf{s\_inf}\}$
Use_def $(\bigcup)$ $\Rightarrow$ $\quad$ $\mathsf{y} \notin \{\mathsf{w} : \mathsf{u} \in \{\mathsf{membs\_x}(\mathsf{s}, \mathsf{v}) : \mathsf{v} \in \mathsf{s\_inf}\}, \mathsf{w} \in \mathsf{u}\}$
SIMPLF $\Rightarrow$ $\quad$ *Stat6* : $\mathsf{y} \notin \{\mathsf{w} : \mathsf{v} \in \mathsf{s\_inf}, \mathsf{w} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{v})\}$
$\langle \{\mathsf{d}\}, \mathsf{y} \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $\quad$ $\mathsf{y} \notin \mathsf{membs\_x}(\mathsf{s}, \{\mathsf{d}\})$
$\langle \mathsf{d}, \mathsf{s} \rangle \hookrightarrow$ *T19* $\Rightarrow$ $\quad$ $\mathsf{membs\_x}(\mathsf{s}, \{\mathsf{d}\}) = \mathsf{membs\_x}(\mathsf{s}, \mathsf{d}) \cup \bigcup \mathsf{membs\_x}(\mathsf{s}, \mathsf{d})$
ELEM $\Rightarrow$ $\quad$ $\mathsf{y} \notin \bigcup \mathsf{membs\_x}(\mathsf{s}, \mathsf{d})$
Use_def $(\bigcup)$ $\Rightarrow$ $\quad$ *Stat7* : $\mathsf{y} \notin \{\mathsf{u} : \mathsf{v} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{d}), \mathsf{u} \in \mathsf{v}\}$
$\langle \mathsf{x}, \mathsf{y} \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $\quad$ $\neg(\mathsf{x} \in \mathsf{membs\_x}(\mathsf{s}, \mathsf{d}) \ \& \ \mathsf{y} \in \mathsf{x})$
ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Next we state our preliminary form of the principle of transfinite induction which simply asserts that if there is any n satisfying a predicate P, there is a minimal m such $\mathsf{P}(\mathsf{m})$. Since an 'arbitrary predicate' is involved, we formulate this as a theory providing just two theorems.

THEORY transfinite_induction $(\mathsf{n}, \mathsf{P}(\mathsf{x}))$
$\mathsf{P}(\mathsf{n})$
END transfinite_induction

ENTER_THEORY transfinite_induction

DEF transfinite_induction $\cdot$ 0. $\quad \mathsf{mt}_\Theta \quad =_{\text{Def}} \quad \mathbf{arb}(\{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\})$

Theorem 29 (transfinite_induction $\cdot$ 1) $\quad \mathsf{P}(\mathsf{mt}_\Theta) \;\&\; \big(\mathsf{K} \in \mathsf{mt}_\Theta \rightarrow \neg \mathsf{P}(\mathsf{K})\big)$. PROOF:

Suppose_not(k) $\Rightarrow \quad \neg \mathsf{P}(\mathsf{mt}_\Theta) \vee \big(\mathsf{k} \in \mathsf{mt}_\Theta \;\&\; \mathsf{P}(\mathsf{k})\big)$

-- Proceed by contradiction, first noting that $\{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\}$ cannot be empty since n belongs to it.

Suppose $\Rightarrow \quad$ *Stat1* : $\{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\} = \emptyset$
$\big\langle \{\mathsf{n}\}, \mathsf{junk}, \mathsf{bunk} \big\rangle \hookrightarrow T20 \Rightarrow \quad \mathsf{n} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\})$
Assump $\Rightarrow \quad \mathsf{P}(\mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\} \neq \emptyset$

-- The axiom of choice now tells us that there is a minimal element mt_Θ of $\{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\}$ This necessarily satisfies $\mathsf{mt}_\Theta \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \;\&\; \mathsf{P}(\mathsf{m})$

$\big\langle \{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\} \big\rangle \hookrightarrow T0 \Rightarrow \quad \mathbf{arb}(\{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\}) \in \{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\} \;\&$
$\quad \mathbf{arb}(\{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\}) \cap \{\mathsf{m} :\ \mathsf{m} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{m})\} = \emptyset$
Use_def$(\mathsf{mt}_\Theta) \Rightarrow \quad$ *Stat2* : $\mathsf{mt}_\Theta \in \{\mathsf{u} :\ \mathsf{u} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{u})\} \;\&\; \mathsf{mt}_\Theta \cap \{\mathsf{u} :\ \mathsf{u} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{u})\} = \emptyset$
$\langle \mathsf{mt}_\Theta \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{mt}_\Theta \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \;\&\; \mathsf{P}(\mathsf{mt}_\Theta)$

-- The negative of our theorem now tells us that there is a $\mathsf{k} \in \mathsf{mt}_\Theta$ such that $\mathsf{P}(\mathsf{k})$; but such a k would clearly belong to $\{\mathsf{u} :\ \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{u})\}$, and so contradict the minimality of mt_Θ. This contradiction proves our theorem.

$\big\langle \{\mathsf{n}\}, \mathsf{mt}_\Theta, \mathsf{k} \big\rangle \hookrightarrow T20 \Rightarrow \quad \mathsf{k} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\})$
Suppose $\Rightarrow \quad$ *Stat3* : $\mathsf{k} \notin \{\mathsf{u} :\ \mathsf{u} \in \mathsf{Ult\_memb}_1(\{\mathsf{n}\}) \mid \mathsf{P}(\mathsf{u})\}$

$\langle k \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $k \in \{u : u \in \text{Ult\_memb}_1(\{n\}) \mid P(u)\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- Now we can write the preliminary form of the principle of transfinite induction as the following theory:

DISPLAY transfinite_induction

THEORY transfinite_induction(n, P)
$P(n)$
$\Rightarrow (mt_\Theta)$
$P(mt_\Theta)$ & $\langle \forall k \in mt_\Theta \mid \neg P(k) \rangle$
END transfinite_induction

-- We go on to sharpen the preceding results by proving that the minimal element whose existence is asserted in the preceding theory can actually be taken to be an element of Ult_membs$(\{n\})$. For this, a few preparatory results are needed. The first of these applies transfinite_induction to show that Ult_membs(S) itself is transitively closed under membership.

Theorem 30 (21) $Y \in \text{Ult\_membs}(S) \rightarrow \text{Ult\_membs}(Y) \subseteq \text{Ult\_membs}(S)$. PROOF:

Suppose_not$(yy, s) \Rightarrow$ $yy \in \text{Ult\_membs}(s)$ & $\text{Ult\_membs}(yy) \not\subseteq \text{Ult\_membs}(s)$

-- We proceed by contradiction. If our theorem is false, there exist an s and a yy $\in$ Ult_membs(s) which contradict it, which therefore have the property stated above. But, by transfinite_induction, this implies the existence of a minimal t with the property that there exists a $y \in$ Ult_membs(t) such that $\neg$Ult_membs$(y) \subseteq$ Ult_membs(t). Consider a y related in this way to t.

Suppose $\Rightarrow$ *Stat1* : $\neg\langle \exists y \mid y \in \text{Ult\_membs}(s)$ & $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(s)\rangle$
$\langle yy \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $\langle \exists y \mid y \in \text{Ult\_membs}(s)$ & $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(s)\rangle$
APPLY $\langle mt_\Theta : t \rangle$ transfinite_induction$(n \mapsto s, P(x) \mapsto \langle \exists y \mid y \in \text{Ult_membs}(x)$ & $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(x)\rangle) \Rightarrow$
Stat2 : $\langle \forall x \mid \langle \exists y \mid y \in \text{Ult\_membs}(t)$ & $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(t)\rangle$ & $(x \in t \longrightarrow \neg\langle \exists y \mid y \in \text{Ult\_membs}(x)$ & $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(x)\rangle)\rangle$
$\langle a_0 \rangle \hookrightarrow Stat2 \Rightarrow$ *Stat3* : $\langle \exists y \mid y \in \text{Ult\_membs}(t)$ & $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(t)\rangle$
$\langle y \rangle \hookrightarrow Stat3 \Rightarrow$ $y \in \text{Ult\_membs}(t)$ & *Stat4* : $\text{Ult\_membs}(y) \not\subseteq \text{Ult\_membs}(t)$

-- Plainly, there must exist a $c \in$ Ult_membs(y) which is not in Ult_membs(t). Since Ult_membs$(t) \supseteq \{v : u \in t, v \in \text{Ult_membs}(u)\}$ by Theorem 14, it follows that c is not in this latter set either.

$\langle t \rangle \hookrightarrow T14 \Rightarrow$ $\mathsf{Ult\_membs}(t) = t \cup \{v : u \in t, v \in \mathsf{Ult\_membs}(u)\}$
$\langle c \rangle \hookrightarrow Stat4 \Rightarrow$ $c \in \mathsf{Ult\_membs}(y)\ \&\ c \notin \mathsf{Ult\_membs}(t)\ \&\ Stat5 : c \notin \{v : u \in t, v \in \mathsf{Ult\_membs}(u)\}$

-- y cannot be in t, since this would contradict Stat6 6; hence y must be in $\{v : u \in t, v \in \mathsf{Ult\_membs}(u)\}$, and so there must exist a $u \in t$ such that $y \in \mathsf{Ult\_membs}(u)$

Suppose $\Rightarrow$ $y \in t$
$\langle y, c \rangle \hookrightarrow Stat5 \Rightarrow$ $c \notin \mathsf{Ult\_membs}(y)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat7 : y \in \{v : u \in t, v \in \mathsf{Ult\_membs}(u)\}$
$\langle u, v \rangle \hookrightarrow Stat7 \Rightarrow$ $u \in t\ \&\ y \in \mathsf{Ult\_membs}(u)$

-- But now, by the minimality of t, Ult_membs(y) must be included in Ult_membs(u), and therefore Ult_membs(u) cannot be included in Ult_membs(t), from which it follows that there must exist a $d \in \mathsf{Ult\_membs}(u)$ which is not in $\{v : w \in t, v \in \mathsf{Ult\_membs}(w)\}$; hence, since $u \in t$, d cannot be in Ult_membs(u), contradicting its definition. This contradiction proves our theorem.

$\langle u \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat8 : \neg\langle \exists y \mid y \in \mathsf{Ult\_membs}(u)\ \&\ \mathsf{Ult\_membs}(y) \not\subseteq \mathsf{Ult\_membs}(u)\rangle$
$\langle y \rangle \hookrightarrow Stat8 \Rightarrow$ $\mathsf{Ult\_membs}(y) \subseteq \mathsf{Ult\_membs}(u)$
ELEM $\Rightarrow$ $Stat9 : \mathsf{Ult\_membs}(u) \not\subseteq \mathsf{Ult\_membs}(t)$
$\langle d \rangle \hookrightarrow Stat9 \Rightarrow$ $d \in \mathsf{Ult\_membs}(u)\ \&\ Stat10 : d \notin \{v : w \in t, v \in \mathsf{Ult\_membs}(w)\}$
$\langle u, d \rangle \hookrightarrow Stat10 \Rightarrow$ $\neg(u \in t\ \&\ d \in \mathsf{Ult\_membs}(u))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Since we know by Theorem 14 that $s \subseteq \mathsf{Ult\_membs}(s)$, it follows immediately from the preceding theorem that Ult_membs(s) includes every one of its members.

Theorem 31 (22) $Y \in \mathsf{Ult\_membs}(S) \rightarrow Y \subseteq \mathsf{Ult\_membs}(S)$. PROOF:

Suppose_not(y, s) $\Rightarrow$ $y \in \mathsf{Ult\_membs}(s)\ \&\ y \not\subseteq \mathsf{Ult\_membs}(s)$
$\langle y, s \rangle \hookrightarrow T21 \Rightarrow$ $\mathsf{Ult\_membs}(y) \subseteq \mathsf{Ult\_membs}(s)$
$\langle y \rangle \hookrightarrow T14 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The preceding results easily yield the fact, captured in the following theory, that if there exists an n with a given property P then either n (if it is minimal) or some other minimal element of the set of ultimate members of n has the property P.

THEORY transfinite_member_induction(n, P(x))
P(n)
END transfinite_member_induction

ENTER_THEORY transfinite_member_induction

-- To derive the result stated just below, we first define the minimal element that we want, whose properties are then established in the theorem following the definition.

DEF $00h$. $\quad \mathsf{mt}_\Theta \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{\mathsf{k} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{k})\})$

Theorem 32 (transfinite_member_induction$_1$) $\mathsf{P}(\mathsf{mt}_\Theta) \,\&\, \mathsf{mt}_\Theta \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,\&\, \big(\mathsf{K} \in \mathsf{mt}_\Theta \rightarrow \neg\mathsf{P}(\mathsf{K})\big)$. PROOF:

Suppose_not$(\mathsf{n},\mathsf{k}) \Rightarrow \quad \neg\mathsf{P}(\mathsf{mt}_\Theta) \vee \mathsf{mt}_\Theta \notin \mathsf{Ult_membs}(\{\mathsf{n}\}) \vee \big(\mathsf{k} \in \mathsf{mt}_\Theta \,\&\, \mathsf{P}(\mathsf{k})\big)$

-- We proceed by contradiction. If our theorem is false, there exists a $\mathsf{k} \in \mathsf{m}$ such that $\mathsf{P}(\mathsf{k})$. But, since n is clearly a member of $\{\mathsf{j} : \mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\}$, this set cannot be null, so by the axiom of choice mt_Θ must belong to it, but not intersect it.

Suppose $\Rightarrow \quad$ *Stat1* : $\{\mathsf{j} : \mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\} = \emptyset$
$\langle \mathsf{n} \rangle \hookrightarrow T17 \Rightarrow \quad$ *Stat2* : $\mathsf{Ult\_membs}(\{\mathsf{n}\}) = \{\mathsf{n}\} \cup \mathsf{Ult\_membs}(\mathsf{n})$
$\langle Stat2 \rangle$ ELEM $\Rightarrow \quad \mathsf{n} \in \mathsf{Ult\_membs}(\{\mathsf{n}\})$
Assump $\Rightarrow \quad \mathsf{P}(\mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \{\mathsf{j} : \mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\} \neq \emptyset$
Use_def$(\mathsf{mt}_\Theta) \Rightarrow \quad \mathsf{mt}_\Theta = \mathbf{arb}(\{\mathsf{k} \in \mathsf{Ult_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{k})\})$
$\big\langle \{\mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\} \big\rangle \hookrightarrow T0 \Rightarrow \quad$ *Stat3* :
$\mathsf{mt}_\Theta \in \{\mathsf{j} : \mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\} \,\&\, \mathsf{mt}_\Theta \cap \{\mathsf{j} : \mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\} = \emptyset$
$\langle \mathsf{mt}_\Theta \rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{mt}_\Theta \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,\&\, \mathsf{P}(\mathsf{mt}_\Theta)$

-- Thus $\mathsf{P}(\mathsf{mt}_\Theta)$ and $\mathsf{mt}_\Theta \in \mathsf{Ult\_membs}(\{\mathsf{n}\})$ are both true, so that $\mathsf{k} \in \mathsf{mt}_\Theta$ and $\mathsf{P}(\mathsf{k})$ must both be true. Since $\mathsf{Ult\_membs}(\mathsf{mt}_\Theta) \supseteq \mathsf{mt}_\Theta$ by definition, k must belong to $\mathsf{Ult\_membs}(\mathsf{mt}_\Theta)$, and hence to $\mathsf{Ult\_membs}(\{\mathsf{n}\})$ by Theorem 22, which tells us that $\mathsf{Ult\_membs}(\mathsf{mt}_\Theta)$ is a subset of $\mathsf{Ult\_membs}(\{\mathsf{n}\})$

ELEM $\Rightarrow \quad \mathsf{k} \in \mathsf{mt}_\Theta \,\&\, \mathsf{P}(\mathsf{k})$
Suppose $\Rightarrow \quad \mathsf{k} \notin \mathsf{Ult\_membs}(\mathsf{mt}_\Theta)$
Use_def(Ult_membs) $\Rightarrow \quad \mathsf{Ult\_membs}(\mathsf{mt}_\Theta) = \mathsf{mt}_\Theta \cup \{\mathsf{w} : \mathsf{u} \in \{\mathsf{Ult\_membs}(\mathsf{v}) : \mathsf{v} \in \mathsf{mt}_\Theta\}, \mathsf{w} \in \mathsf{u}\}$
ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \mathsf{k} \in \mathsf{Ult\_membs}(\mathsf{mt}_\Theta)$
$\langle \mathsf{mt}_\Theta, \{\mathsf{n}\} \rangle \hookrightarrow T22 \Rightarrow \quad \mathsf{k} \in \mathsf{Ult\_membs}(\{\mathsf{n}\})$
Suppose $\Rightarrow \quad$ *Stat4* : $\mathsf{k} \notin \{\mathsf{j} : \mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\}$

-- Hence k belongs to $\{\mathsf{j} \in \mathsf{Ult\_membs}(\{\mathsf{n}\}) \,|\, \mathsf{P}(\mathsf{j})\}$, a contradiction which proves our theorem.

$\langle k \rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ $k \in \{j \in \text{Ult\_membs}(\{n\}) \mid P(j)\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- Formulated as a theory, the preceding result appears as follows.

DISPLAY transfinite_member_induction

THEORY transfinite_member_induction(n, P)
$P(n)$
$\Rightarrow (m)$
$P(m)$ & $m \in \text{Ult\_membs}(\{n\})$ & $\langle \forall k \in m \mid \neg P(k) \rangle$
END transfinite_member_induction

THEORY orderedGroups$\big(\text{In_domain}(x), x \oplus y, e, \text{rvz}(x), \text{nneg}(x), \text{leq}(x, y)\big)$
In_domain(e)

-- closure axiom

$\langle \forall x, y \mid \text{In\_domain}(x)$ & $\text{In\_domain}(y) \longrightarrow \text{In\_domain}(x \oplus y) \rangle$

-- closure axiom

$\langle \forall x \mid \text{In\_domain}(x) \longrightarrow \text{In\_domain}(\text{rvz}(x)) \rangle$

-- closure axiom

$\langle \forall x, y, z \mid \text{In\_domain}(x)$ & $\text{In\_domain}(y)$ & $\text{In\_domain}(z) \longrightarrow (x \oplus y) \oplus z = x \oplus (y \oplus z) \rangle$

-- associativity

$\langle \forall x \mid \text{In\_domain}(x) \longrightarrow x \oplus e = x \rangle$

-- right unit

$\langle \forall x \mid \text{In\_domain}(x) \longrightarrow x \oplus \text{rvz}(x) = e \rangle$

-- right inverse

$\langle \forall x, y \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \longrightarrow x \oplus y = y \oplus x \rangle$

-- commutativity

$\langle \forall x, y \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \longrightarrow \mathsf{nneg}(x) \;\&\; \mathsf{nneg}(y) \longrightarrow \mathsf{nneg}(x \oplus y) \rangle$
$\langle \forall x \mid \mathsf{In\_domain}(x) \longrightarrow \mathsf{nneg}(x) \vee \mathsf{nneg}(\mathsf{rvz}(x)) \rangle$
$\langle \forall x \mid \mathsf{In\_domain}(x) \longrightarrow \mathsf{nneg}(x) \;\&\; \mathsf{nneg}(\mathsf{rvz}(x)) \longrightarrow x = e \rangle$
$\langle \forall x, y \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \longrightarrow \mathsf{leq}(x, y) = \mathsf{nneg}(y \oplus \mathsf{rvz}(x)) \rangle$

extdfn $\Rrightarrow$ $\mathsf{abs}_\Theta(X) =_{\mathrm{Def}}$ **if** $\mathsf{nneg}(X)$ **then** X **else** $\mathsf{rvz}(X)$ **fi**

$\Rightarrow$

$\langle \forall x, y, z \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \;\&\; \mathsf{In\_domain}(z) \longrightarrow x \oplus y = x \oplus z \longrightarrow y = z \rangle$

-- cancellation law

$\langle \forall x, y \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \longrightarrow \mathsf{rvz}(x \oplus \mathsf{rvz}(y)) = y \oplus \mathsf{rvz}(x) \rangle$
$\langle \forall x, y \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \longrightarrow \mathsf{leq}(x, y) \vee \mathsf{leq}(y, x) \rangle$

-- totality

$\langle \forall x \mid \mathsf{In\_domain}(x) \longrightarrow \mathsf{leq}(x, x) \rangle$

-- reflexivity

$\langle \forall x, y, z \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \;\&\; \mathsf{In\_domain}(z) \longrightarrow \mathsf{leq}(x, y) \;\&\; \mathsf{leq}(y, z) \longrightarrow \mathsf{leq}(x, z) \rangle$

-- transitivity

$\langle \forall x, y, z \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \;\&\; \mathsf{In\_domain}(z) \longrightarrow \mathsf{leq}(x, y) \;\&\; x \neq y \;\&\; \mathsf{leq}(y, z) \longrightarrow x \neq z \rangle$

-- transitivity

$\langle \forall x, y, z \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \;\&\; \mathsf{In\_domain}(z) \longrightarrow \mathsf{leq}(x, y) \;\&\; \mathsf{leq}(y, z) \;\&\; y \neq z \longrightarrow x \neq z \rangle$

-- transitivity

$\langle \forall x, y, z \mid \mathsf{In\_domain}(x) \;\&\; \mathsf{In\_domain}(y) \;\&\; \mathsf{In\_domain}(z) \longrightarrow \mathsf{leq}(x, y) \longrightarrow \mathsf{leq}(x \oplus z, y \oplus z) \rangle$

-- isotony

$$\left\langle \forall x, y, z \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \;\&\; \mathrm{In\_domain}(z) \longrightarrow x \oplus z = y \oplus z \longrightarrow x = y \right\rangle$$

-- cancellation law

$$\left\langle \forall x, y, z \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \;\&\; \mathrm{In\_domain}(z) \longrightarrow \mathrm{leq}(x, y) \;\&\; x \neq y \longrightarrow x \oplus z \neq y \oplus z \right\rangle$$

-- strictness of isotony

$$\left\langle \forall x \mid \mathrm{In\_domain}(x) \longrightarrow \mathrm{abs}_{\ominus}\big(x \oplus \mathrm{rvz}(x)\big) = e \right\rangle$$
$$\left\langle \forall x \mid \mathrm{In\_domain}(x) \longrightarrow \mathrm{leq}\big(x, \mathrm{abs}_{\ominus}(x)\big) \right\rangle$$
$$\left\langle \forall x \mid \mathrm{In\_domain}(x) \longrightarrow \mathrm{abs}_{\ominus}\big(\mathrm{abs}_{\ominus}(x)\big) = \mathrm{abs}_{\ominus}(x) \right\rangle$$
$$\left\langle \forall x \mid \mathrm{In\_domain}(x) \longrightarrow \big(\mathrm{abs}_{\ominus}(x) = e \leftrightarrow x = e\big) \right\rangle$$
$$\left\langle \forall x \mid \mathrm{In\_domain}(x) \longrightarrow \mathrm{abs}_{\ominus}\big(\mathrm{rvz}(x)\big) = \mathrm{abs}_{\ominus}(x) \right\rangle$$
$$\left\langle \forall x, y \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \longrightarrow \mathrm{leq}\big(x \oplus y, \mathrm{abs}_{\ominus}(x) \oplus \mathrm{abs}_{\ominus}(y)\big) \right\rangle$$
$$\left\langle \forall x, y \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \longrightarrow \mathrm{leq}\big(\mathrm{abs}_{\ominus}(x \oplus y), \mathrm{abs}_{\ominus}(x) \oplus \mathrm{abs}_{\ominus}(y)\big) \right\rangle$$
$$\left\langle \forall x, y \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \longrightarrow \neg\mathrm{nneg}(x) \longrightarrow \mathrm{leq}\big(x, \mathrm{abs}_{\ominus}(y)\big) \;\&\; x \neq \mathrm{abs}_{\ominus}(y) \right\rangle$$
$$\left\langle \forall x, y, z \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \;\&\; \mathrm{In\_domain}(z) \longrightarrow \mathrm{leq}\big(\#x \oplus \mathrm{rvz}(y)_{\mathbb{R}}, z\big) \longrightarrow \mathrm{leq}(y, x \oplus z) \right\rangle$$

--

$$\left\langle \forall x, y, z \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \;\&\; \mathrm{In\_domain}(z) \longrightarrow x \oplus z = x \oplus \mathrm{rvz}(y) \oplus (y \oplus z) \right\rangle$$

[10a]

$$\left\langle \forall x, y, z \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \;\&\; \mathrm{In\_domain}(z) \longrightarrow \mathrm{leq}\Big(\mathrm{abs}_{\ominus}\big(x \oplus \mathrm{rvz}(z)\big), \mathrm{abs}_{\ominus}\big(x \oplus \mathrm{rvz}(y)\big) \oplus \mathrm{abs}_{\ominus}\big(y \oplus \mathrm{rvz}(z)\big)\Big) \right\rangle$$

-- (proved sans axioms)

$$\left\langle \forall x, y \mid \mathrm{In\_domain}(x) \;\&\; \mathrm{In\_domain}(y) \longrightarrow \mathrm{nneg}(y) \longrightarrow \mathrm{leq}\big(x \oplus \mathrm{rvz}(y), x \oplus y\big) \right\rangle$$

--

$\langle \forall x, y \,|\, \mathsf{In\_domain}(x) \,\&\, \mathsf{In\_domain}(y) \rightarrow \mathsf{nneg}(x) \,\&\, \neg\mathsf{nneg}(y) \rightarrow \mathsf{leq}\Big(\mathsf{abs}_\ominus\Big(\mathsf{abs}_\ominus(x) \oplus \mathsf{rvz}(\mathsf{abs}_\ominus(y))\Big), \mathsf{abs}_\ominus(x \oplus \mathsf{rvz}(y))\Big)\rangle$

[12a]

$\langle \forall x, y \,|\, \mathsf{In\_domain}(x) \,\&\, \mathsf{In\_domain}(y) \rightarrow \mathsf{nneg}(x) \,\&\, \mathsf{nneg}(y) \rightarrow \mathsf{abs}_\ominus\Big(\mathsf{abs}_\ominus(x) \oplus \mathsf{rvz}(\mathsf{abs}_\ominus(y))\Big) = \mathsf{abs}_\ominus(x \oplus \mathsf{rvz}(y))\rangle$

[12b]

$\langle \forall x, y \,|\, \mathsf{In\_domain}(x) \,\&\, \mathsf{In\_domain}(y) \rightarrow \neg\mathsf{nneg}(x) \,\&\, \neg\mathsf{nneg}(y) \rightarrow \mathsf{abs}_\ominus\Big(\mathsf{abs}_\ominus(\mathsf{rvz}(x)) \oplus \mathsf{rvz}\Big(\mathsf{abs}_\ominus(\mathsf{rvz}(y))\Big)\Big) = \mathsf{abs}_\ominus(\mathsf{rvz}(x) \oplus y)\rangle$

[12c]

$\langle \forall x, y \mid \mathsf{In\_domain}(x) \,\&\, \mathsf{In\_domain}(y) \longrightarrow \mathsf{leq}\Big(\mathsf{abs}_\ominus\Big(\mathsf{abs}_\ominus(x) \oplus \mathsf{rvz}(\mathsf{abs}_\ominus(y))\Big), \mathsf{abs}_\ominus(x \oplus \mathsf{rvz}(y))\Big)\rangle$

$\langle \forall x, y \mid \mathsf{In\_domain}(x) \,\&\, \mathsf{In\_domain}(y) \longrightarrow \mathsf{leq}\Big(\mathsf{abs}_\ominus(x) \oplus \mathsf{rvz}\Big(\mathsf{abs}_\ominus\Big(\mathsf{abs}_\ominus(y) \oplus \mathsf{rvz}(\mathsf{abs}_\ominus(x))\Big)\Big), \mathsf{abs}_\ominus(y)\Big)\rangle$

END orderedGroups

3 Additional basic operations of set theory; properties of setformers

-- Next we define various familiar notions of set theory: (possibly multivalued) maps, their ranges and domains, and the subclasses of single valued and 1-1 maps. After giving these definitions we build up a few small utility theories which ease subsequent work with these predicates.

DEF 4. $\mathsf{Is\_map}(X) \quad \leftrightarrow_{\mathrm{Def}} \quad X = \{[x^{[1]}, x^{[2]}] : x \in X\}$

DEF 5. $\mathbf{domain}(X) \quad =_{\mathrm{Def}} \quad \{x^{[1]} : x \in X\}$

DEF 6. $\mathbf{range}(X) \quad =_{\mathrm{Def}} \quad \{x^{[2]} : x \in X\}$

DEF 7. $\mathsf{Svm}(X) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{Is\_map}(X) \,\&\, \langle \forall x \in X, y \in X \mid x^{[1]} = y^{[1]} \rightarrow x = y\rangle$

DEF 8. $\mathsf{1{-}1}(X) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{Svm}(X) \,\&\, \langle \forall x \in X, y \in X \mid x^{[2]} = y^{[2]} \rightarrow x = y\rangle$

-- Our next elementary result states that a set f is a map if and only if every element in it is an ordered pair.

Theorem 33 (23) $\mathsf{Is\_map}(\mathsf{F}) \leftrightarrow \mathsf{F} \subseteq \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{F}\}$. PROOF:

Suppose_not(f) $\Rightarrow$ $\neg\big(\mathsf{Is\_map}(\mathsf{f}) \leftrightarrow \mathsf{f} \subseteq \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\}\big)$

-- For if we suppose the contrary and use the definition of Is_map, we see that some element $\mathsf{c} \notin \mathsf{f}$ must have the form $\mathsf{c} = \left[\mathsf{d}^{[1]}, \mathsf{d}^{[2]}\right]$, where $\mathsf{d} \in \mathsf{f}$ and so in turn must have the form $\mathsf{d} = [\mathsf{e}^{[1]}, \mathsf{e}^{[2]}]$ with $\mathsf{e} \in \mathsf{f}$, implying $\mathsf{d} = \mathsf{e}$ and so leading to an immediate contradiction.

Use_def(Is_map) $\Rightarrow$ $\mathit{Stat1}: \mathsf{f} \subseteq \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\}$ & $\mathit{Stat2}: \mathsf{f} \neq \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\}$

$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat2}([\mathit{Stat1}, \cap]) \Rightarrow$ $\mathit{Stat4}: \mathsf{c} \notin \mathsf{f}$ & $\mathit{Stat5}: \mathsf{c} \in \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\}$

$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat5}([\mathit{Stat1}, \cap]) \Rightarrow$ $\mathit{Stat6}: \mathsf{d} \in \mathsf{f}$ & $\mathsf{c} = \left[\mathsf{d}^{[1]}, \mathsf{d}^{[2]}\right]$ & $\mathit{Stat7}: \mathsf{d} \in \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\}$

$\langle \mathsf{e} \rangle \hookrightarrow \mathit{Stat7}([\mathit{Stat6}, \cap]) \Rightarrow$ $\mathsf{e} \in \mathsf{f}$ & $\mathsf{d} = [\mathsf{e}^{[1]}, \mathsf{e}^{[2]}]$

$\langle \mathit{Stat4} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The next small theory simply tells us that any setformer of the form $\{[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\}$ is a map. The proof of the one theorem it provides is an elementary consequence of the definition of 'Is_map'

THEORY Iz_map$(\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x}), \mathsf{s})$
END Iz_map

ENTER_THEORY Iz_map

Theorem 34 (iz_map · 1) $\mathsf{Is\_map}(\{[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\})$. PROOF:

Suppose_not(s) $\Rightarrow$ $\neg\mathsf{Is\_map}(\{[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\})$

Use_def(Is_map) $\Rightarrow$ $\{[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\} \neq \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \{[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\}\}$

SIMPLF $\Rightarrow$ $\mathit{Stat1}: \{[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\} \neq \left\{\left[[\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})]^{[1]}, [\mathsf{a}(\mathsf{x}), \mathsf{b}(\mathsf{x})]^{[2]}\right] : \mathsf{x} \in \mathsf{s}\right\}$

$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathsf{c} \in \mathsf{s}$ & $[\mathsf{a}(\mathsf{c}), \mathsf{b}(\mathsf{c})] \neq \left[[\mathsf{a}(\mathsf{c}), \mathsf{b}(\mathsf{c})]^{[1]}, [\mathsf{a}(\mathsf{c}), \mathsf{b}(\mathsf{c})]^{[2]}\right]$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY Iz_map

THEORY Iz_map$(\mathsf{a}, \mathsf{b}, \mathsf{s})$

$\Rightarrow$

$\quad \mathsf{Is\_map}\big(\{[\mathsf{a}(\mathsf{x}),\mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\}\big)$

END Iz_map

-- The two small theories which follow extend 'Iz_map' to the 2-and 3-variable cases respectively.

THEORY $\mathsf{Iz\_map}_2\big(\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y}),\mathsf{s},\mathsf{t},\mathsf{P}(\mathsf{x},\mathsf{y})\big)$

END $\mathsf{Iz\_map}_2$

ENTER_THEORY $\mathsf{Iz\_map}_2$

-- The proof of the one theorem of this theory is very close to, and just as elementary as, the corresponding one-variable result.

Theorem 35 ($\mathsf{iz\_map}_2 \cdot 1$) $\mathsf{Is\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\}\big)$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\neg\mathsf{Is\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\}\big)$

Use_def(Is_map) $\Rightarrow$ $\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\} \neq$
$\quad \big\{\big[\mathsf{x}^{[1]},\mathsf{x}^{[2]}\big] : \mathsf{x} \in \{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\}\big\}$

SIMPLF $\Rightarrow$ *Stat1* :
$\quad \{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\} \neq$
$\quad\quad \Big\{\Big[[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})]^{[1]},[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})]^{[2]}\Big] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\Big\}$

$\langle \mathsf{c},\mathsf{d}\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $[\mathsf{a}(\mathsf{c},\mathsf{d}),\mathsf{b}(\mathsf{c},\mathsf{d})] \neq \Big[[\mathsf{a}(\mathsf{c},\mathsf{d}),\mathsf{b}(\mathsf{c},\mathsf{d})]^{[1]},[\mathsf{a}(\mathsf{c},\mathsf{d}),\mathsf{b}(\mathsf{c},\mathsf{d})]^{[2]}\Big]$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY $\mathsf{Iz\_map}_2$

THEORY $\mathsf{Iz\_map}_2(\mathsf{a},\mathsf{b},\mathsf{s},\mathsf{t},\mathsf{P})$

$\Rightarrow$

$\quad \mathsf{Is\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t} \,|\, \mathsf{P}(\mathsf{x},\mathsf{y})\}\big)$

END $\mathsf{Iz\_map}_2$

THEORY $\mathsf{Iz\_map}_3\big(\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{z}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{z}),\mathsf{s},\mathsf{t},\mathsf{u},\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})\big)$

END $\mathsf{Iz\_map}_3$

ENTER_THEORY $\mathsf{lz\_map}_3$

-- The proof of the one theorem of this theory is very close to, and just as elementary as, the corresponding one-variable result.

Theorem 36 ($\mathsf{iz\_map}_3 \cdot 1$) $\mathsf{ls\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\}\big)$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s},\mathsf{t},\mathsf{u}) \Rightarrow \quad \neg\mathsf{ls\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\}\big)$

$\mathsf{Use\_def}(\mathsf{ls\_map}) \Rightarrow$

$$\{[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\} \neq \Big\{\big[\mathsf{x}^{[1]},\mathsf{x}^{[2]}\big] : \mathsf{x} \in \{[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\}\Big\}$$

$\mathsf{SIMPLF} \Rightarrow \quad$ *Stat1* :

$$\{[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\} \neq \Big\{\Big[[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})]^{[1]},[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})]^{[2]}\Big] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\Big\}$$

$\big\langle \mathsf{c},\mathsf{d},\mathsf{dd}\big\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad [\mathsf{a}(\mathsf{c},\mathsf{d},\mathsf{dd}),\mathsf{b}(\mathsf{c},\mathsf{d},\mathsf{dd})] \neq$

$$\Big[[\mathsf{a}(\mathsf{c},\mathsf{d},\mathsf{dd}),\mathsf{b}(\mathsf{c},\mathsf{d},\mathsf{dd})]^{[1]},[\mathsf{a}(\mathsf{c},\mathsf{d},\mathsf{dd}),\mathsf{b}(\mathsf{c},\mathsf{d},\mathsf{dd})]^{[2]}\Big]$$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

ENTER_THEORY Set_theory

DISPLAY $\mathsf{lz\_map}_3$

THEORY $\mathsf{lz\_map}_3(\mathsf{a},\mathsf{b},\mathsf{s},\mathsf{t},\mathsf{u},\mathsf{P})$

$\Rightarrow$

$\mathsf{ls\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y},\mathsf{yy}),\mathsf{b}(\mathsf{x},\mathsf{y},\mathsf{yy})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{yy} \in \mathsf{u} \mid \mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{yy})\}\big)$

END $\mathsf{lz\_map}_3$

-- Our next utility theory tells us that a setformer of the form $\{[\mathsf{a}(\mathsf{x}),\mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\}$ is a single valued map unless there are x and y in s such that $\mathsf{a}(\mathsf{x}) = \mathsf{a}(\mathsf{y})$ does not imply $\mathsf{b}(\mathsf{x}) = \mathsf{b}(\mathsf{y})$, and that $\{[\mathsf{x},\mathsf{b}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\}$ is always a single valued map.

THEORY $\mathsf{Svm\_test}\big(\mathsf{a}(\mathsf{x}),\mathsf{b}(\mathsf{x}),\mathsf{s}\big)$

END Svm_test

ENTER_THEORY Svm_test

DEF $\mathsf{Svm\_test} \cdot 0a$. $\quad \mathsf{xy}_\ominus =_{\mathrm{Def}} \mathbf{arb}(\{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{s} \mid \mathsf{a}(\mathsf{x}) = \mathsf{a}(\mathsf{y}) \,\&\, \mathsf{b}(\mathsf{x}) \neq \mathsf{b}(\mathsf{y})\})$

DEF $\mathsf{Svm\_test} \cdot 0b$. $\quad \mathsf{x}_\ominus =_{\mathrm{Def}} {\mathsf{xy}_\ominus}^{[1]}$

DEF $\mathsf{Svm\_test} \cdot 0c$. $\quad \mathsf{y}_\ominus =_{\mathrm{Def}} {\mathsf{xy}_\ominus}^{[2]}$

Theorem 37 (Svm_test · 1) $(x_\Theta, y_\Theta \in s \,\&\, a(x_\Theta) = a(y_\Theta) \,\&\, b(x_\Theta) \neq b(y_\Theta)) \vee \mathrm{Svm}(\{[a(x), b(x)] : x \in s\})$. PROOF:

Suppose_not(s) $\Rightarrow$ $\neg(x_\Theta, y_\Theta \in s \,\&\, a(x_\Theta) = a(y_\Theta) \,\&\, b(x_\Theta) \neq b(y_\Theta)) \,\&\, \neg\mathrm{Svm}(\{[a(x), b(x)] : x \in s\})$

-- By definition, the contrary of our assertion can only be true if $\{[a(x), b(x)] : x \in s\}$ is either not a map or fails the single-valuedness test. But the preceding theory Iz_map tells us that the first case is impossible, and an elementary simplification shows that the second case is impossible also.

Use_def(Svm) $\Rightarrow$ $\neg\mathrm{Is\_map}(\{[a(x), b(x)] : x \in s\}) \vee$
$\neg\langle\forall u \in \{[a(x), b(x)] : x \in s\}, v \in \{[a(x), b(x)] : x \in s\} \mid u^{[1]} = v^{[1]} \rightarrow u = v\rangle$

APPLY $\langle\,\rangle$ Iz_map$(a(x) \mapsto a(x), b(x) \mapsto b(x), s \mapsto s)$ $\Rightarrow$ $\mathrm{Is\_map}(\{[a(x), b(x)] : x \in s\})$

SIMPLF $\Rightarrow$ *Stat1*: $\neg\langle\forall x \in s, y \in s \mid [a(x), b(x)]^{[1]} = [a(y), b(y)]^{[1]} \rightarrow [a(x), b(x)] = [a(y), b(y)]\rangle$

$\langle x, y\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x, y \in s \,\&\, [a(x), b(x)]^{[1]} = [a(y), b(y)]^{[1]} \,\&\, [a(x), b(x)] \neq [a(y), b(y)]$

Suppose $\Rightarrow$ *Stat2*: $\{[x, y] : x \in s, y \in s \mid a(x) = a(y) \,\&\, b(x) \neq b(y)\} = \emptyset$

$\langle x, y\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\neg(x, y \in s \,\&\, a(x) = a(y) \,\&\, b(x) \neq b(y))$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\{[x, y] : x \in s, y \in s \mid a(x) = a(y) \,\&\, b(x) \neq b(y)\} \neq \emptyset$

$\langle\{[x, y] : x \in s, y \in s \mid a(x) = a(y) \,\&\, b(x) \neq b(y)\}\rangle \hookrightarrow$ *T0* $\Rightarrow$ $\mathbf{arb}(\{[x, y] : x \in s, y \in s \mid a(x) = a(y) \,\&\, b(x) \neq b(y)\}) \in$
$\{[x, y] : x \in s, y \in s \mid a(x) = a(y) \,\&\, b(x) \neq b(y)\}$

Use_def(xy_Θ) $\Rightarrow$ *Stat3*: $xy_\Theta \in \{[x, y] : x \in s, y \in s \mid a(x) = a(y) \,\&\, b(x) \neq b(y)\}$

$\langle xx, yy\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat4*: $xx, yy \in s \,\&\, xy_\Theta = [xx, yy] \,\&\, a(xx) = a(yy) \,\&\, b(xx) \neq b(yy)$

$\langle$*Stat4*$\rangle$ ELEM $\Rightarrow$ $xx = xy_\Theta^{[1]} \,\&\, yy = xy_\Theta^{[2]}$

Use_def(x_Θ) $\Rightarrow$ $x_\Theta = xy_\Theta^{[1]}$

Use_def(y_Θ) $\Rightarrow$ $y_\Theta = xy_\Theta^{[2]}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY Svm_test

THEORY Svm_test(a, b, s)
$\Rightarrow$
$(x_\Theta, y_\Theta \in s \,\&\, a(x_\Theta) = a(y_\Theta) \,\&\, b(x_\Theta) \neq b(y_\Theta)) \vee \mathrm{Svm}(\{[a(x), b(x)] : x \in s\})$
END Svm_test

-- As in the case of Iz_map, we give the two and 3-variable versions of the preceding theory.

THEORY $\mathsf{Svm\_test_2}\big(\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y}),\mathsf{s},\mathsf{t},\mathsf{P}(\mathsf{x},\mathsf{y})\big)$
END $\mathsf{Svm\_test_2}$

ENTER_THEORY $\mathsf{Svm\_test_2}$

DEF Svm_test $\cdot\, 0a.$ $\quad \mathsf{xy}_\Theta =_{\text{Def}} \mathbf{arb}(\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}',\mathsf{y}']] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t},\mathsf{x}'\in\mathsf{s},\mathsf{y}'\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\ \&\ \mathsf{P}(\mathsf{x}',\mathsf{y}')\ \&\ \mathsf{a}(\mathsf{x},\mathsf{y})=\mathsf{a}(\mathsf{x}',\mathsf{y}')\ \&\ \mathsf{b}(\mathsf{x},\mathsf{y})\neq\mathsf{b}(\mathsf{x}',\mathsf{y}')\})$
DEF Svm_test $\cdot\, 0b.$ $\quad \mathsf{x}_\Theta =_{\text{Def}} {\mathsf{xy}_\Theta}^{[1]\,[1]}$
DEF Svm_test $\cdot\, 0c.$ $\quad \mathsf{y}_\Theta =_{\text{Def}} {\mathsf{xy}_\Theta}^{[1]\,[2]}$
DEF Svm_test $\cdot\, 0d.$ $\quad \mathsf{xp}_\Theta =_{\text{Def}} {\mathsf{xy}_\Theta}^{[2]\,[1]}$
DEF Svm_test $\cdot\, 0h.$ $\quad \mathsf{yp}_\Theta =_{\text{Def}} {\mathsf{xy}_\Theta}^{[2]\,[2]}$

Theorem 38 (Svm_test$_2 \cdot 1$) $\big(\mathsf{x}_\Theta\in\mathsf{s}\ \&\ \mathsf{y}_\Theta\in\mathsf{t}\ \&\ \mathsf{xp}_\Theta\in\mathsf{s}\ \&\ \mathsf{yp}_\Theta\in\mathsf{t}\ \&\ \mathsf{a}(\mathsf{x}_\Theta,\mathsf{y}_\Theta)=\mathsf{a}(\mathsf{xp}_\Theta,\mathsf{yp}_\Theta)\ \&\ \mathsf{b}(\mathsf{x}_\Theta,\mathsf{y}_\Theta)\neq\mathsf{b}(\mathsf{xp}_\Theta,\mathsf{yp}_\Theta)\big)\vee$ $\mathsf{Svm}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\}\big)$. PROOF:

Suppose_not(s, t) $\Rightarrow$ *Stat1* :
$\neg\big(\mathsf{x}_\Theta\in\mathsf{s}\ \&\ \mathsf{y}_\Theta\in\mathsf{t}\ \&\ \mathsf{xp}_\Theta\in\mathsf{s}\ \&\ \mathsf{yp}_\Theta\in\mathsf{t}\ \&\ \mathsf{a}(\mathsf{x}_\Theta,\mathsf{y}_\Theta)=\mathsf{a}(\mathsf{xp}_\Theta,\mathsf{yp}_\Theta)\ \&\ \mathsf{b}(\mathsf{x}_\Theta,\mathsf{y}_\Theta)\neq\mathsf{b}(\mathsf{xp}_\Theta,\mathsf{yp}_\Theta)\big)\ \&\ \neg\mathsf{Svm}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\}\big)$

-- By definition, the negative of our assertion can only be true if

$$\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\}$$

is either not a map or fails the single-valuedness test. But Iz_map_2 tells us that the first case is impossible, so that this set must fail the single-valuedness test.

Use_def(Svm) $\Rightarrow$
$\neg\mathsf{Is\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\}\big)\vee$
$\neg\big\langle\forall\mathsf{u}\in\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\},\mathsf{v}\in\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\}\,|\,\mathsf{u}^{[1]}=\mathsf{v}^{[1]}\longrightarrow\mathsf{u}=\mathsf{v}\big\rangle$
APPLY $\langle\,\rangle$ $\mathsf{Iz\_map_2}\big(\mathsf{a}(\mathsf{x},\mathsf{y})\mapsto\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})\mapsto\mathsf{b}(\mathsf{x},\mathsf{y}),\mathsf{s}\mapsto\mathsf{s},\mathsf{t}\mapsto\mathsf{t},\mathsf{P}(\mathsf{x},\mathsf{y})\mapsto\mathsf{P}(\mathsf{x},\mathsf{y})\big)\Rightarrow$
$\mathsf{Is\_map}\big(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\}\big)$
ELEM $\Rightarrow$ *Stat2* :
$\neg\big\langle\forall\mathsf{u}\in\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\},\mathsf{v}\in\{[\mathsf{a}(\mathsf{xx},\mathsf{yy}),\mathsf{b}(\mathsf{xx},\mathsf{yy})] : \mathsf{xx}\in\mathsf{s},\mathsf{yy}\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{xx},\mathsf{yy})\}\,|\,\mathsf{u}^{[1]}=\mathsf{v}^{[1]}\longrightarrow\mathsf{u}=\mathsf{v}\big\rangle$

-- Hence there must exist elements x, y, xx, yy violating the single-valuedness condition, and so implying that the set

$$\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}',\mathsf{y}']] : \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{t},\mathsf{x}'\in\mathsf{s},\mathsf{y}'\in\mathsf{t}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\ \&\ \mathsf{P}(\mathsf{x}',\mathsf{y}')\ \&\ \mathsf{a}(\mathsf{x},\mathsf{y})=\mathsf{a}(\mathsf{x}',\mathsf{y}')\ \&\ \mathsf{b}(\mathsf{x},\mathsf{y})\neq\mathsf{b}(\mathsf{x}',\mathsf{y}')\}$$

is non-empty, which by the axiom of choice and definition of xy_Θ implies that xy_thryvar belongs to this set.

SIMPLF $\langle Stat2 \rangle \Rightarrow$ $Stat3:$

$$\neg\langle \forall x \in s, y \in t, xx \in s, yy \in t \mid P(x,y) \,\&\, P(xx,yy) \rightarrow [a(x,y), b(x,y)]^{[1]} = [a(xx,yy), b(xx,yy)]^{[1]} \rightarrow [a(x,y), b(x,y)] = [a(xx,yy), b(xx,yy)] \rangle$$

$\langle x, y, xx, yy \rangle \hookrightarrow Stat3 \Rightarrow$ $x \in s \,\&\, y \in t \,\&\, xx \in s \,\&\, yy \in t \,\&\, P(x,y) \,\&\, P(xx,yy) \,\&$
$a(x,y) = a(xx,yy) \,\&\, b(x,y) \neq b(xx,yy)$

Suppose $\Rightarrow$ $Stat4:$ $\{[[x,y],[x',y']] : x \in s, y \in t, x' \in s, y' \in t \mid P(x,y) \,\&\, P(x',y') \,\&\, a(x,y) = a(x',y') \,\&\, b(x,y) \neq b(x',y')\} =$
$\emptyset$

$\langle x, y, xx, yy \rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ $\{[[x,y],[x',y']] : x \in s, y \in t, x' \in s, y' \in t \mid P(x,y) \,\&\, P(x',y') \,\&\, a(x,y) = a(x',y') \,\&\, b(x,y) \neq b(x',y')\} \neq \emptyset$

$\langle \{[[x,y],[x',y']] : x \in s, y \in t, x' \in s, y' \in t \mid P(x,y) \,\&\, P(x',y') \,\&\, a(x,y) = a(x',y') \,\&\, b(x,y) \neq b(x',y')\} \rangle \hookrightarrow T0 \Rightarrow$

$\mathbf{arb}(\{[[x,y],[x',y']] : x \in s, y \in t, x' \in s, y' \in t \mid P(x,y) \,\&\, P(x',y') \,\&\, a(x,y) = a(x',y') \,\&\, b(x,y) \neq b(x',y')\}) \in$
$\{[[x,y],[x',y']] : x \in s, y \in t, x' \in s, y' \in t \mid P(x,y) \,\&\, P(x',y') \,\&\, a(x,y) = a(x',y') \,\&\, b(x,y) \neq b(x',y')\}$

Use_def $(xy_\Theta) \Rightarrow$ $Stat5:$
$xy_\Theta \in \{[[x,y],[x',y']] : x \in s, y \in t, x' \in s, y' \in t \mid P(x,y) \,\&\, P(x',y') \,\&\, a(x,y) = a(x',y') \,\&\, b(x,y) \neq b(x',y')\}$

-- Hence there exist elements x_2, y_2, xp_2, yp_2 satisfying the condition seen just below, and it is an elementary consequence of the definition of x_Θ, y_Θ, etc. that these must be x_Θ, y_Θ, xp_Θ, yp_Θ, leading to an immediate contradiction with our hypothesis, and so proving our theorem.

$\langle x_2, y_2, xp_2, yp_2 \rangle \hookrightarrow Stat5 \Rightarrow$ $Stat6:$
$x_2 \in s \,\&\, y_2 \in t \,\&\, xp_2 \in s \,\&\, yp_2 \in t \,\&\, P(x_2,y_2) \,\&\, P(xp_2,yp_2) \,\&\, xy_\Theta = [[x_2,y_2],[xp_2,yp_2]] \,\&\, a(x_2,y_2) = a(xp_2,yp_2) \,\&\, b(x_2,y_2) \neq b(xp_2,yp_2)$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $x_2 = {xy_\Theta}^{[1]\,[1]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $y_2 = {xy_\Theta}^{[1]\,[2]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $xp_2 = {xy_\Theta}^{[2]\,[1]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $yp_2 = {xy_\Theta}^{[2]\,[2]}$

Use_def $(x_\Theta) \Rightarrow$ $x_\Theta = {xy_\Theta}^{[1]\,[1]}$

Use_def $(y_\Theta) \Rightarrow$ $y_\Theta = {xy_\Theta}^{[1]\,[2]}$

Use_def $(xp_\Theta) \Rightarrow$ $xp_\Theta = {xy_\Theta}^{[2]\,[1]}$

Use_def $(yp_\Theta) \Rightarrow$ $yp_\Theta = {xy_\Theta}^{[2]\,[2]}$

EQUAL $\Rightarrow$ $Stat7:$
$x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, P(x_\Theta,y_\Theta) \,\&\, P(xp_\Theta,yp_\Theta) \,\&\, xy_\Theta = [[x_\Theta,y_\Theta],[xp_\Theta,yp_\Theta]] \,\&\, a(x_\Theta,y_\Theta) = a(xp_\Theta,yp_\Theta) \,\&\, b(x_\Theta,y_\Theta) \neq b(xp_\Theta,yp_\Theta)$

$\langle Stat7, Stat1, * \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY Svm_test$_2$

THEORY Svm_test$_2(a,b,s,t,P)$
$\Rightarrow$

$(x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, a(x_\Theta, y_\Theta) = a(xp_\Theta, yp_\Theta) \,\&\, b(x_\Theta, y_\Theta) \neq b(xp_\Theta, yp_\Theta)) \vee \mathsf{Svm}(\{[a(x,y), b(x,y)] : x \in s, y \in t \mid P(x,y)\})$

END $\mathsf{Svm\_test}_2$

-- We will occasionally require the three-variable version of the single-valued map principle, which is given by the following variant THEORY.

THEORY $\mathsf{Svm\_test}_3(a(x,y,z), b(x,y,z), s, t, u, P(x,y,z))$

END $\mathsf{Svm\_test}_3$

ENTER_THEORY $\mathsf{Svm\_test}_3$

DEF Svm_test · 0a. $xy_\Theta =_{\text{Def}}$
$\mathbf{arb}(\{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\})$

DEF Svm_test · 0b. $x_\Theta =_{\text{Def}} {xy_\Theta}^{[1]}{}^{[1]}$

DEF Svm_test · 0c. $y_\Theta =_{\text{Def}} {xy_\Theta}^{[1]}{}^{[2]}{}^{[1]}$

DEF Svm_test · 0c. $z_\Theta =_{\text{Def}} {xy_\Theta}^{[1]}{}^{[2]}{}^{[2]}$

DEF Svm_test · 0d. $xp_\Theta =_{\text{Def}} {xy_\Theta}^{[2]}{}^{[1]}$

DEF Svm_test · 0h. $yp_\Theta =_{\text{Def}} {xy_\Theta}^{[2]}{}^{[2]}{}^{[1]}$

DEF Svm_test · 0f. $zp_\Theta =_{\text{Def}} {xy_\Theta}^{[2]}{}^{[2]}{}^{[2]}$

Theorem 39 ($\mathsf{Svm\_test}_3 \cdot 1$)
$(x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, z_\Theta \in u \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, zp_\Theta \in u \,\&\, P(x_\Theta, y_\Theta, z_\Theta) \,\&\, P(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, a(x_\Theta, y_\Theta, z_\Theta) = a(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, b(x_\Theta, y_\Theta, z_\Theta) \neq b(xp_\Theta, yp_\Theta, zp_\Theta)) \vee$
$\mathsf{Svm}(\{[a(x,y,w), b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\})$. PROOF:

Suppose_not(s, t, u) ⇒ *Stat1* :
$\neg(x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, z_\Theta \in u \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, zp_\Theta \in u \,\&\, P(x_\Theta, y_\Theta, z_\Theta) \,\&\, P(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, a(x_\Theta, y_\Theta, z_\Theta) = a(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, b(x_\Theta, y_\Theta, z_\Theta) \neq b(xp_\Theta, yp_\Theta, zp_\Theta)) \,\&$
$\neg\mathsf{Svm}(\{[a(x,y,w), b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\})$

-- By definition, the negative of our assertion can only be true if

$$\{[a(x,y,w), b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\}$$

is either not a map or fails the single-valuedness test. But Iz_map_3 tells us that the first case is impossible, so that this set must fail the single-valuedness test.

Use_def(Svm) ⇒
$\neg\mathsf{Is\_map}(\{[a(x,y,w), b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\}) \vee$

$\neg\langle\forall z_1 \in \{[a(x,y,w),b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\}, z_2 \in \{[a(x,y,w),b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\} \mid {z_1}^{[1]} = {z_2}^{[1]} \to z_1 = z_2\rangle$

APPLY $\langle\,\rangle$ $\mathsf{Iz\_map_3}(a(x,y,w) \mapsto a(x,y,w), b(x,y,w) \mapsto b(x,y,w), s \mapsto s, t \mapsto t, u \mapsto u, P(x,y,w) \mapsto P(x,y,w)) \Rightarrow$

$\mathsf{Is\_map}(\{[a(x,y,w),b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\})$

ELEM $\Rightarrow$

$\neg\langle\forall z_1 \in \{[a(x,y,w),b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\}, z_2 \in \{[a(x,y,w),b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\} \mid {z_1}^{[1]} = {z_2}^{[1]} \to z_1 = z_2\rangle$

SIMPLF $\Rightarrow$ *Stat2* :

$\neg\langle\forall x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid P(x,y,zz) \,\&\, P(x',y',zz') \to [a(x,y,zz),b(x,y,zz)]^{[1]} = [a(x',y',zz'),b(x',y',zz')]^{[1]} \to [a(x,y,zz),b(x,y,zz)] = [a(x',y',z$

-- Hence there must exist elements x, y, zz, x′, y′, zz′ violating the single-valuedness condition, and so implying that the set

$\{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\} \neq \emptyset$

is non-empty, which by the axiom of choice and definition of xy_Θ implies that xy_thryvar belongs to this set.

$\langle x,y,zz,x',y',zz'\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3* :

$x \in s \,\&\, y \in t \,\&\, zz \in u \,\&\, x' \in s \,\&\, y' \in t \,\&\, zz' \in u \,\&$

$P(x,y,zz) \,\&\, P(x',y',zz') \,\&\, [a(x,y,zz),b(x,y,zz)]^{[1]} = [a(x',y',zz'),b(x',y',zz')]^{[1]} \,\&\, b(x,y,zz) \neq b(x',y',zz')$

$\langle$*Stat3*$\rangle$ ELEM $\Rightarrow$ $a(x,y,zz) = a(x',y',zz')$

Suppose $\Rightarrow$ *Stat4* :

$\{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\} = \emptyset$

$\langle x,y,zz,x',y',zz'\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$

$\{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\} \neq \emptyset$

$\langle \{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\} \rangle \hookrightarrow$ *T0* $\Rightarrow$

$\mathbf{arb}(\{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\}) \in$

$\{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\}$

Use_def (xy_Θ) $\Rightarrow$ *Stat5* :

$xy_\Theta \in \{[[x,[y,zz]],[x',[y',zz']]] : x \in s, y \in t, zz \in u, x' \in s, y' \in t, zz' \in u \mid a(x,y,zz) = a(x',y',zz') \,\&\, b(x,y,zz) \neq b(x',y',zz') \,\&\, P(x,y,zz) \,\&\, P(x',y',zz')\}$

-- Hence there exist elements x_2, y_2, z_2, xp_2, yp_2, zp_2 satisfying the condition seen just below, and it is an elementary consequence of the definition of x_Θ, y_Θ, etc. that these must be x_Θ, y_Θ, z_Θ, xp_Θ, yp_Θ, zp_Θ, leading to an immediate contradiction with our hypothesis, and so proving our theorem.

$\langle x_2,y_2,z_2,xp_2,yp_2,zp_2\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ *Stat6* :

$x_2 \in s \,\&\, y_2 \in t \,\&\, z_2 \in u \,\&\, xp_2 \in s \,\&\, yp_2 \in t \,\&\, zp_2 \in u \,\&$

$P(x_2,y_2,z_2) \,\&\, P(xp_2,yp_2,zp_2) \,\&\, xy_\Theta = [[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]] \,\&\, a(x_2,y_2,z_2) = a(xp_2,yp_2,zp_2) \,\&\, b(x_2,y_2,z_2) \neq b(xp_2,yp_2,zp_2)$

$\langle$*Stat6*$\rangle$ ELEM $\Rightarrow$ $x_2 = {{xy_\Theta}^{[1]}}^{[1]}$

$\langle$*Stat6*$\rangle$ ELEM $\Rightarrow$ $y_2 = {{{xy_\Theta}^{[1]}}^{[2]}}^{[1]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $z_2 = xy_\Theta{}^{[1]}{}^{[2]}{}^{[2]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $xp_2 = xy_\Theta{}^{[2]}{}^{[1]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $yp_2 = xy_\Theta{}^{[2]}{}^{[2]}{}^{[1]}$

$\langle Stat6 \rangle$ ELEM $\Rightarrow$ $zp_2 = xy_\Theta{}^{[2]}{}^{[2]}{}^{[2]}$

Use_def (x_Θ) $\Rightarrow$ $x_\Theta = xy_\Theta{}^{[1]}{}^{[1]}$

Use_def (y_Θ) $\Rightarrow$ $y_\Theta = xy_\Theta{}^{[1]}{}^{[2]}{}^{[1]}$

Use_def (z_Θ) $\Rightarrow$ $z_\Theta = xy_\Theta{}^{[1]}{}^{[2]}{}^{[2]}$

Use_def (xp_Θ) $\Rightarrow$ $xp_\Theta = xy_\Theta{}^{[2]}{}^{[1]}$

Use_def (yp_Θ) $\Rightarrow$ $yp_\Theta = xy_\Theta{}^{[2]}{}^{[2]}{}^{[1]}$

Use_def (zp_Θ) $\Rightarrow$ $zp_\Theta = xy_\Theta{}^{[2]}{}^{[2]}{}^{[2]}$

EQUAL $\Rightarrow$ *Stat7* :

$x_\Theta \in s \;\&\; y_\Theta \in t \;\&\; z_\Theta \in u \;\&\; xp_\Theta \in s \;\&\; yp_\Theta \in t \;\&\; zp_\Theta \in u \;\&$
$P(x_\Theta, y_\Theta, z_\Theta) \;\&\; P(xp_\Theta, yp_\Theta, zp_\Theta) \;\&\; xy_\Theta = [[x_\Theta, [y_\Theta, z_\Theta]], [xp_\Theta, [yp_\Theta, zp_\Theta]]] \;\&\; a(x_\Theta, y_\Theta, z_\Theta) = a(xp_\Theta, yp_\Theta, zp_\Theta) \;\&\; b(x_\Theta, y_\Theta, z_\Theta) \neq b(xp_\Theta, yp_\Theta, zp_\Theta)$

$\langle Stat7, Stat1 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY $\mathsf{Svm\_test}_3$

THEORY $\mathsf{Svm\_test}_3(a(x,y,z), b(x,y,z), s, t, u, P(x,y,z))$
$\Rightarrow (x_\Theta, y_\Theta, z_\Theta, xp_\Theta, yp_\Theta, zp_\Theta)$
$(x_\Theta \in s \;\&\; y_\Theta \in t \;\&\; z_\Theta \in u \;\&\; xp_\Theta \in s \;\&\; yp_\Theta \in t \;\&\; zp_\Theta \in u \;\&\; P(x_\Theta, y_\Theta, z_\Theta) \;\&\; P(xp_\Theta, yp_\Theta, zp_\Theta) \;\&\; a(x_\Theta, y_\Theta, z_\Theta) = a(xp_\Theta, yp_\Theta, z_\Theta) \;\&\; b(x_\Theta, y_\Theta, z_\Theta) \neq b(xp_\Theta, yp_\Theta, z_\Theta)) \vee$
$\mathsf{Svm}(\{[a(x,y,w), b(x,y,w)] : x \in s, y \in t, w \in u \mid P(x,y,w)\})$
END $\mathsf{Svm\_test}_3$

-- The next mini-theory simply specializes Svm_test to the form in which it is most commonly used. The proof required is completely elementary.

THEORY Must_be_svm $(b, s(x))$
END Must_be_svm

ENTER_THEORY Must_be_svm

Theorem 40 (Must_be_svm · 1) $\mathsf{Svm}(\{[x, b(x)] : x \in s\})$. PROOF:

$\mathsf{Suppose\_not}(s) \Rightarrow \quad \neg \mathsf{Svm}\big(\{[x, b(x)] : x \in s\}\big)$

$\mathsf{APPLY}\ \langle x_\Theta : x, y_\Theta : y\rangle\ \mathsf{Svm\_test}\big(a(x) \mapsto x, b(x) \mapsto b(x), s \mapsto s\big) \Rightarrow$

$\quad \big(x, y \in s \ \&\ x = y \ \&\ b(x) \neq b(y)\big) \vee \mathsf{Svm}\big(\{[x, b(x)] : x \in s\}\big)$

$\mathsf{ELEM} \Rightarrow \quad x, y \in s \ \&\ x = y \ \&\ b(x) \neq b(y)$

$\mathsf{EQUAL} \Rightarrow \quad b(x) = b(y)$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

ENTER_THEORY Set_theory

DISPLAY Must_be_svm

THEORY Must_be_svm(b, s)

$\Rightarrow$

$\quad \mathsf{Svm}\big(\{[x, b(x)] : x \in s\}\big)$

END Must_be_svm

-- The preceding mini-theory has the following obvious 2-variable analog. The proof of the one theorem it provides is completely elementary.

THEORY Must_be_svm$_2\big(b(x, y), s, t, P(x, y)\big)$

END Must_be_svm$_2$

ENTER_THEORY Must_be_svm$_2$

Theorem 41 (Must_be_svm$_2 \cdot 1$) $\mathsf{Svm}\big(\{[[x, y], b(x, y)] : x \in s, y \in t \mid P(x, y)\}\big)$. PROOF:

$\mathsf{Suppose\_not}(s, t) \Rightarrow \quad \neg \mathsf{Svm}\big(\{[[x, y], b(x, y)] : x \in s, y \in t \mid P(x, y)\}\big)$

$\mathsf{APPLY}\ \langle x_\Theta : x, y_\Theta : y, xp_\Theta : xx, yp_\Theta : yy\rangle\ \mathsf{Svm\_test}_2\big(a(x, y) \mapsto [x, y], b(x, y) \mapsto b(x, y), P(x, y) \mapsto P(x, y), s \mapsto s, t \mapsto t\big) \Rightarrow$

$\quad \big(x \in s \ \&\ y \in t \ \&\ xx \in s \ \&\ yy \in t \ \&\ [x, y] = [xx, yy] \ \&\ b(x, y) \neq b(xx, yy)\big) \vee \mathsf{Svm}\big(\{[[x, y], b(x, y)] : x \in s, y \in t \mid P(x, y)\}\big)$

$\mathsf{ELEM} \Rightarrow \quad x = xx \ \&\ y = yy$

$\mathsf{EQUAL} \Rightarrow \quad b(x, y) = b(xx, yy)$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

ENTER_THEORY Set_theory

DISPLAY Must_be_svm$_2$

THEORY Must_be_svm$_2\big(b(x, y), s, t, P(x, y)\big)$

$\Rightarrow$

$\mathsf{Svm}\big(\{[[x,y],b(x,y)] :\ x\in s, y\in t \mid P(x,y)\}\big)$

END $\mathsf{Must\_be\_svm_2}$

-- The following final small theories in the present utility series adapt Svm_test and its multivariable analogs to the form more conveniently use in proving that a map is 1-1. Once more the sole theorem provided has an easy proof.

THEORY $\mathsf{one\_1\_test}\big(a(x), b(x), s\big)$

END one_1_test

ENTER_THEORY one_1_test

DEF one_1_test · $0a$. $\quad xy_\Theta \quad =_{\text{Def}} \quad \mathbf{arb}\big(\{[x,y] :\ x\in s, y\in s \mid \neg\big(a(x)=a(y)\leftrightarrow b(x)=b(y)\big)\}\big)$

DEF one_1_test · $0b$. $\quad x_\Theta \quad =_{\text{Def}} \quad {xy_\Theta}^{[1]}$

DEF one_1_test · $0c$. $\quad y_\Theta \quad =_{\text{Def}} \quad {xy_\Theta}^{[2]}$

Theorem 42 (one_1_test · 1) $\Big(x_\Theta, y_\Theta\in s \mathbin{\&} \neg\big(a(x_\Theta)=a(y_\Theta)\leftrightarrow b(x_\Theta)=b(y_\Theta)\big)\Big)\vee 1\text{–}1(\{[a(x),b(x)] :\ x\in s\})$. PROOF:

Suppose_not $\Rightarrow \quad \neg\Big(x_\Theta, y_\Theta\in s \mathbin{\&} \neg\big(a(x_\Theta)=a(y_\Theta)\leftrightarrow b(x_\Theta)=b(y_\Theta)\big)\Big) \mathbin{\&} \neg 1\text{–}1(\{[a(u),b(u)] :\ u\in s\})$

-- For let s be a counterexample to our assertion. Then the set (*) $\big\{[x,y] :\ x\in s, y\in s \mid \neg\big(a(x)=a(y)\leftrightarrow b(x)=b(y)\big)\big\}$ cannot be empty, since if it were $\{[a(x),b(x)] :\ x\in s\}$ would necessarily be single valued, in which case there would have to exist two elements xx, yy of s for which $b(xx)=b(yy) \mathbin{\&} a(xx)\neq a(yy)$, an impossibility given that the set (*) seen above is empty.

Use_def(1–1) $\Rightarrow \quad Stat1:\ \neg\mathsf{Svm}\big(\{[a(u),b(u)] :\ u\in s\}\big)\vee$
$\quad \neg\big\langle\forall x\in\{[a(u),b(u)] :\ u\in s\}, y\in\{[a(v),b(v)] :\ v\in s\}\mid x^{[2]}=y^{[2]}\rightarrow x=y\big\rangle$

Suppose $\Rightarrow \quad Stat2:\ \big\{[x,y] :\ x\in s, y\in s \mid \neg\big(a(x)=a(y)\leftrightarrow b(x)=b(y)\big)\big\}=\emptyset$

Suppose $\Rightarrow \quad \neg\mathsf{Svm}\big(\{[a(x),b(x)] :\ x\in s\}\big)$

APPLY $\langle x_\Theta :\ x, y_\Theta :\ y\rangle$ Svm_test$\big(a(x)\mapsto a(x), b(x)\mapsto b(x), s\mapsto s\big)\Rightarrow$
$\quad \big(x,y\in s \mathbin{\&} a(x)=a(y) \mathbin{\&} b(x)\neq b(y)\big)\vee\mathsf{Svm}\big(\{[a(x),b(x)] :\ x\in s\}\big)$

ELEM $\Rightarrow \quad x,y\in s \mathbin{\&} a(x)=a(y) \mathbin{\&} b(x)\neq b(y)$

$\langle x,y\rangle\hookrightarrow Stat2\Rightarrow \quad a(x)=a(y)\leftrightarrow b(x)=b(y)$

ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \neg\big\langle\forall x\in\{[a(u),b(u)] :\ u\in s\}, y\in\{[a(v),b(v)] :\ v\in s\}\mid x^{[2]}=y^{[2]}\rightarrow x=y\big\rangle$

SIMPLF $\Rightarrow \quad Stat3:\ \neg\big\langle\forall u\in s, v\in s\mid [a(u),b(u)]^{[2]}=[a(v),b(v)]^{[2]}\rightarrow [a(u),b(u)]=[a(v),b(v)]\big\rangle$

$\langle xx,yy\rangle\hookrightarrow Stat3\Rightarrow \quad xx,yy\in s \mathbin{\&} [a(xx),b(xx)]^{[2]}=[a(yy),b(yy)]^{[2]} \mathbin{\&} [a(xx),b(xx)]\neq[a(yy),b(yy)]$

ELEM $\Rightarrow \quad xx,yy\in s \mathbin{\&} b(xx)=b(yy) \mathbin{\&} a(xx)\neq a(yy)$

$\langle \mathsf{xx}, \mathsf{yy} \rangle \hookrightarrow Stat2 \Rightarrow \quad a(xx) = a(yy) \leftrightarrow b(xx) = b(yy)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\{[x, y] : x \in s, y \in s \mid \neg(a(x) = a(y) \leftrightarrow b(x) = b(y))\} \neq \emptyset$

-- It therefore follows by the axiom of choice that xy_Θ, as defined above, is an element of the set (*), and thus its two components x_Θ and y_Θ stand in contradiction to the hypotheses of the present theorem. This contradiction proves our assertion.

$\langle \{[x, y] : x \in s, y \in s \mid \neg(a(x) = a(y) \leftrightarrow b(x) = b(y))\} \rangle \hookrightarrow T0 \Rightarrow \quad \mathbf{arb}(\{[x, y] : x \in s, y \in s \mid \neg(a(x) = a(y) \leftrightarrow b(x) = b(y))\}) \in \{[x, y] : x \in s, y \in s \mid \neg(a(x) = a(y) \leftrightarrow b(x) = b(y))\}$
Use_def $(xy_\Theta) \Rightarrow \quad Stat4 : xy_\Theta \in \{[x, y] : x \in s, y \in s \mid \neg(a(x) = a(y) \leftrightarrow b(x) = b(y))\}$
$\langle x_2, y_2 \rangle \hookrightarrow Stat4 \Rightarrow \quad xy_\Theta = [x_2, y_2] \;\&\; x_2, y_2 \in s \;\&\; \neg(a(x_2) = a(y_2) \leftrightarrow b(x_2) = b(y_2))$
ELEM $\Rightarrow \quad xy_\Theta^{[1]} = x_2 \;\&\; xy_\Theta^{[2]} = y_2$
Use_def $(x_\Theta) \Rightarrow \quad x_\Theta = xy_\Theta^{[1]}$
Use_def $(y_\Theta) \Rightarrow \quad y_\Theta = xy_\Theta^{[2]}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The utility theory just developed can be summarized as follows.

DISPLAY one_1_test

THEORY one_1_test $(a(x), b(x), s)$
$\Rightarrow (x_\Theta, y_\Theta)$
$\left(x_\Theta, y_\Theta \in s \;\&\; \neg(a(x_\Theta) = a(y_\Theta) \leftrightarrow b(x_\Theta) = b(y_\Theta))\right) \vee 1\text{–}1(\{[a(x), b(x)] : x \in s\})$
END one_1_test

THEORY one_1_test$_2$ $(a(x, y), b(x, y), s, t)$
END one_1_test$_2$

ENTER_THEORY one_1_test$_2$

-- Next we give the two variable analog of the THEORY given just above. The proof of the one theorem it provides differs little from that seen above.

DEF one_1_test$_2 \cdot 0a$. $\quad xy_\Theta =_{\mathrm{Def}} \mathbf{arb}(\{[[x, y], [x_2, y_2]] : x \in s, y \in t, x_2 \in s, y_2 \in t \mid \neg(a(x, y) = a(x_2, y_2) \leftrightarrow b(x, y) = b(x_2, y_2))\})$
DEF one_1_test$_2 \cdot 0b$. $\quad x_\Theta =_{\mathrm{Def}} xy_\Theta^{[1]^{[1]}}$
DEF one_1_test$_2 \cdot 0c$. $\quad y_\Theta =_{\mathrm{Def}} xy_\Theta^{[1]^{[2]}}$
DEF one_1_test$_2 \cdot 0d$. $\quad x2_\Theta =_{\mathrm{Def}} xy_\Theta^{[2]^{[1]}}$

DEF one_1_test$_2 \cdot 0h$. $\mathsf{y2}_\Theta =_{\text{Def}} \mathsf{xy}_\Theta{}^{[2]}{}^{[2]}$

Theorem 43 (one_1_test$_2 \cdot 1$) $\big(\mathsf{x}_\Theta \in \mathsf{s} \,\&\, \mathsf{y}_\Theta \in \mathsf{t} \,\&\, \mathsf{x2}_\Theta \in \mathsf{s} \,\&\, \mathsf{y2}_\Theta \in \mathsf{t} \,\&\, \neg\big(\mathsf{a}(\mathsf{x}_\Theta,\mathsf{y}_\Theta) = \mathsf{a}(\mathsf{x2}_\Theta,\mathsf{y2}_\Theta) \leftrightarrow \mathsf{b}(\mathsf{x}_\Theta,\mathsf{y}_\Theta) = \mathsf{b}(\mathsf{x2}_\Theta,\mathsf{y2}_\Theta)\big)\big) \vee 1\text{–}1(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}\})$. PROOF

Suppose_not ⇒ $\neg\big(\mathsf{x}_\Theta \in \mathsf{s} \,\&\, \mathsf{y}_\Theta \in \mathsf{t} \,\&\, \mathsf{x2}_\Theta \in \mathsf{s} \,\&\, \mathsf{y2}_\Theta \in \mathsf{t} \,\&\, \neg\big(\mathsf{a}(\mathsf{x}_\Theta,\mathsf{y}_\Theta) = \mathsf{a}(\mathsf{x2}_\Theta,\mathsf{y2}_\Theta) \leftrightarrow \mathsf{b}(\mathsf{x}_\Theta,\mathsf{y}_\Theta) = \mathsf{b}(\mathsf{x2}_\Theta,\mathsf{y2}_\Theta)\big)\big) \,\&\, \neg 1\text{–}1(\{[\mathsf{a}(\mathsf{x},\mathsf{y}),\mathsf{b}(\mathsf{x},\mathsf{y})] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}\})$

-- For let s be a counterexample to our assertion. Then the set (*) $\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\}$ cannot be empty, since if it were $\{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}\}$ would necessarily be single valued, in which case there would have to exist elements x, x' of s and elements y, y' of t for which $\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}',\mathsf{y}') \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}',\mathsf{y}')$, an impossibility given that the set (*) seen above is empty.

Use_def (1–1) ⇒ *Stat1* :
$\neg\mathsf{Svm}\big(\{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}\}\big) \vee$
$\neg\langle\forall \mathsf{x} \in \{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}\}, \mathsf{y} \in \{[\mathsf{a}(\mathsf{u}',\mathsf{v}'),\mathsf{b}(\mathsf{u}',\mathsf{v}')] : \mathsf{u}' \in \mathsf{s}, \mathsf{v}' \in \mathsf{t}\} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y}\rangle$
Suppose ⇒ *Stat2* : $\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\} = \emptyset$
Suppose ⇒ $\neg\mathsf{Svm}\big(\{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}\}\big)$
APPLY $\langle \mathsf{x}_\Theta : \mathsf{x}, \mathsf{y}_\Theta : \mathsf{y}, \mathsf{xp}_\Theta : \mathsf{x}', \mathsf{yp}_\Theta : \mathsf{y}'\rangle$ Svm_test$_2(\mathsf{a}(\mathsf{x},\mathsf{y}) \mapsto \mathsf{a}(\mathsf{x},\mathsf{y}), \mathsf{b}(\mathsf{x},\mathsf{y}) \mapsto \mathsf{b}(\mathsf{x},\mathsf{y}), \mathsf{s} \mapsto \mathsf{s}, \mathsf{t} \mapsto \mathsf{t}, \mathsf{P}(\mathsf{x},\mathsf{y}) \mapsto \mathsf{true})$ ⇒
$(\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{y} \in \mathsf{t} \,\&\, \mathsf{x}' \in \mathsf{s} \,\&\, \mathsf{y}' \in \mathsf{t} \,\&\, \mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}',\mathsf{y}') \,\&\, \mathsf{b}(\mathsf{x},\mathsf{y}) \neq \mathsf{b}(\mathsf{x}',\mathsf{y}')) \vee \mathsf{Svm}\big(\{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t} \mid \mathsf{true}\}\big)$
$\langle \mathsf{x},\mathsf{y},\mathsf{x}',\mathsf{y}'\rangle \hookrightarrow$ *Stat2* ⇒ $\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}',\mathsf{y}') \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}',\mathsf{y}')$
ELEM ⇒ false; Discharge ⇒ $\neg\langle\forall \mathsf{x} \in \{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}\}, \mathsf{y} \in \{[\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] : \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}\} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y}\rangle$
SIMPLF ⇒ *Stat3* :
$\neg\langle\forall \mathsf{u} \in \mathsf{s}, \mathsf{v} \in \mathsf{t}, \mathsf{u}' \in \mathsf{s}, \mathsf{v}' \in \mathsf{t} \mid [\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})]^{[2]} = [\mathsf{a}(\mathsf{u}',\mathsf{v}'),\mathsf{b}(\mathsf{u}',\mathsf{v}')]^{[2]} \rightarrow [\mathsf{a}(\mathsf{u},\mathsf{v}),\mathsf{b}(\mathsf{u},\mathsf{v})] = [\mathsf{a}(\mathsf{u}',\mathsf{v}'),\mathsf{b}(\mathsf{u}',\mathsf{v}')]\rangle$
$\langle \mathsf{xx},\mathsf{yy},\mathsf{xx}',\mathsf{yy}'\rangle \hookrightarrow$ *Stat3* ⇒ $\mathsf{xx} \in \mathsf{s} \,\&\, \mathsf{yy} \in \mathsf{t} \,\&\, \mathsf{xx}' \in \mathsf{s} \,\&\, \mathsf{yy}' \in \mathsf{t} \,\&\, [\mathsf{a}(\mathsf{xx},\mathsf{yy}),\mathsf{b}(\mathsf{xx},\mathsf{yy})]^{[2]} = [\mathsf{a}(\mathsf{xx}',\mathsf{yy}'),\mathsf{b}(\mathsf{xx}',\mathsf{yy}')]^{[2]} \,\&$
$[\mathsf{a}(\mathsf{xx},\mathsf{yy}),\mathsf{b}(\mathsf{xx},\mathsf{yy})] \neq [\mathsf{a}(\mathsf{xx}',\mathsf{yy}'),\mathsf{b}(\mathsf{xx}',\mathsf{yy}')]$
$\langle \mathsf{xx},\mathsf{yy},\mathsf{xx}',\mathsf{yy}'\rangle \hookrightarrow$ *Stat2* ⇒ $\mathsf{a}(\mathsf{xx},\mathsf{yy}) = \mathsf{a}(\mathsf{xx}',\mathsf{yy}') \leftrightarrow \mathsf{b}(\mathsf{xx},\mathsf{yy}) = \mathsf{b}(\mathsf{xx}',\mathsf{yy}')$
ELEM ⇒ false; Discharge ⇒ $\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\} \neq \emptyset$

-- It therefore follows by the axiom of choice that xy_Θ, as defined above, is an element of the set (*), and thus its subcomponents x_Θ, y_Θ, $\mathsf{x2}_\Theta$, $\mathsf{y2}_\Theta$ stand in contradiction to the hypotheses of the present theorem. This contradiction proves our assertion.

$\langle \{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\}\rangle \hookrightarrow$ *T0* ⇒
$\mathbf{arb}\big(\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\}\big) \in$
$\{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\}$
Use_def (xy_Θ) ⇒ *Stat4* : $\mathsf{xy}_\Theta \in \{[[\mathsf{x},\mathsf{y}],[\mathsf{x}_2,\mathsf{y}_2]] : \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{t}, \mathsf{x}_2 \in \mathsf{s}, \mathsf{y}_2 \in \mathsf{t} \mid \neg\big(\mathsf{a}(\mathsf{x},\mathsf{y}) = \mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) \leftrightarrow \mathsf{b}(\mathsf{x},\mathsf{y}) = \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2)\big)\}$
$\langle \mathsf{x}_2,\mathsf{y}_2,\mathsf{xp}_2,\mathsf{yp}_2\rangle \hookrightarrow$ *Stat4* ⇒ $\mathsf{x}_2 \in \mathsf{s} \,\&\, \mathsf{y}_2 \in \mathsf{t} \,\&\, \mathsf{xp}_2 \in \mathsf{s} \,\&\, \mathsf{yp}_2 \in \mathsf{t} \,\&\, \neg\big(\mathsf{a}(\mathsf{x}_2,\mathsf{y}_2) = \mathsf{a}(\mathsf{xp}_2,\mathsf{yp}_2) \leftrightarrow \mathsf{b}(\mathsf{x}_2,\mathsf{y}_2) = \mathsf{b}(\mathsf{xp}_2,\mathsf{yp}_2)\big) \,\&$

$xy_\Theta = [[x_2, y_2], [xp_2, yp_2]]$

ELEM $\Rightarrow$ $[[x_2, y_2], [xp_2, yp_2]]^{[1]^{[1]}} = x_2$ & $[[x_2, y_2], [xp_2, yp_2]]^{[1]^{[2]}} = y_2$

ELEM $\Rightarrow$ $[[x_2, y_2], [xp_2, yp_2]]^{[2]^{[1]}} = xp_2$ & $[[x_2, y_2], [xp_2, yp_2]]^{[2]^{[2]}} = yp_2$

EQUAL $\Rightarrow$ $xy_\Theta^{[1]^{[1]}} = x_2$ & $xy_\Theta^{[1]^{[2]}} = y_2$

EQUAL $\Rightarrow$ $xy_\Theta^{[2]^{[1]}} = xp_2$ & $xy_\Theta^{[2]^{[2]}} = yp_2$

Use_def (x_Θ) $\Rightarrow$ $x_\Theta = xy_\Theta^{[1]^{[1]}}$

Use_def (y_Θ) $\Rightarrow$ $y_\Theta = xy_\Theta^{[1]^{[2]}}$

Use_def $(x2_\Theta)$ $\Rightarrow$ $x2_\Theta = xy_\Theta^{[2]^{[1]}}$

Use_def $(y2_\Theta)$ $\Rightarrow$ $y2_\Theta = xy_\Theta^{[2]^{[2]}}$

$\langle x_2, y_2, xp_2, yp_2\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $a(x_2, y_2) = a(xp_2, yp_2) \leftrightarrow b(x_2, y_2) = b(xp_2, yp_2)$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The utility theory just developed can be summarized as follows.

DISPLAY one_1_test

THEORY $\text{one\_1\_test}_2(a(x, y), b(x, y), s, t)$

$\Rightarrow (x_\Theta, y_\Theta, x2_\Theta, y2_\Theta)$

$\Big(x_\Theta \in s \;\&\; y_\Theta \in t \;\&\; x2_\Theta \in s \;\&\; y2_\Theta \in t \;\&\; \neg\big(a(x_\Theta, y_\Theta) = a(x2_\Theta, y2_\Theta) \leftrightarrow b(x_\Theta, y_\Theta) = b(x2_\Theta, y2_\Theta)\big)\Big) \vee 1\text{–}1(\{[a(x, y), b(x, y)] : x \in s, y \in t\})$

END one_1_test_2

THEORY $\text{one\_1\_test}_3(a(x, y, zz), b(x, y, zz), s, t, r, P(x, y, zz))$

END one_1_test_3

ENTER_THEORY one_1_test_3

-- Next we give the three variable analog of the THEORY given just above. The proof of the one theorem it provides differs little from that seen above.

DEF $\text{one\_1\_test}_3 \cdot 0a.$ $xyz_\Theta =_{\text{Def}}$
$\mathbf{arb}\big(\{[[x, [y, zz]], [x_2, [y_2, z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid P(x, y, zz) \;\&\; P(x_2, y_2, z_2) \;\&\; \big(a(x, y, zz) \neq a(x_2, y_2, z_2) \leftrightarrow b(x, y, zz) = b(x_2, y_2, z_2)\big)\}\big)$

DEF $\text{one\_1\_test}_3 \cdot 0b.$ $x_\Theta =_{\text{Def}} xyz_\Theta^{[1]^{[1]}}$

DEF $\text{one\_1\_test}_3 \cdot 0c.$ $y_\Theta =_{\text{Def}} xyz_\Theta^{[1]^{[2]^{[1]}}}$

DEF $\text{one\_1\_test}_3 \cdot 0c.$ $z_\Theta =_{\text{Def}} xyz_\Theta^{[1]^{[2]^{[2]}}}$

$\text{DEF one\_1\_test}_3 \cdot 0d.$ $\quad xp_\Theta =_{\text{Def}} xyz_\Theta{}^{[2]}{}^{[1]}$

$\text{DEF one\_1\_test}_3 \cdot 0h.$ $\quad yp_\Theta =_{\text{Def}} xyz_\Theta{}^{[2]}{}^{[2]}{}^{[1]}$

$\text{DEF one\_1\_test}_3 \cdot 0f.$ $\quad zp_\Theta =_{\text{Def}} xyz_\Theta{}^{[2]}{}^{[2]}{}^{[2]}$

Theorem 44 (one_1_test$_3 \cdot 1$)
$\big(x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, z_\Theta \in r \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, zp_\Theta \in r \,\&\, P(x_\Theta, y_\Theta, z_\Theta) \,\&\, P(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, \big(a(x_\Theta, y_\Theta, z_\Theta) \neq a(xp_\Theta, yp_\Theta, zp_\Theta) \leftrightarrow b(x_\Theta, y_\Theta, z_\Theta) = b(xp_\Theta, yp_\Theta, zp_\Theta)\big)\big) \vee$
$1\text{–}1(\{[a(x,y,zz), b(x,y,zz)] : x \in s, y \in t, zz \in r \mid P(x,y,zz)\})$. PROOF:

Suppose_not $\Rightarrow$
$\neg\big(x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, z_\Theta \in r \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, zp_\Theta \in r \,\&\, P(x_\Theta, y_\Theta, z_\Theta) \,\&\, P(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, \big(a(x_\Theta, y_\Theta, z_\Theta) \neq a(xp_\Theta, yp_\Theta, zp_\Theta) \leftrightarrow b(x_\Theta, y_\Theta, z_\Theta) = b(xp_\Theta, yp_\Theta, zp_\Theta)\big)\big)$
$\neg 1\text{–}1(\{[a(x,y,zz), b(x,y,zz)] : x \in s, y \in t, zz \in r \mid P(x,y,zz)\})$

-- For let s be a counterexample to our assertion. Then the set (*) $\{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid P(x,y,zz) \,\&\, P(x_2,y_2,z_2) \,\&\, \neg\big(a(x,y) = a(x_2,y_2) \leftrightarrow b(x,y) = b(x_2,y_2)\big)\}$ cannot be empty, since if it were $\{[a(u,v,w), b(u,v,w)] : u \in s, v \in t, w \in r \mid P(u,v,w)\}$ would necessarily be single valued, in which case there would have to exist elements x, x′ of s, elements y, y′ of t, and elements z, z′ of w for which $a(x,y,w) = a(x',y',w') \leftrightarrow b(x,y,w) = b(x',y',w)$ and $P(x,y,zz)$, $P(x_2,y_2,z_2)$ an impossibility given that the set (*) seen above is empty.

Use_def(1–1) $\Rightarrow$ *Stat1* :
$\neg \mathrm{Svm}\big(\{[a(u,v,w), b(u,v,w)] : u \in s, v \in t, w \in r \mid P(u,v,w)\}\big) \vee$
$\neg\langle \forall x \in \{[a(u,v,w), b(u,v,w)] : u \in s, v \in t, w \in r \mid P(u,v,w)\}, y \in \{[a(u',v',w'), b(u',v',w')] : u' \in s, v' \in t, w' \in r \mid P(u',v',w')\} \mid x^{[2]} = y^{[2]} \rightarrow x = y\rangle$
Suppose $\Rightarrow$ *Stat2* :
$\big\{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid P(x,y,zz) \,\&\, P(x_2,y_2,z_2) \,\&\, \big(a(x,y,zz) \neq a(x_2,y_2,z_2) \leftrightarrow b(x,y,zz) = b(x_2,y_2,z_2)\big)\big\} = \emptyset$
Suppose $\Rightarrow$ $\neg \mathrm{Svm}\big(\{[a(u,v,w), b(u,v,w)] : u \in s, v \in t, w \in r \mid P(u,v,w)\}\big)$
APPLY $\langle x_\Theta : x, y_\Theta : y, z_\Theta : z_1, xp_\Theta : x', yp_\Theta : y', zp_\Theta : z'\rangle$ $\mathrm{Svm\_test}_3\big(a(x,y,zz) \mapsto a(x,y,zz), b(x,y,zz) \mapsto b(x,y,zz), s \mapsto s, t \mapsto t, u \mapsto r, P(x,y,zz) \mapsto P(x,y,zz)\big) \Rightarrow$

$\big(x \in s \,\&\, y \in t \,\&\, z_1 \in r \,\&\, x' \in s \,\&\, y' \in t \,\&\, z' \in r \,\&\, P(x,y,z_1) \,\&\, P(x',y',z') \,\&\, a(x,y,z_1) = a(x',y',z') \,\&\, b(x,y,z_1) \neq b(x',y',z')\big) \vee \mathrm{Svm}\big(\{[a(x,y,w), b(x,y,w)] : x \in s, y \in t,$
$\langle x, y, z_1, x', y', z'\rangle \hookrightarrow$ *Stat2* $\Rightarrow$
$a(x,y,z_1) = a(x',y',z') \leftrightarrow b(x,y,z_1) = b(x',y',z') \,\&\, P(x,y,z_1) \,\&\, P(x',y',z')$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$
$\neg\langle \forall x \in \{[a(u,v,w), b(u,v,w)] : u \in s, v \in t, w \in r \mid P(u,v,w)\}, y \in \{[a(u,v,w), b(u,v,w)] : u \in s, v \in t, w \in r \mid P(u,v,w)\} \mid x^{[2]} = y^{[2]} \rightarrow x = y\rangle$
SIMPLF $\Rightarrow$ *Stat3* :
$\neg\langle \forall u \in s, v \in t, w \in r, u' \in s, v' \in t, w' \in r \mid P(u,v,w) \,\&\, P(u',v',w') \rightarrow [a(u,v,w), b(u,v,w)]^{[2]} = [a(u',v',w'), b(u',v',w')]^{[2]} \rightarrow [a(u,v,w), b(u,v,w)] = [a(u',v',w'),$
$\langle xx, yy, zz, xx', yy', zz'\rangle \hookrightarrow$ *Stat3* $\Rightarrow$
$xx \in s \,\&\, yy \in t \,\&\, zz \in r \,\&\, xx' \in s \,\&\, yy' \in t \,\&\, zz' \in r \,\&$

$\mathsf{P}(xx,yy,zz) \,\&\, \mathsf{P}(xx',yy',zz') \,\&\, [\mathsf{a}(xx,yy,zz),\mathsf{b}(xx,yy,zz)]^{[2]} = [\mathsf{a}(xx',yy',zz'),\mathsf{b}(xx',yy',zz')]^{[2]} \,\&\, [\mathsf{a}(xx,yy,zz),\mathsf{b}(xx,yy,zz)] \neq [\mathsf{a}(xx',yy',zz'),\mathsf{b}(xx',yy',zz')]$

$\langle xx,yy,zz,xx',yy',zz'\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$

$\mathsf{a}(xx,yy,zz) = \mathsf{a}(xx',yy',zz') \leftrightarrow \mathsf{b}(xx,yy,zz) = \mathsf{b}(xx',yy',zz') \vee \neg\mathsf{P}(xx,yy,zz) \vee \neg\mathsf{P}(xx',yy',zz')$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$

$\{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid \mathsf{P}(x,y,zz) \,\&\, \mathsf{P}(x_2,y_2,z_2) \,\&\, (\mathsf{a}(x,y,zz) \neq \mathsf{a}(x_2,y_2,z_2) \leftrightarrow \mathsf{b}(x,y,zz) = \mathsf{b}(x_2,y_2,z_2))\} \neq \emptyset$

-- It therefore follows by the axiom of choice that xy_Θ, as defined above, is an element of the set (*), and thus its subcomponents x_Θ, y_Θ, xp_Θ, yp_Θ stand in contradiction to the hypotheses of the present theorem. This contradiction proves our assertion.

$\langle \{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid \mathsf{P}(x,y,zz) \,\&\, \mathsf{P}(x_2,y_2,z_2) \,\&\, (\mathsf{a}(x,y,zz) \neq \mathsf{a}(x_2,y_2,z_2) \leftrightarrow \mathsf{b}(x,y,zz) = \mathsf{b}(x_2,y_2,z_2))\}\rangle \hookrightarrow \mathit{T0} \Rightarrow$

$\mathbf{arb}(\{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid \mathsf{P}(x,y,zz) \,\&\, \mathsf{P}(x_2,y_2,z_2) \,\&\, (\mathsf{a}(x,y,zz) \neq \mathsf{a}(x_2,y_2,z_2) \leftrightarrow \mathsf{b}(x,y,zz) = \mathsf{b}(x_2,y_2,z_2))\}) \in \{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid \mathsf{P}(x,y,zz) \,\&\, \mathsf{P}(x_2,y_2,z_2) \,\&\, (\mathsf{a}(x,y,zz) \neq \mathsf{a}(x_2,y_2,z_2) \leftrightarrow \mathsf{b}(x,y,zz) = \mathsf{b}(x_2,y_2,z_2))\}$

Use_def (xyz_Θ) $\Rightarrow$ *Stat4* :

$xyz_\Theta \in \{[[x,[y,zz]],[x_2,[y_2,z_2]]] : x \in s, y \in t, zz \in r, x_2 \in s, y_2 \in t, z_2 \in r \mid \mathsf{P}(x,y,zz) \,\&\, \mathsf{P}(x_2,y_2,z_2) \,\&\, (\mathsf{a}(x,y,zz) \neq \mathsf{a}(x_2,y_2,z_2) \leftrightarrow \mathsf{b}(x,y,zz) = \mathsf{b}(x_2,y_2,z_2))\}$

$\langle x_2,y_2,z_2,xp_2,yp_2,zp_2\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$

$x_2 \in s \,\&\, y_2 \in t \,\&\, z_2 \in r \,\&\, xp_2 \in s \,\&\, yp_2 \in t \,\&\, zp_2 \in r \,\&\, \mathsf{P}(x_2,y_2,z_2) \,\&\, \mathsf{P}(xp_2,yp_2,zp_2) \,\&\, (\mathsf{a}(x_2,y_2,z_2) \neq \mathsf{a}(xp_2,yp_2,zp_2) \leftrightarrow \mathsf{b}(x_2,y_2,z_2) = \mathsf{b}(xp_2,yp_2,zp_2)) \,\&\, xyz_\Theta = [[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]$

$\langle x_2,y_2,z_2,xp_2,yp_2,zp_2\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathsf{a}(x_2,y_2,z_2) = \mathsf{a}(xp_2,yp_2,zp_2) \leftrightarrow \mathsf{b}(x_2,y_2,z_2) = \mathsf{b}(xp_2,yp_2,zp_2)$

ELEM $\Rightarrow$

$[[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]^{[1]^{[1]}} = x_2 \,\&\, [[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]^{[1]^{[2]^{[1]}}} = y_2 \,\&\, [[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]^{[1]^{[2]^{[2]}}} = z_2$

ELEM $\Rightarrow$

$[[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]^{[2]^{[1]}} = xp_2 \,\&\, [[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]^{[2]^{[2]^{[1]}}} = yp_2 \,\&\, [[x_2,[y_2,z_2]],[xp_2,[yp_2,zp_2]]]^{[2]^{[2]^{[2]}}} = zp_2$

EQUAL $\Rightarrow$ $xyz_\Theta{}^{[1]^{[1]}} = x_2 \,\&\, xyz_\Theta{}^{[1]^{[2]^{[1]}}} = y_2 \,\&\, xyz_\Theta{}^{[1]^{[2]^{[2]}}} = z_2$

EQUAL $\Rightarrow$ $xyz_\Theta{}^{[2]^{[1]}} = xp_2 \,\&\, xyz_\Theta{}^{[2]^{[2]^{[1]}}} = yp_2 \,\&\, xyz_\Theta{}^{[2]^{[2]^{[2]}}} = zp_2$

Use_def (x_Θ) $\Rightarrow$ $x_\Theta = xyz_\Theta{}^{[1]^{[1]}}$

Use_def (y_Θ) $\Rightarrow$ $y_\Theta = xyz_\Theta{}^{[1]^{[2]^{[1]}}}$

Use_def (z_Θ) $\Rightarrow$ $z_\Theta = xyz_\Theta{}^{[1]^{[2]^{[2]}}}$

Use_def (xp_Θ) $\Rightarrow$ $xp_\Theta = xyz_\Theta{}^{[2]^{[1]}}$

Use_def (yp_Θ) $\Rightarrow$ $yp_\Theta = xyz_\Theta{}^{[2]^{[2]^{[1]}}}$

Use_def (zp_Θ) $\Rightarrow$ $zp_\Theta = xyz_\Theta{}^{[2]^{[2]^{[2]}}}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The utility theory just developed can be summarized as follows.

DISPLAY one_1_test

THEORY $\mathsf{one\_1\_test}_3\big(\mathsf{a}(x,y,zz), \mathsf{b}(x,y,zz), s, t, r, \mathsf{P}(x,y,zz)\big)$
$\Rightarrow (x_\Theta, y_\Theta, z_\Theta, xp_\Theta, yp_\Theta, zp_\Theta)$
$\big(x_\Theta \in s \,\&\, y_\Theta \in t \,\&\, z_\Theta \in u \,\&\, xp_\Theta \in s \,\&\, yp_\Theta \in t \,\&\, zp_\Theta \in u \,\&\, \mathsf{P}(x_\Theta, y_\Theta, z_\Theta) \,\&\, \mathsf{P}(xp_\Theta, yp_\Theta, zp_\Theta) \,\&\, \neg\big(\mathsf{a}(x_\Theta, y_\Theta, z_\Theta) = \mathsf{a}(xp_\Theta, yp_\Theta, zp_\Theta) \leftrightarrow \mathsf{b}(x_\Theta, y_\Theta, z_\Theta) = \mathsf{b}(xp_\Theta, yp_\Theta, zp_\Theta)\big)\big) \vee$
$\mathsf{1\text{–}1}(\{[\mathsf{a}(x,y,zz), \mathsf{b}(x,y,zz)] : x \in s, y \in t, zz \in u\})$
END $\mathsf{one\_1\_test}_3$

4 The ordinal enumerability theorem

-- Now we begin more serious development of the theory of ordinals, along von Neumann's line. Our first theorem uses induction to show that if one ordinal t is included in another ordinal s but not equal to s, then t must be a member of s, and in fact must be the smallest element of s-t.

Theorem 45 (24) $\mathcal{O}(S) \,\&\, \mathcal{O}(T) \,\&\, T \subseteq S \rightarrow T = S \vee T = \mathbf{arb}(S \setminus T)$. PROOF:

Suppose_not$(s,t) \Rightarrow$ $\mathcal{O}(s) \,\&\, \mathcal{O}(t) \,\&\, t \subseteq s \,\&\, t \neq s \,\&\, t \neq \mathbf{arb}(s \setminus t)$

-- For if our assertion is false, s must have a proper subset t, in which case the axiom of choice tells us that s\t has a minimal element **arb**(s\t) disjoint from s\t. Plainly **arb**(s\t) is also a member of the superset s of s\t. (s-t) $\hookrightarrow$T0 $\Rightarrow$ ((s-t = 0) & (arb (s-t) = 0)) or ((arb (s-t) in (s-t)) & (arb (s-t) * (s-t) = 0))

ELEM $\Rightarrow$ $\mathbf{arb}(s \setminus t) \in s \,\&\, \mathbf{arb}(s \setminus t) \cap (s \setminus t) = \emptyset$
Use_def$(\mathcal{O}) \Rightarrow$ $\mathit{Stat1}: \big\langle \forall x \in s \mid x \subseteq s \big\rangle \,\&\, \mathit{Stat2}: \big\langle \forall x \in s, y \in s \mid x \in y \vee y \in x \vee x = y \big\rangle$

-- But then, by definition of ordinal, **arb**(s\t) must be a subset of $s \cap t$, since it is disjoint from s\t. Therefore **arb**(s\t) cannot include t, otherwise the initial assumption $t \neq \mathbf{arb}(s \setminus t)$ would be contradicted.

$\big\langle \mathbf{arb}(s \setminus t) \big\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathbf{arb}(s \setminus t) \subseteq s$
ELEM $\Rightarrow$ $\mathbf{arb}(s \setminus t) \subseteq s \cap t \,\&\, \mathit{Stat3}: \mathbf{arb}(s \setminus t) \not\supseteq t$

-- Since **arb**(s\t) fails to include t, there must be some b in t but not in **arb**(s\t). By the definition of ordinals, this implies that **arb**(s\t) = b ∨ **arb**(s\t) ∈ b.

$\langle b \rangle \hookrightarrow Stat3 \Rightarrow \quad b \in t \,\&\, b \notin \mathbf{arb}(s\backslash t)$
$\langle \mathbf{arb}(s\backslash t), b \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathbf{arb}(s\backslash t) \in b \vee \mathbf{arb}(s\backslash t) = b$
$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad Stat4: \langle \forall x \in t \,|\, x \subseteq t \rangle \,\&\, \langle \forall x \in t, y \in t \,|\, x \in y \vee y \in x \vee x = y \rangle$

-- Using the definition of ordinals once more, this time for t, we see that b must be a subset of t, which rules out both **arb**(s\t) ∈ b and **arb**(s\t) = b, because either of these would yield **arb**(s\t) ∈ t which is impossible. We have contradicted our original assumption, and so proved our theorem.

$\langle b \rangle \hookrightarrow Stat4 \Rightarrow \quad b \subseteq t$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that the intersection of any two ordinals is an ordinal. (Indeed, as Theorem 26 shows, this intersection must be the smaller of the two ordinals.)

Theorem 46 (25) $\mathcal{O}(S) \,\&\, \mathcal{O}(T) \rightarrow \mathcal{O}(S \cap T)$. PROOF:

$\mathsf{Suppose\_not}(s, t) \Rightarrow \quad \mathcal{O}(s) \,\&\, \mathcal{O}(t) \,\&\, \neg\mathcal{O}(s \cap t)$

-- For suppose the contrary. Then by definition of ordinals there must exist a, b, c such that a is a member of s ∩ t but not included in it, or b and c are both members or s ∩ t, but are unrelated by membership.

$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad Stat0: \neg(\langle \forall x \in s \cap t \,|\, x \subseteq s \cap t \rangle \,\&\, \langle \forall x \in s \cap t, y \in s \cap t \,|\, x \in y \vee y \in x \vee x = y \rangle)$
$\langle a, b, c \rangle \hookrightarrow Stat0 \Rightarrow \quad Stat1: (a \in s \cap t \,\&\, a \not\subseteq s \cap t) \vee$
$b, c \in s \cap t \,\&\, \neg(b \in c \vee c \in b \vee c = b)$

-- However, since s and t are both ordinals the first case is clearly impossible, so we need only consider the second case.

$\mathsf{Suppose} \Rightarrow \quad Stat1a: a \in s \cap t \,\&\, a \not\subseteq s \cap t$
$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad Stat2: \langle \forall x \in s \,|\, x \subseteq s \rangle \,\&\, Stat3: \langle \forall x \in s, y \in s \,|\, x \in y \vee y \in x \vee x = y \rangle$
$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad Stat4: \langle \forall x \in t \,|\, x \subseteq t \rangle$
$\langle a \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat2a: a \in s \rightarrow a \subseteq s$
$\langle a \rangle \hookrightarrow Stat4 \Rightarrow \quad Stat4a: a \in t \rightarrow a \subseteq t$
$\langle Stat1a, Stat2a, Stat4a, * \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat5: b, c \in s \,\&\, \neg(b \in c \vee c \in b \vee c = b)$

-- But using the definition of ordinal once more we see that this case is impossible also.

$\langle \mathsf{b}, \mathsf{c} \rangle \hookrightarrow Stat3([Stat5, Stat5]) \Rightarrow \quad \mathsf{b}, \mathsf{c} \in \mathsf{s} \to \mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{c} = \mathsf{b}$
$\langle Stat5 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now we prove the related but slightly less elementary result that one of any pair of ordinals must include the other.

Theorem 47 (26) $\mathcal{O}(\mathsf{S}) \,\&\, \mathcal{O}(T) \to \mathsf{S} \subseteq T \vee T \subseteq \mathsf{S}$. PROOF:

Suppose_not(s, t) $\Rightarrow \quad \mathcal{O}(\mathsf{s}) \,\&\, \mathcal{O}(\mathsf{t}) \,\&\, \mathsf{s} \not\subseteq \mathsf{t} \,\&\, \mathsf{t} \not\subseteq \mathsf{s}$

-- For if not, neither of these ordinals is included in the other, so neither can equal the intersection of the two, which is an ordinal by Theorem 25.

$\langle \mathsf{s}, \mathsf{t} \rangle \hookrightarrow T25 \Rightarrow \quad \mathcal{O}(\mathsf{s} \cap \mathsf{t})$

-- It now follows, using Theorem 24 twice, that s * t is equal to both $\mathbf{arb}(\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t})$ and $\mathbf{arb}(\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t})$, and so, since neither of these sets is empty, is a member of both $\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t}$ and $\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t}$, which is impossible since the intersection of these two sets is empty. This contradiction proves our theorem.

$\langle \mathsf{s}, \mathsf{s} \cap \mathsf{t} \rangle \hookrightarrow T24 \Rightarrow \quad \mathsf{s} \cap \mathsf{t} = \mathbf{arb}(\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t})$
$\langle \mathsf{t}, \mathsf{s} \cap \mathsf{t} \rangle \hookrightarrow T24 \Rightarrow \quad \mathsf{s} \cap \mathsf{t} = \mathbf{arb}(\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t})$
ELEM $\Rightarrow \quad \mathsf{s} \backslash \mathsf{s} \cap \mathsf{t} \neq \emptyset$
ELEM $\Rightarrow \quad \mathsf{t} \backslash \mathsf{s} \cap \mathsf{t} \neq \emptyset$
$\langle \mathsf{s} \backslash \mathsf{s} \cap \mathsf{t} \rangle \hookrightarrow T0 \Rightarrow \quad (\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t} = \emptyset \,\&\, \mathbf{arb}(\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t}) = \emptyset) \vee$
$\mathbf{arb}(\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t}) \in \mathsf{s} \backslash \mathsf{s} \cap \mathsf{t} \,\&\, \mathbf{arb}(\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t}) \cap (\mathsf{s} \backslash \mathsf{s} \cap \mathsf{t}) = \emptyset$
ELEM $\Rightarrow \quad \mathsf{s} \cap \mathsf{t} \in \mathsf{s}$
$\langle \mathsf{t} \backslash \mathsf{s} \cap \mathsf{t} \rangle \hookrightarrow T0 \Rightarrow \quad (\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t} = \emptyset \,\&\, \mathbf{arb}(\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t}) = \emptyset) \vee$
$\mathbf{arb}(\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t}) \in \mathsf{t} \backslash \mathsf{s} \cap \mathsf{t} \,\&\, \mathbf{arb}(\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t}) \cap (\mathsf{t} \backslash \mathsf{s} \cap \mathsf{t}) = \emptyset$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following corollary to the preceding theorem asserts that the union of two ordinals is an ordinal and that the intersection of two ordinals is an ordinal.

Theorem 48 (27) $\mathcal{O}(\mathsf{S}) \,\&\, \mathcal{O}(T) \to \mathcal{O}(\mathsf{S} \cup T) \,\&\, \mathcal{O}(\mathsf{S} \cap T)$. PROOF:

Suppose_not(s, t) $\Rightarrow \quad \mathcal{O}(\mathsf{s}) \,\&\, \mathcal{O}(\mathsf{t}) \,\&\, \neg\big(\mathcal{O}(\mathsf{s} \cup \mathsf{t}) \,\&\, \mathcal{O}(\mathsf{s} \cap \mathsf{t})\big)$
$\langle \mathsf{s}, \mathsf{t} \rangle \hookrightarrow T26 \Rightarrow \quad \mathsf{s} \subseteq \mathsf{t} \vee \mathsf{t} \subseteq \mathsf{s}$
Suppose $\Rightarrow \quad \mathsf{s} \subseteq \mathsf{t}$
ELEM $\Rightarrow \quad \mathsf{s} \cup \mathsf{t} = \mathsf{t} \,\&\, \mathsf{s} \cap \mathsf{t} = \mathsf{s}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $s \cup t = s \;\&\; s \cap t = t$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the family of all ordinals (we will see shortly that this is not a set) is linearly ordered by membership.

Theorem 49 (28) $\mathcal{O}(S) \;\&\; \mathcal{O}(T) \rightarrow S \in T \vee T \in S \vee S = T$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\mathcal{O}(s) \;\&\; \mathcal{O}(t) \;\&\; \neg(s \in t \vee t \in s \vee s = t)$

-- For if we suppose the contrary, and note that by Theorem 26 one must include the other but not be equal to it, it follows (by the axiom of choice) that one must be a member of the other, a contradiction which proves our theorem.

$\langle s, t\rangle \hookrightarrow T26 \Rightarrow$ $Stat1: \; s \subseteq t \vee t \subseteq s$

$\langle s, t\rangle \hookrightarrow T24 \Rightarrow$ $t \subseteq s \rightarrow t = \mathbf{arb}(s \backslash t)$

$\langle t, s\rangle \hookrightarrow T24 \Rightarrow$ $s \subseteq t \rightarrow s = \mathbf{arb}(t \backslash s)$

-- ELEM $\Rightarrow$ (t = arb (s-t) or s = arb (t-s))

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The successor of an ordinal has a simple and very general definition:

DEF 11. $\mathsf{next}(X) \quad =_{\text{Def}} \quad X \cup \{X\}$

-- It is easy to show that this successor is an ordinal.

Theorem 50 (29) $\mathcal{O}(S) \rightarrow \mathcal{O}\big(\mathsf{next}(S)\big)$. PROOF:

Suppose_not(s) $\Rightarrow$ $\mathcal{O}(s) \;\&\; \neg\mathcal{O}\big(\mathsf{next}(s)\big)$

-- For if we suppose the contrary, and use the definition of ordinals, we see that there must exist a, b, c such that a is a member of $s \cup \{s\}$ but not included in it, or b and c are both members of $s \cup \{s\}$, but are unrelated by membership.

Use_def(next) $\Rightarrow$ $\mathcal{O}(s) \;\&\; \neg\mathcal{O}(s \cup \{s\})$

Use_def($\mathcal{O}$) $\Rightarrow$ $Stat1: \; \neg(\langle \forall x \in s \cup \{s\} \mid x \subseteq s \cup \{s\}\rangle \;\&\; \langle \forall x \in s \cup \{s\}, y \in s \cup \{s\} \mid x \in y \vee y \in x \vee x = y\rangle)$

$\langle a, b, c\rangle \hookrightarrow Stat1 \Rightarrow$ $(a \in s \cup \{s\} \;\&\; a \not\subseteq s \cup \{s\}) \vee$

$\mathsf{b}, \mathsf{c} \in \mathsf{s} \cup \{\mathsf{s}\} \;\&\; \neg(\mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{b} = \mathsf{c})$

-- Since the cases $\mathsf{a} = \mathsf{s}$, $\mathsf{b} = \mathsf{s}$, and $\mathsf{c} = \mathsf{s}$ are all impossible, we must either have $\mathsf{a} \in \mathsf{s}$ or $\mathsf{b} \in \mathsf{s}$ and $\mathsf{c} \in \mathsf{s}$.

ELEM $\Rightarrow$ $(\mathsf{a} \in \mathsf{s} \;\&\; \mathsf{a} \not\subseteq \mathsf{s}) \vee \big(\mathsf{b}, \mathsf{c} \in \mathsf{s} \;\&\; \neg(\mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{b} = \mathsf{c})\big)$

-- But both of these cases are impossible since s is an ordinal, a contradiction proving our theorem.

Use_def$(\mathcal{O})$ $\Rightarrow$ *Stat2* : $\big\langle \forall \mathsf{x} \in \mathsf{s} \,|\, \mathsf{x} \subseteq \mathsf{s} \big\rangle \;\&\; \big\langle \forall \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{s} \,|\, \mathsf{x} \in \mathsf{y} \vee \mathsf{y} \in \mathsf{x} \vee \mathsf{x} = \mathsf{y} \big\rangle$
$\big\langle \mathsf{a}, \mathsf{b}, \mathsf{c} \big\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{b} \in \mathsf{c} \vee \mathsf{c} \in \mathsf{b} \vee \mathsf{b} = \mathsf{c}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Sometimes it is useful to have the preceding theorem in the following more direct form.

Theorem 51 (30) $\mathcal{O}(\mathsf{S}) \rightarrow \mathcal{O}(\mathsf{S} \cup \{\mathsf{S}\})$. PROOF:

Suppose_not(s) $\Rightarrow$ $\mathcal{O}(\mathsf{s}) \;\&\; \neg\mathcal{O}(\mathsf{s} \cup \{\mathsf{s}\})$
Use_def(next) $\Rightarrow$ $\mathcal{O}(\mathsf{s}) \;\&\; \neg\mathcal{O}\big(\mathsf{next}(\mathsf{s})\big)$
$\big\langle \mathsf{s} \big\rangle \hookrightarrow$ *T29* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem shows that for ordinals inclusion is equivalent to the disjunction of identity and membership.

Theorem 52 (31) $\mathcal{O}(\mathsf{S}) \;\&\; \mathcal{O}(T) \rightarrow (T \subseteq \mathsf{S} \leftrightarrow T \in \mathsf{S} \vee T = \mathsf{S})$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\mathcal{O}(\mathsf{s}) \;\&\; \mathcal{O}(\mathsf{t}) \;\&\; (\mathsf{t} \not\subseteq \mathsf{s} \;\&\; \mathsf{t} \in \mathsf{s} \vee \mathsf{t} = \mathsf{s}) \vee \big(\mathsf{t} \subseteq \mathsf{s} \;\&\; \neg(\mathsf{t} \in \mathsf{s} \vee \mathsf{t} = \mathsf{s})\big)$

-- For in the contrary case there must exist two ordinals s and t such that either t is a member but not a subset of s, or t is a subset of s but neither a member of, or equal to, s;

ELEM $\Rightarrow$ $(\mathsf{t} \not\subseteq \mathsf{s} \;\&\; \mathsf{t} \in \mathsf{s}) \vee \big(\mathsf{t} \subseteq \mathsf{s} \;\&\; \neg(\mathsf{t} \in \mathsf{s} \vee \mathsf{t} = \mathsf{s})\big)$

-- but the first case is ruled out by definition of ordinal and the second case by Theorem 24, proving our theorem.

$\big\langle \mathsf{s}, \mathsf{t} \big\rangle \hookrightarrow$ *T12* $\Rightarrow$ $\mathsf{t} \subseteq \mathsf{s} \;\&\; \mathsf{t} \notin \mathsf{s} \;\&\; \mathsf{t} \neq \mathsf{s}$
$\big\langle \mathsf{s}, \mathsf{t} \big\rangle \hookrightarrow$ *T24* $\Rightarrow$ $\mathsf{t} = \mathbf{arb}(\mathsf{s} \backslash \mathsf{t})$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is sometimes convenient to use theorem in the following modified form.

Theorem 53 (32) $\mathcal{O}(\mathsf{S})\ \&\ \mathcal{O}(T) \rightarrow (T \notin \mathsf{S} \leftrightarrow \mathsf{S} \subseteq T)$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\mathcal{O}(\mathsf{s})\ \&\ \mathcal{O}(\mathsf{t})\ \&\ \mathsf{t} \notin \mathsf{s}\ \&\ \mathsf{s} \not\subseteq \mathsf{t}$

-- Since $\mathsf{t} \in \mathsf{s}\ \&\ \mathsf{s} \subseteq \mathsf{t}$ is impossible, a counterexample to our assertion must satisfy $\mathsf{t} \notin \mathsf{s}\ \&\ \neg\mathsf{s} \subseteq \mathsf{t}$. But by theorem 28 we then have $\mathsf{s} \subseteq \mathsf{t}$, a contradiction which proves the present corollary.

$\langle \mathsf{s}, \mathsf{t} \rangle \hookrightarrow T28 \Rightarrow$ $\mathsf{s} \in \mathsf{t} \vee \mathsf{s} = \mathsf{t}$
$\langle \mathsf{t}, \mathsf{s} \rangle \hookrightarrow T31 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now we start to prepare for proof of the basic ordinal enumerability theorem, Theorem 41 below. Our first step is to prove that both the collection of all sets and the collection of all ordinals are 'too large' to be sets. The following theorem gives the first of these results.

-- The class of all sets is not a set

Theorem 54 (33) $\neg\langle \exists \mathsf{x}, \forall \mathsf{y} \mid \mathsf{y} \in \mathsf{x} \rangle$. PROOF:

Suppose_not $\Rightarrow$ *Stat1* : $\langle \exists \mathsf{x}, \forall \mathsf{y} \mid \mathsf{y} \in \mathsf{x} \rangle$
$\langle \mathsf{u} \rangle \hookrightarrow Stat1 \Rightarrow$ *Stat2* : $\langle \forall \mathsf{y} \mid \mathsf{y} \in \mathsf{u} \rangle$

-- For, in the contrary case, consider the set u of all sets. This u would be a member of itself, whereas the axiom of choice forbids membership loops. A derivation of this fact from 'first principles' would be: $\{\mathsf{u}\} \hookrightarrow$T0 $\Rightarrow \mathbf{arb}(\{\mathsf{u}\}) \in \{\mathsf{u}\}\ \&\ \mathbf{arb}(\{\mathsf{u}\}) \cap \{\mathsf{u}\} = \emptyset$ ELEM $\Rightarrow \mathbf{arb}(\{\mathsf{u}\}) = \mathsf{u}\ \&\ \mathsf{u} \notin \mathsf{u}$ u $\hookrightarrow$Stat2 $\Rightarrow$ false; Discharge $\Rightarrow$ QED In our inference environment, the following abridged proof suffices:

$\langle \mathsf{u} \rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{u} \in \mathsf{u}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- There is no antinomic Russell 's set

Theorem 55 (34) $\neg\langle \exists \mathsf{x}, \forall \mathsf{y} \mid \mathsf{y} \in \mathsf{x} \leftrightarrow \mathsf{y} \notin \mathsf{y} \rangle$. PROOF:

$\mathsf{Suppose\_not} \Rightarrow \quad \mathit{Stat1}: \ \langle \exists x, \forall y \mid y \in x \leftrightarrow y \notin y \rangle$
$\langle a \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathit{Stat2}: \ \langle \forall y \mid y \in a \leftrightarrow y \notin y \rangle$

-- For in the contrary case consider the set a of all sets which are not members of themselves. Then a cannot be a member of itself, or fail to be a member of itself. This impossibility proves our theorem.

$\langle a \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad a \in a \leftrightarrow a \notin a$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that the class of ordinals is not a set

-- The class of ordinals is not a set

Theorem 56 (35) $\neg \langle \forall x \mid x \in \mathsf{OS} \leftrightarrow \mathcal{O}(x) \rangle$. PROOF:

$\mathsf{Suppose\_not}(o) \Rightarrow \quad \mathit{Stat1}: \ \langle \forall x \mid \mathcal{O}(x) \leftrightarrow x \in o \rangle$

-- For suppose the contrary, so that there is a set o consisting of all ordinals. But we can show that o must be an ordinal. Indeed, if it were not, then by the definition of ordinals there would exist a, b, c such that either a was a member but not a subset of o, or b and c are two members of o not related by membership.

$\mathsf{Suppose} \Rightarrow \quad \neg \mathcal{O}(o)$
$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad \mathit{Stat2}: \ \neg(\langle \forall x \in o \mid x \subseteq o \rangle \ \& \ \langle \forall x \in o, y \in o \mid x \in y \vee y \in x \vee x = y \rangle)$
$\langle a, b, c \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad (a \in o \ \& \ a \nsubseteq o) \vee \big(b, c \in o \ \& \ \neg(b \in c \vee c \in b \vee b = c)\big)$

-- In the second of these cases b and c are both plainly ordinals. so that this case is ruled out by Theorem 28. Hence only the first case need be considered.

$\mathsf{Suppose} \Rightarrow \quad b, c \in o \ \& \ \neg(b \in c \vee c \in b \vee b = c)$
$\langle b \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathcal{O}(b)$
$\langle c \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathcal{O}(c)$
$\langle b, c \rangle \hookrightarrow \mathit{T28} \Rightarrow \quad \mathit{Stat3}: \ a \in o \ \& \ a \nsubseteq o$

-- In this second case the set a, which must plainly be an ordinal, must have a member d which is not in o, and hence not an ordinal, which is impossible by Stat4 4 above, so our theorem is proved.

$\langle a \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathcal{O}(a)$
$\langle d \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad d \in a \ \& \ d \notin o$

$\langle a, d \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(d)$
$\langle d \rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- We now give the following transfinite recursive definition, which introduces the function that will be shown to put every set in 1-1 correspondence with some ordinal.

-- The enumeration of a set

DEF 9. $\quad \mathsf{enum}(X, Y) \;=_{\mathrm{Def}}\;$ **if** $Y \subseteq \{\mathsf{enum}(y, Y) : y \in X\}$ **then** Y **else** $\mathbf{arb}(Y \setminus \{\mathsf{enum}(y, Y) : y \in X\})$ **fi**

-- To begin our work toward the culminating Theorem 41 seen below, we first show that if a set s is a member of $b =_{\mathrm{Def}} \{\mathsf{enum}(y, s) : y \in x\}$ for some ordinal x, so is every one of the members of s.

Theorem 57 (36) $\mathcal{O}(X) \;\&\; S \in \{\mathsf{enum}(y, S) : y \in X\} \rightarrow S \subseteq \{\mathsf{enum}(y, S) : y \in X\}$. PROOF:

Suppose_not$(x, s) \Rightarrow \quad \mathcal{O}(x) \;\&\; Stat1 : s \in \{\mathsf{enum}(y, s) : y \in x\} \;\&\; s \not\subseteq \{\mathsf{enum}(u, s) : u \in x\}$

-- For suppose not. Then there is a v in x such that $s = \mathsf{enum}(v, s)$, but s cannot be a subset of $\{\mathsf{enum}(u, s) : u \in v\}$, so by definition of enum, $\mathsf{enum}(v, s) = \mathbf{arb}(s \setminus \{\mathsf{enum}(z, s) : z \in v\})$, which is impossible, since it wold imply $s \in s$.

$\langle v \rangle \hookrightarrow Stat1 \Rightarrow \quad s = \mathsf{enum}(v, s) \;\&\; v \in x$
$\langle x, v \rangle \hookrightarrow T12 \Rightarrow \quad v \subseteq x$
Set_monot $\Rightarrow \quad \{\mathsf{enum}(u, s) : u \in x\} \supseteq \{\mathsf{enum}(u, s) : u \in v\}$
ELEM $\Rightarrow \quad s \not\subseteq \{\mathsf{enum}(z, s) : z \in v\}$
Use_def(enum) $\Rightarrow \quad \mathsf{enum}(v, s) =$ **if** $s \subseteq \{\mathsf{enum}(z, s) : z \in v\}$ **then** s **else** $\mathbf{arb}(s \setminus \{\mathsf{enum}(z, s) : z \in v\})$ **fi**
ELEM $\Rightarrow \quad \mathsf{enum}(v, s) = \mathbf{arb}(s \setminus \{\mathsf{enum}(z, s) : z \in v\})$
ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- It is also easy to show that for any x, $\mathsf{enum}(x, s)$ is either s or a member of s.

Theorem 58 (37) $\mathsf{enum}(X, S) = S \vee \mathsf{enum}(X, S) \in S$. PROOF:

Suppose_not$(x, s) \Rightarrow \quad \mathsf{enum}(x, s) \neq s \;\&\; \mathsf{enum}(x, s) \notin s$

-- For in the contrary case, $s \subseteq \{\mathsf{enum}(y, s) : y \in x\}$ must be false, so $\mathsf{enum}(x, s) \in s$ must be true by definition of enum and by the axiom of choice.

Use_def(enum) $\Rightarrow \quad \mathsf{enum}(x, s) =$ **if** $s \subseteq \{\mathsf{enum}(y, s) : y \in x\}$ **then** s **else** $\mathbf{arb}(s \setminus \{\mathsf{enum}(y, s) : y \in x\})$ **fi**
ELEM $\Rightarrow \quad s \setminus \{\mathsf{enum}(y, s) : y \in x\} \neq \emptyset$

ELEM $\Rightarrow$ $\mathsf{enum}(x,s) = \mathbf{arb}(s \setminus \{\mathsf{enum}(y,s) : y \in x\})$
$\langle s \setminus \{\mathsf{enum}(y,s) : y \in x\} \rangle \hookrightarrow T0 \Rightarrow$ $\mathbf{arb}(s \setminus \{\mathsf{enum}(y,s) : y \in x\}) \in s \setminus \{\mathsf{enum}(y,s) : y \in x\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that if $\mathsf{enum}(x,s)$ equals s for any x, it also equals s for any larger y:

Theorem 59 (38) $\mathsf{enum}(X,S) = S \;\&\; Y \supseteq X \rightarrow \mathsf{enum}(Y,S) = S$. PROOF:

Suppose_not(x, s, w) $\Rightarrow$ $\mathsf{enum}(x,s) = s \;\&\; w \supseteq x \;\&\; \mathsf{enum}(w,s) \neq s$

-- For in the contrary case, we must have $s \subseteq \{\mathsf{enum}(u,s) : u \in x\}$ by definition of enum and the axiom of choice.

Use_def(enum) $\Rightarrow$ $\mathsf{enum}(x,s) = \mathbf{if}\ s \subseteq \{\mathsf{enum}(u,s) : u \in x\}\ \mathbf{then}\ s\ \mathbf{else}\ \mathbf{arb}(s \setminus \{\mathsf{enum}(u,s) : u \in x\})\ \mathbf{fi}$
Suppose $\Rightarrow$ $s \not\subseteq \{\mathsf{enum}(u,s) : u \in x\}$
ELEM $\Rightarrow$ $s \setminus \{\mathsf{enum}(y,s) : y \in x\} \neq \emptyset$
$\langle s \setminus \{\mathsf{enum}(u,s) : u \in x\} \rangle \hookrightarrow T0 \Rightarrow$ $\mathbf{arb}(s \setminus \{\mathsf{enum}(u) : u \in x\}) \in s \setminus \{\mathsf{enum}(u,s) : u \in x\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $s \subseteq \{\mathsf{enum}(u,s) : u \in x\}$

-- Thus $s \subseteq \{\mathsf{enum}(u,s) : u \in w\}$ by set monotonicity, and so $\mathsf{enum}(w,s) = s$, a contradiction which proves our theorem.

Set_monot $\Rightarrow$ $\{\mathsf{enum}(u,s) : u \in x\} \subseteq \{\mathsf{enum}(u,s) : u \in w\}$
ELEM $\Rightarrow$ $s \subseteq \{\mathsf{enum}(u,s) : u \in w\}$
Use_def(enum) $\Rightarrow$ $\mathsf{enum}(w,s) = \mathbf{if}\ s \subseteq \{\mathsf{enum}(u,s) : u \in w\}\ \mathbf{then}\ s\ \mathbf{else}\ \mathbf{arb}(s \setminus \{\mathsf{enum}(u,s) : u \in w\})\ \mathbf{fi}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following result tells us that, if we confine ourselves to the range of ordinals x for which $s \notin \{\mathsf{enum}(u,s) : u \in x\}$, the map $x \mapsto \mathsf{enum}(x,s)$ is one-to-one:

-- The enumeration of a set is 1 - 1

Theorem 60 (39) $\mathcal{O}(X) \;\&\; \mathcal{O}(W) \;\&\; X \neq W \rightarrow S \in \{\mathsf{enum}(u,S) : u \in X\} \vee S \in \{\mathsf{enum}(u,S) : u \in W\} \vee \mathsf{enum}(X,S) \neq \mathsf{enum}(W,S)$. PROOF:

Suppose_not(x, zz, s) $\Rightarrow$ $\mathcal{O}(x) \;\&\; \mathcal{O}(zz) \;\&\; x \neq zz \;\&\; \mathit{Stat1} : s \notin \{\mathsf{enum}(u,s) : u \in x\} \;\&\; \mathit{Stat2} : s \notin \{\mathsf{enum}(u,s) : u \in zz\} \;\&\; \mathsf{enum}(x,s) = \mathsf{enum}(zz,s)$

-- For if not, there are two distinct ordinals x and zz in the stated range such that $\mathsf{enum}(x,s) = \mathsf{enum}(zz,s)$. Theorem 28 tells us that one of the ordinals x and zz must be a member of the other. First suppose that $zz \in x$, so that, by Stat3 3, $\mathsf{enum}(zz,s) \neq s$, and therefore $\mathsf{enum}(x,s) \neq s$.

$\langle x, zz\rangle \hookrightarrow T28 \Rightarrow \quad x \in zz \vee zz \in x$
Suppose $\Rightarrow \quad zz \in x$
$\langle zz\rangle \hookrightarrow Stat1 \Rightarrow \quad enum(zz, s) \neq s$

-- Then it follows from the definition of enum that s cannot be a member of $\{enum(u, s) : u \in x\}$, so that $enum(x, s) = \mathbf{arb}(s \setminus \{enum(y, s) : y \in x\})$, in which case the axiom of choice tells us that $enum(x, s)$ is in $s \setminus \{enum(y, s) : y \in x\}$. But then, since $enum(zz, s)$ is clearly a member of $\{enum(y, s) : y \in x\}$, $enum(x, s)$ and $enum(zz, s)$ must be different, contrary to assumption.

Use_def(enum) $\Rightarrow \quad enum(x, s) = \mathbf{if}\ s \subseteq \{enum(u, s) : u \in x\}\ \mathbf{then}\ s\ \mathbf{else}\ \mathbf{arb}(s \setminus \{enum(y, s) : y \in x\})\ \mathbf{fi}$
ELEM $\Rightarrow \quad s \not\subseteq \{enum(u, s) : u \in x\}\ \&\ enum(x, s) = \mathbf{arb}(s \setminus \{enum(y, s) : y \in x\})$
$\langle s \setminus \{enum(u, s) : u \in x\}\rangle \hookrightarrow T0 \Rightarrow \quad \mathbf{arb}(s \setminus \{enum(u, s) : u \in x\}) \in s \setminus \{enum(u, s) : u \in x\}$
Suppose $\Rightarrow \quad Stat4 : enum(zz, s) \notin \{enum(y, s) : y \in x\}$
$\langle zz\rangle \hookrightarrow Stat4 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad enum(zz, s) \in \{enum(u, s) : u \in x\}$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad x \in zz$

-- This leaves us with the case $x \in zz$ to consider: this can be treated symmetrically, and another contradiction derived, thereby proving the present theorem.

$\langle x\rangle \hookrightarrow Stat2 \Rightarrow \quad enum(x, s) \neq s$
Use_def(enum) $\Rightarrow \quad enum(zz, s) = \mathbf{if}\ s \subseteq \{enum(u, s) : u \in zz\}\ \mathbf{then}\ s\ \mathbf{else}\ \mathbf{arb}(s \setminus \{enum(y, s) : y \in zz\})\ \mathbf{fi}$
ELEM $\Rightarrow \quad s \not\subseteq \{enum(u, s) : u \in zz\}\ \&\ enum(zz, s) = \mathbf{arb}(s \setminus \{enum(y, s) : y \in zz\})$
$\langle s \setminus \{enum(u, s) : u \in zz\}\rangle \hookrightarrow T0 \Rightarrow \quad \mathbf{arb}(s \setminus \{enum(u, s) : u \in zz\}) \in s \setminus \{enum(u, s) : u \in zz\}$
Suppose $\Rightarrow \quad Stat5 : enum(x, s) \notin \{enum(y, s) : y \in zz\}$
$\langle x\rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad enum(x, s) \in \{enum(u, s) : u \in zz\}$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad$ QED

-- Next we prove that, for each set s, there must exist some ordinal x for which s belongs to $\{enum(y, s) : y \in x\}$. This is done by showing that in the contrary case the collection of all ordinals would be a set, which we have already shown to be false.

-- Enumeration Lemma

Theorem 61 (40) $\langle \exists x \mid \mathcal{O}(x)\ \&\ S \in \{enum(y, S) : y \in x\}\rangle$. PROOF:

Suppose_not(s) $\Rightarrow \quad Stat1 : \neg\langle \exists x \mid \mathcal{O}(x)\ \&\ s \in \{enum(u, S) : u \in x\}\rangle$

-- We proceed by contradiction. If our theorem is false, there must exist a set s such that $s \notin \{enum(y, s) : y \in x\}$ for every ordinal x. In this case, Theorem 39 tells us that $enum(x, s) \neq enum(y, s)$ for every distinct pair x, y of ordinals.

$\text{Suppose} \Rightarrow \quad Stat2: \neg\langle\forall x, y \mid \mathcal{O}(x) \,\&\, \mathcal{O}(y) \rightarrow \mathsf{enum}(x,s), \mathsf{enum}(y,s) \in s \,\&\, x \neq y \rightarrow \mathsf{enum}(x,s) \neq \mathsf{enum}(y,s)\rangle$
$\langle x, y\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathcal{O}(x) \,\&\, \mathcal{O}(y) \,\&\, \mathsf{enum}(x,s), \mathsf{enum}(y,s) \in s \,\&\, x \neq y \,\&\, \mathsf{enum}(x,s) = \mathsf{enum}(y,s)$
$\langle x\rangle \hookrightarrow Stat1 \Rightarrow \quad s \notin \{\mathsf{enum}(u,s) : u \in x\}$
$\langle y\rangle \hookrightarrow Stat1 \Rightarrow \quad s \notin \{\mathsf{enum}(u,s) : u \in y\}$
$\langle x, y, s\rangle \hookrightarrow T39 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat3: \langle\forall x, y \mid \mathcal{O}(x) \,\&\, \mathcal{O}(y) \rightarrow \mathsf{enum}(x,s), \mathsf{enum}(y,s) \in s \,\&\, x \neq y \rightarrow \mathsf{enum}(x,s) \neq \mathsf{enum}(y,s)\rangle$

-- We shall now show that s is not a subset of $\{\mathsf{enum}(y,s) : y \in o_1\}$ for any ordinal o_1. For if $s \subseteq \{\mathsf{enum}(y,s) : y \in o_1\}$, then by definition of enum, $\mathsf{enum}(o_1, s) = s$, and hence s in {enum (x, s): s in next (o1)}, contradicting Stat4 4 above. It follows by a second use of the definition of enum that $\mathsf{enum}(o_1,s) = \mathbf{arb}(s \setminus \{\{\mathsf{enum}(y,s) : y \in o_1\}\})$, so that $\mathsf{enum}(o_1,s) \in s$ for every ordinal o_1.

$\text{Suppose} \Rightarrow \quad Stat5: \neg\langle\forall o_1 \mid \mathcal{O}(o_1) \rightarrow s \not\subseteq \{\mathsf{enum}(y,s) : y \in o_1\}\rangle$
$\langle o_2\rangle \hookrightarrow Stat5 \Rightarrow \quad \mathcal{O}(o_2) \,\&\, s \subseteq \{\mathsf{enum}(y,s) : y \in o_2\}$
$\text{Use\_def}(\mathsf{enum}) \Rightarrow \quad \mathsf{enum}(o_2,s) = \textbf{if } s \subseteq \{\mathsf{enum}(y,s) : y \in o_2\} \textbf{ then } s \textbf{ else } \mathbf{arb}(s \setminus \{\mathsf{enum}(y,s) : y \in o_2\}) \textbf{ fi}$
$\text{ELEM} \Rightarrow \quad \mathsf{enum}(o_2,s) = s$
$\langle \mathsf{next}(o_2)\rangle \hookrightarrow Stat1 \Rightarrow \quad \neg\Big(\mathcal{O}(\mathsf{next}(o_2)) \,\&\, s \in \{\mathsf{enum}(u,S) : u \in \mathsf{next}(o_2)\}\Big)$
$\langle o_2\rangle \hookrightarrow T29 \Rightarrow \quad Stat6: s \notin \{\mathsf{enum}(u,S) : u \in \mathsf{next}(o_2)\}$
$\langle o_2\rangle \hookrightarrow Stat6 \Rightarrow \quad o_2 \notin \mathsf{next}(o_2)$
$\text{Use\_def}(\mathsf{next}) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat7: \langle\forall o_1 \mid \mathcal{O}(o_1) \rightarrow s \not\subseteq \{\mathsf{enum}(y,s) : y \in o_1\}\rangle$
$\text{Suppose} \Rightarrow \quad Stat8: \neg\langle\forall o_1 \mid \mathcal{O}(o_1) \rightarrow \mathsf{enum}(o_1,s) \in s\rangle$
$\langle o_3\rangle \hookrightarrow Stat8 \Rightarrow \quad \mathcal{O}(o_3) \,\&\, \mathsf{enum}(o_3,s) \notin s$
$\langle o_3\rangle \hookrightarrow Stat7 \Rightarrow \quad s \not\subseteq \{\mathsf{enum}(y,s) : y \in o_3\}$
$\text{Use\_def}(\mathsf{enum}) \Rightarrow \quad \mathsf{enum}(o_3,s) = \textbf{if } s \subseteq \{\mathsf{enum}(y,s) : y \in o_3\} \textbf{ then } s \textbf{ else } \mathbf{arb}(s \setminus \{\mathsf{enum}(y,s) : y \in o_3\}) \textbf{ fi}$
$\text{ELEM} \Rightarrow \quad \mathsf{enum}(o_3,s) = \mathbf{arb}(s \setminus \{\mathsf{enum}(y,s) : y \in o_3\})$
$\langle s \setminus \{\mathsf{enum}(y,s) : y \in o_3\}\rangle \hookrightarrow T0 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat9: \langle\forall o_1 \mid \mathcal{O}(o_1) \rightarrow \mathsf{enum}(o_1,s) \in s\rangle$

-- Now consider the set t of all $x \in s$ having the form $\mathsf{enum}(o,s)$ for some ordinal o, so that for each $x \in t$ there is an ordinal o such that $x = \mathsf{enum}(o,s)$.

$\text{Loc\_def} \Rightarrow \quad t = \{x \in s \mid \langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = x\rangle\}$
$\text{Suppose} \Rightarrow \quad Stat10: \neg\langle\forall x \mid x \in t \rightarrow \langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = x\rangle\rangle$
$\langle a\rangle \hookrightarrow Stat10 \Rightarrow \quad a \in t \,\&\, \neg\langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = a\rangle$
$\text{ELEM} \Rightarrow \quad Stat11: a \in \{x \in s \mid \langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = x\rangle\}$
$\langle\,\rangle \hookrightarrow Stat11 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat12: \langle\forall x \mid x \in t \rightarrow \langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = x\rangle\rangle$
$\text{Suppose} \Rightarrow \quad Stat13: \neg\langle\forall x, \exists o \mid x \in t \rightarrow \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = x\rangle$
$\langle xa\rangle \hookrightarrow Stat13 \Rightarrow \quad Stat14: \neg\langle\exists o \mid xa \in t \rightarrow \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = xa\rangle$
$\langle xa\rangle \hookrightarrow Stat12 \Rightarrow \quad Stat15: xa \in t \rightarrow \langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = xa\rangle$
$\langle oa\rangle \hookrightarrow Stat14 \Rightarrow \quad \neg(xa \in t \rightarrow \mathcal{O}(oa) \,\&\, \mathsf{enum}(oa,s) = xa)$
$\langle Stat15\rangle \text{ ELEM} \Rightarrow \quad xa \in t \,\&\, Stat16: \langle\exists o \mid \mathcal{O}(o) \,\&\, \mathsf{enum}(o,s) = xa\rangle$

$\langle \mathsf{ob} \rangle \hookrightarrow Stat16 \Rightarrow \quad \mathcal{O}(\mathsf{ob})\ \&\ \mathsf{enum}(\mathsf{ob},\mathsf{s}) = \mathsf{xa}$
$\langle \mathsf{ob} \rangle \hookrightarrow Stat14 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat17:\ \langle \forall \mathsf{x}, \exists \mathsf{o} \mid \mathsf{x} \in \mathsf{t} \rightarrow \mathcal{O}(\mathsf{o})\ \&\ \mathsf{enum}(\mathsf{o},\mathsf{s}) = \mathsf{x} \rangle$

-- Skolemize this last statement, rewriting it in the following form:

APPLY $\langle \mathsf{v1}_\Theta : \mathsf{ord\_for} \rangle$ Skolem $\Rightarrow$
$Stat18:\ \langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{t} \rightarrow \mathcal{O}(\mathsf{ord\_for}(\mathsf{x}))\ \&\ \mathsf{enum}(\mathsf{ord\_for}(\mathsf{x}),\mathsf{s}) = \mathsf{x} \rangle$

-- We will show that every ordinal belongs to the set $\{\mathsf{ord\_for}(\mathsf{x}) : \mathsf{x} \in \mathsf{t}\}$. For suppose the contrary, and consider an ordinal o not in this set, which then plainly differs from $\mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s}))$;

$\mathsf{Suppose} \Rightarrow \quad Stat19:\ \neg\langle \forall \mathsf{o} \mid \mathcal{O}(\mathsf{o}) \rightarrow \mathsf{o} \in \{\mathsf{ord\_for}(\mathsf{x}) : \mathsf{x} \in \mathsf{t}\} \rangle$
$\langle \mathsf{o} \rangle \hookrightarrow Stat19 \Rightarrow \quad \mathcal{O}(\mathsf{o})\ \&\ Stat20:\ \mathsf{o} \notin \{\mathsf{ord\_for}(\mathsf{x}) : \mathsf{x} \in \mathsf{t}\}$
$\langle \mathsf{o} \rangle \hookrightarrow Stat3 \Rightarrow \quad \neg\langle \exists \mathsf{y} \mid \neg(\mathcal{O}(\mathsf{o})\ \&\ \mathcal{O}(\mathsf{y}) \rightarrow \mathsf{enum}(\mathsf{o},\mathsf{s}), \mathsf{enum}(\mathsf{y},\mathsf{s}) \in \mathsf{s}\ \&\ \mathsf{o} \neq \mathsf{y} \rightarrow \mathsf{enum}(\mathsf{o},\mathsf{s}) \neq \mathsf{enum}(\mathsf{y},\mathsf{s})) \rangle$
$\langle \mathsf{o} \rangle \hookrightarrow Stat9 \Rightarrow \quad Stat21:\ \mathsf{enum}(\mathsf{o},\mathsf{s}) \in \mathsf{s}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{enum}(\mathsf{o},\mathsf{s}) \notin \mathsf{t}$
$\mathsf{ELEM} \Rightarrow \quad Stat22:\ \mathsf{enum}(\mathsf{o},\mathsf{s}) \notin \{\mathsf{x} \in \mathsf{s} \mid \langle \exists \mathsf{oo} \mid \mathcal{O}(\mathsf{oo})\ \&\ \mathsf{enum}(\mathsf{oo},\mathsf{s}) = \mathsf{x} \rangle\}$
$\langle\ \rangle \hookrightarrow Stat22 \Rightarrow \quad Stat23:\ \neg\langle \exists \mathsf{oo} \mid \mathcal{O}(\mathsf{oo})\ \&\ \mathsf{enum}(\mathsf{oo},\mathsf{s}) = \mathsf{enum}(\mathsf{o},\mathsf{s}) \rangle$
$\langle \mathsf{o} \rangle \hookrightarrow Stat23 \Rightarrow \quad \neg(\mathcal{O}(\mathsf{o})\ \&\ \mathsf{enum}(\mathsf{o},\mathsf{s}) = \mathsf{enum}(\mathsf{o},\mathsf{s}))$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{enum}(\mathsf{o},\mathsf{s}) \in \mathsf{t}$
$\langle \mathsf{enum}(\mathsf{o},\mathsf{s}) \rangle \hookrightarrow Stat20 \Rightarrow \quad \mathsf{o} \neq \mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s}))$

-- by definition of the set t, $\mathsf{enum}(\mathsf{o},\mathsf{s})$ must belong to it, and hence to s.

$\mathsf{Suppose} \Rightarrow \quad \mathsf{enum}(\mathsf{o},\mathsf{s}) \notin \mathsf{t}$
$\mathsf{ELEM} \Rightarrow \quad Stat24:\ \mathsf{enum}(\mathsf{o},\mathsf{s}) \notin \{\mathsf{x} \in \mathsf{s} \mid \langle \exists \mathsf{o} \mid \mathcal{O}(\mathsf{o})\ \&\ \mathsf{enum}(\mathsf{o},\mathsf{s}) = \mathsf{x} \rangle\}$
$\langle \mathsf{enum}(\mathsf{o},\mathsf{s}) \rangle \hookrightarrow Stat24 \Rightarrow \quad Stat25:\ \neg(\mathsf{enum}(\mathsf{o},\mathsf{s}) \in \mathsf{s}\ \&\ \langle \exists \mathsf{oo} \mid \mathcal{O}(\mathsf{oo})\ \&\ \mathsf{enum}(\mathsf{oo},\mathsf{s}) = \mathsf{enum}(\mathsf{o},\mathsf{s}) \rangle)$
$\langle Stat21, Stat25, * \rangle\ \mathsf{ELEM} \Rightarrow \quad Stat26:\ \neg\langle \exists \mathsf{oo} \mid \mathcal{O}(\mathsf{oo})\ \&\ \mathsf{enum}(\mathsf{oo},\mathsf{s}) = \mathsf{enum}(\mathsf{o},\mathsf{s}) \rangle$
$\langle \mathsf{o} \rangle \hookrightarrow Stat26 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{enum}(\mathsf{o},\mathsf{s}) \in \mathsf{t}$
$\mathsf{ELEM} \Rightarrow \quad Stat27:\ \mathsf{enum}(\mathsf{o},\mathsf{s}) \in \{\mathsf{x} \in \mathsf{s} \mid \langle \exists \mathsf{o} \mid \mathcal{O}(\mathsf{o})\ \&\ \mathsf{enum}(\mathsf{o},\mathsf{s}) = \mathsf{x} \rangle\}$
$\langle\ \rangle \hookrightarrow Stat27 \Rightarrow \quad \mathsf{enum}(\mathsf{o},\mathsf{s}) \in \mathsf{s}$

-- Since the definition of $\mathsf{ord\_for}$ implies the following formula, the fact that enum is one-to-one on ordinals (Stat28) implies that $\mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s})) = \mathsf{o}$.

$\langle \mathsf{enum}(\mathsf{o},\mathsf{s}) \rangle \hookrightarrow Stat18 \Rightarrow \quad \mathcal{O}\Big(\mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s}))\Big)\ \&\ \mathsf{enum}\Big(\mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s})),\mathsf{s}\Big) = \mathsf{enum}(\mathsf{o},\mathsf{s})$
$\langle \mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s})), \mathsf{o} \rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{ord\_for}(\mathsf{enum}(\mathsf{o},\mathsf{s})) = \mathsf{o}$

-- This contradiction shows that every ordinal belongs to the set $\{\mathsf{ord\_for}(x) : x \in t\}$, contradicting the fact, proved as Theorem 35, that there can be no set to which all ordinals belong. This final contradiction proves the present theorem.

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat29*: $\langle \forall o \mid \mathcal{O}(o) \rightarrow o \in \{\mathsf{ord\_for}(u) : u \in t\}\rangle$
Suppose $\Rightarrow$ *Stat30*: $\neg\langle \forall o \mid \mathcal{O}(o) \leftrightarrow o \in \{\mathsf{ord\_for}(x) : x \in t\}\rangle$
$\langle os\rangle \hookrightarrow Stat30 \Rightarrow$ $\neg(\mathcal{O}(os) \leftrightarrow os \in \{\mathsf{ord\_for}(x) : x \in t\})$
$\langle os\rangle \hookrightarrow Stat29 \Rightarrow$ *Stat31*: $os \in \{\mathsf{ord\_for}(x) : x \in t\}$ & $\neg\mathcal{O}(os)$
$\langle xx\rangle \hookrightarrow Stat31 \Rightarrow$ $xx \in t$ & $os = \mathsf{ord\_for}(xx)$
$\langle xx\rangle \hookrightarrow Stat18 \Rightarrow$ $\mathcal{O}(\mathsf{ord\_for}(xx))$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\langle \forall o \mid \mathcal{O}(o) \leftrightarrow o \in \{\mathsf{ord\_for}(u) : u \in t\}\rangle$
$\langle \{\mathsf{ord\_for}(x) : x \in t\}\rangle \hookrightarrow T35 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We are now in position to prove the key fact that the function $\mathsf{enum}(x, s)$ puts any set s in one-to-one correspondence with an ordinal. The following theorem states this result formally.

-- Enumeration theorem

Theorem 62 (41) $\langle \exists x \mid \mathcal{O}(x)$ & $S = \{\mathsf{enum}(y, S) : y \in x\}$ & $\langle \forall y \in x, z \in x \mid y \neq z \rightarrow \mathsf{enum}(y, S) \neq \mathsf{enum}(z, S)\rangle\rangle$. PROOF:

Suppose_not(s) $\Rightarrow$ *Stat1*: $\neg\langle \exists x \mid \mathcal{O}(x)$ & $s = \{\mathsf{enum}(y, s) : y \in x\}$ & $\langle \forall y \in x, z \in x \mid y \neq z \rightarrow \mathsf{enum}(y, s) \neq \mathsf{enum}(z, s)\rangle\rangle$

-- We proceed by contradiction. Suppose that our theorem is false, so that no ordinal having the properties considered in the theorem exists. By Theorem 40, there exists an ordinal x such that s is in $\{\mathsf{enum}(y, s) : y \in x\}$, and hence an ordinal y such that $s = \mathsf{enum}(y, s)$.

$\langle s\rangle \hookrightarrow T40 \Rightarrow$ *Stat2*: $\langle \exists x \mid \mathcal{O}(x)$ & $s \in \{\mathsf{enum}(y, s) : y \in x\}\rangle$
$\langle x\rangle \hookrightarrow Stat2 \Rightarrow$ $\mathcal{O}(x)$ & *Stat3*: $s \in \{\mathsf{enum}(y, s) : y \in x\}$
$\langle w\rangle \hookrightarrow Stat3 \Rightarrow$ $s = \mathsf{enum}(w, s)$ & $w \in x$
$\langle x, w\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(w)$

-- By definition of enum, $\mathsf{enum}(w, s)$ is a member of s unless $s \subseteq \{\mathsf{enum}(u, s) : u \in w\}$ is true; since $s = \{\mathsf{enum}(u, s) : u \in w\}$ this is impossible, so we must have $s \subseteq \{\mathsf{enum}(u, s) : u \in w\}$.

Use_def(enum) $\Rightarrow$ $\mathsf{enum}(w, s) = \textbf{if } s \subseteq \{\mathsf{enum}(u, s) : u \in w\} \textbf{ then } s \textbf{ else arb}(s \setminus \{\mathsf{enum}(u, s) : u \in w\}) \textbf{ fi}$
Suppose $\Rightarrow$ $s \not\subseteq \{\mathsf{enum}(u, s) : u \in w\}$
ELEM $\Rightarrow$ $s \setminus \{\mathsf{enum}(u, s) : u \in w\} \neq \emptyset$
$\langle s \setminus \{\mathsf{enum}(u, s) : u \in w\}\rangle \hookrightarrow T0 \Rightarrow$ $\mathsf{enum}(w, s) \in s \setminus \{\mathsf{enum}(u, s) : u \in w\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $s \subseteq \{enum(u,s) : u \in w\}$

-- The principle of transfinite induction now tells us that there exists a minimal ordinal b such that $s \subseteq \{enum(y,s) : y \in b\}$. Since our theorem is false, either s must be different from $\{enum(y,s) : y \in b\}$, or the function $enum(.,s)$ cannot be 1-1 on b.

APPLY $\langle mt_\Theta : b\rangle$ transfinite_induction$\Big(n \mapsto w, P(x) \mapsto (\mathcal{O}(x) \,\&\, s \subseteq \{enum(u,s) : u \in x\})\Big) \Rightarrow$

Stat4 : $\langle \forall u \,|\, (\mathcal{O}(b) \,\&\, s \subseteq \{enum(y,s) : y \in b\}) \,\&\, (u \in b \rightarrow \neg\mathcal{O}(u) \,\&\, s \subseteq \{enum(vv,s) : vv \in u\})\rangle$

$\langle a_0\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\mathcal{O}(b) \,\&\, s \subseteq \{enum(y,s) : y \in b\}$

$\langle b\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $s \neq \{enum(y,s) : y \in b\} \vee \neg\langle \forall y \in b, z \in b \,|\, y \neq z \rightarrow enum(y,s) \neq enum(z,s)\rangle$

-- First suppose that $\{enum(y,s) : y \in b\} \neq s$, i. e. that the second of these sets does not include the first, so that there is a c in the first but not the second, and so a $d \in b$ such that $c = enum(d,s)$.

Suppose $\Rightarrow$ *Stat5* : $\{enum(y,s) : y \in b\} \not\subseteq s$

$\langle c\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ *Stat6* : $c \in \{enum(y,s) : y \in b\} \,\&\, c \notin s$

$\langle d\rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $c = enum(d,s) \,\&\, d \in b \,\&\, c \notin s$

-- Since b is an ordinal, d is also an ordinal, so Stat7 7 tells us that s is not a subset of $\{enum(y,s) : y \in d\}$, and thus by definition $c = enum(d,s)$ must be a member of s, a contradiction which tells us that $\{enum(y,s) : y \in b\}$ must be equal to s.

$\langle d\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\neg\mathcal{O}(d) \vee s \not\subseteq \{enum(y,s) : y \in d\}$

$\langle b,d\rangle \hookrightarrow$ *T11* $\Rightarrow$ $s \not\subseteq \{enum(y,s) : y \in d\} \,\&\, s \setminus \{enum(y,s) : y \in d\} \neq \emptyset$

Use_def(enum) $\Rightarrow$ $enum(d,s) = \textbf{if } s \subseteq \{enum(u,s) : u \in d\} \textbf{ then } s \textbf{ else arb}(s \setminus \{enum(u,s) : u \in d\}) \textbf{ fi}$

ELEM $\Rightarrow$ $enum(d,s) = \textbf{arb}(s \setminus \{enum(u,s) : u \in d\})$

$\langle s \setminus \{enum(u,s) : u \in d\}\rangle \hookrightarrow$ *T0* $\Rightarrow$ $enum(d,s) \in s$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $s = \{enum(y,s) : y \in b\}$

-- This leaves only the possibility that the function $enum(.,s)$ is not 1-1 on b, in which case there must exist two distinct ordinals v and w', both in b, such that $enum(v,s) = enum(w',s)$.

ELEM $\Rightarrow$ *Stat8* : $\neg\langle \forall y \in b, zz \in b \,|\, y \neq zz \rightarrow enum(y,s) \neq enum(zz,s)\rangle$

$\langle v,w'\rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $v,w' \in b \,\&\, v \neq w' \,\&\, enum(v,s) = enum(w',s)$

$\langle b,v\rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(v)$

$\langle b,w'\rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(w')$

-- But, by Theorem 39, this can only happen if s is in one of the sets $\{enum(y,s) : y \in v\}$ and $\{enum(y,s) : y \in w'\}$. However, both of these possibilities are ruled out by Stat7 7 if we take Theorem 36 into account. So we have a contradiction proving our theorem.

$\langle v, w', s\rangle \hookrightarrow T39 \Rightarrow \quad s \in \{\text{enum}(y, s) : y \in v\} \vee s \in \{\text{enum}(y, s) : y \in w'\}$
$\langle v\rangle \hookrightarrow Stat4 \Rightarrow \quad s \not\subseteq \{\text{enum}(x, s) : x \in v\}$
$\langle w'\rangle \hookrightarrow Stat4 \Rightarrow \quad s \not\subseteq \{\text{enum}(x, s) : x \in w'\}$
$\langle v, s\rangle \hookrightarrow T36 \Rightarrow \quad s \in \{\text{enum}(y, s) : y \in v\} \rightarrow s \subseteq \{\text{enum}(y, s) : y \in v\}$
$\langle w', s\rangle \hookrightarrow T36 \Rightarrow \quad s \in \{\text{enum}(y, s) : y \in w'\} \rightarrow s \subseteq \{\text{enum}(y, s) : y \in w'\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

5 Maps, map restrictions, and Cardinality

-- Our next main goal is to introduce the notion of cardinality and prove its properties. In working with this notion we will find use for the familiar mathematical ideas appearing in the following auxiliary definitions.

-- Map Restriction
DEF 12. $X_{|Y} =_{\text{Def}} \{p \in X \mid p^{[1]} \in Y\}$
-- Value of single - valued function
DEF 13. $X \upharpoonright Y =_{\text{Def}} \mathbf{arb}(X_{|\{Y\}})^{[2]}$

-- Map Product
DEF 14. $X \bullet Y =_{\text{Def}} \{[x^{[1]}, y^{[2]}] : x \in Y, y \in X \mid x^{[2]} = y^{[1]}\}$
-- Inverse Map
DEF 14a. $X^{\leftarrow} =_{\text{Def}} \{[x^{[2]}, x^{[1]}] : x \in X\}$
-- Identity Map
DEF 14b. $\iota_X =_{\text{Def}} \{[x, x] : x \in X\}$

-- We define the notion of 'the enumerating ordinal of a set' by Skolemizing Theorem 41. The formal definition is as follows.

APPLY $\langle v1_\Theta : \text{enum\_Ord}\rangle$ Skolem$\Rightarrow$

Theorem 63 (42) $\langle \forall s \mid \mathcal{O}(\text{enum\_Ord}(s))\ \&\ s = \{\text{enum}(y, s) : y \in \text{enum\_Ord}(s)\}\ \&\ \langle \forall y \in \text{enum\_Ord}(s), z \in \text{enum\_Ord}(s) \mid y \neq z \longrightarrow \text{enum}(y, s) \neq \text{enum}(z, s)\rangle\rangle$.

-- Using the preceding definition, we can define the Cardinality $\#s$ of a set s as the smallest ordinal in one-one correspondence with s. (The existence of such an ordinal follows from the fact that the set next (enum_Ord (s)) must contain at least one such.)

-- Cardinality

DEF 15. $\#X \quad =_{Def} \quad \mathbf{arb}\big(\{x :\ x \in \mathsf{next}\big(\mathsf{enum\_Ord}(X)\big) \mid \langle \exists f \mid 1\text{–}1(f) \ \&\ \mathbf{domain}(f) = x \ \&\ \mathbf{range}(f) = X \rangle\}\big)$

-- An ordinal c is said to be a cardinal if it cannot be seen as the range of any single valued map on a smaller ordinal. We shall see below that this is equivalent to the condition that c cannot be put into 1-1 correspondence with any smaller ordinal.

-- Cardinal

DEF 16. $\mathsf{Card}(X) \quad \leftrightarrow_{Def} \quad \mathcal{O}(X) \ \&\ \langle \forall y \in X, f \mid \mathbf{domain}(f) \neq y \vee \mathbf{range}(f) \neq X \vee \neg \mathsf{Svm}(f) \rangle$

-- In preparation with our work with cardinals we prove various small utility lemmas having to do with map restrictions, single-valued maps, 1-1 maps, map products and inverses, identity maps, etc. The first of these simply says that a restriction of a map f is a subset of f, a fact immediate if we use proof by monotonicity.

Theorem 64 (43) $F_{|A} \subseteq F$. PROOF:

Suppose_not$(f, a) \Rightarrow \quad f_{|a} \not\subseteq f$
Use_def$(|) \Rightarrow \quad \{p :\ p \in f \mid p^{[1]} \in a\} \not\subseteq f$
Set_monot $\Rightarrow \quad \{p :\ p \in f \mid p^{[1]} \in a\} \subseteq \{p :\ p \in f\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we note the even more elementary fact that the intersection of two sets can be written as a setformer.

Theorem 65 (44) $S \cap T = \{x \in S \mid x \in T\}$. PROOF:

Suppose_not$(s, t) \Rightarrow \quad$ *Stat1* : $s \cap t \neq \{x \in s \mid x \in t\}$
$\langle c \rangle \hookrightarrow$ *Stat1* $\Rightarrow \quad (c \in s \cap t \ \&\ c \notin \{x \in s \mid x \in t\}) \vee (c \notin s \cap t \ \&\ c \in \{x \in s \mid x \in t\})$
Suppose $\Rightarrow \quad$ *Stat2* : $c \notin \{x \in s \mid x \in t\} \ \&\ c \in s \cap t$
$\langle c \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3* : $c \in \{x \in s \mid x \in t\} \ \&\ c \notin s \cap t$
$\langle\, \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The existence of a similar setformer defining the difference of two sets is equally elementary.

Theorem 66 (45) $S \setminus T = \{x \in S \mid x \notin T\}$. PROOF:

$\mathsf{Suppose\_not}(s,t) \Rightarrow \quad \mathit{Stat1}:\ s\backslash t \neq \{x \in s \mid x \notin t\}$

$\langle c \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad (c \in s\backslash t \;\&\; c \notin \{x \in s \mid x \notin t\}) \vee (c \notin s\backslash t \;\&\; c \in \{x \in s \mid x \notin t\})$

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat2}:\ c \notin \{x \in s \mid x \notin t\} \;\&\; c \in s\backslash t$

$\langle c \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad c \notin s\backslash t \;\&\; \mathit{Stat3}:\ c \in \{x \in s \mid x \notin t\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next utility lemma puts the definition of the Is_map predicate into a sightly more convenient form by showing that every element of a map f is a pair, and that every set all of whose elements are pairs must be a map.

Theorem 67 (46) $\big(\mathsf{Is\_map}(F) \rightarrow X \in F \rightarrow X = \left[X^{[1]}, X^{[2]}\right]\big) \;\&\; \big(\langle \forall x \in F \mid x = \left[x^{[1]}, x^{[2]}\right] \rangle \rightarrow \mathsf{Is\_map}(F)\big)$. PROOF:

$\mathsf{Suppose\_not}(f,x) \Rightarrow \quad \big(\mathsf{Is\_map}(f) \;\&\; x \in f \;\&\; x \neq \left[x^{[1]}, x^{[2]}\right]\big) \vee \neg\big(\langle \forall x \in f \mid x = \left[x^{[1]}, x^{[2]}\right] \rangle \rightarrow \mathsf{Is\_map}(f)\big)$

-- We must consider one of two contrary cases. Start with the first, in which a map has an element which is not a pair. This is impossible by definition, only the second case need be considered.

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat0}:\ \mathsf{Is\_map}(f) \;\&\; x \in f \;\&\; x \neq \left[x^{[1]}, x^{[2]}\right]$

$\mathsf{Use\_def}(\mathsf{Is\_map}) \Rightarrow \quad \mathit{Stat1}:\ x \in \left\{\left[y^{[1]}, y^{[2]}\right] : y \in f\right\}$

$\langle c \rangle \hookrightarrow \mathit{Stat1}([\mathit{Stat1}, \cap]) \Rightarrow \quad \mathit{Stat2}:\ x = \left[c^{[1]}, c^{[2]}\right]$

$\langle \mathit{Stat2} \rangle\ \mathsf{ELEM} \Rightarrow \quad x = \left[x^{[1]}, x^{[2]}\right]$

$\langle \mathit{Stat0}, * \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat3}:\ \neg\mathsf{Is\_map}(f)$

-- In this second case we have a set of pairs, which by definition differs from the collection of all pairs in it, another impossibility. So our theorem is proved.

$\langle f \rangle \hookrightarrow T23(\langle \mathit{Stat3} \rangle) \Rightarrow \quad \mathit{Stat4}:\ f \not\subseteq \left\{\left[x^{[1]}, x^{[2]}\right] : x \in f\right\} \;\&\; \mathit{Stat5}:\ \langle \forall x \in f \mid x = \left[x^{[1]}, x^{[2]}\right] \rangle$

$\langle d \rangle \hookrightarrow \mathit{Stat4}(\langle \mathit{Stat4} \rangle) \Rightarrow \quad \mathit{Stat6}:\ d \in f \;\&\; \mathit{Stat7}:\ d \notin \left\{\left[x^{[1]}, x^{[2]}\right] : x \in f\right\}$

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat8}:\ \neg\langle \forall x \in f \mid d \neq \left[x^{[1]}, x^{[2]}\right] \rangle$

$\langle dd \rangle \hookrightarrow \mathit{Stat8}(\langle \mathit{Stat8} \rangle) \Rightarrow \quad \mathit{Stat9}:\ dd \in f \;\&\; d = \left[dd^{[1]}, dd^{[2]}\right]$

$\langle dd \rangle \hookrightarrow \mathit{Stat7}(\langle \mathit{Stat9} \rangle) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat10}:\ \langle \forall x \in f \mid d \neq \left[x^{[1]}, x^{[2]}\right] \rangle$

$\langle d \rangle \hookrightarrow \mathit{Stat10}([\mathit{Stat6}, \mathit{Stat6}]) \Rightarrow \quad \mathit{Stat11}:\ d \neq \left[d^{[1]}, d^{[2]}\right]$

$\langle d \rangle \hookrightarrow \mathit{Stat5}([\mathit{Stat6}, \mathit{Stat11}]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- As stated next, any subset of a map is a map. Once more we have only to use the definition, and then monotonicity.

Theorem 68 (47) $G \subseteq F \,\&\, \mathsf{Is\_map}(F) \rightarrow \mathsf{Is\_map}(G)$. PROOF:

$\mathsf{Suppose\_not}(g, f) \Rightarrow \quad g \subseteq f \,\&\, \mathsf{Is\_map}(f) \,\&\, \neg \mathsf{Is\_map}(g)$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat1}: \neg\langle \forall x \in f \,|\, x = [x^{[1]}, x^{[2]}] \rangle$
$\langle y \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad y \in f \,\&\, y \neq [y^{[1]}, y^{[2]}]$
$\langle f, y \rangle \hookrightarrow T46 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \langle \forall x \in f \,|\, x = [x^{[1]}, x^{[2]}] \rangle$
$\langle g, \mathsf{junk} \rangle \hookrightarrow T46 \Rightarrow \quad \neg\langle \forall x \in g \,|\, x = [x^{[1]}, x^{[2]}] \rangle$
$\mathsf{Pred\_monot} \Rightarrow \quad \langle \forall x \in f \,|\, x = [x^{[1]}, x^{[2]}] \rangle \rightarrow \langle \forall x \in g \,|\, x = [x^{[1]}, x^{[2]}] \rangle$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Similarly, any subset of a single-valued map is a single-valued map. Again, the proof is by monotonicity.

Theorem 69 (48) $G \subseteq F \,\&\, \mathsf{Svm}(F) \rightarrow \mathsf{Svm}(G)$. PROOF:

$\mathsf{Suppose\_not}(g, f) \Rightarrow \quad g \subseteq f \,\&\, \mathsf{Svm}(f) \,\&\, \neg \mathsf{Svm}(g)$
$\mathsf{Use\_def}(\mathsf{Svm}) \Rightarrow \quad \mathsf{Is\_map}(f) \,\&\, \langle \forall x \in f, y \in f \,|\, x^{[1]} = y^{[1]} \rightarrow x = y \rangle \,\&\, \neg(\mathsf{Is\_map}(g) \,\&\, \langle \forall x \in g, y \in g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y \rangle)$
$\langle g, f \rangle \hookrightarrow T47 \Rightarrow \quad \neg\langle \forall x \in g, y \in g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\mathsf{Pred\_monot} \Rightarrow \quad \langle \forall x \in f, y \in f \,|\, x^{[1]} = y^{[1]} \rightarrow x = y \rangle \rightarrow \langle \forall x \in g, y \in g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Proof by monotonicity also suffices to show that any subset of a 1-1 map is also a 1-1 map.

Theorem 70 (49) $G \subseteq F \,\&\, 1\text{–}1(F) \rightarrow 1\text{–}1(G)$. PROOF:

$\mathsf{Suppose\_not}(g, f) \Rightarrow \quad g \subseteq f \,\&\, 1\text{–}1(f) \,\&\, \neg 1\text{–}1(g)$
$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad \mathsf{Svm}(f) \,\&\, \langle \forall x \in f, y \in f \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle \,\&\, \neg(\mathsf{Svm}(g) \,\&\, \langle \forall x \in g, y \in g \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle)$
$\langle g, f \rangle \hookrightarrow T48 \Rightarrow \quad \neg\langle \forall x \in g, y \in g \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle$
$\mathsf{Pred\_monot} \Rightarrow \quad \langle \forall x \in f, y \in f \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle \rightarrow \langle \forall x \in g, y \in g \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- To show that a map product is always a map, we have only to expand the definition and simplify.

Theorem 71 (50) $\mathsf{Is\_map}(F \bullet G)$. PROOF:

$\mathsf{Suppose\_not}(f,g) \Rightarrow\quad \neg \mathsf{Is\_map}(f \bullet g)$

$\mathsf{Use\_def}(\bullet) \Rightarrow\quad \neg \mathsf{Is\_map}(\{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\})$

$\mathsf{Use\_def}(\mathsf{Is\_map}) \Rightarrow\quad \{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \neq$
$\{[u^{[1]}, u^{[2]}] : u \in \{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}\}$

$\mathsf{SIMPLF} \Rightarrow$
$\{[u^{[1]}, u^{[2]}] : u \in \{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}\} =$
$\left\{\left[[x^{[1]}, y^{[2]}]^{[1]}, [x^{[1]}, y^{[2]}]^{[2]}\right] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\right\}$

$\mathsf{Set\_monot} \Rightarrow\quad \left\{\left[[x^{[1]}, y^{[2]}]^{[1]}, [x^{[1]}, y^{[2]}]^{[2]}\right] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\right\} =$
$\{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}$

$\mathsf{ELEM} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

-- The next three theorems are simple corollaries of theorems 27, 28, and 29 respectively.

Theorem 72 (51) $\mathsf{Is\_map}(F) \rightarrow \mathsf{Is\_map}(F_{|S})$. PROOF:

$\mathsf{Suppose\_not}(f,s) \Rightarrow\quad \mathsf{Is\_map}(f) \,\&\, \neg \mathsf{Is\_map}(f_{|s})$

$\langle f, s\rangle \hookrightarrow T43 \Rightarrow\quad f_{|s} \subseteq f$

$\langle f_{|s}, f\rangle \hookrightarrow T47 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

Theorem 73 (52) $\mathsf{Svm}(F) \rightarrow \mathsf{Svm}(F_{|S})$. PROOF:

$\mathsf{Suppose\_not}(f,s) \Rightarrow\quad \mathsf{Svm}(f) \,\&\, \neg \mathsf{Svm}(f_{|s})$

$\langle f, s\rangle \hookrightarrow T43 \Rightarrow\quad f_{|s} \subseteq f$

$\langle f_{|s}, f\rangle \hookrightarrow T48 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

Theorem 74 (53) $1\text{–}1(F) \rightarrow 1\text{–}1(F_{|S})$. PROOF:

$\mathsf{Suppose\_not}(f,s) \Rightarrow\quad 1\text{–}1(f) \,\&\, \neg 1\text{–}1(f_{|s})$

$\langle f, s\rangle \hookrightarrow T43 \Rightarrow\quad f_{|s} \subseteq f$

$\langle f_{|s}, f\rangle \hookrightarrow T49 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

-- Next we note the elementary fact that the empty set is a single-valued map, a 1-1 map, and that its domain and range are both the empty set.

Theorem 75 (54) $\mathsf{Is\_map}(\emptyset)\ \&\ \mathsf{Svm}(\emptyset)\ \&\ \mathsf{1\text{–}1}(\emptyset)\ \&\ \mathbf{range}(\emptyset) = \emptyset\ \&\ \mathbf{domain}(\emptyset) = \emptyset$. PROOF:

Suppose $\Rightarrow$ $\neg\big(\mathsf{Is\_map}(\emptyset)\ \&\ \mathsf{Svm}(\emptyset)\ \&\ \mathsf{1\text{–}1}(\emptyset)\ \&\ \mathbf{range}(\emptyset) = \emptyset\ \&\ \mathbf{domain}(\emptyset) = \emptyset\big)$

-- Indeed, all the facts follow immediately by application of our utility fcn_symbol theory to $\emptyset = \{[x, x] : x \in \emptyset\}$.

Use_def(Svm) $\Rightarrow$ $\mathsf{Svm}(\emptyset) \leftrightarrow \mathsf{Is\_map}(\emptyset)\ \&\ \big\langle \forall x \in \emptyset, y \in \emptyset \mid x^{[1]} = y^{[1]} \rightarrow x = y\big\rangle$
ELEM $\Rightarrow$ $\neg\big(\mathsf{Svm}(\emptyset)\ \&\ \mathsf{1\text{–}1}(\emptyset)\ \&\ \mathbf{range}(\emptyset) = \emptyset\ \&\ \mathbf{domain}(\emptyset) = \emptyset\big)$
Loc_def $\Rightarrow$ $g = \{[x, x] : x \in \emptyset\}$
APPLY $\big\langle x_\Theta : a, y_\Theta : b\big\rangle$ fcn_symbol$\big(f(x) \mapsto x, g \mapsto g, s \mapsto \emptyset\big) \Rightarrow$
$\mathsf{Svm}(g)\ \&\ \mathbf{domain}(g) = \emptyset\ \&\ \mathbf{range}(g) = \{x : x \in \emptyset\}\ \&\ a, b \in \emptyset \vee \mathsf{1\text{–}1}(g)$
Suppose $\Rightarrow$ *Stat1* : $\neg\big\langle \forall x \in \emptyset, y \in \emptyset \mid x = x \rightarrow x = y\big\rangle$
$\langle c\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{1\text{–}1}(g)$
Set_monot $\Rightarrow$ $\{x : x \in \emptyset\} = \{[x, x] : x \in \emptyset\}$
Set_monot $\Rightarrow$ $\emptyset = \{x : x \in \emptyset\}$
EQUAL $\Rightarrow$ $\mathsf{Svm}(\emptyset)\ \&\ \mathbf{domain}(\emptyset) = \emptyset\ \&\ \mathbf{range}(\emptyset) = \emptyset\ \&\ \mathsf{1\text{–}1}(\emptyset)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we state two entirely elementary facts concerning the range and domain of a map f: for each $x \in f$, $x^{[1]}$ is in $\mathbf{domain}(f)$ and $x^{[2]}$ is in $\mathbf{range}(f)$.

Theorem 76 (55) $X \in F \rightarrow X^{[1]} \in \mathbf{domain}(F)$. PROOF:

Suppose_not(c, f) $\Rightarrow$ $c \in f\ \&\ c^{[1]} \notin \mathbf{domain}(f)$
Use_def($\mathbf{domain}$) $\Rightarrow$ *Stat1* : $c^{[1]} \notin \big\{x^{[1]} : x \in f\big\}$
$\langle c\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\neg(c^{[1]} = c^{[1]}\ \&\ c \in f)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 77 (56) $X \in F \rightarrow X^{[2]} \in \mathbf{range}(F)$. PROOF:

Suppose_not(c, f) $\Rightarrow$ $c \in f\ \&\ c^{[2]} \notin \mathbf{range}(f)$
Use_def($\mathbf{range}$) $\Rightarrow$ *Stat1* : $c^{[2]} \notin \big\{x^{[2]} : x \in f\big\}$
$\langle c\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\neg(c^{[2]} = c^{[2]}\ \&\ c \in f)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is also an elementary consequence of the definition that the union of two maps is a map.

Theorem 78 (57) $\text{Is\_map}(F) \,\&\, \text{Is\_map}(G) \rightarrow \text{Is\_map}(F \cup G)$. PROOF:

Suppose_not$(f, g) \Rightarrow$ $\text{Is\_map}(f) \,\&\, \text{Is\_map}(g) \,\&\, \neg\text{Is\_map}(f \cup g)$

Use_def(Is_map) $\Rightarrow$ $f = \left\{\left[x^{[1]}, x^{[2]}\right] : x \in f\right\} \,\&\, g = \left\{\left[x^{[1]}, x^{[2]}\right] : x \in g\right\} \,\&\, f \cup g \neq \left\{\left[x^{[1]}, x^{[2]}\right] : x \in f \cup g\right\}$

Set_monot $\Rightarrow$ $\left\{\left[x^{[1]}, x^{[2]}\right] : x \in f \cup g\right\} = \left\{\left[x^{[1]}, x^{[2]}\right] : x \in f\right\} \cup \left\{\left[x^{[1]}, x^{[2]}\right] : x \in g\right\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the map restriction operation is additive in its second argument. Again, this is an entirely elementary consequence of the definition, by set monotonicity.

Theorem 79 (58) $F_{|A \cup B} = F_{|A} \cup F_{|B}$. PROOF:

Suppose_not$(f, a, b) \Rightarrow$ $f_{|a \cup b} \neq f_{|a} \cup f_{|b}$

Use_def$(|) \Rightarrow$ $\left\{p \in f \,|\, p^{[1]} \in a \cup b\right\} \neq \left\{p \in f \,|\, p^{[1]} \in a\right\} \cup \left\{p \in f \,|\, p^{[1]} \in b\right\}$

Set_monot $\Rightarrow$ $\left\{p \in f \,|\, p^{[1]} \in a \cup b\right\} = \left\{p \in f \,|\, p^{[1]} \in a \vee p^{[1]} \in b\right\}$

Set_monot $\Rightarrow$ $\left\{p \in f \,|\, p^{[1]} \in a \vee p^{[1]} \in b\right\} = \left\{p \in f \,|\, p^{[1]} \in a\right\} \cup \left\{p \in f \,|\, p^{[1]} \in b\right\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The map restriction operation is also additive in its first argument the argument being similar and equally elementary.

Theorem 80 (59) $(F \cup G)_{|A} = F_{|A} \cup G_{|A}$. PROOF:

Suppose_not$(f, g, a) \Rightarrow$ $(f \cup g)_{|a} \neq f_{|a} \cup g_{|a}$

Use_def$(|) \Rightarrow$ $\left\{p \in f \cup g \,|\, p^{[1]} \in a\right\} \neq \left\{p \in f \,|\, p^{[1]} \in a\right\} \cup \left\{p \in g \,|\, p^{[1]} \in a\right\}$

Set_monot $\Rightarrow$ $\left\{p \in f \cup g \,|\, p^{[1]} \in a\right\} = \left\{p \in f \,|\, p^{[1]} \in a\right\} \cup \left\{p \in g \,|\, p^{[1]} \in a\right\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The fact that the range and domain of a map f are both monotone increasing functions of f also follows immediately by set monotonicity.

Theorem 81 (60) $F \subseteq G \rightarrow \mathbf{range}(F) \subseteq \mathbf{range}(G) \,\&\, \mathbf{domain}(F) \subseteq \mathbf{domain}(G)$. PROOF:

Suppose_not$(f, g) \Rightarrow$ $f \subseteq g \,\&\, \neg(\mathbf{range}(f) \subseteq \mathbf{range}(g) \,\&\, \mathbf{domain}(f) \subseteq \mathbf{domain}(g))$

Suppose $\Rightarrow$ $\mathbf{range}(f) \not\subseteq \mathbf{range}(g)$

Use_def(range) $\Rightarrow$ $\left\{x^{[2]} : x \in f\right\} \not\subseteq \left\{x^{[2]} : x \in g\right\}$

Set_monot $\Rightarrow$ $\left\{x^{[2]} : x \in f\right\} \subseteq \left\{x^{[2]} : x \in g\right\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{domain}(f) \not\subseteq \mathbf{domain}(g)$

Use_def (**domain**) $\Rightarrow$ $\{x^{[1]} : x \in f\} \not\subseteq \{x^{[1]} : x \in g\}$
Set_monot $\Rightarrow$ $\{x^{[1]} : x \in f\} \subseteq \{x^{[1]} : x \in g\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states the important but elementary fact that map composition is associative.
-- Associativity of map multiplication

Theorem 82 (61) $F\bullet(G\bullet H) = (F\bullet G)\bullet H$. PROOF:

Suppose_not (f, g, h) $\Rightarrow$ $f\bullet(g\bullet h) \neq f\bullet g\bullet h$
Use_def ($\bullet$) $\Rightarrow$ $f\bullet(g\bullet h) = \{[x^{[1]}, v^{[2]}] : x \in \{[x^{[1]}, y^{[2]}] : x \in h, y \in g \mid x^{[2]} = y^{[1]}\}, v \in f \mid x^{[2]} = v^{[1]}\}$
Use_def ($\bullet$) $\Rightarrow$ $f\bullet g\bullet h = \{[x^{[1]}, y^{[2]}] : x \in h, y \in \{[y^{[1]}, v^{[2]}] : y \in g, v \in f \mid y^{[2]} = v^{[1]}\} \mid x^{[2]} = y^{[1]}\}$
ELEM $\Rightarrow$
$$\{[x^{[1]}, v^{[2]}] : x \in \{[x^{[1]}, y^{[2]}] : x \in h, y \in g \mid x^{[2]} = y^{[1]}\}, v \in f \mid x^{[2]} = v^{[1]}\} \neq \{[x^{[1]}, y^{[2]}] : x \in h, y \in \{[y^{[1]}, v^{[2]}] : y \in g, v \in f \mid y^{[2]} = v^{[1]}\} \mid x^{[2]} = y^{[1]}\}$$

-- For if not, simplification after using the definition of map composition gives us the following inequality, and so the elementary inequality seen just below it. But since this is impossible our lemma follows.

SIMPLF $\Rightarrow$ *Stat1* :
$$\left\{\left[[x^{[1]}, y^{[2]}]^{[1]}, v^{[2]}\right] : x \in h, y \in g, v \in f \mid x^{[2]} = y^{[1]} \,\&\, [x^{[1]}, y^{[2]}]^{[2]} = v^{[1]}\right\} \neq \left\{\left[x^{[1]}, [y^{[1]}, v^{[2]}]^{[2]}\right] : x \in h, y \in g, v \in f \mid y^{[2]} = v^{[1]} \,\&\, x^{[2]} = [y^{[1]}, v^{[2]}]^{[1]}\right\}$$
$\langle x, y, v\rangle \hookrightarrow$ *Stat1* $\Rightarrow$
$$\left[[x^{[1]}, y^{[2]}]^{[1]}, v^{[2]}\right] \neq \left[x^{[1]}, [y^{[1]}, v^{[2]}]^{[2]}\right] \vee \neg(x^{[2]} = y^{[1]} \,\&\, [x^{[1]}, y^{[2]}]^{[2]} = v^{[1]} \leftrightarrow y^{[2]} = v^{[1]} \,\&\, x^{[2]} = [y^{[1]}, v^{[2]}]^{[1]})$$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now we show that the restriction of a map f to its own domain is simply f.

Theorem 83 (62) $F_{|\mathbf{domain}(F)} = F$. PROOF:

Suppose_not (f) $\Rightarrow$ $f_{|\mathbf{domain}(f)} \neq f$

-- For if not, it follows by Theorem 43 that there must be some element $a \in f$ which is not in $\{p \in f \mid p^{[1]} \in \mathbf{domain}(f)\}$, which is clearly impossible by Theorem 55.

$\langle \mathsf{f}, \mathbf{domain}(\mathsf{f})\rangle \hookrightarrow T43 \Rightarrow \quad \mathsf{f}_{|\mathbf{domain}(\mathsf{f})} \subseteq \mathsf{f}$
ELEM $\Rightarrow \quad \mathsf{f}_{|\mathbf{domain}(\mathsf{f})} \not\supseteq \mathsf{f}$
Use_def ($|$) $\Rightarrow \quad$ *Stat1* : $\{\mathsf{p} \in \mathsf{f} \,|\, \mathsf{p}^{[1]} \in \mathbf{domain}(\mathsf{f})\} \not\supseteq \mathsf{f}$
$\langle \mathsf{a}\rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{a} \in \mathsf{f}$ & *Stat2* : $\mathsf{a} \notin \{\mathsf{p} \in \mathsf{f} \,|\, \mathsf{p}^{[1]} \in \mathbf{domain}(\mathsf{f})\}$
$\langle \mathsf{a}\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{a}^{[1]} \notin \mathbf{domain}(\mathsf{f})$
$\langle \mathsf{a}, \mathsf{f}\rangle \hookrightarrow T55 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The following easy lemma generalizes the fact that the restriction of any map f to its own domain is f itself.

Theorem 84 (63) $\mathsf{F}_{|\mathbf{domain}(\mathsf{F}) \cap T} = \mathsf{F}_{|T}$. PROOF:

Suppose_not $(\mathsf{f}, \mathsf{t}) \Rightarrow \quad \mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \cap \mathsf{t}} \neq \mathsf{f}_{|\mathsf{t}}$

-- For if not, the additivity of map restriction would imply that $\mathsf{f}_{|\mathsf{t} \setminus \mathbf{domain}(\mathsf{f})} \neq \emptyset$, which is easily seen to be imposible.

TELEM $\Rightarrow \quad \mathsf{t} = \mathbf{domain}(\mathsf{f}) \cap \mathsf{t} \cup (\mathsf{t} \setminus \mathbf{domain}(\mathsf{f}))$
EQUAL $\Rightarrow \quad \mathsf{f}_{|\mathsf{t}} = \mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \cap \mathsf{t} \cup (\mathsf{t} \setminus \mathbf{domain}(\mathsf{f}))}$
$\langle \mathsf{f}, \mathbf{domain}(\mathsf{f}) \cap \mathsf{t}, \mathsf{t} \setminus \mathbf{domain}(\mathsf{f})\rangle \hookrightarrow T58 \Rightarrow \quad \mathsf{f}_{|\mathsf{t} \setminus \mathbf{domain}(\mathsf{f})} \neq \emptyset$
Use_def ($|$) $\Rightarrow \quad$ *Stat1* : $\{\mathsf{p} \in \mathsf{f} \,|\, \mathsf{p}^{[1]} \in \mathsf{t} \setminus \mathbf{domain}(\mathsf{f})\} \neq \emptyset$
$\langle \mathsf{a}\rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{a} \in \mathsf{f}$ & $\mathsf{a}^{[1]} \in \mathsf{t} \setminus \mathbf{domain}(\mathsf{f})$
$\langle \mathsf{a}, \mathsf{f}\rangle \hookrightarrow T55 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The following theorem tells us that, as we have defined it, the map value $\mathsf{f} \upharpoonright \mathsf{x}$ always belongs to the range of f, provided that x belongs to the domain of f; this is true even if f is not single-valued, or even if f is not a map.

Theorem 85 (64) $\mathsf{X} \in \mathbf{domain}(\mathsf{F}) \rightarrow \mathsf{F} \upharpoonright \mathsf{X} \in \mathbf{range}(\mathsf{F})$. PROOF:

Suppose_not $(\mathsf{x}, \mathsf{f}) \Rightarrow \quad \mathsf{x} \in \mathbf{domain}(\mathsf{f})$ & $\mathsf{f} \upharpoonright \mathsf{x} \notin \mathbf{range}(\mathsf{f})$

-- For if not, then by definition there must be some $\mathsf{c} \in \mathsf{f}$ such that $\mathbf{arb}(\{\mathsf{p} \in \mathsf{f} \,|\, \mathsf{p}^{[1]} \in \{\mathsf{c}^{[1]}\}\})^{[2]}$ does not belong to $\{\mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{f}\}$.

Use_def (**range**) $\Rightarrow \quad \mathsf{f} \upharpoonright \mathsf{x} \notin \{\mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{f}\}$
Use_def ($\upharpoonright$) $\Rightarrow \quad \mathbf{arb}(\mathsf{f}_{|\{\mathsf{x}\}})^{[2]} \notin \{\mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{f}\}$
Use_def ($|$) $\Rightarrow \quad$ *Stat1* : $\mathbf{arb}(\{\mathsf{p} \in \mathsf{f} \,|\, \mathsf{p}^{[1]} \in \{\mathsf{x}\}\})^{[2]} \notin \{\mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{f}\}$

Use_def (**domain**) ⇒ $\quad$ *Stat2*: $x \in \{y^{[1]} : y \in f\}$
$\langle c \rangle \hookrightarrow$ *Stat2* ⇒ $\quad x = c^{[1]} \;\&\; c \in f$

-- But c clearly belongs to the set $\{p \in f \mid p^{[1]} \in \{c^{[1]}\}\}$, which is therefore nonempty, from which it follows by the axiom of choice that arrb $=_{\text{Def}}$ $\mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\})$ belongs to this same set.

Suppose ⇒ $\quad$ *Stat3*: $c \notin \{p \in f \mid p^{[1]} \in \{x\}\}$
Suppose ⇒ $\quad$ *Stat4*: $\neg\langle \forall p \in f \mid p^{[1]} \in \{x\} \to p \neq c\rangle$
$\langle q \rangle \hookrightarrow$ *Stat4* ⇒ $\quad q \in f \;\&\; q^{[1]} \in \{x\} \;\&\; q = c$
$\langle q \rangle \hookrightarrow$ *Stat3* ⇒ $\quad$ false; $\qquad$ Discharge ⇒ $\quad$ *Stat5*: $\langle \forall p \in f \mid p^{[1]} \in \{x\} \to p \neq c\rangle$
$\langle c \rangle \hookrightarrow$ *Stat5* ⇒ $\quad c^{[1]} \in \{x\} \to c \neq c$
ELEM ⇒ $\quad$ false; $\qquad$ Discharge ⇒ $\quad c \in \{p \in f \mid p^{[1]} \in \{x\}\}$
$\langle \{p \in f \mid p^{[1]} \in \{x\}\} \rangle \hookrightarrow$ *T0* ⇒ $\quad \mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\}) \in \{p \in f \mid p^{[1]} \in \{x\}\}$

-- Hence arrb must belong to the larger set f. By Stat6 6, this implies the impossible inequality seen below, proving our theorem.

Set_monot ⇒ $\quad \{p \in f \mid p^{[1]} \in \{x\}\} \subseteq \{p : p \in f\}$
ELEM ⇒ $\quad \mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\}) \in \{p : p \in f\}$
SIMPLF ⇒ $\quad \mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\}) \in f$
$\langle \mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\}) \rangle \hookrightarrow$ *Stat1* ⇒ $\quad \mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\})^{[2]} \neq \mathbf{arb}(\{p \in f \mid p^{[1]} \in \{x\}\})^{[2]}$
ELEM ⇒ $\quad$ false; $\qquad$ Discharge ⇒ $\quad$ QED

-- It is convenient to summarize some of the key results derived above in the following auxiliary THEORY, which specializes them and eases their use. We focus on maps of the form $\{[x, f(x)] : x \in s\}$, which are always single-valued.

THEORY fcn_symbol $(f(x), g, s)$
-- Contains some elementary lemmas about single - valued functions
$g = \{[x, f(x)] : x \in s\}$
END fcn_symbol

ENTER_THEORY fcn_symbol
-- Note : till we return from 'fcn_symbol' to set theory , we are

-- reasoning within the theory , so $g = [x, f(x)] : xins$ is available as an axiom , and all theorems proved are added to the set of conclusions of the theory , rather than to the set of conclusions of the top - level set - theory . First we show that the domain of g is simply s.

Theorem 86 (fcn_symbol · 1) $\mathbf{domain}(g) = s$. PROOF:

Suppose_not$(g, s) \Rightarrow$ $\mathbf{domain}(g) \neq s$

-- For in the contrary case we would have $\{x^{[1]} : x \in \{[x, f(x)] : x \in s\}\} \neq s$ by definition, so there would exist an $x \in s$ such that $[x, f(x)]^{[1]} \neq x$, which is impossible.

Use_def$(\mathbf{domain}) \Rightarrow$ $\{x^{[1]} : x \in g\} \neq s$
Assump $\Rightarrow$ $g = \{[y, f(y)] : y \in s\}$
EQUAL $\Rightarrow$ $\{x^{[1]} : x \in g\} = \{x^{[1]} : x \in \{[y, f(y)] : y \in s\}\}$
ELEM $\Rightarrow$ $\{x^{[1]} : x \in \{[y, f(y)] : y \in s\}\} \neq s$
SIMPLF $\Rightarrow$ $\left\{[y, f(y)]^{[1]} : y \in s\right\} \neq \{x : x \in s\}$
Set_monot $\Rightarrow$ $\left\{[y, f(y)]^{[1]} : y \in s\right\} = \{y : y \in s\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that $g \restriction x = f(x)$ for any $x \in s$.

Theorem 87 (fcn_symbol · 2) $XX \in s \rightarrow g \restriction XX = f(XX)$. PROOF:

Suppose_not$(c, s, g) \Rightarrow$ $c \in s \;\&\; g \restriction c \neq f(c)$

-- For suppose not, and let $c \in s$ be a counterexample, so that by definition of functional application (and map restriction) we would have $\mathbf{arb}\left(\left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\}\right)^{[2]} \neq f(c)$.

Use_def$(\restriction) \Rightarrow$ $\mathbf{arb}\left(g_{|\{c\}}\right)^{[2]} \neq f(c)$
Use_def$(|) \Rightarrow$ $\mathbf{arb}\left(\{p \in g \mid p^{[1]} \in \{c\}\}\right)^{[2]} \neq f(c)$
Assump $\Rightarrow$ $g = \{[x, f(x)] : x \in s\}$
EQUAL $\Rightarrow$ $\mathbf{arb}\left(\{p \in \{[x, f(x)] : x \in s\} \mid p^{[1]} \in \{c\}\}\right)^{[2]} \neq f(c)$
SIMPLF $\Rightarrow$ $\mathbf{arb}\left(\left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\}\right)^{[2]} \neq f(c)$

-- We can simplify $\left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\}$ to $\{[x, f(x)] : x \in s \mid x \in \{c\}\}$, for if these sets were different there would be a $d \in s$ such that the conditions $[d, f(d)]^{[1]} \in \{c\}$ and $d \in c$ were inequivalent, which is impossible.

Suppose $\Rightarrow$ $\quad$ *Stat1* : $\left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\} \neq \{[x, f(x)] : x \in s \mid x \in \{c\}\}$

$\langle d \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad d \in s \;\&\; \neg\left([d, f(d)]^{[1]} \in \{c\} \leftrightarrow d \in \{c\}\right)$

ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\} = \{[x, f(x)] : x \in s \mid x \in \{c\}\}$

-- But $\{[x, f(x)] : x \in s \mid x \in \{c\}\}$ simplifies in two steps to $\{[x, f(x)] : x \in \{c\}\}$, which is the same as $\{[c, f(c)]\}$. Hence if our theorem is false we would have $\mathbf{arb}(\{[c, f(c)]\})^{[2]} \neq f(c)$, a contradiction proving the theorem.

Suppose $\Rightarrow$ $\quad$ *Stat2* : $\{[x, f(x)] : x \in s \mid x \in \{c\}\} \neq \{[x, f(x)] : x \in \{c\}\}$

$\langle e \rangle \hookrightarrow$ *Stat2* $\Rightarrow$

$\quad \left(e \in \{[x, f(x)] : x \in s \mid x \in \{c\}\} \;\&\; e \notin \{[x, f(x)] : x \in \{c\}\}\right) \vee$

$\quad\quad e \notin \{[x, f(x)] : x \in s \mid x \in \{c\}\} \;\&\; e \in \{[x, f(x)] : x \in \{c\}\}$

Suppose $\Rightarrow$ $\quad$ *Stat3* : $e \in \{[x, f(x)] : x \in s \mid x \in \{c\}\}$ & *Stat4* : $e \notin \{[x, f(x)] : x \in \{c\}\}$

$\langle e_1 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad e = [e_1, f(e_1)] \;\&\; e_1 \in s \;\&\; e_1 \in \{c\}$

$\langle e_1 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ *Stat5* : $e \notin \{[x, f(x)] : x \in s \mid x \in \{c\}\}$ & *Stat6* : $e \in \{[x, f(x)] : x \in \{c\}\}$

$\langle e_2 \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $\quad e = [e_2, f(e_2)] \;\&\; e_2 \in \{c\}$

$\langle e_2 \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \{[x, f(x)] : x \in s \mid x \in \{c\}\} = \{[x, f(x)] : x \in \{c\}\}$

SIMPLF $\Rightarrow$ $\quad \{[x, f(x)] : x \in \{c\}\} = \{[c, f(c)]\}$

EQUAL $\Rightarrow$ $\quad \mathbf{arb}(\{[c, f(c)]\})^{[2]} \neq f(c)$

ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Our next theorem rounds out the preceding result by showing that $g \restriction x = \emptyset$ for $x \notin s$.

Theorem 88 (fcn_symbol · 3) $\quad XX \notin s \rightarrow g \restriction XX = \emptyset$. PROOF:

Suppose_not(c, s, g) $\Rightarrow$ $\quad$ *Stat0* : $c \notin s \;\&\; g \restriction c \neq \emptyset$

-- For suppose not, and let c in s be a counterexample. Then by definition of functional application (and map restriction) the value $\mathbf{arb}\left(\left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\}\right)$ must be nonzero, and then by the axiom of choice so is the set $\left\{[x, f(x)] : x \in s \mid [x, f(x)]^{[1]} \in \{c\}\right\}$.

Use_def$(\restriction)$ $\Rightarrow$ $\quad c \notin s \;\&\; \mathbf{arb}\left(g_{|\{c\}}\right)^{[2]} \neq \emptyset$

Suppose $\Rightarrow$ $\quad \mathbf{arb}\left(g_{|\{c\}}\right) = \emptyset$

EQUAL $\Rightarrow$ $\quad \emptyset^{[2]} \neq \emptyset$

Use_def$([\cdot, 2])$ $\Rightarrow$ $\quad \mathbf{arb}(\mathbf{arb}(\mathbf{arb}(\emptyset \setminus \{\mathbf{arb}(\emptyset)\}) \setminus \{\mathbf{arb}(\emptyset)\})) \neq \emptyset$

$\mathit{TELEM} \Rightarrow \quad \emptyset \setminus \{\mathbf{arb}(\emptyset)\} = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathbf{arb}(\mathbf{arb}(\mathbf{arb}(\emptyset) \setminus \{\mathbf{arb}(\emptyset)\})) \neq \emptyset$
$\langle \emptyset \rangle \hookrightarrow \mathit{T0}(\langle \cap \rangle) \Rightarrow \quad \mathbf{arb}(\emptyset) = \emptyset$
$\mathit{TELEM} \Rightarrow \quad \mathbf{arb}(\emptyset) \setminus \{\mathbf{arb}(\emptyset)\} = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathbf{arb}(\mathbf{arb}(\emptyset)) \neq \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathbf{arb}(\emptyset) \neq \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathbf{arb}\Big(\mathsf{g}_{|\{\mathsf{c}\}}\Big) \neq \emptyset$
$\mathsf{Use\_def}(|) \Rightarrow \quad \mathsf{g}_{|\{\mathsf{c}\}} = \{\mathsf{p} \in \mathsf{g} \mid \mathsf{p}^{[1]} \in \{\mathsf{c}\}\}$
$\mathsf{EQUAL} \Rightarrow \quad \mathbf{arb}(\{\mathsf{p} \in \mathsf{g} \mid \mathsf{p}^{[1]} \in \{\mathsf{c}\}\}) \neq \emptyset$
$\mathsf{Assump} \Rightarrow \quad \mathsf{g} = \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\}$
$\mathsf{EQUAL} \Rightarrow \quad \mathbf{arb}(\{\mathsf{p} \in \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}\} \mid \mathsf{p}^{[1]} \in \{\mathsf{c}\}\}) \neq \emptyset$
$\mathsf{SIMPLF} \Rightarrow \quad \mathit{Stat1}: \ \mathbf{arb}\Big(\Big\{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s} \mid [\mathsf{x}, \mathsf{f}(\mathsf{x})]^{[1]} \in \{\mathsf{c}\}\Big\}\Big) \neq \emptyset$
$\Big\langle \Big\{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s} \mid [\mathsf{x}, \mathsf{f}(\mathsf{x})]^{[1]} \in \{\mathsf{c}\}\Big\} \Big\rangle \hookrightarrow \mathit{T0}([\mathit{Stat1}, \cap]) \Rightarrow \quad \mathit{Stat2}: \ \Big\{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s} \mid [\mathsf{x}, \mathsf{f}(\mathsf{x})]^{[1]} \in \{\mathsf{c}\}\Big\} \neq \emptyset$

-- Hence there would exist a $\mathsf{d} \in \mathsf{s}$ such that $[\mathsf{d}, \mathsf{f}(\mathsf{d})]^{[1]} \in \{\mathsf{c}\}$, implying $\mathsf{c} \in \mathsf{s}$, a contradiction which proves our assertion.

$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat2}([]) \Rightarrow \quad \mathit{Stat3}: \ \mathsf{d} \in \mathsf{s} \ \& \ [\mathsf{d}, \mathsf{f}(\mathsf{d})]^{[1]} \in \{\mathsf{c}\}$
$\langle \mathit{Stat0}, \mathit{Stat3} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following result summarizes the two which precede it.

Theorem 89 (fcn_symbol · 4) $\mathsf{g} \upharpoonright \mathsf{XX} = \textbf{if } \mathsf{XX} \in \mathsf{s} \textbf{ then } \mathsf{f}(\mathsf{XX}) \textbf{ else } \emptyset \textbf{ fi}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{g}, \mathsf{c}, \mathsf{s}) \Rightarrow \quad \mathsf{g} \upharpoonright \mathsf{c} \neq \textbf{if } \mathsf{c} \in \mathsf{s} \textbf{ then } \mathsf{f}(\mathsf{c}) \textbf{ else } \emptyset \textbf{ fi}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Tfcn\_symbol} \cdot 3 \Rightarrow \quad \mathsf{c} \in \mathsf{s}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Tfcn\_symbol} \cdot 2 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- It is easy to derive the following formula for the range of a map g:

Theorem 90 (fcn_symbol · 5) $\mathbf{range}(\mathsf{g}) = \{\mathsf{f}(\mathsf{xx}) : \mathsf{xx} \in \mathsf{s}\}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{g}, \mathsf{s}) \Rightarrow \quad \mathbf{range}(\mathsf{g}) \neq \{\mathsf{f}(\mathsf{x}) : \mathsf{x} \in \mathsf{s}\}$

-- For if not, it follows by definition of range that $\left\{[x, f(x)]^{[2]} : x \in s\right\} \neq \{f(x) : x \in s\}$, implying the existence of an x such that $[x, f(x)]^{[2]} \neq f(x)$, which is impossible.

Use_def(**range**) $\Rightarrow$ $\quad \mathbf{range}(g) = \{x^{[2]} : x \in g\}$
Assump $\Rightarrow$ $\quad g = \{[x, f(x)] : x \in s\}$
EQUAL $\Rightarrow$ $\quad \{x^{[2]} : x \in \{[x, f(x)] : x \in s\}\} \neq \{f(x) : x \in s\}$
SIMPLF $\Rightarrow$ $\quad \left\{[x, f(x)]^{[2]} : x \in s\right\} \neq \{f(x) : x \in s\}$
Set_monot $\Rightarrow$ $\quad \left\{[x, f(x)]^{[2]} : x \in s\right\} = \{f(x) : x \in s\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- and also easy to derive the following criterion for a map g to be 1-1:

DEF fcn_symbol · 0*a*. $\quad xy_\Theta =_{\text{Def}} \mathbf{arb}(\{[xx, yy] : xx \in s, yy \in s \mid f(xx) = f(yy) \;\&\; xx \neq yy\})$
DEF fcn_symbol · 0*b*. $\quad x_\Theta =_{\text{Def}} {xy_\Theta}^{[1]}$
DEF fcn_symbol · 0*c*. $\quad y_\Theta =_{\text{Def}} {xy_\Theta}^{[2]}$

Theorem 91 (fcn_symbol · 6) $\left(x_\Theta, y_\Theta \in s \;\&\; f(x_\Theta) = f(y_\Theta) \;\&\; x_\Theta \neq y_\Theta\right) \vee 1\text{–}1(g)$. PROOF:

Suppose_not(s, g) $\Rightarrow$ $\quad \neg\left(x_\Theta, y_\Theta \in s \;\&\; f(x_\Theta) = f(y_\Theta) \;\&\; x_\Theta \neq y_\Theta\right) \;\&\; \neg 1\text{–}1(g)$

-- For if not, then it easily using our previously derived THEORY one_1_test that the set $\{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\}$ must be non-null, so that by the axiom of choice and the definition of x_Θ, y_Θ the pair x_Θ, y_Θ must belong to it.

APPLY $\langle x_\Theta : x, y_\Theta : y\rangle$ one_1_test$(a(x) \mapsto x, b(x) \mapsto f(x), s \mapsto s) \Rightarrow$
$\quad \Big(x, y \in s \;\&\; \neg\big(x = y \leftrightarrow f(x) = f(y)\big)\Big) \vee 1\text{–}1(\{[x, f(x)] : x \in s\})$
Assump $\Rightarrow$ $\quad g = \{[x, f(x)] : x \in s\}$
EQUAL $\Rightarrow$ $\quad \neg 1\text{–}1(\{[x, f(x)] : x \in s\})$
Suppose $\Rightarrow$ $\quad$ *Stat2* : $\{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\} = \emptyset$
$\langle x, y\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\quad \neg(x, y \in s \;\&\; f(x) = f(y) \;\&\; x \neq y)$
Suppose $\Rightarrow$ $\quad x = y$
EQUAL $\Rightarrow$ $\quad f(x) = f(y)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad x \neq y$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad \{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\} \neq \emptyset$
$\langle \{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\} \rangle \hookrightarrow$ *T0* $\Rightarrow$ $\quad \mathbf{arb}(\{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\}) \in$
$\quad \{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\}$
Use_def(xy_Θ) $\Rightarrow$ $\quad$ *Stat3* : $xy_\Theta \in \{[x, y] : x \in s, y \in s \mid f(x) = f(y) \;\&\; x \neq y\}$

-- Hence there must exist elements xx and yy satisfying the condition seen below, and since it is easily seen that these must be the two components x_Θ, y_Θ of xy_Θ, we have a contradiction with our hypothesis and so a proof of our assertion.

$\langle xx, yy \rangle \hookrightarrow Stat3 \Rightarrow \quad xx, yy \in s \;\&\; xy_\Theta = [xx, yy] \;\&\; f(xx) = f(yy) \;\&\; xx \neq yy$
ELEM $\Rightarrow \quad xx = xy_\Theta{}^{[1]} \;\&\; yy = xy_\Theta{}^{[2]}$
Use_def$(x_\Theta) \Rightarrow \quad x_\Theta = xx$
Use_def$(y_\Theta) \Rightarrow \quad y_\Theta = yy$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It follows immediately using our previously derived Svm_test THEORY that g must be single-valued.

Theorem 92 (fcn_symbol · 7) Svm(g). PROOF:

Suppose_not $\Rightarrow \quad \neg Svm(g)$
APPLY $\langle x_\Theta : x, y_\Theta : y \rangle$ Svm_test$(a(x) \mapsto x, b(x) \mapsto f(x), s \mapsto s) \Rightarrow$
$\big(x = y \;\&\; f(x) \neq f(y)\big) \vee Svm\big(\{[x, f(x)] : x \in s\}\big)$
Assump $\Rightarrow \quad g = \{[x, f(x)] : x \in s\}$
EQUAL $\Rightarrow \quad \big(x = y \;\&\; f(x) \neq f(y)\big) \vee Svm(g)$
ELEM $\Rightarrow \quad x = y \;\&\; f(x) \neq f(y)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- Note: this exits the subtheory 'fcn_symbol', and re-enters the main Set_theory] Note: if we used 'DISPLAY fcn_symbol' at this point, the result would be as follows:

DISPLAY fcn_symbol

THEORY fcn_symbol$(f(x), g, s)$
$g = \{[x, f(x)] : x \in s\}$
$\Rightarrow (x_\Theta, y_\Theta)$
$\mathbf{domain}(g) = s$
$\langle \forall x \mid x \in s \rightarrow g \upharpoonright x = f(x) \rangle$
$\langle \forall x \mid x \notin s \rightarrow g \upharpoonright X = \emptyset \rangle$
$\langle \forall x \mid g \upharpoonright x = \textbf{if } x \in s \textbf{ then } f(x) \textbf{ else } \emptyset \textbf{ fi} \rangle$
$\mathbf{range}(g) = \{f(x) : x \in s\}$
$\big(x_\Theta, y_\Theta \in s \;\&\; f(x_\Theta) = f(y_\Theta) \;\&\; y_\Theta \neq y_\Theta\big) \vee 1\text{–}1(g)$
$Svm(g)$
END fcn_symbol

ENTER_THEORY Set_theory

-- Note: this exits the subtheory 'fcn_symbol', and re-enters the main Set_theory] Note: if we used 'DISPLAY fcn_symbol' at this point, the result would be as follows:

DISPLAY fcn_symbol

THEORY fcn_symbol$\big(f(x), g, s\big)$
$g = \{[x, f(x)] : x \in s\}$
$\Rightarrow (x_\Theta, y_\Theta)$
$\mathbf{domain}(g) = s$
$\big\langle \forall x \mid x \in s \rightarrow g \upharpoonright x = f(x) \big\rangle$
$\big\langle \forall x \mid x \notin s \rightarrow g \upharpoonright X = \emptyset \big\rangle$
$\big\langle \forall x \mid g \upharpoonright x = \textbf{if } x \in s \textbf{ then } f(x) \textbf{ else } \emptyset \textbf{ fi} \big\rangle$
$\mathbf{range}(g) = \{f(x) : x \in s\}$
$\big(x_\Theta, y_\Theta \in s \,\&\, f(x_\Theta) = f(y_\Theta) \,\&\, y_\Theta \neq y_\Theta\big) \vee 1\text{–}1(g)$
$\mathsf{Svm}(g)$
END fcn_symbol

-- Single valued maps can always be represented in the convenient form required by the preceding theory and repeated in the following theorem. Conversely, the preceding theory tells us that any set of this form is a single-valued map.

Theorem 93 (65) $\mathsf{Svm}(F) \leftrightarrow F = \{[x, F\upharpoonright x] : x \in \mathbf{domain}(F)\}$. PROOF:

Suppose_not(f) $\Rightarrow$ $\big(\mathsf{Svm}(f) \,\&\, f \neq \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}\big) \vee \big(\neg\mathsf{Svm}(f) \,\&\, f = \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}\big)$

-- For the preceding theory tells us that $\{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}$ is single valued, leaving only the first of the two above contrary possibilities.

Suppose $\Rightarrow$ *Stat0*: $\neg\mathsf{Svm}(f) \,\&\, f = \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}$
APPLY $\langle\ \rangle$ fcn_symbol$\big(f(x) \mapsto f\upharpoonright x, g \mapsto f, s \mapsto \mathbf{domain}(f)\big) \Rightarrow$ $\mathsf{Svm}(f)$
$\langle$*Stat0*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Svm}(f) \,\&\, f \neq \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}$

-- By definition of domain, map application, and map restriction, the preceding inequality simplifies to $f \neq \Big\{\Big[y^{[1]}, \mathbf{arb}\big(\{u \in f \mid u^{[1]} \in \{y^{[1]}\}\}\big)^{[2]}\Big] : y \in f\Big\}$.

Use_def(**domain**) $\Rightarrow$ $f \neq \big\{[x, f\upharpoonright x] : x \in \{y^{[1]} : y \in f\}\big\}$
SIMPLF $\Rightarrow$ $f \neq \big\{[y^{[1]}, f\upharpoonright y^{[1]}] : y \in f\big\}$

Use_def ($\upharpoonright$) $\Rightarrow$ $\quad f \neq \left\{\left[y^{[1]}, \mathbf{arb}\left(f_{|\{y^{[1]}\}}\right)^{[2]}\right] : y \in f\right\}$

Use_def ($|$) $\Rightarrow$ $\quad f \neq \left\{\left[y^{[1]}, \mathbf{arb}\left(\{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\}\right)^{[2]}\right] : y \in f\right\}$

-- Since f is a single-valued map and hence a map, this implies the inequality seen below:

Use_def (Svm) $\Rightarrow$ $\quad \mathsf{Is\_map}(f)$ & *Stat1* : $\langle \forall x \in f, y_1 \in f \,|\, x^{[1]} = y_1{}^{[1]} \rightarrow x = y_1 \rangle$

Use_def (Is_map) $\Rightarrow$ $\quad f = \left\{\left[y^{[1]}, y^{[2]}\right] : y \in f\right\}$

EQUAL $\Rightarrow$ $\quad$ *Stat2* : $\left\{\left[y^{[1]}, y^{[2]}\right] : y \in f\right\} \neq \left\{\left[y^{[1]}, \mathbf{arb}\left(\{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\}\right)^{[2]}\right] : y \in f\right\}$

-- Hence there must exist a $y \in f$ for which $y^{[2]}$ and $\mathbf{arb}\left(\{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\}\right)^{[2]}$ are different,

$\langle y \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\quad$ *Stat3* : $y \in f$ & $\left[y^{[1]}, y^{[2]}\right] \neq \left[y^{[1]}, \mathbf{arb}\left(\{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\}\right)^{[2]}\right]$

-- But it is easily seen that $\{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\} = \{y\}$.

Suppose $\Rightarrow$ $\quad$ *Stat4* : $\{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\} \neq \{y\}$

$\langle d \rangle \hookrightarrow$ *Stat4*$(\langle \cap \rangle)$ $\Rightarrow$ $\quad$ *Stat20* : $\neg(d \in \{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\} \leftrightarrow d \in \{y\})$

Set_monot $\Rightarrow$ $\quad \{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\} = \{u \in f \,|\, u^{[1]} = y^{[1]}\}$

$\langle$*Stat20*, $*\rangle$ ELEM $\Rightarrow$ $\quad \neg(d \in \{u \in f \,|\, u^{[1]} = y^{[1]}\} \leftrightarrow d = y)$

Suppose $\Rightarrow$ $\quad$ *Stat5* : $d = y$ & *Stat6* : $d \notin \{u \in f \,|\, u^{[1]} = y^{[1]}\}$

$\langle d \rangle \hookrightarrow$ *Stat6*([*Stat5*, *Stat3*]) $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ *Stat8* : $d \in \{u \in f \,|\, u^{[1]} = y^{[1]}\}$

$\langle\ \rangle \hookrightarrow$ *Stat8*([]) $\Rightarrow$ $\quad d \in f$ & $d^{[1]} = y^{[1]}$

$\langle d, y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \{u \in f \,|\, u^{[1]} \in \{y^{[1]}\}\} = \{y\}$

EQUAL $\Rightarrow$ $\quad$ *Stat9* : $\left\{\left[y^{[1]}, y^{[2]}\right] : y \in f\right\} \neq \left\{\left[y^{[1]}, \mathbf{arb}(\{y\})^{[2]}\right] : y \in f\right\}$

$\langle e \rangle \hookrightarrow$ *Stat9* $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- The following corollary simply adds the formula for **range**(f) to the preceding result.

Theorem 94 (66) $\quad \mathsf{Svm}(F) \rightarrow F = \{[x, F\upharpoonright x] : x \in \mathbf{domain}(F)\}$ & $\mathbf{range}(F) = \{F\upharpoonright x : x \in \mathbf{domain}(F)\}$. PROOF:

Suppose_not (f) $\Rightarrow$ $\quad \mathsf{Svm}(f)$ & $\neg(f = \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}$ & $\mathbf{range}(f) = \{f\upharpoonright x : x \in \mathbf{domain}(f)\})$

$\langle f \rangle \hookrightarrow$ *T65* $\Rightarrow$ $\quad f = \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}$ & $\mathbf{range}(f) \neq \{f\upharpoonright x : x \in \mathbf{domain}(f)\}$

Use_def (**range**) $\Rightarrow$ $\quad \{x^{[2]} : x \in f\} \neq \{f\upharpoonright x : x \in \mathbf{domain}(f)\}$

EQUAL $\Rightarrow$ $\quad \{x^{[2]} : x \in \{[x, f\upharpoonright x] : x \in \mathbf{domain}(f)\}\} \neq \{f\upharpoonright x : x \in \mathbf{domain}(f)\}$

SIMPLF $\Rightarrow$ $\quad Stat1: \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f)\right\} \neq \{f\restriction x : x \in \mathbf{domain}(f)\}$

$\langle x\rangle \hookrightarrow Stat1 \Rightarrow \quad [x, f\restriction x]^{[2]} \neq f\restriction x$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following is a variant of Theorem 65.

Theorem 95 (67) $\mathsf{Svm}(F) \,\&\, X \in F \rightarrow F\restriction X^{[1]} = X^{[2]}$. PROOF:

Suppose_not(f, a) $\Rightarrow$ $\quad \mathsf{Svm}(f) \,\&\, a \in f \,\&\, f\restriction a^{[1]} \neq a^{[2]}$

-- For if our theorem is false, then by Theorem 65 there is an a in $\left\{\left[x^{[1]}, f\restriction x^{[1]}\right] : x \in f\right\}$ such that $f\restriction a^{[1]} \neq a^{[2]}$.

$\langle f\rangle \hookrightarrow T65 \Rightarrow \quad a \in \{[x, f\restriction x] : x \in \mathbf{domain}(f)\}$

Use_def(**domain**) $\Rightarrow$ $\quad a \in \left\{[x, f\restriction x] : x \in \left\{x^{[1]} : x \in f\right\}\right\}$

SIMPLF $\Rightarrow$ $\quad Stat1: a \in \left\{\left[x^{[1]}, f\restriction x^{[1]}\right] : x \in f\right\}$

-- Such an a must have the form $\left[b^{[1]}, f\restriction b^{[1]}\right]$ where $b \in f$, so that $a^{[1]} = b^{[1]}$; but then, since f is single valued, we must have $a = b$, so $a^{[2]} = f\restriction a^{[1]}$.

$\langle b\rangle \hookrightarrow Stat1 \Rightarrow \quad a = \left[b^{[1]}, f\restriction b^{[1]}\right] \,\&\, b \in f$

ELEM $\Rightarrow$ $\quad a^{[1]} = b^{[1]}$

Use_def(Svm) $\Rightarrow$ $\quad Stat2: \left\langle \forall x \in f, y \in f \mid x^{[1]} = y^{[1]} \rightarrow x = y\right\rangle$

$\langle a, b\rangle \hookrightarrow Stat2 \Rightarrow \quad a = b$

EQUAL $\Rightarrow$ $\quad f\restriction b^{[1]} = f\restriction a^{[1]}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma simply states the elementary fact that every element of a map is a pair.

Theorem 96 (68) $\mathsf{Is\_map}(F) \,\&\, U \in F \rightarrow U = \left[U^{[1]}, U^{[2]}\right]$. PROOF:

Suppose_not(g, u) $\Rightarrow$ $\quad \mathsf{Is\_map}(g) \,\&\, u \in g \,\&\, u \neq \left[u^{[1]}, u^{[2]}\right]$

Use_def(Is_map) $\Rightarrow$ $\quad Stat1: u \in \left\{\left[x^{[1]}, x^{[2]}\right] : x \in g\right\}$

$\langle a\rangle \hookrightarrow Stat1 \Rightarrow \quad a \in g \,\&\, u = \left[a^{[1]}, a^{[2]}\right]$

EQUAL $\Rightarrow$ $\quad Stat2: \left[a^{[1]}, a^{[2]}\right] \neq \left[\left[a^{[1]}, a^{[2]}\right]^{[1]}, \left[a^{[1]}, a^{[2]}\right]^{[2]}\right]$

$\langle Stat2 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following is another lemma sometimes useful in connection with maps.

Theorem 97 (69) $\text{Is\_map}(F) \to (X \in \mathbf{domain}(F) \leftrightarrow [X, F\restriction X] \in F)$. PROOF:

Suppose_not$(f, x) \Rightarrow$ $\text{Is\_map}(f) \,\&\, \neg(x \in \mathbf{domain}(f) \leftrightarrow [x, f\restriction x] \in f)$

-- For suppose the contrary, and first consider the case $x \in \mathbf{domain}(f)$, so that there exists a $y \in f$ such that $x = y^{[1]}$.

Suppose $\Rightarrow$ $x \in \mathbf{domain}(f) \,\&\, [x, f\restriction x] \notin f$
Use_def$(\mathbf{domain}) \Rightarrow$ $Stat1: \; x \in \{y^{[1]} : y \in f\} \,\&\, [x, f\restriction x] \notin f$
$\langle y \rangle \hookrightarrow Stat1 \Rightarrow$ $x = y^{[1]} \,\&\, y \in f$

-- By definition of the operators involved, $\text{is}(f\restriction x, yy^{[2]})$ for some $yy \in f$ such that $yy^{[1]} = x$.

Use_def$(\restriction) \Rightarrow$ $f\restriction x = \mathbf{arb}(f_{|\{x\}})^{[2]}$
Use_def$(|) \Rightarrow$ $f\restriction x = \mathbf{arb}(\{u : u \in f \,|\, u^{[1]} \in \{x\}\})^{[2]}$
Suppose $\Rightarrow$ $Stat2: \; y \notin \{u : u \in f \,|\, u^{[1]} \in \{x\}\}$
$\langle y \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ $y \in \{u : u \in f \,|\, u^{[1]} \in \{x\}\}$
$\langle \{u : u \in f \,|\, u^{[1]} \in \{x\}\} \rangle \hookrightarrow T0 \Rightarrow$ $Stat3:$
$\mathbf{arb}(\{u : u \in f \,|\, u^{[1]} \in \{x\}\}) \in \{u : u \in f \,|\, u^{[1]} \in \{x\}\}$
$\langle yy \rangle \hookrightarrow Stat3 \Rightarrow$ $\mathbf{arb}(\{u : u \in f \,|\, u^{[1]} \in \{x\}\}) = yy \,\&\, yy \in f \,\&\, yy^{[1]} = x$
EQUAL $\Rightarrow$ $f\restriction x = yy^{[2]} \,\&\, yy \in f \,\&\, yy^{[1]} = x$

-- But by Theorem 68, this implies that $yy \notin f$, a contradiction which excludes the case $x \in \mathbf{domain}(f)$ of our initial assumption, leaving only the case $[x, f\restriction x] \in f$.

EQUAL $\Rightarrow$ $[yy^{[1]}, yy^{[2]}] \notin f$
$\langle f, yy \rangle \hookrightarrow T68 \Rightarrow$ $yy \notin f$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $x \notin \mathbf{domain}(f) \,\&\, [x, f\restriction x] \in f$

-- But then by definition we have an immediate contradiction which proves our theorem.

Use_def$(\mathbf{domain}) \Rightarrow$ $Stat4: \; x \notin \{v^{[1]} : v \in f\}$
$\langle [x, f\restriction x] \rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The additivity of domain and range, stated in the following theorems, both follow immediately by application of set monotonicity.

Theorem 98 (70) $\mathbf{domain}(\mathsf{F} \cup \mathsf{G}) = \mathbf{domain}(\mathsf{F}) \cup \mathbf{domain}(\mathsf{G})$. PROOF:

Suppose_not(f, g) ⇒ $\mathbf{domain}(\mathsf{f} \cup \mathsf{g}) \neq \mathbf{domain}(\mathsf{f}) \cup \mathbf{domain}(\mathsf{g})$
Use_def(**domain**) ⇒ $\{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{f} \cup \mathsf{g}\} \neq \{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{f}\} \cup \{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{g}\}$
Set_monot ⇒ $\{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{f} \cup \mathsf{g}\} = \{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{f}\} \cup \{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{g}\}$
ELEM ⇒ false; Discharge ⇒ QED

Theorem 99 (71) $\mathbf{range}(\mathsf{F} \cup \mathsf{G}) = \mathbf{range}(\mathsf{F}) \cup \mathbf{range}(\mathsf{G})$. PROOF:

Suppose_not(f, g) ⇒ $\mathbf{range}(\mathsf{f} \cup \mathsf{g}) \neq \mathbf{range}(\mathsf{f}) \cup \mathbf{range}(\mathsf{g})$
Use_def(**range**) ⇒ $\{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{f} \cup \mathsf{g}\} \neq \{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{f}\} \cup \{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{g}\}$
Set_monot ⇒ $\{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{f} \cup \mathsf{g}\} = \{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{f}\} \cup \{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{g}\}$
ELEM ⇒ false; Discharge ⇒ QED

-- The following is a corollary of Theorems 23 and 43.

Theorem 100 (72) $\mathbf{range}(\mathsf{F}_{|\mathsf{S}}) \subseteq \mathbf{range}(\mathsf{F})$. PROOF:

Suppose_not(f, s) ⇒ $\mathbf{range}(\mathsf{f}_{|\mathsf{s}}) \not\subseteq \mathbf{range}(\mathsf{f})$
$\langle \mathsf{f}, \mathsf{s} \rangle \hookrightarrow T43$ ⇒ $\mathsf{f} = \mathsf{f}_{|\mathsf{s}} \cup (\mathsf{f} \backslash \mathsf{f}_{|\mathsf{s}})$
$\langle \mathsf{f}_{|\mathsf{s}}, \mathsf{f} \backslash \mathsf{f}_{|\mathsf{s}} \rangle \hookrightarrow T71$ ⇒ $\mathbf{range}(\mathsf{f}_{|\mathsf{s}} \cup (\mathsf{f} \backslash \mathsf{f}_{|\mathsf{s}})) = \mathbf{range}(\mathsf{f}_{|\mathsf{s}}) \cup \mathbf{range}(\mathsf{f} \backslash \mathsf{f}_{|\mathsf{s}})$
EQUAL ⇒ $\mathbf{range}(\mathsf{f}) = \mathbf{range}(\mathsf{f}_{|\mathsf{s}}) \cup \mathbf{range}(\mathsf{f} \backslash \mathsf{f}_{|\mathsf{s}})$
ELEM ⇒ false; Discharge ⇒ QED

-- The following elementary lemmas concerning the additivity, monotonicity of the range function, and some easy additional properties specific to 1-1 maps, are also useful.

Theorem 101 (73) $\mathbf{range}(\mathsf{F}_{|\mathsf{S} \cup T}) = \mathbf{range}(\mathsf{F}_{|\mathsf{S}}) \cup \mathbf{range}(\mathsf{F}_{|T})$. PROOF:

Suppose_not(f, s, t) ⇒ $\mathbf{range}(\mathsf{f}_{|\mathsf{s} \cup \mathsf{t}}) \neq \mathbf{range}(\mathsf{f}_{|\mathsf{s}}) \cup \mathbf{range}(\mathsf{f}_{|\mathsf{t}})$
$\langle \mathsf{f}, \mathsf{s}, \mathsf{t} \rangle \hookrightarrow T58$ ⇒ $\mathsf{f}_{|\mathsf{s} \cup \mathsf{t}} = \mathsf{f}_{|\mathsf{s}} \cup \mathsf{f}_{|\mathsf{t}}$
$\langle \mathsf{f}_{|\mathsf{s}}, \mathsf{f}_{|\mathsf{t}} \rangle \hookrightarrow T71$ ⇒ $\mathbf{range}(\mathsf{f}_{|\mathsf{s}} \cup \mathsf{f}_{|\mathsf{t}}) = \mathbf{range}(\mathsf{f}_{|\mathsf{s}}) \cup \mathbf{range}(\mathsf{f}_{|\mathsf{t}})$
EQUAL ⇒ false; Discharge ⇒ QED

Theorem 102 (74) $S \supseteq T \rightarrow \mathbf{range}(F_{|S}) \supseteq \mathbf{range}(F_{|T})$. PROOF:

Suppose_not$(s,t,f) \Rightarrow$ $s \supseteq t \;\&\; \mathbf{range}(f_{|s}) \not\supseteq \mathbf{range}(f_{|t})$

$\langle f, t, s \backslash t \rangle \hookrightarrow T73 \Rightarrow$ $\mathbf{range}(f_{|t \,\cup\, (s \backslash t)}) \supseteq \mathbf{range}(f_{|t})$

ELEM $\Rightarrow$ $t \cup (s \backslash t) = s$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that the range of a 1-1 map f on the intersection of two sets is the intersection of the restrictions of f to each of these two sets.

Theorem 103 (75) $1\text{–}1(F) \rightarrow \mathbf{range}(F_{|S \,\cap\, T}) = \mathbf{range}(F_{|S}) \cap \mathbf{range}(F_{|T})$. PROOF:

Suppose_not$(f,s,t) \Rightarrow$ $1\text{–}1(f) \;\&\; \mathbf{range}(f_{|s \,\cap\, t}) \neq \mathbf{range}(f_{|s}) \cap \mathbf{range}(f_{|t})$

-- For suppose that f, s, and t are a counterexample to our assertion. Since by monotonicity $\mathbf{range}(f_{|s \,\cap\, t})$ must be a subset of both $\mathbf{range}(f_{|s})$ and $\mathbf{range}(f_{|t})$, and hence of their intersection, it follows that the intersection I of these two latter sets must contain $\mathbf{range}(f_{|s \,\cap\, t})$, and so if our theorem is false $\mathbf{range}(f_{|s \,\cap\, t})$ cannot be a subset of I.

$\langle s, s \cap t, f \rangle \hookrightarrow T74 \Rightarrow$ $\mathbf{range}(f_{|s}) \supseteq \mathbf{range}(f_{|s \,\cap\, t})$

$\langle t, s \cap t, f \rangle \hookrightarrow T74 \Rightarrow$ $\mathbf{range}(f_{|t}) \supseteq \mathbf{range}(f_{|s \,\cap\, t})$

ELEM $\Rightarrow$ *Stat1* : $\mathbf{range}(f_{|s \,\cap\, t}) \not\supseteq \mathbf{range}(f_{|s}) \cap \mathbf{range}(f_{|t})$

-- Hence there must exit an element c of I which does not belong to $\mathbf{range}(f_{|s \,\cap\, t})$, and therefore elements $d \in f$, $d \in f$ such that $c = d^{[2]}$, $c = e^{[2]}$ such that $d^{[1]}$ and $e^{[1]}$ are members of s, t respectively.

$\langle c \rangle \hookrightarrow Stat1 \Rightarrow$ $c \in \mathbf{range}(f_{|s}) \cap \mathbf{range}(f_{|t}) \;\&\; c \notin \mathbf{range}(f_{|s \,\cap\, t})$

Use_def(range) $\Rightarrow$ $c \in \{x^{[2]} : x \in f_{|s}\} \;\&\; c \in \{x^{[2]} : x \in f_{|t}\} \;\&\; c \notin \{x^{[2]} : x \in f_{|s \,\cap\, t}\}$

Use_def(|) $\Rightarrow$ $c \in \{x^{[2]} : x \in \{y \in f \,|\, y^{[1]} \in s\}\} \;\&\; c \in \{x^{[2]} : x \in \{y \in f \,|\, y^{[1]} \in t\}\} \;\&$
$c \notin \{x^{[2]} : x \in \{y \in f \,|\, y^{[1]} \in s \cap t\}\}$

SIMPLF $\Rightarrow$ *Stat2* : $c \in \{x^{[2]} : x \in f \,|\, x^{[1]} \in s\} \;\&$ *Stat3* : $c \in \{x^{[2]} : x \in f \,|\, x^{[1]} \in t\} \;\&$ *Stat4* : $c \notin \{x^{[2]} : x \in f \,|\, x^{[1]} \in s \cap t\}$

$\langle d \rangle \hookrightarrow Stat2 \Rightarrow$ $c = d^{[2]} \;\&\; d \in f \;\&\; d^{[1]} \in s$

$\langle e \rangle \hookrightarrow Stat3 \Rightarrow$ $c = e^{[2]} \;\&\; e \in f \;\&\; e^{[1]} \in t$

-- But now, since f is 1-1, we must have $d = e$, so that $d^{[1]} \in s \cap t$, which contradicts $c \notin \mathbf{range}(f_{|s \,\cap\, t})$ and so proves our theorem.

Use_def(1–1) $\Rightarrow$ *Stat5* : $\langle \forall x \in f, y \in f \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle$

$\langle d, e \rangle \hookrightarrow Stat5 \Rightarrow$ $d = e$

ELEM $\Rightarrow$ $d^{[1]} \in s \cap t$
$\langle d \rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemma gives the entirely elementary fact that the restriction of a map to the null set, and the range of this restriction, are both null.

Theorem 104 (76) $F_{|\emptyset} = \emptyset \,\&\, \mathbf{range}(F_{|\emptyset}) = \emptyset$. PROOF:

Suppose_not(f) $\Rightarrow$ $f_{|\emptyset} \neq \emptyset \vee \mathbf{range}(f_{|\emptyset}) \neq \emptyset$
Suppose $\Rightarrow$ $f_{|\emptyset} \neq \emptyset$
Use_def(|) $\Rightarrow$ $Stat1: \{y \in f \mid y^{[1]} \in \emptyset\} \neq \emptyset$
$\langle d \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $f_{|\emptyset} = \emptyset$
EQUAL $\Rightarrow$ $\mathbf{range}(\emptyset) \neq \emptyset$
$T54 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 105 (77) $1\text{–}1(F) \,\&\, S \cap T = \emptyset \rightarrow \mathbf{range}(F_{|S}) \cap \mathbf{range}(F_{|T}) = \emptyset$. PROOF:

Suppose_not(f, s, t) $\Rightarrow$ $1\text{–}1(f) \,\&\, s \cap t = \emptyset \,\&\, \mathbf{range}(f_{|s}) \cap \mathbf{range}(f_{|t}) \neq \emptyset$
$\langle f, s, t \rangle \hookrightarrow T75 \Rightarrow$ $\mathbf{range}(f_{|s \cap t}) \neq \emptyset$
EQUAL $\Rightarrow$ $\mathbf{range}(f_{|\emptyset}) \neq \emptyset$
$\langle f \rangle \hookrightarrow T76 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that if either of the domain and range of a set f is empty, so is the other (either of these conditions is equivalent to the condition that f is empty).

Theorem 106 (78) $\mathbf{domain}(F) = \emptyset \leftrightarrow \mathbf{range}(F) = \emptyset$. PROOF:

Suppose_not(f) $\Rightarrow$ $\neg(\mathbf{domain}(f) = \emptyset \leftrightarrow \mathbf{range}(f) = \emptyset)$
Use_def(**domain**) $\Rightarrow$ $\neg(\{x^{[1]} : x \in f\} = \emptyset \leftrightarrow \mathbf{range}(f) = \emptyset)$
Use_def(**range**) $\Rightarrow$ $\neg(\{x^{[1]} : x \in f\} = \emptyset \leftrightarrow \{x^{[2]} : x \in f\} = \emptyset)$

-- For if not, one of these sets must be empty and the other not. Suppose first that $\{x^{[2]} : x \in f\}$ is nonempty, so that there exists a d in f. Then the first set must also be nonempty. The same argument applies in the case that $\{x^{[1]} : x \in f\}$ is nonempty, and so our assertion holds in each case.

Suppose $\Rightarrow$ $Stat1: \{x^{[1]} : x \in f\} = \emptyset \,\&\, Stat2: \{x^{[2]} : x \in f\} \neq \emptyset$
$\langle d \rangle \hookrightarrow Stat2 \Rightarrow$ $d \in f$

$\langle d\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat3: \{x^{[1]} : x \in f\} \neq \emptyset \;\&\; Stat4: \{x^{[2]} : x \in f\} = \emptyset$
$\langle a\rangle \hookrightarrow Stat3 \Rightarrow$ $a \in f$
$\langle a\rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The union of two single valued maps need not always be single valued, but the following theorem tells us that it is if the domains of the two maps are disjoint.

-- Union of single_valued maps

Theorem 107 (79) $\mathsf{Svm}(F) \;\&\; \mathsf{Svm}(G) \;\&\; \mathbf{domain}(F) \cap \mathbf{domain}(G) = \emptyset \rightarrow \mathsf{Svm}(F \cup G)$. PROOF:

Suppose_not$(f, g) \Rightarrow$ $\mathsf{Svm}(f) \;\&\; \mathsf{Svm}(g) \;\&\; \mathbf{domain}(f) \cap \mathbf{domain}(g) = \emptyset \;\&\; \neg\mathsf{Svm}(f \cup g)$

-- For suppose not, use the definition of Svm to expand the negative of our theorem, and use the fact, which follows from Theorem 57, that $f \cup g$ must be a map. from which it follows that $f \cup g$ must have two distinct elements a and b such that $a^{[1]} = b^{[1]}$.

Use_def(Svm) $\Rightarrow$ $\mathsf{Is\_map}(f) \;\&\; Stat1: \langle \forall x \in f, y \in f \,|\, x^{[1]} = y^{[1]} \rightarrow x = y\rangle$
Use_def(Svm) $\Rightarrow$ $\mathsf{Is\_map}(g) \;\&\; Stat2: \langle \forall x \in g, y \in g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y\rangle$
Use_def(Svm) $\Rightarrow$ $\neg\mathsf{Is\_map}(f \cup g) \vee \neg\langle \forall x \in f \cup g, y \in f \cup g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y\rangle$
$\langle f, g\rangle \hookrightarrow T57 \Rightarrow$ $Stat3: \neg\langle \forall x \in f \cup g, y \in f \cup g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y\rangle$
$\langle a, b\rangle \hookrightarrow Stat3 \Rightarrow$ $a, b \in f \cup g \;\&\; a^{[1]} = b^{[1]} \;\&\; a \neq b$

-- Then, since f and g are both single valued, it follows that one of a and b must belong to f and the other to g. Suppose for definiteness that a belongs to f and b to g. Then by Theorem 55 $a^{[1]}$ belongs to $\mathbf{domain}(f)$ and $b^{[1]}$ to the disjoint set $\mathbf{domain}(g)$, which is impossible since $a^{[1]} = b^{[1]}$.

$\langle a, b\rangle \hookrightarrow Stat1 \Rightarrow$ $a, b \in f \rightarrow a^{[1]} = b^{[1]} \rightarrow a = b$
$\langle a, b\rangle \hookrightarrow Stat2 \Rightarrow$ $a, b \in g \rightarrow a^{[1]} = b^{[1]} \rightarrow a = b$
ELEM $\Rightarrow$ $(a \in f \;\&\; b \in g) \vee (a \in g \;\&\; b \in f)$
Suppose $\Rightarrow$ $a \in f \;\&\; b \in g$
$\langle a, f\rangle \hookrightarrow T55 \Rightarrow$ $a^{[1]} \in \mathbf{domain}(f)$
$\langle b, g\rangle \hookrightarrow T55 \Rightarrow$ $b^{[1]} \in \mathbf{domain}(g)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $a \in g \;\&\; b \in f$

-- This leaves only the symmetric case $a \in g \;\&\; b \in f$, which can be treated in the same way, proving our theorem.

$\langle a, g\rangle \hookrightarrow T55 \Rightarrow$ $a^{[1]} \in \mathbf{domain}(g)$
$\langle b, f\rangle \hookrightarrow T55 \Rightarrow$ $b^{[1]} \in \mathbf{domain}(f)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- A very similar argument shows that the union of two 1-1 maps with disjoint ranges and domains must be 1-1.

-- Union of 1 - 1 maps

Theorem 108 (80) $1\text{–}1(F) \,\&\, 1\text{–}1(G) \,\&\, \mathbf{range}(F) \cap \mathbf{range}(G) = \emptyset \,\&\, \mathbf{domain}(F) \cap \mathbf{domain}(G) = \emptyset \rightarrow 1\text{–}1(F \cup G)$. PROOF:

Suppose_not$(f, g) \Rightarrow$ $(1\text{–}1(f) \,\&\, 1\text{–}1(g) \,\&\, \mathbf{range}(f) \cap \mathbf{range}(g) = \emptyset \,\&\, \mathbf{domain}(f) \cap \mathbf{domain}(g) = \emptyset) \,\&\, \neg 1\text{–}1(f \cup g)$

-- For suppose not, use the definition of 'one_1_map' to expand the negative of our theorem, and use the fact, following by Theorem 79, that $f \cup g$ must be a single valued map. from which it follows that $f \cup g$ must have two distinct elements a and b such that $a^{[2]} = b^{[2]}$.

Use_def$(1\text{–}1) \Rightarrow$ $\mathsf{Svm}(f) \,\&\, \mathit{Stat1}:$
$\langle \forall x \in f, y \in f \mid x^{[2]} = y^{[2]} \longrightarrow x = y \rangle \,\&\, \mathsf{Svm}(g) \,\&\, \mathit{Stat2}:$
$\langle \forall x \in g, y \in g \mid x^{[2]} = y^{[2]} \longrightarrow x = y \rangle \,\&\, \neg\mathsf{Svm}(f \cup g) \vee \neg\langle \forall x \in f \cup g, y \in f \cup g \mid x^{[2]} = y^{[2]} \longrightarrow x = y \rangle$
$\langle f, g \rangle \hookrightarrow T79 \Rightarrow$ $\mathit{Stat3}:\ \neg\langle \forall x \in f \cup g, y \in f \cup g \mid x^{[2]} = y^{[2]} \longrightarrow x = y \rangle$
$\langle a, b \rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $\mathit{Stat4}:\ a, b \in f \cup g \,\&\, a^{[2]} = b^{[2]} \,\&\, a \neq b$

-- Then, since f and g are both 1-1 maps, it follows that one of a and b must belong to f and the other to g. Suppose for definiteness that a belongs to f and b to g. Then by Theorem 56 $a^{[2]}$ belongs to $\mathbf{range}(f)$ and $a^{[2]}$ to the disjoint set $\mathbf{range}(g)$, which is impossible since $a^{[2]} = b^{[2]}$.

$\langle a, b \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $a, b \in f \rightarrow a^{[2]} = b^{[2]} \longrightarrow a = b$
$\langle a, b \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $a, b \in g \rightarrow a^{[2]} = b^{[2]} \longrightarrow a = b$
ELEM $\Rightarrow$ $(a \in f \,\&\, b \in g) \vee (a \in g \,\&\, b \in f)$
Suppose $\Rightarrow$ $a \in f \,\&\, b \in g$
$\langle a, f \rangle \hookrightarrow T56 \Rightarrow$ $a^{[2]} \in \mathbf{range}(f)$
$\langle b, g \rangle \hookrightarrow T56 \Rightarrow$ $b^{[2]} \in \mathbf{range}(g)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $a \in g \,\&\, b \in f$

-- This leaves only the symmetric case $a \in g \,\&\, b \in f$, which can be treated in the same way, proving our theorem.

$\langle a, g \rangle \hookrightarrow T56 \Rightarrow$ $a^{[2]} \in \mathbf{range}(g)$
$\langle b, f \rangle \hookrightarrow T56 \Rightarrow$ $b^{[2]} \in \mathbf{range}(f)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following simple lemma will be useful in our later work with map inverses.

Theorem 109 (81) $\mathsf{Is\_map}(F) \rightarrow ([X, Y] \in F \leftrightarrow [Y, X] \in F^{\leftarrow})$. PROOF:

Suppose_not(f, x, y) $\Rightarrow$ $\mathsf{Is\_map}(f)$ & $\neg([x, y] \in f \leftrightarrow [y, x] \in f^{\leftarrow})$

-- In the contrary case, and first considering the subcase in which $[x, y]$ belongs to f, use of the definition leads to an immediate contradiction, so that we must have $[x, y] \notin f$ and $[y, x] \in f^{\leftarrow}$.

Suppose $\Rightarrow$ $[x, y] \in f$ & $[y, x] \notin f^{\leftarrow}$
Use_def$(^{\leftarrow})$ $\Rightarrow$ $\mathit{Stat1}:\ [y, x] \notin \{[u^{[2]}, u^{[1]}] : u \in f\}$
$\langle [x, y] \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $[y, x] \neq \left[[x, y]^{[2]}, [x, y]^{[1]}\right]$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $[x, y] \notin f$ & $[y, x] \in f^{\leftarrow}$

-- But in this case use of the definition also leads, via Theorem 68, to an immediate contradiction, which completes the proof of the present lemma.

Use_def$(^{\leftarrow})$ $\Rightarrow$ $[x, y] \notin f$ & $\mathit{Stat2}:\ [y, x] \in \{[u^{[2]}, u^{[1]}] : u \in f\}$
$\langle u \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $[y, x] = [u^{[2]}, u^{[1]}]$ & $u \in f$
ELEM $\Rightarrow$ $[u^{[1]}, u^{[2]}] \neq u$
$\langle f, u \rangle \hookrightarrow T68 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The two following elementary lemmas note other ways in which small 1-1 maps can be constructed. First we observe that the singleton $\{[x, y]\}$ is always a 1-1 map.

Theorem 110 (82) $\mathsf{Svm}(\{[X, Y]\})$ & $1\text{–}1(\{[X, Y]\})$ & $\{[X, Y]\} \restriction X = Y$. PROOF:

Suppose_not(x_1, y_1) $\Rightarrow$ $\neg\big(\mathsf{Svm}(\{[x_1, y_1]\})$ & $1\text{–}1(\{[x_1, y_1]\})$ & $\{[x_1, y_1]\} \restriction x_1 = y_1\big)$

-- We prove the various clauses of this theorem successively. The fact that $\{[x_1, y_1]\}$ is a map follows readily by use of our utility lz_map theory, and by simplification.

SIMPLF $\Rightarrow$ $\{[u^{[1]}, u^{[2]}] : u \in \{[x_1, y_1]\}\} = \left\{\left[[x_1, y_1]^{[1]}, [x_1, y_1]^{[2]}\right]\right\}$
ELEM $\Rightarrow$ $\{[u^{[1]}, u^{[2]}] : u \in \{[x_1, y_1]\}\} = \{[x_1, y_1]\}$
APPLY $\langle\,\rangle$ lz_map$(a(u) \mapsto u^{[1]}, b(u) \mapsto u^{[2]}, s \mapsto \{[x_1, y_1]\})$ $\Rightarrow$

$\mathsf{Is\_map}(\{[u^{[1]}, u^{[2]}] : u \in \{[x_1, y_1]\}\})$

EQUAL $\Rightarrow$ $\mathsf{Is\_map}(\{[x_1, y_1]\})$

-- Likewise, to prove that $\{[x_1, y_1]\}$ is a single valued map we have only to use the definition of Svm.

Suppose $\Rightarrow$ $\neg\mathsf{Svm}(\{[x_1, y_1]\})$

Use_def (Svm) $\Rightarrow$ $\neg\big(\mathsf{Is\_map}(\{[x_1, y_1]\}) \,\&\, \langle\forall u \in \{[x_1, y_1]\}, v \in \{[x_1, y_1]\} \mid u^{[1]} = v^{[1]} \rightarrow u = v\rangle\big)$

ELEM $\Rightarrow$ *Stat1* : $\neg\langle\forall u \in \{[x_1, y_1]\}, v \in \{[x_1, y_1]\} \mid u^{[1]} = v^{[1]} \rightarrow u = v\rangle$

$\langle a, b\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $a, b \in \{[x_1, y_1]\} \,\&\, a^{[1]} = b^{[1]} \,\&\, a \neq b$

ELEM $\Rightarrow$ $a = [x_1, y_1] \,\&\, b = [x_1, y_1]$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Svm}(\{[x_1, y_1]\})$

-- The proof that $\{[x_1, y_1]\}$ is 1-1 is equally elementary. Once this is done it only remains to prove the final clause of our theorem.

Suppose $\Rightarrow$ $\neg 1\text{–}1(\{[x_1, y_1]\})$

Use_def (1–1) $\Rightarrow$ $\neg\big(\mathsf{Svm}(\{[x_1, y_1]\}) \,\&\, \langle\forall u \in \{[x_1, y_1]\}, v \in \{[x_1, y_1]\} \mid u^{[2]} = v^{[2]} \rightarrow u = v\rangle\big)$

ELEM $\Rightarrow$ *Stat2* : $\neg\langle\forall u \in \{[x_1, y_1]\}, v \in \{[x_1, y_1]\} \mid u^{[2]} = v^{[2]} \rightarrow u = v\rangle$

$\langle c, d\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3* : $c, d \in \{[x_1, y_1]\} \,\&\, c^{[2]} = d^{[2]} \,\&\, c \neq d$

$\langle$*Stat3*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat4* : $\{[x_1, y_1]\} \upharpoonright x_1 \neq y_1$

-- For this we simply use the definitions of map application, restriction, and range successively, and simplify the resulting formula, to obtain a final contradiction which proves our theorem.

Use_def ($\upharpoonright$) $\Rightarrow$ $\{[x_1, y_1]\} \upharpoonright x_1 = \mathbf{arb}\Big(\{[x_1, y_1]\}_{|\{x_1\}}\Big)^{[2]}$

Use_def ($|$) $\Rightarrow$ $\{[x_1, y_1]\} \upharpoonright x_1 = \mathbf{arb}\big(\{u \in \{[x_1, y_1]\} \mid u^{[1]} \in \{x_1\}\}\big)^{[2]}$

SIMPLF $\Rightarrow$ $\{[x_1, y_1]\} \upharpoonright x_1 = \mathbf{arb}\Big(\textbf{if } [x_1, y_1]^{[1]} \in \{x_1\} \textbf{ then } \{[x_1, y_1]\} \textbf{ else } \emptyset \textbf{ fi}\Big)^{[2]}$

ELEM $\Rightarrow$ $\textbf{if } [x_1, y_1]^{[1]} \in \{x_1\} \textbf{ then } \{[x_1, y_1]\} \textbf{ else } \emptyset \textbf{ fi} = \{[x_1, y_1]\}$

EQUAL $\Rightarrow$ $\mathbf{arb}(\{[x_1, y_1]\})^{[2]} = \mathbf{arb}\Big(\textbf{if } [x_1, y_1]^{[1]} \in \{x_1\} \textbf{ then } \{[x_1, y_1]\} \textbf{ else } \emptyset \textbf{ fi}\Big)^{[2]}$

ELEM $\Rightarrow$ $\{[x_1, y_1]\} \upharpoonright x = \mathbf{arb}(\{[x_1, y_1]\})^{[2]}$

ELEM $\Rightarrow$ $\mathbf{arb}(\{[x_1, y_1]\})^{[2]} = y_1$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we observe that the doubleton $\{[x, y], [zz, w]\}$ is a single-valued map if $x \neq zz$, in which case $\{[x, y], [zz, w]\} \upharpoonright x = y$.

Theorem 111 (83) $X \neq ZZ \to \{[X,Y],[ZZ,W]\} \upharpoonright X = Y$. PROOF:

Suppose_not$(x,zz,y,w) \Rightarrow$ *Stat0* : $x \neq zz \ \& \ \{[x,y],[zz,w]\} \upharpoonright x \neq y$

-- To show this, use the definitions of map application, restriction, and range, and then simplify, obtaining the equality seen below:

Use_def$(\upharpoonright) \Rightarrow$ $\{[x,y],[zz,w]\} \upharpoonright x = \mathbf{arb}\Big(\{[x,y],[zz,w]\}_{|\{x\}}\Big)^{[2]}$

Use_def$(|) \Rightarrow$ $\{[x,y],[zz,w]\} \upharpoonright x = \mathbf{arb}\big(\{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\}\big)^{[2]}$

-- It is now easy to see, since $x \neq zz$ m that $\{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\} \neq \{[x,y]\}$

Suppose $\Rightarrow$ *Stat1* : $\{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\} \neq \{[x,y]\}$
$\langle ww \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $ww \in \{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\} \leftrightarrow ww \neq [x,y]$
Suppose $\Rightarrow$ *Stat3* : $ww \in \{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\}$
$\langle\,\rangle \hookrightarrow$ *Stat3*$(\langle$*Stat2*$\rangle) \Rightarrow$ *Stat4* : $ww = [zz,w] \ \& \ ww^{[1]} = x$
$\langle$*Stat0, Stat4*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6* : $ww = [x,y] \ \&$ *Stat5* : $ww \notin \{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\}$
$\langle$*Stat6*$\rangle$ ELEM $\Rightarrow$ $ww^{[1]} \in \{x\}$
$\langle ww \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\neg(ww \in \{[x,y],[zz,w]\} \ \& \ ww^{[1]} \in \{x\})$
$\langle$*Stat6*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\{v \in \{[x,y],[zz,w]\} \mid v^{[1]} \in \{x\}\} = \{[x,y]\}$

-- Hence, substituting into the third line of our proof, we find that $\{[x,y],[zz,w]\} \upharpoonright x = \mathbf{arb}(\{[x,y]\})^{[2]}$, which simplifies easily to the assertion of our theorem.

EQUAL $\Rightarrow$ $\{[x,y],[zz,w]\} \upharpoonright x = \mathbf{arb}(\{[x,y]\})^{[2]}$
ELEM $\Rightarrow$ $\mathbf{arb}(\{[x,y]\}) = [x,y]$
EQUAL $\Rightarrow$ $\{[x,y],[zz,w]\} \upharpoonright x = [x,y]^{[2]}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we give a simple formula for the restriction of a map.

Theorem 112 (84) $\mathbf{domain}(F_{|S}) = \mathbf{domain}(F) \cap S$. PROOF:

Suppose_not$(f,s) \Rightarrow$ $\mathbf{domain}(f_{|s}) \neq \mathbf{domain}(f) \cap s$

-- For if we expand the negative of our theorem using the definitions of the functions involved, we see that the two sets displayed below must differ:

Use_def (**domain**) $\Rightarrow$ $\{x^{[1]} : x \in f_{|s}\} \neq \{x^{[1]} : x \in f\} \cap s$
Use_def (|) $\Rightarrow$ $\{x^{[1]} : x \in \{x \in f \,|\, x^{[1]} \in s\}\} \neq \{x^{[1]} : x \in f\} \cap s$
SIMPLF $\Rightarrow$ *Stat1*: $\{x^{[1]} : x \in f \,|\, x^{[1]} \in s\} \neq \{x^{[1]} : x \in f\} \cap s$
$\langle c \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $(c \in \{x^{[1]} : x \in f \,|\, x^{[1]} \in s\}$ & $c \notin \{x^{[1]} : x \in f\} \cap s) \vee$
$c \notin \{x^{[1]} : x \in f \,|\, x^{[1]} \in s\}$ & $c \in \{x^{[1]} : x \in f\} \cap s$

-- This is a disjunction. The first of its cases is impossible, since it would imply that c had the form $d^{[1]}$ and was both in s and not in s.

Suppose $\Rightarrow$ *Stat2*: $c \in \{x^{[1]} : x \in f \,|\, x^{[1]} \in s\}$ & *Stat3*: $c \notin \{x^{[1]} : x \in f\} \vee c \notin s$
$\langle d \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $c = d^{[1]}$ & $d \in f$ & $d^{[1]} \in s$
$\langle d \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; -

-- Hence we need only consider the second case, but this leads immediately to a similar impossibility, proving our theorem.

Discharge $\Rightarrow$ *Stat4*: $c \notin \{x^{[1]} : x \in f \,|\, x^{[1]} \in s\}$ & *Stat5*: $c \in \{x^{[1]} : x \in f\}$ & $c \in s$
$\langle e \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $c = e^{[1]}$ & $e \in f$
$\langle e \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem gives formulae for the range and domain of a product map $f \bullet g$, under the simplifying hypothesis that the range of g is included in the domain of f.

Theorem 113 (85) $\mathbf{range}(G) \subseteq \mathbf{domain}(F) \rightarrow \mathbf{range}(F \bullet G) = \mathbf{range}(F_{|\mathbf{range}(G)})$ & $\mathbf{domain}(F \bullet G) = \mathbf{domain}(G)$. PROOF:

Suppose_not (g, f) $\Rightarrow$ *Stat1*: $\mathbf{range}(g) \subseteq \mathbf{domain}(f)$ & $\mathbf{range}(f \bullet g) \neq \mathbf{range}(f_{|\mathbf{range}(g)}) \vee \mathbf{domain}(f \bullet g) \neq \mathbf{domain}(g)$

-- Proceeding by contradiction, we have two cases to consider. First suppose that the two ranges appearing in the theorem are different. Use the definitions of the functions involved and simplify.

Suppose $\Rightarrow$ $\mathbf{range}(f \bullet g) \neq \mathbf{range}(f_{|\mathbf{range}(g)})$
Use_def (**range**) $\Rightarrow$ $\mathbf{range}(f \bullet g) = \{x^{[2]} : x \in f \bullet g\}$
Use_def ($\bullet$) $\Rightarrow$ $\mathbf{range}(f \bullet g) = \{x^{[2]} : x \in \{[x^{[1]}, y^{[2]}] : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\}\}$
SIMPLF $\Rightarrow$ $\mathbf{range}(f \bullet g) = \left\{[x^{[1]}, y^{[2]}]^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\}$

-- But $\mathbf{range}(f \bullet g)$ can be simplified further to $\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\}$.

Suppose $\Rightarrow$ $\quad Stat2: \ \left\{\left[x^{[1]}, y^{[2]}\right]^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\} \neq \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\}$

$\langle a', b' \rangle \hookrightarrow Stat2 \Rightarrow$ $\quad \left[a'^{[1]}, b'^{[2]}\right]^{[2]} \neq b'^{[2]}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad \mathbf{range}(f \bullet g) = \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\}$

-- From this, using the definitions of **range** and |, we get the set inequality seen below.

Use_def (**range**) $\Rightarrow$ $\quad \mathbf{range}(f_{|\mathbf{range}(g)}) = \left\{x^{[2]} : x \in f_{|\mathbf{range}(g)}\right\}$

Use_def (|) $\Rightarrow$ $\quad \mathbf{range}(f_{|\mathbf{range}(g)}) = \left\{x^{[2]} : x \in \left\{x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}\right\}$

SIMPLF $\Rightarrow$ $\quad \mathbf{range}(f_{|\mathbf{range}(g)}) = \left\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}$

ELEM $\Rightarrow$ $\quad Stat3: \ \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\} \neq \left\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}$

-- Hence there is a c which is in one of these sets but not the other. Suppose first that c is in the first of these sets, and so has the form seen below.

$\langle c \rangle \hookrightarrow Stat3 \Rightarrow$

$\quad (c \in \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\} \ \& \ c \notin \left\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}) \vee$

$\quad\quad c \notin \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\} \ \& \ c \in \left\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}$

Suppose $\Rightarrow$ $\quad Stat4: \ c \in \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\} \ \& \ Stat5: \ c \notin \left\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}$

$\langle a, b \rangle \hookrightarrow Stat4 \Rightarrow$ $\quad c = b^{[2]} \ \& \ a \in g \ \& \ b \in f \ \& \ a^{[2]} = b^{[1]}$

-- Then by Stat6 6 $b^{[1]}$ is not in **range**(g), which, using the definition of **range**, leads to an immediate contradiction with Stat6 7.

$\langle b \rangle \hookrightarrow Stat5 \Rightarrow$ $\quad b^{[1]} \notin \mathbf{range}(g)$

Use_def (**range**) $\Rightarrow$ $\quad Stat7: \ b^{[1]} \notin \left\{x^{[2]} : x \in g\right\}$

$\langle a \rangle \hookrightarrow Stat7 \Rightarrow$ $\quad$ false; -

-- Hence c must be in the second of the sets considered above, but not in the first, and so must have the form seen below.

Discharge $\Rightarrow$ $\quad Stat8: \ c \notin \left\{y^{[2]} : x \in g, y \in f \,|\, x^{[2]} = y^{[1]}\right\} \ \& \ Stat9: \ c \in \left\{x^{[2]} : x \in f \,|\, x^{[1]} \in \mathbf{range}(g)\right\}$

$\langle u \rangle \hookrightarrow Stat9 \Rightarrow$ $\quad c = u^{[2]} \ \& \ u \in f \ \& \ u^{[1]} \in \mathbf{range}(g)$

Use_def (**range**) $\Rightarrow$ $\quad Stat10: \ u^{[1]} \in \left\{x^{[2]} : x \in g\right\}$

$\langle v \rangle \hookrightarrow Stat10 \Rightarrow$ $\quad u^{[1]} = v^{[2]} \ \& \ v \in g$

-- But this leads to an immediate contradiction with Stat6 8, and so rules out the first of our two main cases, leaving the only case $\mathbf{domain}(f \bullet g) \neq \mathbf{domain}(g)$ to be considered.

$\langle v, u \rangle \hookrightarrow Stat8 \Rightarrow$ $\quad \neg(c = u^{[2]} \ \& \ u \in f \ \& \ v \in g \ \& \ v^{[2]} = u^{[1]})$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad Stat11: \ \mathbf{range}(f \bullet g) = \mathbf{range}(f_{|\mathbf{range}(g)})$

$\langle Stat11, Stat1\rangle$ ELEM $\Rightarrow$ $\quad \mathbf{domain}(f\bullet g) \neq \mathbf{domain}(g)$

-- This can be handled in much the same way as the case just analyzed. Using the definitions of the functions involved, we see that the two sets displayed below must differ

Use_def($\mathbf{domain}$) $\Rightarrow$ $\quad \{x^{[1]} : x \in f\bullet g\} \neq \{x^{[1]} : x \in g\}$
Use_def($\bullet$) $\Rightarrow$ $\quad \{x^{[1]} : x \in \{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}\} \neq \{x^{[1]} : x \in g\}$
SIMPLF $\Rightarrow$ $\quad \left\{[x^{[1]}, y^{[2]}]^{[1]} : x \in g, y \in f \mid x^{[2]} = y^{[1]}\right\} \neq \{x^{[1]} : x \in g\}$
$\langle X^{[1]}, Y^{[2]}\rangle \hookrightarrow T7 \Rightarrow$ $\quad \left[X^{[1]}, Y^{[2]}\right]^{[1]} = X^{[1]}$
EQUAL $\Rightarrow$ $\quad Stat12: \{x^{[1]} : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \neq \{x^{[1]} : x \in g\}$

-- Hence there is a ca which is in one of these sets but not the other. Suppose first that ca is in the first of these sets, and so has the form seen below.

$\langle ca\rangle \hookrightarrow Stat12 \Rightarrow$ $\quad (ca \in \{x^{[1]} : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; ca \notin \{x^{[1]} : x \in g\}) \vee$
$ca \notin \{x^{[1]} : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; ca \in \{x^{[1]} : x \in g\}$
Suppose $\Rightarrow$ $\quad Stat13: ca \in \{x^{[1]} : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; Stat14: ca \notin \{x^{[1]} : x \in g\}$

-- Then by Stat6 21 $ba^{[1]}$ is not in $\{x^{[1]} : x \in f\}$, which leads to an immediate contradiction.

$\langle aa, ba\rangle \hookrightarrow Stat13 \Rightarrow$ $\quad ca = aa^{[1]} \;\&\; aa \in g \;\&\; ba \in f \;\&\; aa^{[2]} = ba^{[1]}$
$\langle aa\rangle \hookrightarrow Stat14 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad Stat15: ca \notin \{x^{[1]} : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; Stat16: ca \in \{x^{[1]} : x \in g\}$

-- Hence ca must be in the second of the sets considered above, but not in the first. Thus ca must have the form $x^{[1]}$ for some $x \in g$. Since $x^{[2]}$ belongs to $\mathbf{range}(g) \subseteq \mathbf{domain}(f)$, it follows that $x^{[2]} = y^{[1]}$ for some $y \in f$. Substitution of x and y into Stat6 31 now leads immediately to a contradiction which completes our proof.

$\langle x\rangle \hookrightarrow Stat16 \Rightarrow$ $\quad ca = x^{[1]} \;\&\; x \in g$
Suppose $\Rightarrow$ $\quad Stat17: x^{[2]} \notin \{x^{[2]} : x \in g\}$
$\langle x\rangle \hookrightarrow Stat17 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad x^{[2]} \in \{x^{[2]} : x \in g\}$
Use_def($\mathbf{range}$) $\Rightarrow$ $\quad x^{[2]} \in \mathbf{range}(g)$
ELEM $\Rightarrow$ $\quad x^{[2]} \in \mathbf{domain}(f)$
Use_def($\mathbf{domain}$) $\Rightarrow$ $\quad Stat18: x^{[2]} \in \{y^{[1]} : y \in f\}$
$\langle y\rangle \hookrightarrow Stat18 \Rightarrow$ $\quad x^{[2]} = y^{[1]} \;\&\; y \in f$
$\langle x, y\rangle \hookrightarrow Stat15 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

-- If the range of g equals the domain of f, the following slightly stronger corollary to the preceding result applies. The proof simply combines Theorems 51 and 37.

Theorem 114 (86) $\mathrm{range}(G) = \mathbf{domain}(F) \rightarrow \mathbf{range}(F \bullet G) = \mathbf{range}(F) \,\&\, \mathbf{domain}(F \bullet G) = \mathbf{domain}(G)$. PROOF:

Suppose_not$(g, f) \Rightarrow$ $\mathbf{range}(g) = \mathbf{domain}(f) \,\&\, \neg(\mathbf{range}(f \bullet g) = \mathbf{range}(f) \,\&\, \mathbf{domain}(f \bullet g) = \mathbf{domain}(g))$
$\langle g, f \rangle \hookrightarrow T85 \Rightarrow$ $\mathbf{range}(f \bullet g) = \mathbf{range}(f_{|\mathbf{range}(g)}) \,\&\, \mathbf{domain}(f \bullet g) = \mathbf{domain}(g)$
EQUAL $\Rightarrow$ $\mathbf{range}(f \bullet g) = \mathbf{range}(f_{|\mathbf{domain}(f)}) \,\&\, \mathbf{domain}(f \bullet g) = \mathbf{domain}(g)$
$\langle f \rangle \hookrightarrow T62 \Rightarrow$ $f_{|\mathbf{domain}(f)} = f$
EQUAL $\Rightarrow$ $\mathbf{range}(f_{|\mathbf{domain}(f)}) = \mathbf{range}(f)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is sometimes convenient to use the following corollary of Theorem 85 rather than the theorem itself.

Theorem 115 (87) $\mathrm{range}(G) \subseteq \mathbf{domain}(F) \rightarrow \mathbf{range}(F \bullet G) \subseteq \mathbf{range}(F) \,\&\, \mathbf{domain}(F \bullet G) = \mathbf{domain}(G)$. PROOF:

Suppose_not$(g, f) \Rightarrow$ $Stat1:\ \mathbf{range}(g) \subseteq \mathbf{domain}(f) \,\&\, \mathbf{range}(f \bullet g) \not\subseteq \mathbf{range}(f) \vee \mathbf{domain}(f \bullet g) \neq \mathbf{domain}(g)$
$\langle g, f \rangle \hookrightarrow T85 \Rightarrow$ $\mathbf{range}(f \bullet g) = \mathbf{range}(f_{|\mathbf{range}(g)}) \,\&\, \mathbf{domain}(f \bullet g) = \mathbf{domain}(g)$
ELEM $\Rightarrow$ $\mathbf{range}(f_{|\mathbf{range}(g)}) \not\subseteq \mathbf{range}(f)$
$\langle f, \mathbf{range}(g) \rangle \hookrightarrow T72 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next easy theorem tells us that a 1-1 map, combined with the 'range' operator, induces a 1-1 map on the set of subsets of is domain.

Theorem 116 (88) $1\text{–}1(F) \,\&\, S \subseteq \mathbf{domain}(F) \,\&\, S \neq \mathbf{domain}(F) \rightarrow \mathbf{range}(F_{|S}) \subseteq \mathbf{range}(F) \,\&\, \mathbf{range}(F_{|S}) \neq \mathbf{range}(F)$. PROOF:

Suppose_not$(f, s) \Rightarrow$ $1\text{–}1(f) \,\&\, s \subseteq \mathbf{domain}(f) \,\&\, Stat1:\ s \neq \mathbf{domain}(f) \,\&\, \neg(\mathbf{range}(f_{|s}) \subseteq \mathbf{range}(f) \,\&\, \mathbf{range}(f_{|s}) \neq \mathbf{range}(f))$
$\langle f, s \rangle \hookrightarrow T72 \Rightarrow$ $\mathbf{range}(f_{|s}) = \mathbf{range}(f)$
$\langle c \rangle \hookrightarrow Stat1 \Rightarrow$ $c \in \mathbf{domain}(f) \,\&\, c \notin s$
Use_def$(\mathbf{domain}) \Rightarrow$ $Stat2:\ c \in \{x^{[1]} : x \in f\}$
$\langle x \rangle \hookrightarrow Stat2 \Rightarrow$ $c = x^{[1]} \,\&\, x \in f$
Use_def$(\mathbf{range}) \Rightarrow$ $\mathbf{range}(f) = \{x^{[2]} : x \in f\}$
Suppose $\Rightarrow$ $x^{[2]} \notin \mathbf{range}(f)$
ELEM $\Rightarrow$ $Stat3:\ x^{[2]} \notin \{x^{[2]} : x \in f\}$
$\langle x \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $x^{[2]} \in \mathbf{range}(f_{|s})$
Use_def$(\mathbf{range}) \Rightarrow$ $x^{[2]} \in \{x^{[2]} : x \in f_{|s}\}$
Use_def$(|) \Rightarrow$ $x^{[2]} \in \{x^{[2]} : x \in \{y \in f \,|\, y^{[1]} \in s\}\}$
SIMPLF $\Rightarrow$ $Stat4:\ x^{[2]} \in \{y^{[2]} : y \in f \,|\, y^{[1]} \in s\}$
$\langle y \rangle \hookrightarrow Stat4 \Rightarrow$ $x^{[2]} = y^{[2]} \,\&\, y \in f \,\&\, y^{[1]} \in s$

Use_def (1–1) $\Rightarrow$ $\quad$ *Stat5*: $\langle \forall x \in f, y \in f \,|\, x^{[2]} = y^{[2]} \rightarrow x = y \rangle$
$\langle x, y \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\quad x = y$
ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Next we turn to a discussion of map inverses. The following theorem tells us that the inverse of a map f is always a map, whose range and domain are respectively the domain and range of f.

Theorem 117 (89) $\;$ $\mathsf{Is\_map}(F^{\leftarrow})$ & $\mathbf{range}(F^{\leftarrow}) = \mathbf{domain}(F)$ & $\mathbf{domain}(F^{\leftarrow}) = \mathbf{range}(F)$. PROOF:

Suppose_not (f) $\Rightarrow$ $\quad \neg\big(\mathsf{Is\_map}(f^{\leftarrow})$ & $\mathbf{range}(f^{\leftarrow}) = \mathbf{domain}(f)$ & $\mathbf{domain}(f^{\leftarrow}) = \mathbf{range}(f)\big)$

-- For by our utility lz_map theory $f^{\leftarrow}$ must clearly be a map.

Use_def ($^{\leftarrow}$) $\Rightarrow$ $\quad f^{\leftarrow} = \left\{ \left[x^{[2]}, x^{[1]}\right] : x \in f \right\}$
APPLY $\langle\,\rangle$ lz_map$\left(a(x) \mapsto x^{[2]}, b(x) \mapsto x^{[1]}, s \mapsto f\right)$ $\Rightarrow$ $\quad \mathsf{Is\_map}\left(\left\{\left[x^{[2]}, x^{[1]}\right] : x \in f\right\}\right)$
EQUAL $\Rightarrow$ $\quad \mathsf{Is\_map}(f^{\leftarrow})$
ELEM $\Rightarrow$ $\quad \neg(\mathbf{range}(f^{\leftarrow}) = \mathbf{domain}(f)$ & $\mathbf{domain}(f^{\leftarrow}) = \mathbf{range}(f))$

-- If $\mathbf{range}(f^{\leftarrow})$ and $\mathbf{domain}(f)$ are different, then by definition the two sets seen below are different, so that there exists a y such that $\left[y^{[2]}, y^{[1]}\right]^{[2]} \neq y^{[1]}$, another impossibility, leaving only the third alternative $\mathbf{domain}(f^{\leftarrow}) \neq \mathbf{range}(f)$ to be considered.

Suppose $\Rightarrow$ $\quad \mathbf{range}(f^{\leftarrow}) \neq \mathbf{domain}(f)$
Use_def (**range**) $\Rightarrow$ $\quad \left\{x^{[2]} : x \in f^{\leftarrow}\right\} \neq \mathbf{domain}(f)$
Use_def (**domain**) $\Rightarrow$ $\quad \left\{x^{[2]} : x \in f^{\leftarrow}\right\} \neq \left\{x^{[1]} : x \in f\right\}$
EQUAL $\Rightarrow$ $\quad \left\{x^{[2]} : x \in \left\{\left[x^{[2]}, x^{[1]}\right] : x \in f\right\}\right\} \neq \left\{x^{[1]} : x \in f\right\}$
SIMPLF $\Rightarrow$ $\quad$ *Stat1*: $\left\{\left[x^{[2]}, x^{[1]}\right]^{[2]} : x \in f\right\} \neq \left\{x^{[1]} : x \in f\right\}$
$\langle y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad y \in f$ & $\left[y^{[2]}, y^{[1]}\right]^{[2]} \neq y^{[1]}$
ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \mathbf{domain}(f^{\leftarrow}) \neq \mathbf{range}(f)$

-- But $\mathbf{domain}(f^{\leftarrow}) \neq \mathbf{range}(f)$ leads to the third set inequality seen below, and through it to the impossible inequality $\left[u^{[2]}, u^{[1]}\right]^{[1]} \neq u^{[2]}$, a contradiction which proves our theorem.

Use_def (**range**) $\Rightarrow$ $\quad \mathbf{domain}(f^{\leftarrow}) \neq \left\{x^{[2]} : x \in f\right\}$
Use_def (**domain**) $\Rightarrow$ $\quad \left\{x^{[1]} : x \in f^{\leftarrow}\right\} \neq \left\{x^{[2]} : x \in f\right\}$
EQUAL $\Rightarrow$ $\quad \left\{x^{[1]} : x \in \left\{\left[x^{[2]}, x^{[1]}\right] : x \in f\right\}\right\} \neq \left\{x^{[2]} : x \in f\right\}$

SIMPLF $\Rightarrow$ $\quad$ *Stat2* : $\left\{\left[x^{[2]},x^{[1]}\right]^{[1]} : x \in f\right\} \neq \left\{x^{[2]} : x \in f\right\}$

$\langle u \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\quad$ $\left[u^{[2]},u^{[1]}\right]^{[1]} \neq u^{[2]}$

ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Next we show that the iterated inverse of a map is the map itself. This follows in an elementary way from the definitions involved, by an evident set-theoretic simplification.

Theorem 118 (90) $\quad$ $\mathsf{Is\_map}(F) \rightarrow F = F^{\leftarrow\leftarrow}$. PROOF:

Suppose_not(f) $\Rightarrow$ $\quad$ $\mathsf{Is\_map}(f)$ & $f \neq f^{\leftarrow\leftarrow}$

Use_def(Is_map) $\Rightarrow$ $\quad$ $f = \left\{\left[x^{[1]},x^{[2]}\right] : x \in f\right\}$ & $f \neq f^{\leftarrow\leftarrow}$

Use_def($^{\leftarrow}$) $\Rightarrow$ $\quad$ $f = \left\{\left[x^{[1]},x^{[2]}\right] : x \in f\right\}$ & $f \neq \left\{\left[x^{[2]},x^{[1]}\right] : x \in \left\{\left[y^{[2]},y^{[1]}\right] : y \in f\right\}\right\}$

SIMPLF $\Rightarrow$ $\quad$ $\left\{\left[x^{[2]},x^{[1]}\right] : x \in \left\{\left[y^{[2]},y^{[1]}\right] : y \in f\right\}\right\} = \left\{\left[\left[y^{[2]},y^{[1]}\right]^{[2]},\left[y^{[2]},y^{[1]}\right]^{[1]}\right] : y \in f\right\}$

Set_monot $\Rightarrow$ $\quad$ $\left\{\left[\left[y^{[2]},y^{[1]}\right]^{[2]},\left[y^{[2]},y^{[1]}\right]^{[1]}\right] : y \in f\right\} = \left\{\left[y^{[1]},y^{[2]}\right] : y \in f\right\}$

ELEM $\Rightarrow$ $\quad$ $f \neq \left\{\left[x^{[1]},x^{[2]}\right] : x \in f\right\}$

ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- The following theorem tells us that if a map is one-to-one, so is its inverse. The result follows easily by use of the two preceding theorems and use of the one_1_test theory given earlier.

Theorem 119 (91) $\quad$ $1\text{–}1(F) \rightarrow 1\text{–}1(F^{\leftarrow})$ & $F = F^{\leftarrow\leftarrow}$ & $\mathbf{range}(F^{\leftarrow}) = \mathbf{domain}(F)$ & $\mathbf{domain}(F^{\leftarrow}) = \mathbf{range}(F)$. PROOF:

Suppose_not(f) $\Rightarrow$ $\quad$ *Stat1* : $1\text{–}1(f)$ & $\neg(1\text{–}1(f^{\leftarrow})$ & $f = f^{\leftarrow\leftarrow}$ & $\mathbf{range}(f^{\leftarrow}) = \mathbf{domain}(f)$ & $\mathbf{domain}(f^{\leftarrow}) = \mathbf{range}(f))$

Use_def(1–1) $\Rightarrow$ $\quad$ $\mathsf{Svm}(f)$

Use_def(Svm) $\Rightarrow$ $\quad$ $\mathsf{Is\_map}(f)$

-- Suppose that one of the assertions of our theorem is false. Theorems 53 and 54 tell us that this can only be the assertion concerning one-to-one-ness of $f^{\leftarrow}$.

$\langle f \rangle \hookrightarrow$ *T89* $\Rightarrow$ $\quad$ $\neg(1\text{–}1(f^{\leftarrow})$ & $f = f^{\leftarrow\leftarrow})$

Suppose $\Rightarrow$ $\quad$ $1\text{–}1(f^{\leftarrow})$

Use_def(1–1) $\Rightarrow$ $\quad$ $\mathsf{Svm}(f^{\leftarrow})$

$\langle f \rangle \hookrightarrow$ *T90* $\Rightarrow$ $\quad$ $f = f^{\leftarrow\leftarrow}$

ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ $\neg 1\text{–}1(f^{\leftarrow})$

Use_def($^{\leftarrow}$) $\Rightarrow$ $\quad$ $\neg 1\text{–}1\left(\left\{\left[x^{[2]},x^{[1]}\right] : x \in f\right\}\right)$

-- Since $f^{\leftarrow}$ can be expressed as a setformer, the one_1_test theory given earlier tells us that f must have elements x and y for which $x^{[2]} = y^{[2]}$ and $x = y$ are inequivalent.

APPLY $\langle x_{\Theta} : x, y_{\Theta} : y\rangle$ one_1_test $(a(x) \mapsto x^{[1]}, b(x) \mapsto x^{[2]}, s \mapsto f) \Rightarrow$
$\big(x, y \in f \;\&\; \neg(x^{[2]} = y^{[2]} \leftrightarrow x^{[1]} = y^{[1]})\big) \vee 1\text{–}1(\{[x^{[2]}, x^{[1]}] : x \in f\})$
ELEM $\Rightarrow$ $\neg(x^{[2]} = y^{[2]} \leftrightarrow x^{[1]} = y^{[1]})$

-- but using the definition of one_1_map we see at once that this is impossible, a contradiction which proves our theorem.

Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(f) \;\&\; \mathit{Stat2}: \langle \forall x \in f, y \in f \,|\, x^{[2]} = y^{[2]} \rightarrow x = y\rangle$
Use_def(Svm) $\Rightarrow$ $\mathsf{Is\_map}(f) \;\&\; \mathit{Stat3}: \langle \forall x \in f, y \in f \,|\, x^{[1]} = y^{[1]} \rightarrow x = y\rangle$
$\langle x, y\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $x^{[2]} = y^{[2]} \rightarrow x = y$
$\langle x, y\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $x^{[1]} = y^{[1]} \rightarrow x = y$
EQUAL $\Rightarrow$ $x = y \rightarrow x^{[1]} = y^{[1]}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that for any one-to-one map, the inverse map is a functional left inverse.

Theorem 120 (92) $1\text{–}1(F) \;\&\; X \in \mathbf{domain}(F) \rightarrow F^{\leftarrow}\upharpoonright(F\upharpoonright X) = X$. PROOF:

Suppose_not(f, x) $\Rightarrow$ $1\text{–}1(f) \;\&\; x \in \mathbf{domain}(f) \;\&\; f^{\leftarrow}\upharpoonright(f\upharpoonright x) \neq x$

-- We proceed by contradiction. Suppose that the 1-1 map f has a domain element $c^{[1]}$ such that $f^{\leftarrow}\upharpoonright(f\upharpoonright c^{[1]}) = f^{\leftarrow}\upharpoonright c^{[2]} \neq c^{[1]}$, where $c \in f$.

Use_def(**domain**) $\Rightarrow$ $\mathit{Stat1}: x \in \{y^{[1]} : y \in f\}$
$\langle c\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $c \in f \;\&\; x = c^{[1]}$
Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(f)$
$\langle f, c\rangle \hookrightarrow \mathit{T67} \Rightarrow$ $f\upharpoonright c^{[1]} = c^{[2]}$
EQUAL $\Rightarrow$ $\mathit{Stat2}: f^{\leftarrow}\upharpoonright c^{[2]} \neq c^{[1]}$

-- Theorem 91 tells us that $f^{\leftarrow}$ is a 1-1 map, and thus single-valued. $[c^{[2]}, c^{[1]}]$ must clearly belong to $f^{\leftarrow}$. But then Theorem 67 tells us that $f^{\leftarrow}\upharpoonright[c^{[2]}, c^{[1]}]^{[1]} = [c^{[2]}, c^{[1]}]^{[2]}$. This simplifies to $f^{\leftarrow}\upharpoonright c^{[2]} = c^{[1]}$, contradicting our initial assumption, and so proving the present theorem.

$\langle f\rangle \hookrightarrow \mathit{T91} \Rightarrow$ $1\text{–}1(f^{\leftarrow})$
Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(f^{\leftarrow})$
Suppose $\Rightarrow$ $[c^{[2]}, c^{[1]}] \notin f^{\leftarrow}$

$\mathsf{Use\_def}(^{\leftarrow}) \Rightarrow \quad \mathit{Stat3}: \left[c^{[2]}, c^{[1]}\right] \notin \left\{\left[x^{[2]}, x^{[1]}\right] : x \in f\right\}$
$\langle c\rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad c \notin f \vee \left[c^{[2]}, c^{[1]}\right] \neq \left[c^{[2]}, c^{[1]}\right]$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \left[c^{[2]}, c^{[1]}\right] \in f^{\leftarrow}$
$\left\langle f^{\leftarrow}, \left[c^{[2]}, c^{[1]}\right]\right\rangle \hookrightarrow \mathit{T67} \Rightarrow \quad f^{\leftarrow} \upharpoonright \left[c^{[2]}, c^{[1]}\right]^{[1]} = \left[c^{[2]}, c^{[1]}\right]^{[2]}$
$\mathsf{ELEM} \Rightarrow \quad \left[c^{[2]}, c^{[1]}\right]^{[1]} = c^{[2]}$
$\mathsf{EQUAL} \Rightarrow \quad f^{\leftarrow} \upharpoonright c^{[2]} = \left[c^{[2]}, c^{[1]}\right]^{[2]}$
$\langle \mathit{Stat2}\rangle$ $\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next theorem extends the preceding result by showing that for elements in the range of f, the inverse map is a functional right inverse.

Theorem 121 (93) $1\text{–}1(F) \rightarrow \left(X \in \mathbf{domain}(F) \longrightarrow F^{\leftarrow} \upharpoonright (F \upharpoonright X) = X\right) \,\&\, \left(X \in \mathbf{range}(F) \longrightarrow F \upharpoonright (F^{\leftarrow} \upharpoonright X) = X\right)$. PROOF:

$\mathsf{Suppose\_not}(f, c) \Rightarrow \quad 1\text{–}1(f) \,\&\, \mathit{Stat1}: \left(c \in \mathbf{domain}(f) \,\&\, f^{\leftarrow} \upharpoonright (f \upharpoonright c) \neq c\right) \vee \left(c \in \mathbf{range}(f) \,\&\, f \upharpoonright (f^{\leftarrow} \upharpoonright c) \neq c\right)$

-- For, supposing the contrary, there must either be a c in the domain of f for which $f^{\leftarrow} \upharpoonright (f \upharpoonright c) \neq c$, or a c in the range of f for which $f \upharpoonright (f^{\leftarrow} \upharpoonright c) \neq c$.

$\langle f, c\rangle \hookrightarrow \mathit{T92} \Rightarrow \quad c \in \mathbf{domain}(f) \rightarrow f^{\leftarrow} \upharpoonright (f \upharpoonright c) = c$

-- However, Theorem 92 rules out the first possibility, and, combined with Theorem 91, rules out the second possibility also, thereby proving our theorem.

$\langle f\rangle \hookrightarrow \mathit{T91} \Rightarrow \quad 1\text{–}1(f^{\leftarrow}) \,\&\, \mathbf{domain}(f^{\leftarrow}) = \mathbf{range}(f) \,\&\, f = f^{\leftarrow\leftarrow}$
$\langle f^{\leftarrow}, c\rangle \hookrightarrow \mathit{T92} \Rightarrow \quad c \in \mathbf{domain}(f^{\leftarrow}) \rightarrow f^{\leftarrow\leftarrow} \upharpoonright (f^{\leftarrow} \upharpoonright c) = c$
$\mathsf{EQUAL} \Rightarrow \quad 1\text{–}1(f^{\leftarrow}) \rightarrow c \in \mathbf{range}(f) \longrightarrow f \upharpoonright (f^{\leftarrow} \upharpoonright c) = c$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Inverse maps often occur together with identity maps, whose (entirely elementary) properties are collected in the following theorem.

-- Elementary Properties of identity maps

Theorem 122 (94) $1\text{–}1(\iota_S) \,\&\, \mathbf{domain}(\iota_S) = S \,\&\, \mathbf{range}(\iota_S) = S \,\&\, \iota_S{}^{\leftarrow} = \iota_S \,\&\, (X \in S \longrightarrow \iota_S \upharpoonright X = X) \,\&$
$\mathsf{Is\_map}(F) \longrightarrow (\mathbf{domain}(F) \subseteq S \longrightarrow F \bullet \iota_S = F) \,\&\, (\mathbf{range}(F) \subseteq S \longrightarrow \iota_S \bullet F = F)$. PROOF:

$\mathsf{Suppose\_not}(s, x, f) \Rightarrow \quad \mathit{Stat1}:$
$\neg 1\text{–}1(\iota_s) \vee$

$$\mathbf{domain}(\iota_s) \neq s \vee \mathbf{range}(\iota_s) \neq s \vee \iota_s^{\leftarrow} \neq \iota_s \vee (x \in s \;\&\; \iota_s \upharpoonright x \neq x) \vee \left(\mathsf{Is\_map}(f) \;\&\; \mathbf{domain}(f) \subseteq s \;\&\; f \bullet \iota_s \neq f\right) \vee \left(\mathsf{Is\_map}(f) \;\&\; \mathbf{range}(f) \subseteq s \;\&\; \iota_s \bullet f \neq f\right)$$

-- Proceeding by contradiction, we shall show successively that none of the clauses of our theorem can be false. Indeed, it follows immediately using the one_1_test developed earlier that the first clause cannot be false.

Use_def$(\iota) \Rightarrow \quad \iota_s = \{[x,x] : x \in s\}$

APPLY $\langle x_\Theta : xx, y_\Theta : y\rangle$ one_1_test$(a(x) \mapsto x, b(x) \mapsto x, s \mapsto s) \Rightarrow$

$\neg(xx = y \leftrightarrow xx = y) \vee 1\text{–}1(\{[x,x] : x \in s\})$

ELEM $\Rightarrow \quad 1\text{–}1(\{[x,x] : x \in s\})$

EQUAL $\Rightarrow \quad Stat2 : 1\text{–}1(\iota_s)$

-- In equally direct fashion, our 'fcn_symbol' theory tells us that neither the second, the third, or the fifth clause of our theorem can be false. – (FORALL x in OM |(x in s) imp (g [x] = f (x))) (FORALL x in OM |(x notin s) imp (g [X] = 0)) (FORALL x in OM |g [x] = if x in s then f (x) else 0 end if)

APPLY $\langle x_\Theta : u, y_\Theta : v\rangle$ fcn_symbol$(f(x) \mapsto x, g \mapsto \iota_s, s \mapsto s) \Rightarrow$

$\mathsf{Svm}(\iota_s) \;\&\; \mathbf{range}(\iota_s) = \{x : x \in s\} \;\&\; \mathbf{domain}(\iota_s) = s \;\&\; Stat3 : \langle \forall x \mid x \in s \rightarrow \iota_s \upharpoonright x = x\rangle \;\&\; (u, v \in s \;\&\; u = v \;\&\; v \neq v) \vee 1\text{–}1(\iota_s)$

$\langle x\rangle \hookrightarrow Stat3 \Rightarrow \quad x \in s \rightarrow \iota_s \upharpoonright x = x$

SIMPLF $\Rightarrow \quad Stat4 : \mathbf{range}(\iota_s) = s$

$\langle Stat2\rangle$ ELEM $\Rightarrow \quad \mathbf{range}(\iota_s) = s \;\&\; \mathbf{domain}(\iota_s) = s \;\&\; (x \in \mathbf{domain}(\iota_s) \rightarrow \iota_s \upharpoonright x = x) \;\&\; 1\text{–}1(\iota_s)$

-- We hence see that the only clauses of our original assumption which could be false are the fourth clause and the two final clauses.

$\langle Stat2\rangle$ ELEM $\Rightarrow \quad Stat5 : \mathbf{range}(\iota_s) = s \;\&\; \mathbf{domain}(\iota_s) = s \;\&\; (x \in \mathbf{domain}(\iota_s) \rightarrow \iota_s \upharpoonright x = x) \;\&\; 1\text{–}1(\iota_s)$

EQUAL $\Rightarrow \quad Stat6 : \mathbf{range}(\iota_s) = s \;\&\; \mathbf{domain}(\iota_s) = s \;\&\; (x \in s \rightarrow \iota_s \upharpoonright x = x) \;\&\; 1\text{–}1(\iota_s)$

$\langle Stat1, Stat6\rangle$ ELEM $\Rightarrow \quad Stat7 : \iota_s^{\leftarrow} \neq \iota_s \vee \left(\mathsf{Is\_map}(f) \;\&\; \mathbf{domain}(f) \subseteq s \;\&\; f \bullet \iota_s \neq f\right) \vee \left(\mathsf{Is\_map}(f) \;\&\; \mathbf{range}(f) \subseteq s \;\&\; \iota_s \bullet f \neq f\right)$

-- Use of the definition of inv and an elementary simplification shows immediately that the fourth clause cannot be false.

Suppose $\Rightarrow \quad \iota_s^{\leftarrow} \neq \iota_s$

Use_def$(\iota) \Rightarrow \quad \{[x,x] : x \in s\}^{\leftarrow} \neq \{[x,x] : x \in s\}$

Use_def$(^{\leftarrow}) \Rightarrow \quad \left\{\left[y^{[2]}, y^{[1]}\right] : y \in \{[x,x] : x \in s\}\right\} \neq \{[x,x] : x \in s\}$

SIMPLF $\Rightarrow \quad Stat8 : \left\{\left[[x,x]^{[2]}, [x,x]^{[1]}\right] : x \in s\right\} \neq \{[x,x] : x \in s\}$

$\langle d\rangle \hookrightarrow Stat8 \Rightarrow \quad Stat9 : \left[[d,d]^{[2]}, [d,d]^{[1]}\right] \neq [d,d]$

$\langle Stat9\rangle$ ELEM $\Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \iota_s^{\leftarrow} = \iota_s$

-- Thus only one of the last two of our original clauses need be considered. In both of these cases f is a map.

$\langle\mathit{Stat7}\rangle$ ELEM $\Rightarrow$ $\quad \mathit{Stat10}:\ \big(\mathsf{Is\_map}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{s}\ \&\ \mathsf{f}\bullet\boldsymbol{\iota}_{\mathsf{s}} \neq \mathsf{f}\big) \vee \big(\mathsf{Is\_map}(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s}\ \&\ \boldsymbol{\iota}_{\mathsf{s}}\bullet\mathsf{f} \neq \mathsf{f}\big)$
$\langle\mathit{Stat10}\rangle$ ELEM $\Rightarrow$ $\quad \mathit{Stat11}:\ \mathsf{Is\_map}(\mathsf{f})$

-- Consider the first of these possibilities, and expand the definitions involved, thereby showing that there exists a c for which the conditions $\mathsf{c} \in \mathsf{f}$ and $\mathsf{c} \in \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\}$ are inequivalent.

Suppose $\Rightarrow$ $\quad \mathit{Stat12}:\ \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{s}\ \&\ \mathsf{f}\bullet\boldsymbol{\iota}_{\mathsf{s}} \neq \mathsf{f}$
Use_def$(\bullet)$ $\Rightarrow$ $\quad \{[\mathsf{x}^{[1]},\mathsf{y}^{[2]}] : \mathsf{x} \in \boldsymbol{\iota}_{\mathsf{s}}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \neq \mathsf{f}$
Use_def$(\boldsymbol{\iota})$ $\Rightarrow$ $\quad \{[\mathsf{x}^{[1]},\mathsf{y}^{[2]}] : \mathsf{x} \in \{[\mathsf{z},\mathsf{z}] : \mathsf{z} \in \mathsf{s}\}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \neq \mathsf{f}$
SIMPLF $\Rightarrow$
$\quad \{[\mathsf{x}^{[1]},\mathsf{y}^{[2]}] : \mathsf{x} \in \{[\mathsf{z},\mathsf{z}] : \mathsf{z} \in \mathsf{s}\}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} =$
$\quad\quad \Big\{\Big[[\mathsf{z},\mathsf{z}]^{[1]},\mathsf{y}^{[2]}\Big] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid [\mathsf{z},\mathsf{z}]^{[2]} = \mathsf{y}^{[1]}\Big\}$
Set_monot $\Rightarrow$ $\quad \Big\{\Big[[\mathsf{z},\mathsf{z}]^{[1]},\mathsf{y}^{[2]}\Big] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid [\mathsf{z},\mathsf{z}]^{[2]} = \mathsf{y}^{[1]}\Big\} = \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\}$
ELEM $\Rightarrow$ $\quad \mathit{Stat13}:\ \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\} \neq \mathsf{f}$
$\langle\mathsf{c}\rangle\hookrightarrow\mathit{Stat13} \Rightarrow$ $\quad (\mathsf{c} \in \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\}\ \&\ \mathsf{c} \notin \mathsf{f}) \vee$
$\quad \mathsf{c} \notin \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\}\ \&\ \mathsf{c} \in \mathsf{f}$

-- In the first of the two resulting cases it follows, using Theorem 46, that c is both a member and not a member of f, a contradiction which rules out this case, leaving only the case $\mathsf{c} \in \mathsf{f}$.

Suppose $\Rightarrow$ $\quad \mathit{Stat14}:\ \mathsf{c} \in \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\}\ \&\ \mathsf{c} \notin \mathsf{f}$
$\langle\mathsf{a},\mathsf{b}\rangle\hookrightarrow\mathit{Stat14} \Rightarrow$ $\quad \mathit{Stat15}:\ \mathsf{c} = \Big[\mathsf{a},\mathsf{b}^{[2]}\Big]\ \&\ \mathsf{a} \in \mathsf{s}\ \&\ \mathsf{b} \in \mathsf{f}\ \&\ \mathsf{a} = \mathsf{b}^{[1]}\ \&\ \mathsf{c} \notin \mathsf{f}$
$\langle\mathsf{f},\mathsf{b}\rangle\hookrightarrow\mathit{T46} \Rightarrow$ $\quad \mathsf{b} = \Big[\mathsf{b}^{[1]},\mathsf{b}^{[2]}\Big]$
$\langle\mathit{Stat15}\rangle$ ELEM $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad \mathit{Stat16}:\ \mathsf{c} \notin \{[\mathsf{z},\mathsf{y}^{[2]}] : \mathsf{z} \in \mathsf{s}, \mathsf{y} \in \mathsf{f} \mid \mathsf{z} = \mathsf{y}^{[1]}\}\ \&\ \mathsf{c} \in \mathsf{f}$

-- But in this case a contradiction follows immediately from Theorem 46, leaving out for final consideration only the case $\mathbf{range}(\mathsf{f}) \subseteq \mathsf{s}\ \&\ \boldsymbol{\iota}_{\mathsf{s}}\bullet\mathsf{f} \neq \mathsf{f}$.

$\langle\mathsf{f},\mathsf{c}\rangle\hookrightarrow\mathit{T46} \Rightarrow$ $\quad \mathsf{c} = [\mathsf{c}^{[1]},\mathsf{c}^{[2]}]$
$\langle\mathsf{c}^{[1]},\mathsf{c}\rangle\hookrightarrow\mathit{Stat16} \Rightarrow$ $\quad \mathit{Stat17}:\ \mathsf{c} \in \mathsf{f}\ \&\ \mathsf{c}^{[1]} \notin \mathsf{s}$
$\langle\mathit{Stat17},\mathit{Stat12}\rangle$ ELEM $\Rightarrow$ $\quad \mathit{Stat18}:\ \mathsf{c} \in \mathsf{f}\ \&\ \mathsf{c}^{[1]} \notin \mathbf{domain}(\mathsf{f})$
Use_def$(\mathbf{domain})$ $\Rightarrow$ $\quad \mathit{Stat19}:\ \mathsf{c}^{[1]} \notin \{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{f}\}$
$\langle\mathsf{c}\rangle\hookrightarrow\mathit{Stat19} \Rightarrow$ $\quad \neg(\mathsf{c} \in \mathsf{f}\ \&\ \mathsf{c}^{[1]} = \mathsf{c}^{[1]})$
ELEM $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad \mathit{Stat20}:\ \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s}\ \&\ \boldsymbol{\iota}_{\mathsf{s}}\bullet\mathsf{f} \neq \mathsf{f}$

-- By expanding the definitions involved, we see that there must exist an element e for which the conditions $\mathsf{e} \in \mathsf{f}$ and $\mathsf{e} \in \{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\}$ are inequivalent.

Use_def($\bullet$) $\Rightarrow$ $\{[x^{[1]}, y^{[2]}] : x \in f, y \in \iota_s \mid x^{[2]} = y^{[1]}\} \neq f$
Use_def(ι) $\Rightarrow$ $\{[x^{[1]}, y^{[2]}] : x \in f, y \in \{[z,z] : z \in s\} \mid x^{[2]} = y^{[1]}\} \neq f$
SIMPLF $\Rightarrow$
$\{[x^{[1]}, y^{[2]}] : x \in f, y \in \{[z,z] : z \in s\} \mid x^{[2]} = y^{[1]}\} =$
$\left\{\left[x^{[1]}, [z,z]^{[2]}\right] : x \in f, z \in s \mid x^{[2]} = [z,z]^{[1]}\right\}$
Set_monot $\Rightarrow$ $\left\{\left[x^{[1]}, [z,z]^{[2]}\right] : x \in f, z \in s \mid x^{[2]} = [z,z]^{[1]}\right\} = \{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\}$
ELEM $\Rightarrow$ *Stat21* : $\{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\} \neq f$
$\langle e \rangle \hookrightarrow$ *Stat21* $\Rightarrow$ $(e \in \{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\} \;\&\; e \notin f) \vee$
$e \notin \{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\} \;\&\; e \in f$

-- In the first of the two resulting cases it follows, using Theorem 46, that e is both a member and not a member of f, a contradiction which rules out this case, leaving only the case $\mathsf{e} \in \mathsf{f}$.

Suppose $\Rightarrow$ *Stat22* : $e \in \{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\} \;\&\; e \notin f$
$\langle aa, bb \rangle \hookrightarrow$ *Stat22* $\Rightarrow$ $e = [aa^{[1]}, bb] \;\&\; aa \in f \;\&\; bb \in s \;\&\; aa^{[2]} = bb \;\&\; e \notin f$
$\langle f, aa \rangle \hookrightarrow$ *T46* $\Rightarrow$ $aa = [aa^{[1]}, aa^{[2]}]$
$\langle$ *Stat22* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat23* : $e \notin \{[x^{[1]}, z] : x \in f, z \in s \mid x^{[2]} = z\} \;\&\; e \in f$

-- But in this case a contradiction follows immediately from Theorem 46, proving our theorem.

$\langle f, e \rangle \hookrightarrow$ *T46* $\Rightarrow$ $e = [e^{[1]}, e^{[2]}]$
$\langle e, e^{[2]} \rangle \hookrightarrow$ *Stat23* $\Rightarrow$ *Stat24* : $e \in f \;\&\; e^{[2]} \notin s$
$\langle$ *Stat24*, *Stat20* $\rangle$ ELEM $\Rightarrow$ $e \in f \;\&\; e^{[2]} \notin \mathbf{range}(f)$
Use_def(**range**) $\Rightarrow$ *Stat25* : $e^{[2]} \notin \{x^{[2]} : x \in f\}$
$\langle e \rangle \hookrightarrow$ *Stat25* $\Rightarrow$ $\neg(e \in f \;\&\; e^{[2]} = e^{[2]})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that the product of the inverse of a single-valued map f by the map itself is the identity map on the range of f.

Theorem 123 (95) $\mathrm{Svm}(F) \rightarrow F \bullet F^{\leftarrow} = \iota_{\mathbf{range}(F)}$. PROOF:

Suppose_not(f) $\Rightarrow$ $\mathrm{Svm}(f) \;\&\; f \bullet f^{\leftarrow} \neq \iota_{\mathbf{range}(f)}$

-- For suppose that there is a counterexample to our theorem, and then expand and simplify all the definitions involved, getting the set-theoretic inequality seen below.

Use_def (ι) $\Rightarrow$ $\mathsf{f}\bullet\mathsf{f}^{\leftarrow} \neq \{[\mathsf{x},\mathsf{x}] : \mathsf{x} \in \mathbf{range}(\mathsf{f})\}$

Use_def (**range**) $\Rightarrow$ $\mathsf{f}\bullet\mathsf{f}^{\leftarrow} \neq \left\{[\mathsf{x},\mathsf{x}] : \mathsf{x} \in \left\{\mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{f}\right\}\right\}$

SIMPLF $\Rightarrow$ $\mathsf{f}\bullet\mathsf{f}^{\leftarrow} \neq \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$

Use_def ($\bullet$) $\Rightarrow$ $\left\{\left[\mathsf{x}^{[1]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \mathsf{f}^{\leftarrow}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\right\} \neq \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$

Use_def ($^{\leftarrow}$) $\Rightarrow$ $\left\{\left[\mathsf{x}^{[1]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \left\{\left[\mathsf{u}^{[2]},\mathsf{u}^{[1]}\right] : \mathsf{u} \in \mathsf{f}\right\}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\right\} \neq \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$

SIMPLF $\Rightarrow$

$$\left\{\left[\mathsf{x}^{[1]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \left\{\left[\mathsf{u}^{[2]},\mathsf{u}^{[1]}\right] : \mathsf{u} \in \mathsf{f}\right\}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\right\} = \left\{\left[\left[\mathsf{u}^{[2]},\mathsf{u}^{[1]}\right]^{[1]},\mathsf{y}^{[2]}\right] : \mathsf{u} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \left[\mathsf{u}^{[2]},\mathsf{u}^{[1]}\right]^{[2]} = \mathsf{y}^{[1]}\right\}$$

Set_monot $\Rightarrow$ $\left\{\left[\left[\mathsf{u}^{[2]},\mathsf{u}^{[1]}\right]^{[1]},\mathsf{y}^{[2]}\right] : \mathsf{u} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \left[\mathsf{u}^{[2]},\mathsf{u}^{[1]}\right]^{[2]} = \mathsf{y}^{[1]}\right\} = \left\{\left[\mathsf{u}^{[2]},\mathsf{y}^{[2]}\right] : \mathsf{u} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{u}^{[1]} = \mathsf{y}^{[1]}\right\}$

ELEM $\Rightarrow$ *Stat1*: $\left\{\left[\mathsf{x}^{[2]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]}\right\} \neq \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$

-- Since the sets displayed are not equal, there is a c that is in one but not the other. If it is in the first of these two sets but not the second, a contradiction results from the assumed single-valuedness of f, ruling out this case.

$\langle \mathsf{c} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2*:

$$\left(\mathsf{c} \in \left\{\left[\mathsf{x}^{[2]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]}\right\} \;\&\; \mathsf{c} \notin \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}\right) \vee \mathsf{c} \notin \left\{\left[\mathsf{x}^{[2]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]}\right\} \;\&\; \mathsf{c} \in \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$$

Suppose $\Rightarrow$ *Stat3*: $\mathsf{c} \in \left\{\left[\mathsf{x}^{[2]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]}\right\}$ & *Stat4*: $\mathsf{c} \notin \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$

$\langle \mathsf{a},\mathsf{b} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat5*: $\mathsf{a},\mathsf{b} \in \mathsf{f} \;\&\; \mathsf{c} = \left[\mathsf{a}^{[2]},\mathsf{b}^{[2]}\right] \;\&\; \mathsf{a}^{[1]} = \mathsf{b}^{[1]}$

$\langle \mathsf{b} \rangle \hookrightarrow$ *Stat4*([*Stat5*, $\cap$]) $\Rightarrow$ *Stat5a*: $\mathsf{c} \neq \left[\mathsf{b}^{[2]},\mathsf{b}^{[2]}\right] \;\&\; \mathsf{c} = \left[\mathsf{a}^{[2]},\mathsf{b}^{[2]}\right]$

$\langle$ *Stat5a* $\rangle$ ELEM $\Rightarrow$ $\mathsf{a} \neq \mathsf{b}$

Use_def (Svm) $\Rightarrow$ *Stat6*: $\langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$

$\langle \mathsf{a},\mathsf{b} \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat7*: $\mathsf{c} \notin \left\{\left[\mathsf{x}^{[2]},\mathsf{y}^{[2]}\right] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]}\right\}$ & *Stat8*: $\mathsf{c} \in \left\{\left[\mathsf{x}^{[2]},\mathsf{x}^{[2]}\right] : \mathsf{x} \in \mathsf{f}\right\}$

-- But in the remaining case c has the form $\left[\mathsf{d}^{[2]},\mathsf{d}^{[2]}\right]$ for some $\mathsf{d} \in \mathsf{f}$, and a contradiction results in much the same way, proving our theorem.

$\langle \mathsf{d} \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $\mathsf{c} = \left[\mathsf{d}^{[2]},\mathsf{d}^{[2]}\right] \;\&\; \mathsf{d} \in \mathsf{f}$

$\langle \mathsf{d},\mathsf{d} \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we extend Theorem 95: if a single-valued map f is 1-1, the product of its inverse by f is also the identity map on the domain of f.

Theorem 124 (96) $1\text{–}1(\mathsf{F}) \rightarrow \mathsf{F}\bullet\mathsf{F}^{\leftarrow} = \boldsymbol{\iota}_{\mathbf{range}(\mathsf{F})} \;\&\; \mathsf{F}^{\leftarrow}\bullet\mathsf{F} = \boldsymbol{\iota}_{\mathbf{domain}(\mathsf{F})}$. PROOF:

Suppose_not(f) $\Rightarrow$ $1\text{–}1(\mathsf{f}) \;\&\; \mathsf{f}\bullet\mathsf{f}^{\leftarrow} \neq \boldsymbol{\iota}_{\mathbf{range}(\mathsf{f})} \vee \mathsf{f}^{\leftarrow}\bullet\mathsf{f} \neq \boldsymbol{\iota}_{\mathbf{domain}(\mathsf{f})}$

-- This follows by simple algebraic reasoning using Theorems 53-55 and 59.

Use_def(1–1) $\Rightarrow$ Svm(f)
Use_def(Svm) $\Rightarrow$ Is_map(f)
$\langle \mathsf{f} \rangle \hookrightarrow T95 \Rightarrow$ $\mathsf{f}^{\leftarrow}\bullet\mathsf{f} \neq \boldsymbol{\iota}_{\mathbf{domain}(\mathsf{f})}$
$\langle \mathsf{f} \rangle \hookrightarrow T91 \Rightarrow$ $1\text{–}1(\mathsf{f}^{\leftarrow})$
$\langle \mathsf{f} \rangle \hookrightarrow T89 \Rightarrow$ $\mathbf{range}(\mathsf{f}^{\leftarrow}) = \mathbf{domain}(\mathsf{f})$
$\langle \mathsf{f} \rangle \hookrightarrow T90 \Rightarrow$ $\mathsf{f}^{\leftarrow\leftarrow} = \mathsf{f}$
Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(\mathsf{f}^{\leftarrow})$
$\langle \mathsf{f}^{\leftarrow} \rangle \hookrightarrow T95 \Rightarrow$ $\mathsf{f}^{\leftarrow}\bullet\mathsf{f}^{\leftarrow\leftarrow} = \boldsymbol{\iota}_{\mathbf{range}(\mathsf{f}^{\leftarrow})}$
EQUAL $\Rightarrow$ $\mathsf{f}^{\leftarrow}\bullet\mathsf{f} = \boldsymbol{\iota}_{\mathbf{domain}(\mathsf{f})}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next aim, which we will reach in several steps, is to prove a kind of converse to Theorem 96: mutually inverse maps are each other's inverses.

-- Lemma for subsequent theorem

Theorem 125 (97) $\mathsf{Is\_map}(\mathsf{F}) \;\&\; \mathsf{Is\_map}(\mathsf{G}) \;\&\; \mathbf{domain}(\mathsf{F}) \subseteq \mathbf{range}(\mathsf{G}) \;\&\; \mathsf{Svm}(\mathsf{F}\bullet\mathsf{G}) \rightarrow \mathsf{Svm}(\mathsf{F})$. PROOF:

Suppose_not(f, g) $\Rightarrow$ *Stat1* : $\mathsf{Is\_map}(\mathsf{f}) \;\&\; \mathsf{Is\_map}(\mathsf{g}) \;\&\; \mathbf{domain}(\mathsf{f}) \subseteq \mathbf{range}(\mathsf{g}) \;\&\; \mathsf{Svm}(\mathsf{f}\bullet\mathsf{g}) \;\&\; \neg\mathsf{Svm}(\mathsf{f})$

-- First we show that if the product $\mathsf{f}\bullet\mathsf{g}$ of two maps is single valued, and if the range of g includes the domain of f, then the map f must be single valued. For suppose that a counterexample exists, and apply the utility theory Svm_test.

Use_def(Is_map) $\Rightarrow$ $\mathsf{f} = \{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\}$
APPLY $\langle \mathsf{x}_\Theta : \mathsf{x}, \mathsf{y}_\Theta : \mathsf{y} \rangle$ Svm_test$(\mathsf{a}(\mathsf{x}) \mapsto \mathsf{x}^{[1]}, \mathsf{b}(\mathsf{x}) \mapsto \mathsf{x}^{[2]}, \mathsf{s} \mapsto \mathsf{f}) \Rightarrow$
$(\mathsf{x}, \mathsf{y} \in \mathsf{f} \;\&\; \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \;\&\; \mathsf{x}^{[2]} \neq \mathsf{y}^{[2]}) \vee \mathsf{Svm}(\{[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f}\})$
EQUAL $\Rightarrow$ $(\mathsf{x}, \mathsf{y} \in \mathsf{f} \;\&\; \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \;\&\; \mathsf{x}^{[2]} \neq \mathsf{y}^{[2]}) \vee \mathsf{Svm}(\mathsf{f})$

-- This tells us that there are elements x, y in f with $\mathsf{x}^{[1]} = \mathsf{y}^{[1]}$ such that $\mathsf{x}^{[2]} \neq \mathsf{y}^{[2]}$.

ELEM $\Rightarrow$ *Stat2* : $\mathsf{x}, \mathsf{y} \in \mathsf{f} \;\&\; \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \;\&\; \mathsf{x}^{[2]} \neq \mathsf{y}^{[2]}$
Suppose $\Rightarrow$ $\mathsf{x}^{[1]} \notin \mathbf{domain}(\mathsf{f})$
Use_def(**domain**) $\Rightarrow$ *Stat3* : $\mathsf{x}^{[1]} \notin \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f}\}$

$\langle x \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $x^{[1]} \in \mathbf{range}(g)$

-- Therefore $x^{[1]} = y^{[1]}$ must have the form $u^{[2]}$, where u belongs to g. It follows that $[u^{[1]}, x^{[2]}]$ and $[u^{[1]}, y^{[2]}]$ both belong to $f \bullet g$.

Use_def(**range**) $\Rightarrow$ $Stat4: x^{[1]} \in \{u^{[2]} : u \in g\}$
$\langle u \rangle \hookrightarrow Stat4 \Rightarrow$ $Stat5: x^{[1]} = u^{[2]} \,\&\, u \in g$
Suppose $\Rightarrow$ $[u^{[1]}, x^{[2]}] \notin f \bullet g$
Use_def($\bullet$) $\Rightarrow$ $Stat6: [u^{[1]}, x^{[2]}] \notin \{[v^{[1]}, w^{[2]}] : v \in g, w \in f \mid v^{[2]} = w^{[1]}\}$
$\langle u, x \rangle \hookrightarrow Stat6 \Rightarrow$ $Stat7: u \notin g \vee x \notin f \vee [u^{[1]}, x^{[2]}] \neq [u^{[1]}, x^{[2]}] \vee u^{[2]} \neq x^{[1]}$
$\langle Stat7, Stat2, Stat5 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $[u^{[1]}, x^{[2]}] \in f \bullet g$
Suppose $\Rightarrow$ $[u^{[1]}, y^{[2]}] \notin f \bullet g$
Use_def($\bullet$) $\Rightarrow$ $Stat8: [u^{[1]}, y^{[2]}] \notin \{[v^{[1]}, w^{[2]}] : v \in g, w \in f \mid v^{[2]} = w^{[1]}\}$
$\langle u, y \rangle \hookrightarrow Stat8 \Rightarrow$ $Stat9: u \notin g \vee y \notin f \vee [u^{[1]}, y^{[2]}] \neq [u^{[1]}, y^{[2]}] \vee u^{[2]} \neq y^{[1]}$
$\langle Stat9, Stat2, Stat5, * \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $[u^{[1]}, y^{[2]}] \in f \bullet g$

-- But now, since $f \bullet g$ is single-valued, it follows that $[u^{[1]}, x^{[2]}] = [u^{[1]}, y^{[2]}]$ contrary to our initial assumption. This contradiction proves the present theorem.

Use_def(Svm) $\Rightarrow$ $Stat10: \langle \forall x \in f \bullet g, y \in f \bullet g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\langle [u^{[1]}, x^{[2]}], [u^{[1]}, y^{[2]}] \rangle \hookrightarrow Stat10 \Rightarrow$ $Stat11: x^{[1]} = y^{[1]} \rightarrow x = y$
$\langle Stat2, Stat11 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following proof shows that the inverse of a product map is the product of the inverses, taken in the reverse order.

-- Product of Inverses

Theorem 126 (98) $\text{Is\_map}(F) \,\&\, \text{Is\_map}(G) \rightarrow (F \bullet G)^{\leftarrow} = G^{\leftarrow} \bullet F^{\leftarrow}$. PROOF:

Suppose_not(f, g) $\Rightarrow$ $\text{Is\_map}(f) \,\&\, \text{Is\_map}(g) \,\&\, (f \bullet g)^{\leftarrow} \neq g^{\leftarrow} \bullet f^{\leftarrow}$

-- For if we expand all the definitions involved, simplify, and reverse the order of the bound variables in the setformers which appear, we get the set inequality seen below.

Use_def(Is_map) $\Rightarrow$ $f = \{[x^{[1]}, x^{[2]}] : x \in f\} \,\&\, g = \{[x^{[1]}, x^{[2]}] : x \in g\} \,\&\, (f \bullet g)^{\leftarrow} \neq g^{\leftarrow} \bullet f^{\leftarrow}$
Use_def($\bullet$) $\Rightarrow$ $\{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}^{\leftarrow} \neq \{[x^{[1]}, y^{[2]}] : x \in f^{\leftarrow}, y \in g^{\leftarrow} \mid x^{[2]} = y^{[1]}\}$
Use_def($^{\leftarrow}$) $\Rightarrow$
$\{[u^{[2]}, u^{[1]}] : u \in \{[x^{[1]}, y^{[2]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}\} \neq$

$\{[x^{[1]}, y^{[2]}] : x \in \{[u^{[2]}, u^{[1]}] : u \in f\}, y \in \{[v^{[2]}, v^{[1]}] : v \in g\} \mid x^{[2]} = y^{[1]}\}$

SIMPLF $\Rightarrow$

$\{[[x^{[1]}, y^{[2]}]^{[2]}, [x^{[1]}, y^{[2]}]^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \neq$
$\{[[u^{[2]}, u^{[1]}]^{[1]}, [v^{[2]}, v^{[1]}]^{[2]}] : u \in f, v \in g \mid [u^{[2]}, u^{[1]}]^{[2]} = [v^{[2]}, v^{[1]}]^{[1]}\}$

Set_monot $\Rightarrow$ $\{[[x^{[1]}, y^{[2]}]^{[2]}, [x^{[1]}, y^{[2]}]^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} =$
$\{[y^{[2]}, x^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}$

Set_monot $\Rightarrow$

$\{[[u^{[2]}, u^{[1]}]^{[1]}, [v^{[2]}, v^{[1]}]^{[2]}] : u \in f, v \in g \mid [u^{[2]}, u^{[1]}]^{[2]} = [v^{[2]}, v^{[1]}]^{[1]}\} =$
$\{[u^{[2]}, v^{[1]}] : u \in f, v \in g \mid u^{[1]} = v^{[2]}\}$

ELEM $\Rightarrow$ *Stat1* : $\{[y^{[2]}, x^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \neq \{[u^{[2]}, v^{[1]}] : u \in f, v \in g \mid u^{[1]} = v^{[2]}\}$

-- Thus there must exist an element c which belongs to one of these two last sets but not the other, say the first but not the second. his leads immediately to an elementary contradiction, ruling out this case.

$\langle c \rangle \hookrightarrow$ *Stat1* $\Rightarrow$

$(c \in \{[y^{[2]}, x^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; c \notin \{[u^{[2]}, v^{[1]}] : u \in f, v \in g \mid u^{[1]} = v^{[2]}\}) \vee$
$c \notin \{[y^{[2]}, x^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; c \in \{[u^{[2]}, v^{[1]}] : u \in f, v \in g \mid u^{[1]} = v^{[2]}\}$

Suppose $\Rightarrow$ *Stat2* : $c \in \{[y^{[2]}, x^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; c \notin \{[u^{[2]}, v^{[1]}] : u \in f, v \in g \mid u^{[1]} = v^{[2]}\}$

$\langle x, y, y, x \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $x \in g \;\&\; y \in f \;\&$
$c = [y^{[2]}, x^{[1]}] \;\&\; \neg(x \in g \;\&\; y \in f \;\&\; c = [y^{[2]}, x^{[1]}])$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3* : $c \in \{[u^{[2]}, v^{[1]}] : u \in f, v \in g \mid u^{[1]} = v^{[2]}\} \;\&\; c \notin \{[y^{[2]}, x^{[1]}] : x \in g, y \in f \mid x^{[2]} = y^{[1]}\}$

-- But the case in which c belongs to the second but not the first leads to an exactly similar contradiction, thereby proving that our assertion holds in every possible case.

$\langle x_2, y_2, y_2, x_2 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $x_2 \in f \;\&\; y_2 \in g \;\&$
$c = [{x_2}^{[2]}, {y_2}^{[1]}] \;\&\; \neg(x_2 \in f \;\&\; y_2 \in g \;\&\; c = [{x_2}^{[2]}, {y_2}^{[1]}])$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that a map is 1-1 if and only if it and its inverse are both single-valued.

Theorem 127 (99) $1\text{–}1(F) \leftrightarrow \mathrm{Svm}(F) \;\&\; \mathrm{Svm}(F^{\leftarrow})$. PROOF:

Suppose_not(f) $\Rightarrow$ $\neg(1\text{–}1(f) \leftrightarrow \mathrm{Svm}(f) \;\&\; \mathrm{Svm}(f^{\leftarrow}))$

-- Suppose the contrary, and first consider the case in which f and $\mathsf{f}^{\leftarrow}$ are both single-valued, but f is not 1-1, so that by definition there exist distinct elements of the form $[\mathsf{u}^{[1]}, \mathsf{u}^{[2]}]$ with $\mathsf{u} \in \mathsf{f}$ having identical second components but different first components.

$\mathsf{Suppose} \Rightarrow \quad \mathsf{Svm}(\mathsf{f}) \,\&\, \mathsf{Svm}(\mathsf{f}^{\leftarrow}) \,\&\, \neg 1\text{–}1(\mathsf{f})$
$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad Stat1: \ \neg\langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y}\rangle$
$\langle \mathsf{x}, \mathsf{y}\rangle \hookrightarrow Stat1 \Rightarrow \quad Stat2: \ \mathsf{x}, \mathsf{y} \in \mathsf{f} \,\&\, \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \,\&\, \mathsf{x} \neq \mathsf{y}$
$\mathsf{Use\_def}(\mathsf{Svm}) \Rightarrow \quad \mathsf{Is\_map}(\mathsf{f})$
$\mathsf{Use\_def}(\mathsf{Is\_map}) \Rightarrow \quad Stat3: \ \mathsf{x} \in \{[\mathsf{u}^{[1]}, \mathsf{u}^{[2]}] : \mathsf{u} \in \mathsf{f}\}$
$\mathsf{Use\_def}(\mathsf{Is\_map}) \Rightarrow \quad Stat4: \ \mathsf{y} \in \{[\mathsf{u}^{[1]}, \mathsf{u}^{[2]}] : \mathsf{u} \in \mathsf{f}\}$
$\langle \mathsf{u}\rangle \hookrightarrow Stat3 \Rightarrow \quad Stat5: \ \mathsf{x} = [\mathsf{u}^{[1]}, \mathsf{u}^{[2]}] \,\&\, \mathsf{u} \in \mathsf{f}$
$\langle \mathsf{v}\rangle \hookrightarrow Stat4 \Rightarrow \quad Stat6: \ \mathsf{y} = [\mathsf{v}^{[1]}, \mathsf{v}^{[2]}] \,\&\, \mathsf{v} \in \mathsf{f}$
$\langle Stat2, Stat5, Stat6, *\rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \,\&\, \mathsf{x} \neq \mathsf{y} \,\&\, \mathsf{x} = [\mathsf{u}^{[1]}, \mathsf{u}^{[2]}] \,\&\, \mathsf{y} = [\mathsf{v}^{[1]}, \mathsf{v}^{[2]}]$
$\mathsf{EQUAL} \Rightarrow \quad Stat7: \ [\mathsf{u}^{[1]}, \mathsf{u}^{[2]}]^{[2]} = [\mathsf{v}^{[1]}, \mathsf{v}^{[2]}]^{[2]} \,\&\, [\mathsf{u}^{[1]}, \mathsf{u}^{[2]}] \neq [\mathsf{v}^{[1]}, \mathsf{v}^{[2]}]$
$\langle Stat7\rangle \ \mathsf{ELEM} \Rightarrow \quad Stat8: \ \mathsf{u}^{[2]} = \mathsf{v}^{[2]}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{u}^{[1]} = \mathsf{v}^{[1]}$
$\mathsf{EQUAL} \Rightarrow \quad [\mathsf{u}^{[1]}, \mathsf{u}^{[2]}] = [\mathsf{v}^{[1]}, \mathsf{v}^{[2]}]$
$\langle Stat7, *\rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat8a: \ \mathsf{u}^{[1]} \neq \mathsf{v}^{[1]}$

-- But then, by Theorem 81, $[\mathsf{u}^{[2]}, \mathsf{u}^{[1]}]$ and $[\mathsf{v}^{[2]}, \mathsf{v}^{[1]}]$ both belong to $\mathsf{f}^{\leftarrow}$, contradicting its single-valuedness.

$\langle \mathsf{f}, \mathsf{u}^{[1]}, \mathsf{u}^{[2]}\rangle \hookrightarrow T81 \Rightarrow \quad Stat9: \ [\mathsf{u}^{[2]}, \mathsf{u}^{[1]}] \in \mathsf{f}^{\leftarrow}$
$\langle \mathsf{f}, \mathsf{v}^{[1]}, \mathsf{v}^{[2]}\rangle \hookrightarrow T81 \Rightarrow \quad Stat10: \ [\mathsf{v}^{[2]}, \mathsf{v}^{[1]}] \in \mathsf{f}^{\leftarrow}$
$\mathsf{Use\_def}(\mathsf{Svm}) \Rightarrow \quad Stat11: \ \langle \forall \mathsf{x} \in \mathsf{f}^{\leftarrow}, \mathsf{y} \in \mathsf{f}^{\leftarrow} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \rightarrow \mathsf{x} = \mathsf{y}\rangle$
$\langle [\mathsf{u}^{[2]}, \mathsf{u}^{[1]}], [\mathsf{v}^{[2]}, \mathsf{v}^{[1]}]\rangle \hookrightarrow Stat11 \Rightarrow \quad Stat12: \ \neg$
$\quad [\mathsf{u}^{[2]}, \mathsf{u}^{[1]}], [\mathsf{v}^{[2]}, \mathsf{v}^{[1]}] \in \mathsf{f}^{\leftarrow} \,\&\, [\mathsf{u}^{[2]}, \mathsf{u}^{[1]}]^{[1]} = [\mathsf{v}^{[2]}, \mathsf{v}^{[1]}]^{[1]} \,\&\, [\mathsf{u}^{[2]}, \mathsf{u}^{[1]}] \neq [\mathsf{v}^{[2]}, \mathsf{v}^{[1]}]$
$\langle Stat8, Stat8a, Stat9, Stat10, Stat12\rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad 1\text{–}1(\mathsf{f}) \,\&\, \neg(\mathsf{Svm}(\mathsf{f}) \,\&\, \mathsf{Svm}(\mathsf{f}^{\leftarrow}))$

-- Next consider the case in which f is 1-1, but f and $\mathsf{f}^{\leftarrow}$ are not both single-valued. By definition of 'one_1_map', it must be $\mathsf{f}^{\leftarrow}$ that is not single valued, so that there must exist distinct xx and yy in $\mathsf{f}^{\leftarrow}$ with identical first components. Since these are in effect distinct elements of f with identical second components, they violate the fact that f is 1-1, a contradiction which completes the proof of the present theorem.

$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad \neg\mathsf{Svm}(\mathsf{f}^{\leftarrow}) \,\&\, Stat13: \ \langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \,|\, \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y}\rangle$
$\mathsf{Use\_def}(\mathsf{Svm}) \Rightarrow \quad \neg(\mathsf{Is\_map}(\mathsf{f}^{\leftarrow}) \,\&\, \langle \forall \mathsf{x} \in \mathsf{f}^{\leftarrow}, \mathsf{y} \in \mathsf{f}^{\leftarrow} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \rightarrow \mathsf{x} = \mathsf{y}\rangle)$
$\langle \mathsf{f}\rangle \hookrightarrow T89 \Rightarrow \quad Stat14: \ \neg\langle \forall \mathsf{x} \in \mathsf{f}^{\leftarrow}, \mathsf{y} \in \mathsf{f}^{\leftarrow} \,|\, \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \rightarrow \mathsf{x} = \mathsf{y}\rangle$
$\langle \mathsf{xx}, \mathsf{yy}\rangle \hookrightarrow Stat14 \Rightarrow \quad Stat15: \ \mathsf{xx}, \mathsf{yy} \in \mathsf{f}^{\leftarrow} \,\&\, \mathsf{xx}^{[1]} = \mathsf{yy}^{[1]} \,\&\, \mathsf{xx} \neq \mathsf{yy}$

Use_def$(^{\leftarrow}) \Rightarrow$ $\quad Stat16:\ xx \in \left\{\left[u^{[2]}, u^{[1]}\right] : u \in f\right\}$
$\langle vv \rangle \hookrightarrow Stat16 \Rightarrow$ $\quad xx = \left[vv^{[2]}, vv^{[1]}\right]\ \&\ vv \in f$
Use_def$(^{\leftarrow}) \Rightarrow$ $\quad Stat17:\ yy \in \left\{\left[u^{[2]}, u^{[1]}\right] : u \in f\right\}$
$\langle w \rangle \hookrightarrow Stat17 \Rightarrow$ $\quad yy = \left[w^{[2]}, w^{[1]}\right]\ \&\ w \in f$
$\langle Stat15, * \rangle$ ELEM $\Rightarrow$ $\quad Stat18:\ xx \neq yy\ \&\ xx = \left[vv^{[2]}, vv^{[1]}\right]\ \&\ yy = \left[w^{[2]}, w^{[1]}\right]$
Suppose $\Rightarrow$ $\quad vv = w$
EQUAL $\Rightarrow$ $\quad yy = \left[vv^{[2]}, vv^{[1]}\right]$
$\langle Stat18, * \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad vv \neq w$
$\langle Stat15, * \rangle$ ELEM $\Rightarrow$ $\quad xx^{[1]} = yy^{[1]}\ \&\ xx = \left[vv^{[2]}, vv^{[1]}\right]\ \&\ yy = \left[w^{[2]}, w^{[1]}\right]$
EQUAL $\Rightarrow$ $\quad Stat19:\ \left[vv^{[2]}, vv^{[1]}\right]^{[1]} = \left[w^{[2]}, w^{[1]}\right]^{[1]}$
$\langle Stat19 \rangle$ ELEM $\Rightarrow$ $\quad vv^{[2]} = w^{[2]}$
$\langle vv, w \rangle \hookrightarrow Stat13 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem completes our proof that a pair of mutually inverse maps are each other's inverses.

-- An inverse pair of maps must be 1 - 1 and must be each other 's inverses

Theorem 128 (100) $\mathrm{Is\_map}(F)\ \&\ \mathrm{Is\_map}(G)\ \&\ \mathbf{domain}(F) = \mathbf{range}(G)\ \&\ \mathbf{range}(F) = \mathbf{domain}(G)\ \&\ F \bullet G = \iota_{\mathbf{range}(F)}\ \&\ G \bullet F = \iota_{\mathbf{domain}(F)} \rightarrow 1\text{–}1(F)\ \&\ G = F^{\leftarrow}$. PROOF:

Suppose_not$(f, g) \Rightarrow$ $\quad Stat1:\ \left(\mathrm{Is\_map}(f)\ \&\ \mathrm{Is\_map}(g)\ \&\ \mathbf{domain}(f) = \mathbf{range}(g)\ \&\ \mathbf{range}(f) = \mathbf{domain}(g)\ \&\ f \bullet g = \iota_{\mathbf{range}(f)}\ \&\ g \bullet f = \iota_{\mathbf{domain}(f)}\right)\ \&\ \neg(1\text{–}1(f)\ \&\ g = f^{\leftarrow})$

-- For consider a counterexample f, g. By Theorem 97, f and g must both must be single-valued, so either f is not 1-1, or g is not its inverse. But by Theorem 89, g has the same range and domain as the inverse of f,

$\langle \mathbf{range}(f) \rangle \hookrightarrow T94 \Rightarrow$ $\quad 1\text{–}1\left(\iota_{\mathbf{range}(f)}\right)$
$\langle \mathbf{domain}(f) \rangle \hookrightarrow T94 \Rightarrow$ $\quad 1\text{–}1\left(\iota_{\mathbf{domain}(f)}\right)$
EQUAL $\Rightarrow$ $\quad 1\text{–}1(f \bullet g)\ \&\ 1\text{–}1(g \bullet f)$
Use_def$(1\text{–}1) \Rightarrow$ $\quad \mathrm{Svm}(f \bullet g)\ \&\ \mathrm{Svm}(g \bullet f)$
$\langle f, g \rangle \hookrightarrow T97 \Rightarrow$ $\quad \mathrm{Svm}(f)$
$\langle g, f \rangle \hookrightarrow T97 \Rightarrow$ $\quad \mathrm{Svm}(g)$
EQUAL $\Rightarrow$ $\quad \iota_{\mathbf{range}(f)} = \iota_{\mathbf{domain}(g)}$
EQUAL $\Rightarrow$ $\quad \iota_{\mathbf{domain}(f)} = \iota_{\mathbf{range}(g)}$
$\langle f \rangle \hookrightarrow T89 \Rightarrow$ $\quad \mathrm{Is\_map}(f^{\leftarrow})\ \&\ \mathbf{domain}(f^{\leftarrow}) = \mathbf{range}(f)$
$\langle g \rangle \hookrightarrow T89 \Rightarrow$ $\quad \mathrm{Is\_map}(g^{\leftarrow})\ \&\ \mathbf{range}(g^{\leftarrow}) = \mathbf{domain}(g)$

-- and by Theorem 98 $f^{\leftarrow}$ has $g^{\leftarrow}$ as a right inverse, so that by Theorem 97 $f^{\leftarrow}$ must also be single-valued.

$\mathsf{Use\_def(Svm)} \Rightarrow$ $\mathsf{Is\_map(f)}$
$\mathsf{Use\_def(Svm)} \Rightarrow$ $\mathsf{Is\_map(g)}$
$\langle \mathsf{g}, \mathsf{f} \rangle \hookrightarrow T98 \Rightarrow$ $(\mathsf{g} \bullet \mathsf{f})^{\leftarrow} = \mathsf{f}^{\leftarrow} \bullet \mathsf{g}^{\leftarrow}$
$\langle \mathbf{range}(\mathsf{g}) \rangle \hookrightarrow T94 \Rightarrow$ $1\text{–}1(\boldsymbol{\iota}_{\mathbf{range}(\mathsf{g})})$ & $\boldsymbol{\iota}_{\mathbf{range}(\mathsf{g})}{}^{\leftarrow} = \boldsymbol{\iota}_{\mathbf{range}(\mathsf{g})}$
$\mathsf{EQUAL} \Rightarrow$ $\mathsf{f}^{\leftarrow} \bullet \mathsf{g}^{\leftarrow} = \boldsymbol{\iota}_{\mathbf{range}(\mathsf{g})}$
$\mathsf{Use\_def(1\text{–}1)} \Rightarrow$ $\mathsf{Svm}(\boldsymbol{\iota}_{\mathbf{range}(\mathsf{g})})$
$\mathsf{EQUAL} \Rightarrow$ $\mathsf{Svm}(\mathsf{f}^{\leftarrow} \bullet \mathsf{g}^{\leftarrow})$ & $\mathbf{domain}(\mathsf{f}^{\leftarrow}) = \mathbf{range}(\mathsf{g}^{\leftarrow})$
$\langle \mathsf{f}^{\leftarrow}, \mathsf{g}^{\leftarrow} \rangle \hookrightarrow T97 \Rightarrow$ $\mathsf{Svm}(\mathsf{f}^{\leftarrow})$

-- Theorem 99 tells us that f must be 1-1, so only the possibility that $\mathsf{g} \neq \mathsf{f}^{\leftarrow}$ needs to be considered. But since $\mathsf{f} \bullet \mathsf{f}^{\leftarrow}$ and $\mathsf{g} \bullet \mathsf{f}$ are both identity maps, we can reassociate to show that the triple product $\mathsf{g} \bullet \mathsf{f} \bullet \mathsf{f}^{\leftarrow}$ is equal to both g and $\mathsf{f}^{\leftarrow}$, a contradiction which proves our theorem

$\langle \mathsf{f} \rangle \hookrightarrow T99 \Rightarrow$ $Stat2:$ $1\text{–}1(\mathsf{f})$ & $\mathsf{g} \neq \mathsf{f}^{\leftarrow}$
$\langle \mathsf{f} \rangle \hookrightarrow T96 \Rightarrow$ $\mathsf{f} \bullet \mathsf{f}^{\leftarrow} = \boldsymbol{\iota}_{\mathbf{range}(\mathsf{f})}$
$\langle \mathbf{range}(\mathsf{f}), \mathsf{junk}, \mathsf{g} \rangle \hookrightarrow T94 \Rightarrow$ $\mathsf{Is\_map(g)}$ & $\mathbf{domain}(\mathsf{g}) \subseteq \mathbf{range}(\mathsf{f}) \rightarrow \mathsf{g} \bullet \boldsymbol{\iota}_{\mathbf{range}(\mathsf{f})} = \mathsf{g}$
$\mathsf{EQUAL} \Rightarrow$ $\mathsf{g} \bullet (\mathsf{f} \bullet \mathsf{f}^{\leftarrow}) = \mathsf{g}$
$\langle \mathsf{g}, \mathsf{f}, \mathsf{f}^{\leftarrow} \rangle \hookrightarrow T61 \Rightarrow$ $\mathsf{g} \bullet \mathsf{f} \bullet \mathsf{f}^{\leftarrow} = \mathsf{g}$
$\mathsf{EQUAL} \Rightarrow$ $Stat3:$ $\boldsymbol{\iota}_{\mathbf{domain}(\mathsf{f})} \bullet \mathsf{f}^{\leftarrow} = \mathsf{g}$
$\langle \mathsf{f} \rangle \hookrightarrow T89 \Rightarrow$ $Stat4:$ $\mathbf{range}(\mathsf{f}^{\leftarrow}) = \mathbf{domain}(\mathsf{f})$
$\langle \mathbf{domain}(\mathsf{f}), \mathsf{junk}, \mathsf{f}^{\leftarrow} \rangle \hookrightarrow T94 \Rightarrow$ $Stat5:$ $\mathbf{range}(\mathsf{f}^{\leftarrow}) \subseteq \mathbf{domain}(\mathsf{f}) \rightarrow \boldsymbol{\iota}_{\mathbf{domain}(\mathsf{f})} \bullet \mathsf{f}^{\leftarrow} = \mathsf{f}^{\leftarrow}$
$\langle Stat1, Stat3, Stat4, Stat5, * \rangle$ $\mathsf{ELEM} \Rightarrow$ $Stat6:$ $\mathsf{f}^{\leftarrow} = \mathsf{g}$
$\langle Stat1, Stat2, Stat6, * \rangle$ $\mathsf{ELEM} \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- The following elementary lemma expresses the restriction of a single-valued map as a setformer.

Theorem 129 (101) $\mathsf{Svm}(\mathsf{F}) \rightarrow$
$\mathsf{F}_{|\mathsf{S}} = \{[\mathsf{x}, \mathsf{F} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{F}) \mid \mathsf{x} \in \mathsf{S}\}$ & $\mathbf{domain}(\mathsf{F}_{|\mathsf{S}}) = \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{F}) \mid \mathsf{x} \in \mathsf{S}\}$ & $\mathbf{range}(\mathsf{F}_{|\mathsf{S}}) = \{\mathsf{F} \upharpoonright \mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{F}) \mid \mathsf{x} \in \mathsf{S}\}$. PROOF:

$\mathsf{Suppose\_not(f, s)} \Rightarrow$
$\mathsf{Svm(f)}$ &
$\mathsf{f}_{|\mathsf{s}} \neq \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{x} \in \mathsf{s}\} \vee \mathbf{domain}(\mathsf{f}_{|\mathsf{s}}) \neq \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{x} \in \mathsf{s}\} \vee \mathbf{range}(\mathsf{f}_{|\mathsf{s}}) \neq \{\mathsf{f} \upharpoonright \mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{x} \in \mathsf{s}\}$

-- For if we suppose the first clause of our theorem to be false, use the definitions of the operators involved, and simplify, we are led to the impossible inequalities seen below. Thus only the second and third conclusion of the theorem need be considered.

$\langle \mathsf{f} \rangle \hookrightarrow T65 \Rightarrow$ $\mathsf{f} = \{[\mathsf{u}, \mathsf{f} \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbf{domain}(\mathsf{f})\}$
$\mathsf{Use\_def(|)} \Rightarrow$ $\mathsf{f}_{|\mathsf{s}} = \{\mathsf{x} : \mathsf{x} \in \mathsf{f} \mid \mathsf{x}^{[1]} \in \mathsf{s}\}$

EQUAL $\Rightarrow$ $f_{|s} = \{x : x \in \{[u, f\restriction u] : u \in \mathbf{domain}(f)\} \mid x^{[1]} \in s\}$

SIMPLF $\Rightarrow$ $f_{|s} = \left\{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in s\right\}$

Suppose $\Rightarrow$ $f_{|s} \neq \{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid x \in s\}$

ELEM $\Rightarrow$ *Stat1* : $\{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid x \in s\} \neq \left\{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in s\right\}$

$\langle x \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x \in \mathbf{domain}(f)\ \&\ \neg(x \in s \leftrightarrow [x, f\restriction x]^{[1]} \in s)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $f_{|s} = \{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid x \in s\}$

-- Next suppose that our theorem's second conclusion is false. Using the relevant definitions and simplifying much as above, we are led to a second impossible inequality. Hence only the third conclusion of our theorem could be false.

Suppose $\Rightarrow$ $\mathbf{range}(f_{|s}) \neq \{f\restriction x : x \in \mathbf{domain}(f) \mid x \in s\}$

Use_def (**range**) $\Rightarrow$ $\mathbf{range}(f_{|s}) = \{x^{[2]} : x \in f_{|s}\}$

EQUAL $\Rightarrow$ $\mathbf{range}(f_{|s}) = \{x^{[2]} : x \in \{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid x \in s\}\}$

SIMPLF $\Rightarrow$ $\mathbf{range}(f_{|s}) = \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid x \in s\right\}$

ELEM $\Rightarrow$ *Stat2* : $\{f\restriction x : x \in \mathbf{domain}(f) \mid x \in s\} \neq \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid x \in s\right\}$

Set_monot $\Rightarrow$ $\{f\restriction x : x \in \mathbf{domain}(f) \mid x \in s\} = \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid x \in s\right\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{range}(f_{|s}) = \{f\restriction x : x \in \mathbf{domain}(f) \mid x \in s\}$

Suppose $\Rightarrow$ $\mathbf{domain}(f_{|s}) \neq \{x : x \in \mathbf{domain}(f) \mid x \in s\}$

-- But the domain can be handled in much the same way as the range, and so leads us to a final contradiction which completes the proof of the present theorem.

Use_def (**domain**) $\Rightarrow$ $\mathbf{domain}(f_{|s}) = \{x^{[1]} : x \in f_{|s}\}$

EQUAL $\Rightarrow$ $\mathbf{domain}(f_{|s}) = \{x^{[1]} : x \in \{[x, f\restriction x] : x \in \mathbf{domain}(f) \mid x \in s\}\}$

SIMPLF $\Rightarrow$ $\mathbf{domain}(f_{|s}) = \left\{[x, f\restriction x]^{[1]} : x \in \mathbf{domain}(f) \mid x \in s\right\}$

ELEM $\Rightarrow$ *Stat3* : $\{x : x \in \mathbf{domain}(f) \mid x \in s\} \neq \left\{[x, f\restriction x]^{[1]} : x \in \mathbf{domain}(f) \mid x \in s\right\}$

Set_monot $\Rightarrow$ $\{x : x \in \mathbf{domain}(f) \mid x \in s\} = \left\{[x, f\restriction x]^{[1]} : x \in \mathbf{domain}(f) \mid x \in s\right\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemma simply re-expresses the condition that a map should be 1-1 in terms of the element-mapping operator f [x]:

Theorem 130 (102) $1\text{–}1(F)\ \&\ X, Y \in \mathbf{domain}(F)\ \&\ F\restriction X = F\restriction Y \rightarrow X = Y$. PROOF:

$\text{Suppose\_not}(f,x,y) \Rightarrow \quad 1\text{–}1(f) \,\&\, x,y \in \mathbf{domain}(f) \,\&\, f\restriction x = f\restriction y \,\&\, x \neq y$

-- For suppose the contrary, and let f be a 1-1 map, with distinct elements x, y in its domain such that $f\restriction x = f\restriction y$. Since it is easily seen that $[x, f\restriction x]$ and $[y, f\restriction y]$ both belong to f, this would violate the definition of 1–1, a contradiction which completes our proof.

$\text{Use\_def}(1\text{–}1) \Rightarrow \quad \mathsf{Svm}(f) \,\&\, \mathit{Stat1}: \ \langle \forall x \in f, y \in f \mid x^{[2]} = y^{[2]} \rightarrow x = y \rangle$
$\langle f \rangle \hookrightarrow \mathit{T65} \Rightarrow \quad f = \{[x, f\restriction x] : x \in \mathbf{domain}(f)\}$
$\text{Suppose} \Rightarrow \quad \mathit{Stat2}: \ [x, f\restriction x] \notin \{[x, f\restriction x] : x \in \mathbf{domain}(f)\}$
$\langle x \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad [x, f\restriction x] \in f$
$\text{Suppose} \Rightarrow \quad \mathit{Stat3}: \ [y, f\restriction y] \notin \{[x, f\restriction x] : x \in \mathbf{domain}(f)\}$
$\langle y \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad [y, f\restriction y] \in f$
$\langle [x, f\restriction x], [y, f\restriction y] \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad [x, f\restriction x]^{[2]} = [y, f\restriction y]^{[2]} \rightarrow$
$x = y$
$\text{ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- Next we show that the composition of two single-valued maps is single valued.

Theorem 131 (103) $\mathsf{Svm}(F) \,\&\, \mathsf{Svm}(G) \rightarrow \mathsf{Svm}(F \bullet G)$. PROOF:

$\text{Suppose\_not}(f,g) \Rightarrow \quad \mathsf{Svm}(f) \,\&\, \mathsf{Svm}(g) \,\&\, \neg\mathsf{Svm}(f \bullet g)$

-- For suppose the contrary. Then by definition and using Theorem 50 it follows that there exist a, b in $f \bullet g$ with identical first components but distinct second components:

$\text{Use\_def}(\mathsf{Svm}) \Rightarrow \quad \mathsf{Is\_map}(f) \,\&\, \mathit{Stat1}:$
$\langle \forall x \in f, y \in f \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle \,\&\, \mathsf{Is\_map}(g) \,\&\, \mathit{Stat2}:$
$\langle \forall x \in g, y \in g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle \,\&\, \neg\big(\mathsf{Is\_map}(f \bullet g) \,\&\, \langle \forall x \in f \bullet g, y \in f \bullet g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle\big)$
$\langle f, g \rangle \hookrightarrow \mathit{T50} \Rightarrow \quad \mathsf{Is\_map}(f \bullet g)$
$\text{ELEM} \Rightarrow \quad \mathit{Stat3}: \ \neg\langle \forall x \in f \bullet g, y \in f \bullet g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\langle a, b \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad a, b \in f \bullet g \,\&\, a^{[1]} = b^{[1]} \,\&\, a \neq b$

-- Thus, by definition of map multiplication, there exist c, d, u, v, with c, u in g and d, v in f, satisfying the condition displayed below.

$\text{Use\_def}(\bullet) \Rightarrow \quad \mathit{Stat4}: \ a, b \in \left\{ \left[x^{[1]}, y^{[2]}\right] : x \in g, y \in f \mid x^{[2]} = y^{[1]} \right\}$
$\langle c, d, u, v \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad \mathit{Stat5}:$
$c \in g \,\&\, d \in f \,\&\, a = \left[c^{[1]}, d^{[2]}\right] \,\&\, c^{[2]} = d^{[1]} \,\&\, u \in g \,\&\, v \in f \,\&\, b = \left[u^{[1]}, v^{[2]}\right] \,\&\, u^{[2]} = v^{[1]} \,\&\, a^{[1]} = b^{[1]} \,\&\, a \neq b$

-- But then $c^{[1]} = u^{[1]}$, so by Stat6 6 we have $d^{[1]} = v^{[1]}$.

$\langle Stat5, * \rangle$ ELEM $\Rightarrow$ $Stat7: \ c, u \in g \ \& \ a^{[1]} = b^{[1]} \ \& \ a = \left[c^{[1]}, d^{[2]}\right] \ \& \ b = \left[u^{[1]}, v^{[2]}\right]$

EQUAL $\Rightarrow$ $Stat8: \ \left[c^{[1]}, d^{[2]}\right]^{[1]} = \left[u^{[1]}, v^{[2]}\right]^{[1]}$

$\langle Stat8 \rangle$ ELEM $\Rightarrow$ $Stat9: \ c^{[1]} = u^{[1]}$

$\langle c, u \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat10: \ c, u \in g \ \& \ c^{[1]} = u^{[1]} \rightarrow c = u$

$\langle Stat7, Stat10, Stat9, * \rangle$ ELEM $\Rightarrow$ $Stat11: \ c = u$

EQUAL $\Rightarrow$ $c^{[2]} = u^{[2]}$

$\langle Stat5, * \rangle$ ELEM $\Rightarrow$ $d^{[1]} = v^{[1]}$

-- It follows by Stat6 7 that $d^{[2]} = v^{[2]}$, contradicting $a \neq b$ and so proving our theorem.

$\langle d, v \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat12: \ d = v$

$\langle Stat5, * \rangle$ ELEM $\Rightarrow$ $Stat13: \ \left[c^{[1]}, d^{[2]}\right] \neq \left[u^{[1]}, v^{[2]}\right]$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem gives a standard elementary formula for $f \bullet g \restriction x$.

Theorem 132 (104) $Svm(F) \ \& \ Svm(G) \ \& \ X \in \mathbf{domain}(G) \ \& \ \mathbf{range}(G) \subseteq \mathbf{domain}(F) \rightarrow F \bullet G \restriction X = F \restriction (G \restriction X)$. PROOF:

Suppose_not$(f, g, x) \Rightarrow$ $Svm(f) \ \& \ Svm(g) \ \& \ x \in \mathbf{domain}(g) \ \& \ \mathbf{range}(g) \subseteq \mathbf{domain}(f) \ \& \ f \bullet g \restriction x \neq f \restriction (g \restriction x)$

-- For suppose the contrary. By Theorem 69, we have $[x, g \restriction x] \in g$ and $[g \restriction x, f \restriction (g \restriction x)] \in f$.

Use_def(Svm) $\Rightarrow$ Is_map(g)

$\langle g, x \rangle \hookrightarrow T69 \Rightarrow$ $[x, g \restriction x] \in g$

Suppose $\Rightarrow$ $g \restriction x \notin \mathbf{range}(g)$

Use_def($\mathbf{range}$) $\Rightarrow$ $Stat1: \ g \restriction x \notin \{u^{[2]} : \ u \in g\}$

$\langle [x, g \restriction x] \rangle \hookrightarrow Stat1 \Rightarrow$ $\neg(g \restriction x = [x, g \restriction x]^{[2]} \ \& \ [x, g \restriction x] \in g)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $g \restriction x \in \mathbf{range}(g)$

ELEM $\Rightarrow$ $g \restriction x \in \mathbf{domain}(f)$

Use_def(Svm) $\Rightarrow$ Is_map(f)

$\langle f, g \restriction x \rangle \hookrightarrow T69 \Rightarrow$ $[g \restriction x, f \restriction (g \restriction x)] \in f$

-- It follows that $[x, f \restriction (g \restriction x)]$ belongs to $f \bullet g$, and so, since $f \bullet g$ is single valued by Theorem 103, we have $f \restriction (g \restriction x) = f \bullet g \restriction x$.

Suppose $\Rightarrow$ $[x, f\upharpoonright(g\upharpoonright x)] \notin f\bullet g$
Use_def ($\bullet$) $\Rightarrow$ *Stat2* : $[x, f\upharpoonright(g\upharpoonright x)] \notin \{[u^{[1]}, v^{[2]}] : u \in g, v \in f \mid u^{[2]} = v^{[1]}\}$
$\langle [x, g\upharpoonright x], [g\upharpoonright x, f\upharpoonright(g\upharpoonright x)] \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3* :

$$\neg\big([x, f\upharpoonright(g\upharpoonright x)] = \big[[x, g\upharpoonright x]^{[1]}, [g\upharpoonright x, f\upharpoonright(g\upharpoonright x)]^{[2]}\big] \;\&\; [x, g\upharpoonright x]^{[2]} = [g\upharpoonright x, f\upharpoonright(g\upharpoonright x)]^{[1]} \;\&\; [x, g\upharpoonright x] \in g \;\&\; [g\upharpoonright x, f\upharpoonright(g\upharpoonright x)] \in f\big)$$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $[x, f\upharpoonright(g\upharpoonright x)] \in f\bullet g$
$\langle f, g \rangle \hookrightarrow$ *T103* $\Rightarrow$ $\mathsf{Svm}(f\bullet g)$
$\langle f\bullet g, [x, f\upharpoonright(g\upharpoonright x)] \rangle \hookrightarrow$ *T67* $\Rightarrow$ $f\bullet g\upharpoonright[x, f\upharpoonright(g\upharpoonright x)]^{[1]} = [x, f\upharpoonright(g\upharpoonright x)]^{[2]}$
ELEM $\Rightarrow$ $[x, f\upharpoonright(g\upharpoonright x)]^{[1]} = x$
EQUAL $\Rightarrow$ $f\bullet g\upharpoonright x = [x, f\upharpoonright(g\upharpoonright x)]^{[2]}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next result is a corollary of Theorem 104 which adds several useful clauses to it.

Theorem 133 (105) $\mathsf{Svm}(F) \;\&\; \mathsf{Svm}(G) \;\&\; X \in \mathbf{domain}(G) \;\&\; \mathbf{range}(G) \subseteq \mathbf{domain}(F) \rightarrow F\bullet G\upharpoonright X = F\upharpoonright(G\upharpoonright X) \;\&\; F\bullet G = \{[x, F\upharpoonright(G\upharpoonright x)] : x \in \mathbf{domain}(G)\} \;\&\; \mathbf{range}(F\bullet G) = \{F\upharpoonright(G\upharpoonright x) : x \in \mathbf{domain}(G)\}$. PROOF:

Suppose_not(f, g, x) $\Rightarrow$
$\mathsf{Svm}(f) \;\&\; \mathsf{Svm}(g) \;\&\; x \in \mathbf{domain}(g) \;\&\; \mathbf{range}(g) \subseteq \mathbf{domain}(f) \;\&\;$
$\neg\big(f\bullet g\upharpoonright x = f\upharpoonright(g\upharpoonright x) \;\&\; f\bullet g = \{[x, f\upharpoonright(g\upharpoonright x)] : x \in \mathbf{domain}(g)\} \;\&\; \mathbf{range}(f\bullet g) = \{f\upharpoonright(g\upharpoonright x) : x \in \mathbf{domain}(g)\}\big)$

-- For suppose that our statement is false, and let f, g be a counterexample. It follows immediately from Theorems 101, 64, and 83 that the two final clauses of our assertion must be true if the expression $f\upharpoonright(g\upharpoonright x)$ appearing in the setformers seen there are replaced by $f\bullet g\upharpoonright x$.

$\langle f, g \rangle \hookrightarrow$ *T103* $\Rightarrow$ $\mathsf{Svm}(f\bullet g)$
$\langle f\bullet g \rangle \hookrightarrow$ *T66* $\Rightarrow$ $f\bullet g = \{[x, f\bullet g\upharpoonright x] : x \in \mathbf{domain}(f\bullet g)\} \;\&\; \mathbf{range}(f\bullet g) = \{f\bullet g\upharpoonright x : x \in \mathbf{domain}(f\bullet g)\}$
$\langle g, f \rangle \hookrightarrow$ *T85* $\Rightarrow$ $\mathbf{domain}(f\bullet g) = \mathbf{domain}(g)$
EQUAL $\Rightarrow$ $f\bullet g = \{[x, f\bullet g\upharpoonright x] : x \in \mathbf{domain}(g)\} \;\&\; \mathbf{range}(f\bullet g) = \{f\bullet g\upharpoonright x : x \in \mathbf{domain}(g)\}$

-- However, Theorem 104 lets us replace $f\bullet g\upharpoonright x$ by $f\upharpoonright(g\upharpoonright x)$, after which our assertion is immediate.

$\langle f, g, x \rangle \hookrightarrow$ *T104* $\Rightarrow$ $f\bullet g\upharpoonright x = f\upharpoonright(g\upharpoonright x)$
ELEM $\Rightarrow$
$\{[x, f\bullet g\upharpoonright x] : x \in \mathbf{domain}(g)\} \neq \{[x, f\upharpoonright(g\upharpoonright x)] : x \in \mathbf{domain}(g)\} \;\vee$

$\{f \bullet g{\restriction}x : x \in \mathbf{domain}(g)\} \neq \{f{\restriction}(g{\restriction}x) : x \in \mathbf{domain}(g)\}$

Suppose $\Rightarrow$ *Stat2*: $\{[x, f \bullet g{\restriction}x] : x \in \mathbf{domain}(g)\} \neq \{[x, f{\restriction}(g{\restriction}x)] : x \in \mathbf{domain}(g)\}$
$\langle x' \rangle \hookrightarrow Stat2 \Rightarrow$ $f \bullet g{\restriction}x' \neq f{\restriction}(g{\restriction}x')\ \&\ x' \in \mathbf{domain}(g)$
$\langle f, g, x' \rangle \hookrightarrow T104 \Rightarrow$ false; Discharge $\Rightarrow$ *Stat3*: $\{f \bullet g{\restriction}x : x \in \mathbf{domain}(g)\} \neq \{f{\restriction}(g{\restriction}x) : x \in \mathbf{domain}(g)\}$
$\langle xq \rangle \hookrightarrow Stat3 \Rightarrow$ $f \bullet g{\restriction}xq \neq f{\restriction}(g{\restriction}xq)\ \&\ xq \in \mathbf{domain}(g)$
$\langle f, g, xq \rangle \hookrightarrow T104 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 134 (106) $\mathsf{Svm}(F)\ \&\ G \subseteq F\ \&\ X \in \mathbf{domain}(G) \rightarrow F{\restriction}X = G{\restriction}X$. PROOF:

Suppose_not$(f, g, x) \Rightarrow$ *Stat1*: $\mathsf{Svm}(f)\ \&\ g \subseteq f\ \&\ x \in \mathbf{domain}(g)\ \&\ f{\restriction}x \neq g{\restriction}x$
Use_def$({\restriction}) \Rightarrow$ $\mathbf{arb}(f_{|\{x\}})^{[2]} \neq \mathbf{arb}\big(g_{|\{x\}}\big)^{[2]}$
Use_def$(|) \Rightarrow$ $\mathbf{arb}(\{q : q \in f \mid q^{[1]} \in \{x\}\})^{[2]} \neq \mathbf{arb}(\{q : q \in g \mid q^{[1]} \in \{x\}\})^{[2]}$
Use_def$(\mathbf{domain}) \Rightarrow$ *Stat2*: $x \in \{p^{[1]} : p \in g\}$
$\langle q \rangle \hookrightarrow Stat2 \Rightarrow$ $q \in g\ \&\ x = q^{[1]}$
Set_monot $\Rightarrow$ $\{p : p \in g \mid p^{[1]} \in \{x\}\} \subseteq \{p : p \in f \mid p^{[1]} \in \{x\}\}$
Suppose $\Rightarrow$ *Stat3*: $q \notin \{p : p \in g \mid p^{[1]} \in \{x\}\}$
$\langle q \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ *Stat4*: $q \in \{p : p \in g \mid p^{[1]} \in \{x\}\}$
Suppose $\Rightarrow$ *Stat5*: $\{p : p \in g \mid p^{[1]} \in \{x\}\} \not\supseteq \{p : p \in f \mid p^{[1]} \in \{x\}\}$
$\langle y \rangle \hookrightarrow Stat5 \Rightarrow$ *Stat6*: $y \in \{p : p \in f \mid p^{[1]} \in \{x\}\}\ \&\ y \notin \{p : p \in g \mid p^{[1]} \in \{x\}\}$
$\langle p, p \rangle \hookrightarrow Stat6 \Rightarrow$ *Stat8*: $p \in f\ \&\ p^{[1]} = x\ \&\ p \notin g$
$\langle p' \rangle \hookrightarrow Stat4 \Rightarrow$ *Stat9*: $p' \in g\ \&\ p'^{[1]} = x$
Use_def$(\mathsf{Svm}) \Rightarrow$ *Stat7*: $\langle \forall x \in f, y \in f \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\langle p, p' \rangle \hookrightarrow Stat7(\langle Stat8, Stat9, Stat1 \rangle) \Rightarrow$ false; Discharge $\Rightarrow$ $\{p : p \in g \mid p^{[1]} \in \{x\}\} = \{p : p \in f \mid p^{[1]} \in \{x\}\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The fact that a map f and its inverse are both 1-1 if either is results easily from Theorem 99.

Theorem 135 (107) $\mathsf{Is\_map}(F) \rightarrow (\mathsf{1\text{–}1}(F) \leftrightarrow \mathsf{1\text{–}1}(F^{\leftarrow}))$. PROOF:

Suppose_not$(f) \Rightarrow$ $\mathsf{Is\_map}(f)\ \&\ \neg(\mathsf{1\text{–}1}(f) \leftrightarrow \mathsf{1\text{–}1}(f^{\leftarrow}))$
$\langle f \rangle \hookrightarrow T99 \Rightarrow$ $\mathsf{1\text{–}1}(f) \leftrightarrow \mathsf{Svm}(f)\ \&\ \mathsf{Svm}(f^{\leftarrow})$
$\langle f^{\leftarrow} \rangle \hookrightarrow T99 \Rightarrow$ $\mathsf{1\text{–}1}(f^{\leftarrow}) \leftrightarrow \mathsf{Svm}(f^{\leftarrow})\ \&\ \mathsf{Svm}(f^{\leftarrow\leftarrow})$
ELEM $\Rightarrow$ $\neg\big(\mathsf{Svm}(f) \leftrightarrow \mathsf{Svm}(f^{\leftarrow\leftarrow})\big)$
$\langle f \rangle \hookrightarrow T90 \Rightarrow$ $f = f^{\leftarrow\leftarrow}$
Suppose $\Rightarrow$ $\mathsf{Svm}(f)$

EQUAL ⇒ $\mathsf{Svm}(f^{\leftarrow\leftarrow})$
ELEM ⇒ false; Discharge ⇒ $\mathsf{Svm}(f^{\leftarrow\leftarrow}) \,\&\, \neg\mathsf{Svm}(f)$
EQUAL ⇒ $\mathsf{Svm}(f)$
ELEM ⇒ false; Discharge ⇒ QED

-- Theorem 99 also lets us give a purely algebraic argument to show that the product of one-to-one mappings is one-to-one.

Theorem 136 (108) $1\text{–}1(F) \,\&\, 1\text{–}1(G) \rightarrow 1\text{–}1(F \bullet G)$. PROOF:

Suppose_not(f, g) ⇒ $1\text{–}1(f) \,\&\, 1\text{–}1(g) \,\&\, \neg 1\text{–}1(f \bullet g)$

-- For suppose the contrary, in which case it follows by Theorem 99 that f, g and their inverses are single-valued, but f•g is not.

$\langle f\rangle \hookrightarrow T99$ ⇒ $\mathsf{Svm}(f) \,\&\, \mathsf{Svm}(f^{\leftarrow})$
$\langle g\rangle \hookrightarrow T99$ ⇒ $\mathsf{Svm}(g) \,\&\, \mathsf{Svm}(g^{\leftarrow})$
$\langle f\bullet g\rangle \hookrightarrow T99$ ⇒ $\neg\Big(\mathsf{Svm}(f\bullet g) \,\&\, \mathsf{Svm}\big((f\bullet g)^{\leftarrow}\big)\Big)$

-- But Theorem 103 tells us that f•g is single-valued, and Theorem 98 allows $(f\bullet g)^{\leftarrow}$ to be rewritten as a product of inverses which must be single-valued, proving the present theorem.

$\langle f, g\rangle \hookrightarrow T103$ ⇒ $\mathsf{Svm}(f\bullet g)$
Use_def(Svm) ⇒ Is_map(f)
Use_def(Svm) ⇒ Is_map(g)
$\langle f, g\rangle \hookrightarrow T98$ ⇒ $(f\bullet g)^{\leftarrow} = g^{\leftarrow}\bullet f^{\leftarrow}$
$\langle g^{\leftarrow}, f^{\leftarrow}\rangle \hookrightarrow T103$ ⇒ $\mathsf{Svm}(g^{\leftarrow}\bullet f^{\leftarrow})$
EQUAL ⇒ false; Discharge ⇒ QED

-- The following slight variant of the definition of 'one_1_map' is sometimes useful.

Theorem 137 (109) $\mathsf{Svm}(F) \rightarrow (1\text{–}1(F) \leftrightarrow \langle \forall x \in \mathbf{domain}(F), y \in \mathbf{domain}(F) \,|\, F{\upharpoonright}x = F{\upharpoonright}y \rightarrow x = y\rangle)$. PROOF:

Suppose_not(f) ⇒ $\mathsf{Svm}(f) \,\&\, \neg(1\text{–}1(f) \leftrightarrow \langle \forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \,|\, f{\upharpoonright}x = f{\upharpoonright}y \rightarrow x = y\rangle)$

-- We argue by contradiction, and so suppose that f is a counterexample to our theorem. Since f is single-valued, Theorem 65 lets us represent it by the set expression $f = \{[x, f{\upharpoonright}x] : x \in \mathbf{domain}(f)\}$.

$\langle \mathsf{f} \rangle \hookrightarrow T65 \Rightarrow \quad \mathsf{f} = \{[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$

-- We can easily show that f is 1-1. For suppose the contrary. Then the fcn_symbol theory given previously tells us the quantified clause of our theorem must be false, a contradiction which proves our claim.

Suppose $\Rightarrow \quad \neg 1\text{–}1(\mathsf{f})$
ELEM $\Rightarrow \quad Stat1: \langle \forall \mathsf{x} \in \mathbf{domain}(\mathsf{f}), \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{f}\upharpoonright\mathsf{x} = \mathsf{f}\upharpoonright\mathsf{y} \rightarrow \mathsf{x} = \mathsf{y} \rangle$
APPLY $\langle \mathsf{x}_\Theta : \mathsf{x}, \mathsf{y}_\Theta : \mathsf{y} \rangle$ fcn_symbol$(\mathsf{f}(\mathsf{x}) \mapsto \mathsf{f}\upharpoonright\mathsf{x}, \mathsf{g} \mapsto \mathsf{f}, \mathsf{s} \mapsto \mathbf{domain}(\mathsf{f})) \Rightarrow$
$(\mathsf{x}, \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \;\&\; \mathsf{f}\upharpoonright\mathsf{x} = \mathsf{f}\upharpoonright\mathsf{y} \;\&\; \mathsf{x} \neq \mathsf{y}) \vee 1\text{–}1(\mathsf{f})$
ELEM $\Rightarrow \quad \mathsf{x}, \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \;\&\; \mathsf{f}\upharpoonright\mathsf{x} = \mathsf{f}\upharpoonright\mathsf{y} \;\&\; \mathsf{x} \neq \mathsf{y}$
$\langle \mathsf{x}, \mathsf{y} \rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad 1\text{–}1(\mathsf{f}) \;\&\; Stat2: \neg\langle \forall \mathsf{x} \in \mathbf{domain}(\mathsf{f}), \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{f}\upharpoonright\mathsf{x} = \mathsf{f}\upharpoonright\mathsf{y} \rightarrow \mathsf{x} = \mathsf{y} \rangle$

-- But an elementary contradiction with the definition of 1-1 map follows easily in this case also, so our theorem is proved.

$\langle \mathsf{u}, \mathsf{v} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{u}, \mathsf{v} \in \mathbf{domain}(\mathsf{f}) \;\&\; \mathsf{f}\upharpoonright\mathsf{u} = \mathsf{f}\upharpoonright\mathsf{v} \;\&\; \mathsf{u} \neq \mathsf{v}$
Suppose $\Rightarrow \quad [\mathsf{u}, \mathsf{f}\upharpoonright\mathsf{u}] \notin \mathsf{f}$
ELEM $\Rightarrow \quad Stat3: [\mathsf{u}, \mathsf{f}\upharpoonright\mathsf{u}] \notin \{[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$
$\langle \mathsf{u} \rangle \hookrightarrow Stat3 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad [\mathsf{u}, \mathsf{f}\upharpoonright\mathsf{u}] \in \mathsf{f}$
Suppose $\Rightarrow \quad [\mathsf{v}, \mathsf{f}\upharpoonright\mathsf{v}] \notin \mathsf{f}$
ELEM $\Rightarrow \quad Stat4: [\mathsf{v}, \mathsf{f}\upharpoonright\mathsf{v}] \notin \{[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$
$\langle \mathsf{v} \rangle \hookrightarrow Stat4 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad [\mathsf{v}, \mathsf{f}\upharpoonright\mathsf{v}] \in \mathsf{f}$
Use_def$(1\text{–}1) \Rightarrow \quad Stat5: \langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$
$\langle [\mathsf{u}, \mathsf{f}\upharpoonright\mathsf{u}], [\mathsf{v}, \mathsf{f}\upharpoonright\mathsf{v}] \rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad$ QED

-- Next we show that a 1-1 map on a set u induces a 1-1 map on the power set of u.

-- A 1 - 1 map on a set u induces a 1 - 1 map on the power set of u

Theorem 138 (110) $1\text{–}1(\mathsf{F}) \;\&\; \mathsf{S} \subseteq \mathbf{domain}(\mathsf{F}) \;\&\; T \subseteq \mathbf{domain}(\mathsf{F}) \;\&\; \mathsf{S} \neq T \rightarrow \mathbf{range}(\mathsf{F}_{|\mathsf{S}}) \neq \mathbf{range}(\mathsf{F}_{|T})$. PROOF:

Suppose_not$(\mathsf{f}, \mathsf{s}, \mathsf{t}) \Rightarrow \quad 1\text{–}1(\mathsf{f}) \;\&\; \mathsf{s} \subseteq \mathbf{domain}(\mathsf{f}) \;\&\; \mathsf{t} \subseteq \mathbf{domain}(\mathsf{f}) \;\&\; Stat1: \mathsf{s} \neq \mathsf{t} \;\&\; \mathbf{range}(\mathsf{f}_{|\mathsf{s}}) = \mathbf{range}(\mathsf{f}_{|\mathsf{t}})$
$\langle \mathsf{c} \rangle \hookrightarrow Stat1 \Rightarrow \quad (\mathsf{c} \in \mathsf{s} \;\&\; \mathsf{c} \notin \mathsf{t}) \vee (\mathsf{c} \notin \mathsf{s} \;\&\; \mathsf{c} \in \mathsf{t})$

-- For let f be 1-1, and suppose that there are distinct subsets s and t of its domain such that $\mathbf{range}(\mathsf{f}_{|\mathsf{s}}) = \mathbf{range}(\mathsf{f}_{|\mathsf{t}})$. Then there is an element c of $\mathbf{domain}(\mathsf{f})$ which is in one of s and t but not the other. Using the definitions of the functions involved and simplifying, we can rewrite the equality $\mathbf{range}(\mathsf{f}_{|\mathsf{s}}) = \mathbf{range}(\mathsf{f}_{|\mathsf{t}})$ as follows:

Use_def$(1\text{–}1) \Rightarrow \quad \mathsf{Svm}(\mathsf{f}) \;\&\; Stat2: \langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{f} \rangle \hookrightarrow T65 \Rightarrow \quad Stat3: \mathsf{f} = \{[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$

Use_def(**range**) $\Rightarrow$ *Stat4*: $\{x^{[2]} : x \in f_{|s}\} = \{x^{[2]} : x \in f_{|t}\}$
Use_def(|) $\Rightarrow$ $\{x^{[2]} : x \in \{x \in f \mid x^{[1]} \in s\}\} = \{x^{[2]} : x \in \{x \in f \mid x^{[1]} \in t\}\}$
SIMPLF $\Rightarrow$ $\{x^{[2]} : x \in f \mid x^{[1]} \in s\} = \{x^{[2]} : x \in f \mid x^{[1]} \in t\}$
EQUAL $\Rightarrow$ $\{x^{[2]} : x \in \{[x, f\restriction x] : x \in \mathbf{domain}(f)\} \mid x^{[1]} \in s\} = \{x^{[2]} : x \in \{[x, f\restriction x] : x \in \mathbf{domain}(f)\} \mid x^{[1]} \in t\}$
SIMPLF $\Rightarrow$ $\left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in s\right\} =$
$\left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in t\right\}$

-- Suppose for definiteness sake that $c \in s$, $c \notin t$. Since $f\restriction c$ must be in $\mathbf{range}(f_{|s}) = \mathbf{range}(f_{|t})$, it must have the form $f\restriction d$ where $d \in t$ and so $d \neq c$.

Suppose $\Rightarrow$ *Stat12*: $[c, f\restriction c] \notin \{[x, f\restriction x] : x \in \mathbf{domain}(f)\}$
$\langle c \rangle \hookrightarrow$ *Stat12* $\Rightarrow$ false; Discharge $\Rightarrow$ $[c, f\restriction c] \in f$
Suppose $\Rightarrow$ $c \in s \;\&\; c \notin t$
Suppose $\Rightarrow$ *Stat5*: $f\restriction c \notin \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in s\right\}$
$\langle c \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $f\restriction c \neq [c, f\restriction c]^{[2]} \vee c \notin \mathbf{domain}(f) \vee [c, f\restriction c]^{[1]} \notin s$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6*: $f\restriction c \in \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in t\right\}$
$\langle d \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $f\restriction c = [d, f\restriction d]^{[2]} \;\&\; d \in \mathbf{domain}(f) \;\&\; [d, f\restriction d]^{[1]} \in t$
ELEM $\Rightarrow$ $f\restriction c = f\restriction d \;\&\; d \in \mathbf{domain}(f) \;\&\; d \in t$
ELEM $\Rightarrow$ $c \neq d$

-- But since $f\restriction c = f\restriction d$, this contradicts the fact that f is 1-1, and so we must have $c \notin s \;\&\; c \in t$.

Suppose $\Rightarrow$ *Stat11*: $[d, f\restriction d] \notin \{[x, f\restriction x] : x \in \mathbf{domain}(f)\}$
$\langle d \rangle \hookrightarrow$ *Stat11* $\Rightarrow$ false; Discharge $\Rightarrow$ $[d, f\restriction d] \in f$
$\langle [c, f\restriction c], [d, f\restriction d] \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $[c, f\restriction c]^{[2]} = [d, f\restriction d]^{[2]} \rightarrow$
$[c, f\restriction c] = [d, f\restriction d]$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $c \notin s \;\&\; c \in t$

-- However, the case $c \notin s$, $c \in t$ leads to an exactly similar contradiction, thus completing our proof.

Suppose $\Rightarrow$ *Stat7*: $f\restriction c \notin \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in t\right\}$
$\langle c \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $f\restriction c \neq [c, f\restriction c]^{[2]} \vee c \notin \mathbf{domain}(f) \vee [c, f\restriction c]^{[1]} \notin t$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat8*: $f\restriction c \in \left\{[x, f\restriction x]^{[2]} : x \in \mathbf{domain}(f) \mid [x, f\restriction x]^{[1]} \in s\right\}$
$\langle dd \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ *Stat9*: $f\restriction c = [dd, f\restriction dd]^{[2]} \;\&\; dd \in \mathbf{domain}(f) \;\&\; [dd, f\restriction dd]^{[1]} \in s$
$\langle$ *Stat9* $\rangle$ ELEM $\Rightarrow$ $f\restriction c = f\restriction dd \;\&\; dd \in \mathbf{domain}(f) \;\&\; dd \in s$

ELEM $\Rightarrow$ $\mathsf{c} \neq \mathsf{dd}$

Suppose $\Rightarrow$ *Stat10* : $[\mathsf{dd}, \mathsf{f}\restriction\mathsf{dd}] \notin \{[\mathsf{x}, \mathsf{f}\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$

$\langle \mathsf{dd} \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ false; Discharge $\Rightarrow$ $[\mathsf{dd}, \mathsf{f}\restriction\mathsf{dd}] \in \mathsf{f}$

$\langle [\mathsf{c}, \mathsf{f}\restriction\mathsf{c}], [\mathsf{dd}, \mathsf{f}\restriction\mathsf{dd}] \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $[\mathsf{c}, \mathsf{f}\restriction\mathsf{c}]^{[2]} = [\mathsf{dd}, \mathsf{f}\restriction\mathsf{dd}]^{[2]} \rightarrow$

$[\mathsf{c}, \mathsf{f}\restriction\mathsf{c}] = [\mathsf{dd}, \mathsf{f}\restriction\mathsf{dd}]$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The two following results, both elementary consequences by set monotonicity of the definitions of the functions involved, show that map composition is distributive over map union, both on the left and the right.

Theorem 139 (111) $(\mathsf{F} \cup \mathsf{FF}) \bullet \mathsf{G} = \mathsf{F} \bullet \mathsf{G} \cup \mathsf{FF} \bullet \mathsf{G}$. PROOF:

Suppose_not(f, ff, g) $\Rightarrow$ $(\mathsf{f} \cup \mathsf{ff}) \bullet \mathsf{g} \neq \mathsf{f} \bullet \mathsf{g} \cup \mathsf{ff} \bullet \mathsf{g}$

Use_def($\bullet$) $\Rightarrow$

$$\{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{f} \cup \mathsf{ff} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \neq \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \cup \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{ff} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\}$$

Set_monot $\Rightarrow$

$$\{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{f} \cup \mathsf{ff} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} = \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \cup \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{ff} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\}$$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 140 (112) $\mathsf{G} \bullet (\mathsf{F} \cup \mathsf{FF}) = \mathsf{G} \bullet \mathsf{F} \cup \mathsf{G} \bullet \mathsf{FF}$. PROOF:

Suppose_not(g, f, ff) $\Rightarrow$ $\mathsf{g} \bullet (\mathsf{f} \cup \mathsf{ff}) \neq \mathsf{g} \bullet \mathsf{f} \cup \mathsf{g} \bullet \mathsf{ff}$

Use_def($\bullet$) $\Rightarrow$

$$\{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{f} \cup \mathsf{ff}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \neq \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \cup \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{ff}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\}$$

Set_monot $\Rightarrow$

$$\{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{f} \cup \mathsf{ff}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} = \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\} \cup \{[\mathsf{x}^{[1]}, \mathsf{y}^{[2]}] : \mathsf{x} \in \mathsf{ff}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[1]}\}$$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The theorem that now follows tells us that a 1-1 partial inverse can be defined for any single-valued map.

-- Single - valued maps have 1 - 1 partial inverses

Theorem 141 (113) $\mathsf{Svm}(\mathsf{F}) \rightarrow \langle \exists \mathsf{h} \,|\, (\mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{F})\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{F})\ \&\ 1\text{–}1(\mathsf{h}))\ \&\ \langle \forall \mathsf{x} \in \mathbf{range}(\mathsf{F}) \,|\, \mathsf{F}\!\upharpoonright\!(\mathsf{h}\!\upharpoonright\!\mathsf{x}) = \mathsf{x}\rangle\rangle$. Proof:

$\mathsf{Suppose\_not}(\mathsf{f}) \Rightarrow\quad \mathsf{Svm}(\mathsf{f})\ \&\ \mathit{Stat1}:\ \neg\langle \exists \mathsf{h} \,|\, \mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f})\ \&\ 1\text{–}1(\mathsf{h})\ \&\ \langle \forall \mathsf{x} \in \mathbf{range}(\mathsf{f}) \,|\, \mathsf{f}\!\upharpoonright\!(\mathsf{h}\!\upharpoonright\!\mathsf{x}) = \mathsf{x}\rangle\rangle$

-- We will refute the contrary supposition by giving the following explicit definition of the partial inverse whose existence is asserted.

$\mathsf{Loc\_def} \Rightarrow\quad \mathsf{h} = \{[\mathsf{x}, \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\})] : \mathsf{x} \in \mathbf{range}(\mathsf{f})\}$

-- The 'fcn_symbol' theory tells us immediately that this h is a single-valued map with domain equal to $\mathbf{range}(\mathsf{f})$. Thus we have only to consider the three last clauses of our theorem.

$\mathsf{APPLY}\ \langle \mathsf{x}_\Theta : \mathsf{x}_2, \mathsf{y}_\Theta : \mathsf{y}_2\rangle\ \mathsf{fcn\_symbol}\left(\mathsf{f}(\mathsf{x}) \mapsto \mathbf{arb}\left(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\}\right), \mathsf{g} \mapsto \mathsf{h}, \mathsf{s} \mapsto \mathbf{range}(\mathsf{f})\right) \Rightarrow$

$\mathit{Stat2}:\ \mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{f})\ \&\ \mathsf{Svm}(\mathsf{h})\ \&\ \mathit{Stat3}:\ \langle \forall \mathsf{x} \,|\, \mathsf{h}\!\upharpoonright\!\mathsf{x} = \textbf{if } \mathsf{x} \in \mathbf{range}(\mathsf{f}) \textbf{ then } \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\}) \textbf{ else } \emptyset \textbf{ fi}\rangle\ \&\ \mathbf{range}(\mathsf{h}) = \{\mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\}) :$

-- We first show that $\langle \forall \mathsf{x} \in \mathbf{range}(\mathsf{f}) \,|\, \mathsf{h}\!\upharpoonright\!\mathsf{x} \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\}\rangle$, from which it will follow easily that $\mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f})$. Indeed, if we suppose the existence of an $\mathsf{x} \in \mathbf{range}(\mathsf{f})$ for which $\mathsf{h}\!\upharpoonright\!\mathsf{x} \notin \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\}$, use of the axiom of choice leads to an immediate contradiction.

$\mathsf{Suppose} \Rightarrow\quad \mathit{Stat4}:\ \neg\langle \forall \mathsf{v} \in \mathbf{range}(\mathsf{f}) \,|\, \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}) \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}\rangle$
$\langle \mathsf{v}\rangle \hookrightarrow \mathit{Stat4} \Rightarrow\quad \mathsf{v} \in \mathbf{range}(\mathsf{f})\ \&\ \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}) \notin \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}$
$\langle \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}\rangle \hookrightarrow \mathit{T0} \Rightarrow\quad \mathit{Stat5}:\ \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\} = \emptyset$
$\mathsf{Use\_def}(\mathbf{range}) \Rightarrow\quad \mathit{Stat6}:\ \mathsf{v} \in \{\mathsf{x}^{[2]} : \mathsf{x} \in \mathsf{f}\}$
$\langle \mathsf{vv}\rangle \hookrightarrow \mathit{Stat6} \Rightarrow\quad \mathsf{v} = \mathsf{vv}^{[2]}\ \&\ \mathsf{vv} \in \mathsf{f}$
$\langle \mathsf{vv}\rangle \hookrightarrow \mathit{Stat5} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad \mathit{Stat7}:\ \langle \forall \mathsf{v} \in \mathbf{range}(\mathsf{f}) \,|\, \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}) \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{v}\}\rangle$

-- Next we show (by contradiction) that $\mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f})$.

$\mathsf{Suppose} \Rightarrow\quad \mathit{Stat8}:\ \mathbf{range}(\mathsf{h}) \not\subseteq \mathbf{domain}(\mathsf{f})$
$\langle \mathsf{w}\rangle \hookrightarrow \mathit{Stat8} \Rightarrow\quad \mathit{Stat9}:\ \mathsf{w} \in \mathbf{range}(\mathsf{h})\ \&\ \mathsf{w} \notin \mathbf{domain}(\mathsf{f})$
$\langle \mathit{Stat2}, \mathit{Stat9}\rangle\ \mathsf{ELEM} \Rightarrow\quad \mathit{Stat10}:\ \mathsf{w} \in \{\mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{x}\}) : \mathsf{x} \in \mathbf{range}(\mathsf{f})\}$
$\langle \mathsf{ww}\rangle \hookrightarrow \mathit{Stat10} \Rightarrow\quad \mathsf{w} = \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{ww}\})\ \&\ \mathsf{ww} \in \mathbf{range}(\mathsf{f})$
$\langle \mathsf{ww}\rangle \hookrightarrow \mathit{Stat7} \Rightarrow\quad \mathsf{w} \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{ww}\}$
$\mathsf{Use\_def}(\mathbf{domain}) \Rightarrow\quad \mathsf{w} \notin \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f}\}$
$\mathsf{Set\_monot} \Rightarrow\quad \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f}\} \supseteq \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \,|\, \mathsf{u}^{[2]} = \mathsf{ww}\}$
$\mathsf{ELEM} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f})$

-- The fact that h is 1-1 also follows readily, since by repeated use of Stat11 10 its negative would imply the existence of u_1 and u_2 in f with $\mathsf{u_1}^{[1]} = \mathsf{u_2}^{[1]}$ but $\mathsf{u_1}^{[2]} \neq \mathsf{u_2}^{[2]}$, contradicting the single-valuedness of f.

Suppose $\Rightarrow$ *Stat12*: $\neg 1\text{–}1(\mathsf{h})$
ELEM $\Rightarrow$ *Stat13*: $\mathsf{x}_2 \in \mathbf{range}(\mathsf{f})\ \&\ (\mathsf{y}_2 \in \mathbf{range}(\mathsf{f})\ \&\ \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{x}_2\}) = \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{y}_2\}))\ \&\ \mathsf{x}_2 \neq \mathsf{y}_2$
$\langle \mathsf{x}_2 \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ *Stat14*: $\mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{x}_2\}) \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{x}_2\}$
$\langle \mathsf{y}_2 \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ *Stat15*: $\mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{y}_2\}) \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{y}_2\}$
$\langle \mathsf{u}_1 \rangle \hookrightarrow$ *Stat14* $\Rightarrow$ *Stat16*: $\mathsf{u}_1 \in \mathsf{f}\ \&\ \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{x}_2\}) = \mathsf{u_1}^{[1]}\ \&\ \mathsf{u_1}^{[2]} = \mathsf{x}_2$
$\langle \mathsf{u}_2 \rangle \hookrightarrow$ *Stat15* $\Rightarrow$ *Stat17*: $\mathsf{u}_2 \in \mathsf{f}\ \&\ \mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{y}_2\}) = \mathsf{u_2}^{[1]}\ \&\ \mathsf{u_2}^{[2]} = \mathsf{y}_2$
$\langle$*Stat13, Stat16, Stat17*$\rangle$ ELEM $\Rightarrow$ $\mathsf{u}_1, \mathsf{u}_2 \in \mathsf{f}\ \&\ \mathsf{u_1}^{[1]} = \mathsf{u_2}^{[1]}\ \&\ \mathsf{u_1}^{[2]} \neq \mathsf{u_2}^{[2]}$
$\langle \mathsf{f}, \mathsf{u}_1 \rangle \hookrightarrow$ *T67* $\Rightarrow$ $\mathsf{f} \upharpoonright \mathsf{u_1}^{[1]} = \mathsf{u_1}^{[2]}$
$\langle \mathsf{f}, \mathsf{u}_2 \rangle \hookrightarrow$ *T67* $\Rightarrow$ $\mathsf{f} \upharpoonright \mathsf{u_2}^{[1]} = \mathsf{u_2}^{[2]}$
EQUAL $\Rightarrow$ $\mathsf{f} \upharpoonright \mathsf{u_1}^{[1]} = \mathsf{f} \upharpoonright \mathsf{u_2}^{[1]}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $1\text{–}1(\mathsf{h})$

-- It only remains to show that $\mathsf{f} \upharpoonright (\mathsf{h} \upharpoonright \mathsf{x}) = \mathsf{x}$ for all $\mathsf{x} \in \mathbf{range}(\mathsf{f})$. but it is easily seen that any counterexample t to this assertion would have to satisfy $\mathsf{h} \upharpoonright \mathsf{t} = \mathsf{tt}^{[1]}$ where $\mathsf{tt} \in \mathsf{f}$ and $\mathsf{tt}^{[2]} = \mathsf{t}$, and hence $\mathsf{f} \upharpoonright \mathsf{tt}^{[1]} = \mathsf{t}$, contradicting $\mathsf{f} \upharpoonright (\mathsf{h} \upharpoonright \mathsf{t}) \neq \mathsf{t}$, and so completing the proof of the present theorem.

Suppose $\Rightarrow$ *Stat18*: $\neg \langle \forall \mathsf{x} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{f} \upharpoonright (\mathsf{h} \upharpoonright \mathsf{x}) = \mathsf{x} \rangle$
$\langle \mathsf{t} \rangle \hookrightarrow$ *Stat18* $\Rightarrow$ $\mathsf{t} \in \mathbf{range}(\mathsf{f})\ \&\ \mathsf{f} \upharpoonright (\mathsf{h} \upharpoonright \mathsf{t}) \neq \mathsf{t}$
$\langle \mathsf{t} \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $\mathbf{arb}(\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{t}\}) \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{t}\}$
$\langle \mathsf{t} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat19*: $\mathsf{h} \upharpoonright \mathsf{t} \in \{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{f} \mid \mathsf{u}^{[2]} = \mathsf{t}\}$
$\langle \mathsf{tt} \rangle \hookrightarrow$ *Stat19* $\Rightarrow$ $\mathsf{h} \upharpoonright \mathsf{t} = \mathsf{tt}^{[1]}\ \&\ \mathsf{tt} \in \mathsf{f}\ \&\ \mathsf{tt}^{[2]} = \mathsf{t}$
$\langle \mathsf{f}, \mathsf{tt} \rangle \hookrightarrow$ *T67* $\Rightarrow$ $\mathsf{f} \upharpoonright \mathsf{tt}^{[1]} = \mathsf{t}$
EQUAL $\Rightarrow$ $\mathsf{f} \upharpoonright (\mathsf{h} \upharpoonright \mathsf{t}) = \mathsf{t}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The definition of the very useful Cartesian product operator is as follows:

-- Cartesian Product
DEF 17. $\mathsf{X} \times \mathsf{Y} \quad =_{\text{Def}} \quad \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathsf{X}, \mathsf{y} \in \mathsf{Y}\}$

-- We begin our discussion of this important operator by proving the elementary fact that a Cartesian product is empty if either of its factors is empty. The equally easy converse of this result will be proved later.

Theorem 142 (114) $\mathsf{N} \times \emptyset = \emptyset\ \&\ \emptyset \times \mathsf{N} = \emptyset$. PROOF:

$\text{Suppose\_not}(n) \Rightarrow \quad n \times \emptyset \neq \emptyset \vee \emptyset \times n \neq \emptyset$

-- For supposing the negative of either of our conclusions leads immediately to an elementary contradiction:

$\text{Use\_def}(\times) \Rightarrow \quad \{[x,y] : x \in n, y \in \emptyset\} \neq \emptyset \vee \{[x,y] : x \in \emptyset, y \in n\} \neq \emptyset$
$\text{Suppose} \Rightarrow \quad \mathit{Stat1} : \{[x,y] : x \in n, y \in \emptyset\} \neq \emptyset$
$\langle c, d \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad c \in n \,\&\, d \in \emptyset$
$\text{ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \mathit{Stat2} : \{[x,y] : x \in \emptyset, y \in n\} \neq \emptyset$
$\langle d' \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathit{Stat3} : d' \in \{[x,y] : x \in \emptyset, y \in n\}$
$\langle x_1, y_1 \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad d' = [x_1, y_1] \,\&\, y_1 \in n \,\&\, x_1 \in \emptyset$
$\text{ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- = = = = = = = = = = = = = = = = = = Additional laws concerning Cartesian product = It is easy to show that the first component of any member of the Cartesian product $s \times t$ belongs to s, and its second component to t:

Theorem 143 (115) $X \in S \times T \rightarrow X^{[1]} \in S \,\&\, X^{[2]} \in T$. PROOF:

-- This follows trivially from the very definition of Cartesian product.

$\text{Suppose\_not}(x, s, t) \Rightarrow \quad \mathit{Stat1} : x \in s \times t \,\&\, \neg(x^{[1]} \in s \,\&\, x^{[2]} \in t)$
$\text{Use\_def}(\times) \Rightarrow \quad \mathit{Stat2} : x \in \{[u,v] : u \in s, v \in t\}$
$\langle a, b \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathit{Stat3} : a \in s \,\&\, b \in t \,\&\, x = [a,b]$
$\langle \mathit{Stat1}, \mathit{Stat3} \rangle \text{ ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- Next we show that every subset of a Cartesian $S \times T$ product is a map whose domain and range are included in S and T, respectively:

Theorem 144 (116) $Y \subseteq S \times T \leftrightarrow \text{Is\_map}(Y) \,\&\, \mathbf{domain}(Y) \subseteq S \,\&\, \mathbf{range}(Y) \subseteq T$. PROOF:

-- This follows trivially from the fact that a map is a set each of whose elements x is a pair $[x^{[1]}, x^{[2]}]$

$\text{Suppose\_not}(y, s, t) \Rightarrow \quad \big(y \subseteq s \times t \,\&\, \neg\text{Is\_map}(y) \vee \mathbf{domain}(y) \not\subseteq s \vee \mathbf{range}(y) \not\subseteq t\big) \vee \big(y \not\subseteq s \times t \,\&\, \text{Is\_map}(y) \,\&\, \mathbf{domain}(y) \subseteq s \,\&\, \mathbf{range}(y) \subseteq t\big)$
$\text{Suppose} \Rightarrow \quad \mathit{Stat1} : y \subseteq s \times t \,\&\, \neg\text{Is\_map}(y) \vee \mathbf{domain}(y) \not\subseteq s \vee \mathbf{range}(y) \not\subseteq t$
$\text{Use\_def}(\times) \Rightarrow \quad \mathit{Stat2} : y \subseteq \{[u,v] : u \in s, v \in t\}$
$\text{Suppose} \Rightarrow \quad \neg\text{Is\_map}(y)$
$\langle y \rangle \hookrightarrow T46 \Rightarrow \quad \mathit{Stat3} : \neg\langle \forall x \in y \,|\, x = [x^{[1]}, x^{[2]}] \rangle$

$\langle c \rangle \hookrightarrow Stat3 \Rightarrow \quad Stat4: \ c \in y \ \& \ c \neq [c^{[1]}, c^{[2]}]$
$\langle Stat2, Stat4 \rangle$ ELEM $\Rightarrow \quad Stat5: \ c \in \{[u,v] : \ u \in s, v \in t\}$
$\langle d, e \rangle \hookrightarrow Stat5 \Rightarrow \quad Stat6: \ c = [d, e]$
$\langle Stat4, Stat6 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad \mathbf{domain}(y) \not\subseteq s \vee \mathbf{range}(y) \not\subseteq t$
Suppose $\Rightarrow \quad Stat7: \ \mathbf{range}(y) \not\subseteq t$
$\langle e' \rangle \hookrightarrow Stat7 \Rightarrow \quad e' \in \mathbf{range}(y) \ \& \ e' \notin t$
Use_def $(\mathbf{range}) \Rightarrow \quad Stat8: \ e' \in \{w^{[2]} : \ w \in y\}$
$\langle w \rangle \hookrightarrow Stat8 \Rightarrow \quad e' = w^{[2]} \ \& \ Stat9: \ w \in \{[u,v] : \ u \in s, v \in t\}$
$\langle u, v \rangle \hookrightarrow Stat9 \Rightarrow \quad w = [u, v] \ \& \ v \in t$
$\langle Stat7 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad Stat17: \ \mathbf{domain}(y) \not\subseteq s$
$\langle d' \rangle \hookrightarrow Stat17 \Rightarrow \quad d' \in \mathbf{domain}(y) \ \& \ d' \notin s$
Use_def $(\mathbf{domain}) \Rightarrow \quad Stat18: \ d' \in \{z^{[1]} : \ z \in y\}$
$\langle z \rangle \hookrightarrow Stat18 \Rightarrow \quad d' = z^{[1]} \ \& \ Stat19: \ z \in \{[u,v] : \ u \in s, v \in t\}$
$\langle u', v' \rangle \hookrightarrow Stat19 \Rightarrow \quad z = [u', v'] \ \& \ u' \in s$
$\langle Stat17 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad \mathsf{Is\_map}(y) \ \& \ \mathbf{domain}(y) \subseteq s \ \& \ \mathbf{range}(y) \subseteq t \ \& \ Stat21: \ y \not\subseteq s \times t$
$\langle c' \rangle \hookrightarrow Stat21 \Rightarrow \quad c' \in y \ \& \ c' \notin s \times t$
Use_def $(\times) \Rightarrow \quad Stat22: \ c' \notin \{[u,v] : \ u \in s, v \in t\}$
Use_def $(\mathsf{Is\_map}) \Rightarrow \quad Stat23: \ c' \in \{[w^{[1]}, w^{[2]}] : \ w \in y\}$
$\langle w' \rangle \hookrightarrow Stat23 \Rightarrow \quad c' = [w'^{[1]}, w'^{[2]}] \ \& \ w' \in y$
Use_def $(\mathbf{domain}) \Rightarrow \quad \mathbf{domain}(y) = \{w^{[1]} : \ w \in y\}$
Suppose $\Rightarrow \quad Stat24: \ w'^{[1]} \notin \{w^{[1]} : \ w \in y\}$
$\langle w' \rangle \hookrightarrow Stat24 \Rightarrow$ false; Discharge $\Rightarrow \quad w'^{[1]} \in s$
Use_def $(\mathbf{range}) \Rightarrow \quad \mathbf{range}(y) = \{w^{[2]} : \ w \in y\}$
Suppose $\Rightarrow \quad Stat25: \ w'^{[2]} \notin \{w^{[2]} : \ w \in y\}$
$\langle w' \rangle \hookrightarrow Stat25 \Rightarrow$ false; Discharge $\Rightarrow \quad w'^{[2]} \in t$
$\langle w'^{[1]}, w'^{[2]} \rangle \hookrightarrow Stat22 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- = Our next result states another elementary property of the Cartesian product: two such products are disjoint if their second factors are disjoint. (The reader will readily perceive that the same is true for two Cartesian products whose first factors are disjoint).

Theorem 145 (117) $A \cap B = \emptyset \rightarrow X \times A \cap (Y \times B) = \emptyset$. PROOF:

Suppose_not $(a, b, xx, yy) \Rightarrow \quad a \cap b = \emptyset \ \& \ Stat1: \ xx \times a \cap (yy \times b) \neq \emptyset$
$\langle c \rangle \hookrightarrow Stat1 \Rightarrow \quad c \in xx \times a \ \& \ c \in yy \times b$

Use_def($\times$) $\Rightarrow$ *Stat2*: $c \in \{[u,v] : u \in xx, v \in a\}$ & $c \in \{[u,v] : u \in yy, v \in b\}$
$\langle a_1, b_1, a_2, b_2 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $c = [a_1, b_1]$ & $a_1 \in xx$ & $b_1 \in a$ & $c = [a_2, b_2]$ & $a_2 \in yy$ & $b_2 \in b$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem shows that even though the Cartesian product operator is not associative, there always exists a natural 1-1 correspondence between $A \times B \times C$ and $A \times (B \times C)$.

Theorem 146 (118) $F = \{[[[x,y],z],[x,[y,z]]] : x \in A, y \in B, z \in C\} \rightarrow$ $1\text{–}1(F)$ & $\mathbf{domain}(F) = A \times B \times C$ & $\mathbf{range}(F) = A \times (B \times C)$. PROOF:

Suppose_not(f, a, b, c) $\Rightarrow$ $f = \{[[[x,y],z],[x,[y,z]]] : x \in a, y \in b, z \in c\}$ & $\neg 1\text{–}1(f) \vee \mathbf{domain}(f) \neq a \times b \times c \vee \mathbf{range}(f) \neq a \times (b \times c)$

-- For suppose the contrary. Since we can apply one_1_test_3 to show that f must be 1-1, only the theorem clauses concerning the range and domain of f can be false.

Suppose $\Rightarrow$ $\neg 1\text{–}1(f)$
EQUAL $\Rightarrow$ $\neg 1\text{–}1(\{[[[x,y],z],[x,[y,z]]] : x \in a, y \in b, z \in c \mid \mathsf{true}\})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $1\text{–}1(f)$

-- Next suppose that the clause concerning $\mathbf{domain}(f)$ is false. Using the definitions of the operators involved we find that there must exist elements x, y, u satisfying the impossible inequality seen below, a contradiction leaving only the case $\mathbf{range}(f) \neq a \times (b \times c)$ to be considered.

Suppose $\Rightarrow$ $\mathbf{domain}(f) \neq a \times b \times c$
Use_def(**domain**) $\Rightarrow$ $\{x^{[1]} : x \in f\} \neq a \times b \times c$
EQUAL $\Rightarrow$ $\{x^{[1]} : x \in \{[[[x,y],z],[x,[y,z]]] : x \in a, y \in b, z \in c\}\} \neq a \times b \times c$
SIMPLF $\Rightarrow$ $\left\{[[[x,y],z],[x,[y,z]]]^{[1]} : x \in a, y \in b, z \in c\right\} \neq a \times b \times c$
Use_def($\times$) $\Rightarrow$
$\left\{[[[x,y],z],[x,[y,z]]]^{[1]} : x \in a, y \in b, z \in c\right\} \neq$ $\{[u,z] : u \in \{[x,y] : x \in a, y \in b\}, z \in c\}$
SIMPLF $\Rightarrow$ *Stat2*: $\left\{[[[x,y],z],[x,[y,z]]]^{[1]} : x \in a, y \in b, z \in c\right\} \neq$ $\{[[x,y],z] : x \in a, y \in b, z \in c\}$
$\langle x, y, u \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3*: $[[[x,y],u],[x,[y,u]]]^{[1]} \neq [[x,y],u]$
$\langle$ *Stat3* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{range}(f) \neq a \times (b \times c)$

-- Expanding this last case using the definitions of the operators involved we see just as easily that it leads to an impossible elementary inequality, a final contradiction which completes the proof of our theorem.

Use_def(**range**) ⇒ $\{x^{[2]} : x \in f\} \neq a \times (b \times c)$
EQUAL ⇒ $\{x^{[2]} : x \in \{[[[x,y],z],[x,[y,z]]] : x \in a, y \in b, z \in c\}\} \neq a \times (b \times c)$
SIMPLF ⇒ $\left\{[[[x,y],z],[x,[y,z]]]^{[2]} : x \in a, y \in b, z \in c\right\} \neq a \times (b \times c)$
Use_def(×) ⇒
$\left\{[[[x,y],z],[x,[y,z]]]^{[2]} : x \in a, y \in b, z \in c\right\} \neq$
$\{[x,v] : x \in a, v \in \{[y,z] : y \in b, z \in c\}\}$
SIMPLF ⇒ *Stat4* : $\left\{[[[x,y],z],[x,[y,z]]]^{[2]} : x \in a, y \in b, z \in c\right\} \neq$
$\{[x,[y,z]] : x \in a, y \in b, z \in c\}$
$\langle xx, yy, uu\rangle \hookrightarrow$ *Stat4* ⇒ *Stat5* : $[[[xx,yy],uu],[xx,[yy,uu]]]^{[2]} \neq [xx,[yy,uu]]$
$\langle$*Stat5*$\rangle$ ELEM ⇒ false; Discharge ⇒ QED

-- Moreover, even though the Cartesian product operator is not commutative, there always exists a natural 1-1 correspondence between $A \times B$ and $B \times A$.

Theorem 147 (119) $F = \{[[x,y],[y,x]] : x \in A, y \in B\} \to 1\text{–}1(F)\ \&\ \mathbf{domain}(F) = A \times B\ \&\ \mathbf{range}(F) = B \times A$. PROOF:

Suppose_not(f, a, b) ⇒ $f = \{[[x,y],[y,x]] : x \in a, y \in b\}\ \&\ \neg 1\text{–}1(f) \lor \mathbf{domain}(f) \neq a \times b \lor \mathbf{range}(f) \neq b \times a$

-- For suppose the contrary. Since we can apply one_1_test$_2$ to show that f must be 1-1, only the theorem clauses concerning the range and domain of f can be false.

Suppose ⇒ $\neg 1\text{–}1(f)$
APPLY $\langle x_\Theta : x, y_\Theta : y, x2_\Theta : xx, y2_\Theta : yy\rangle$ one_1_test$_2$$(a(x,y) \mapsto [x,y], b(x,y) \mapsto [y,x], s \mapsto a, t \mapsto b)$ ⇒
$\neg([x,y] = [xx,yy] \leftrightarrow [y,x] = [yy,xx]) \lor 1\text{–}1(\{[[x,y],[y,x]] : x \in a, y \in b\})$
EQUAL ⇒ *Stat1* : $\neg([x,y] = [xx,yy] \leftrightarrow [y,x] = [yy,xx])$
$\langle$*Stat1*$\rangle$ ELEM ⇒ false; Discharge ⇒ $1\text{–}1(f)$

-- Next suppose that the clause concerning **domain**(f) is false. Using the definitions of the operators involved we find that there must exist elements x, y, u satisfying the impossible inequality seen below, a contradiction leaving only the case $\mathbf{range}(f) \neq a \times (b \times c)$ to be considered.

Suppose ⇒ $\mathbf{domain}(f) \neq a \times b$
Use_def(**domain**) ⇒ $\{x^{[1]} : x \in f\} \neq a \times b$
EQUAL ⇒ $\{u^{[1]} : u \in \{[[x,y],[y,x]] : x \in a, y \in b\}\} \neq a \times b$

SIMPLF $\Rightarrow$ $\left\{[[x,y],[y,x]]^{[1]} : x \in a, y \in b\right\} \neq a \times b$

Use_def($\times$) $\Rightarrow$ *Stat2*: $\left\{[[x,y],[y,x]]^{[1]} : x \in a, y \in b\right\} \neq \{[x,y] : x \in a, y \in b\}$

$\langle x', y' \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3*: $[[x',y'],[y',x']]^{[1]} \neq [x',y']$

$\langle$*Stat3*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{range}(f) \neq b \times a$

-- Expanding this last case using the definitions of the operators involved, we see just as easily that it leads to an impossible elementary inequality, a final contradiction which completes the proof of our theorem.

Use_def(**range**) $\Rightarrow$ $\{x^{[2]} : x \in f\} \neq b \times a$

EQUAL $\Rightarrow$ $\{u^{[2]} : u \in \{[[x,y],[y,x]] : x \in a, y \in b\}\} \neq b \times a$

SIMPLF $\Rightarrow$ $\left\{[[x,y],[y,x]]^{[2]} : x \in a, y \in b\right\} \neq b \times a$

Use_def($\times$) $\Rightarrow$ $\left\{[[x,y],[y,x]]^{[2]} : x \in a, y \in b\right\} \neq \{[x,y] : x \in b, y \in a\}$

Suppose $\Rightarrow$ *Stat4*: $\{[x,y] : x \in b, y \in a\} \neq \{[y,x] : x \in a, y \in b\}$

$\langle c \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ *Stat5*:

$(c \in \{[x,y] : x \in b, y \in a\} \;\&\; c \notin \{[y,x] : x \in a, y \in b\}) \vee$

$c \notin \{[x,y] : x \in b, y \in a\} \;\&\; c \in \{[y,x] : x \in a, y \in b\}$

Suppose $\Rightarrow$ *Stat6*: $c \in \{[x,y] : x \in b, y \in a\} \;\&\; c \notin \{[y,x] : x \in a, y \in b\}$

$\langle x_1, y_1, y_1, x_1 \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ *Stat7*: $x_1 \in b \;\&\; y_1 \in a \;\&\; c = [x_1, y_1] \;\&\; \neg(y_1 \in a \;\&\; x_1 \in b \;\&\; c = [x_1, y_1])$

$\langle$*Stat7*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat8*: $c \in \{[y,x] : x \in a, y \in b\} \;\&\; c \notin \{[x,y] : x \in b, y \in a\}$

$\langle x_2, y_2, y_2, x_2 \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ *Stat9*: $x_2 \in a \;\&\; y_2 \in b \;\&\; c = [y_2, x_2] \;\&\; \neg(x_2 \in a \;\&\; y_2 \in b \;\&\; c = [y_2, x_2])$

$\langle$*Stat9*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat10*: $\left\{[[x,y],[y,x]]^{[2]} : x \in a, y \in b\right\} \neq \{[y,x] : x \in a, y \in b\}$

$\langle x_3, y_3 \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ *Stat11*: $[[x_3,y_3],[y_3,x_3]]^{[2]} \neq [y_3, x_3]$

$\langle$*Stat11*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The preceding preliminaries now being completed, we return to a more serious discussion of results on cardinality. Our first, preparatory result states that the restriction of the previously introduced enumerating function $\mathsf{enum}(x,s)$ to any ordinal s is the identity function on s.

Theorem 148 (120) $\mathcal{O}(S) \;\&\; X \in S \rightarrow \mathsf{enum}(X,S) = X$. PROOF:

Suppose_not(s,a) $\Rightarrow$ $\mathcal{O}(s) \;\&\; a \in s \;\&\; \mathsf{enum}(a,s) \neq a$

-- For, supposing the contrary, there would necessarily exist a minimal ordinal $b \in s$ such that $\mathsf{enum}(b,s) \neq b$.

APPLY $\langle \mathsf{mt}_\Theta : \mathsf{b}\rangle$ transfinite_induction$(\mathsf{n} \mapsto \mathsf{a}, \mathsf{P}(\mathsf{x}) \mapsto \mathsf{x} \in \mathsf{s} \,\&\, \mathsf{enum}(\mathsf{x},\mathsf{s}) \neq \mathsf{x}) \Rightarrow$

$Stat2: \langle \forall \mathsf{x} \mid (\mathsf{b} \in \mathsf{s} \,\&\, \mathsf{enum}(\mathsf{b},\mathsf{s}) \neq \mathsf{b}) \,\&\, \big(\mathsf{x} \in \mathsf{b} \rightarrow \neg(\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{enum}(\mathsf{x},\mathsf{s}) \neq \mathsf{x})\big)\rangle$

$\langle \mathsf{a}_0\rangle \hookrightarrow Stat2 \Rightarrow \quad Stat1: \mathsf{b} \in \mathsf{s} \,\&\, \mathsf{enum}(\mathsf{b},\mathsf{s}) \neq \mathsf{b}$

$\langle \mathsf{s},\mathsf{b}\rangle \hookrightarrow T12 \Rightarrow \quad Stat21: \mathsf{b} \subseteq \mathsf{s}$

-- ?? Use_def (enum) ⇒ Stat22: enum (b, s) = if (s incin {enum (y, s): y in b}) then s else arb (s-{enum (y, s): y in b}) end if

Use_def(enum) $\Rightarrow \quad \mathsf{enum}(\mathsf{b},\mathsf{s}) = \textbf{if } \mathsf{s} \subseteq \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \textbf{ then } \mathsf{s} \textbf{ else } \mathbf{arb}(\mathsf{s} \setminus \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\}) \textbf{ fi}$

ELEM $\Rightarrow \quad Stat22: \mathsf{enum}(\mathsf{b},\mathsf{s}) = \textbf{if } \mathsf{s} \subseteq \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \textbf{ then } \mathsf{s} \textbf{ else } \mathbf{arb}(\mathsf{s} \setminus \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\}) \textbf{ fi}$

-- But we can show that such a b must satisfy $\{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} = \mathsf{b}$. Indeed, supposing the contrary, there would exist a c which was in one of these sets but not the other. Suppose first that $\mathsf{c} \notin \mathsf{b}$, so that c must be of the form $\mathsf{enum}(\mathsf{d},\mathsf{s})$ where $\mathsf{d} \in \mathsf{b}$, and so $\mathsf{d} \in \mathsf{s}$ since the ordinal s includes each of its members.

Suppose $\Rightarrow \quad Stat3: \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \neq \mathsf{b}$

$\langle \mathsf{c}\rangle \hookrightarrow Stat3 \Rightarrow \quad (\mathsf{c} \in \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \,\&\, \mathsf{c} \notin \mathsf{b}) \vee (\mathsf{c} \notin \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \,\&\, \mathsf{c} \in \mathsf{b})$

Suppose $\Rightarrow \quad Stat4: \mathsf{c} \in \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \,\&\, \mathsf{c} \notin \mathsf{b}$

$\langle \mathsf{d}\rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{c} = \mathsf{enum}(\mathsf{d},\mathsf{s}) \,\&\, \mathsf{d} \in \mathsf{b}$

ELEM $\Rightarrow \quad \mathsf{d} \in \mathsf{s}$

-- However the minimality of b then implies that $\mathsf{enum}(\mathsf{d},\mathsf{s}) = \mathsf{d}$, and so $\mathsf{d} = \mathsf{c}$ which is impossible since $\mathsf{c} \notin \mathsf{b}$. This contradiction proves that we need only consider the second of our two original cases, that in which c belongs to b and $\mathsf{c} \neq \mathsf{enum}(\mathsf{c},\mathsf{s})$.

$\langle \mathsf{d}\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{enum}(\mathsf{d},\mathsf{s}) = \mathsf{d}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad Stat5: \mathsf{c} \notin \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} \,\&\, \mathsf{c} \in \mathsf{b}$

$\langle \mathsf{c}\rangle \hookrightarrow Stat5 \Rightarrow \quad \neg(\mathsf{c} \in \mathsf{b} \,\&\, \mathsf{c} = \mathsf{enum}(\mathsf{c},\mathsf{s}))$

ELEM $\Rightarrow \quad \mathsf{c} \neq \mathsf{enum}(\mathsf{c},\mathsf{s})$

-- However in this case Stat6 0 leads immediately to the contradiction $\mathsf{c} \notin \mathsf{s}$, completing the proof of our claim $\{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} = \mathsf{b}$. Therefore since $\mathsf{b} \in \mathsf{s}$ and so $\neg\mathsf{b} \supseteq \mathsf{s} \,\&\, \mathsf{s} \setminus \mathsf{b} \neq \emptyset$, the previously cited definition of enum tells us that $\mathsf{enum}(\mathsf{b},\mathsf{s}) = \mathbf{arb}(\mathsf{s} \setminus \mathsf{b})$, and so using the axiom of choice we must have $\mathbf{arb}(\mathsf{s} \setminus \mathsf{b}) \in \mathsf{s} \setminus \mathsf{b}$, which implies $\mathbf{arb}(\mathsf{s} \setminus \mathsf{b}) \in \mathsf{s}$.

$\langle \mathsf{c}\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{c} \notin \mathsf{s}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad Stat6: \{\mathsf{enum}(\mathsf{y},\mathsf{s}) : \mathsf{y} \in \mathsf{b}\} = \mathsf{b}$

$\langle Stat1, Stat22, Stat6\rangle$ ELEM $\Rightarrow \quad Stat7: \mathsf{enum}(\mathsf{b},\mathsf{s}) = \mathbf{arb}(\mathsf{s} \setminus \mathsf{b}) \,\&\, \mathsf{s} \not\subseteq \mathsf{b}$

$\langle \mathsf{s} \setminus \mathsf{b}\rangle \hookrightarrow T0([Stat21, Stat7]) \Rightarrow \quad Stat88: \mathbf{arb}(\mathsf{s} \setminus \mathsf{b}) \in \mathsf{s} \setminus \mathsf{b} \,\&\, \mathbf{arb}(\mathsf{s} \setminus \mathsf{b}) \cap (\mathsf{s} \setminus \mathsf{b}) = \emptyset$

$\langle Stat88, *\rangle$ ELEM $\Rightarrow$ $\quad Stat8:\ \mathbf{arb}(s\backslash b) \in s\backslash b \ \&\ \mathbf{arb}(s\backslash b) \cap (s\backslash b) = \emptyset \ \&\ \mathbf{arb}(s\backslash b) \in s$

-- But since s is an ordinal, its elements b and $\mathbf{arb}(s\backslash b)$ are ordered by membership, and since $\mathbf{arb}(s\backslash b) \in s\backslash b$ we must have $b \in \mathbf{arb}(s\backslash b)$ and hence $b \notin s\backslash b$ implying $b \in b$, which is impossible.

Use_def($\mathcal{O}$) $\Rightarrow$ $\quad Stat9:\ \langle \forall x \in s, y \in s \mid x \in y \lor y \in x \lor x = y\rangle$
$\langle b, \mathbf{arb}(s\backslash b)\rangle \hookrightarrow Stat9 \Rightarrow$ $\quad Stat10:\ b \in \mathbf{arb}(s\backslash b) \lor \mathbf{arb}(s\backslash b) \in b \lor \mathbf{arb}(s\backslash b) = b$
$\langle Stat8, Stat1, Stat7, Stat10, *\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove various key properties of the cardinality #s of a set s, showing first that #s is the smallest ordinal in 1-1 correspondence with s.

-- Cardinality Lemma

Theorem 149 (121) $\mathcal{O}(\#S) \ \&\ \langle \exists f \mid 1\text{–}1(f) \ \&\ \mathbf{range}(f) = S \ \&\ \mathbf{domain}(f) = \#S\rangle \ \&\ \neg\langle \exists o \in \#S, g \mid 1\text{–}1(g) \ \&\ \mathbf{range}(g) = S \ \&\ \mathbf{domain}(g) = o\rangle$. PROOF:

-- We proceed by contradiction. Let s be a counterexample to our theorem, and first suppose that either #s is not an ordinal, or that there is no f which puts s into 1-1 correspondence with #s. But then consider the specific f defined as $f = \{[x, \mathrm{enum}(x,s)] : x \in \mathrm{enum\_Ord}(s)\}$.

Suppose_not(s) $\Rightarrow$ $\quad \neg\mathcal{O}(\#s) \lor \neg\langle \exists f \mid 1\text{–}1(f) \ \&\ \mathbf{range}(f) = s \ \&\ \mathbf{domain}(f) = \#s\rangle \lor \langle \exists o \in \#s, g \mid 1\text{–}1(g) \ \&\ \mathbf{range}(g) = s \ \&\ \mathbf{domain}(g) = o\rangle$
$T42 \Rightarrow$ $\quad Stat1:\ \langle \forall s \mid \mathcal{O}(\mathrm{enum\_Ord}(s)) \ \&\ s = \{\mathrm{enum}(y,s) : y \in \mathrm{enum\_Ord}(s)\} \ \&\ \langle \forall y \in \mathrm{enum\_Ord}(s), z \in \mathrm{enum\_Ord}(s) \mid y \neq z \rightarrow \mathrm{enum}(y,s) \neq \mathrm{enum}(z,s)\rangle\rangle$
$\langle s\rangle \hookrightarrow Stat1 \Rightarrow$ $\quad \mathcal{O}(\mathrm{enum\_Ord}(s)) \ \&\ s = \{\mathrm{enum}(y,s) : y \in \mathrm{enum\_Ord}(s)\}$
$\langle \mathrm{enum\_Ord}(s)\rangle \hookrightarrow T29 \Rightarrow$ $\quad \mathcal{O}\Big(\mathrm{next}\big(\mathrm{enum\_Ord}(s)\big)\Big)$
Use_def($\mathcal{O}$) $\Rightarrow$ $\quad Stat2:\ \langle \forall x \in \mathrm{next}(\mathrm{enum\_Ord}(s)) \mid x \subseteq \mathrm{next}(\mathrm{enum\_Ord}(s))\rangle$
Suppose $\Rightarrow$ $\quad \neg\mathcal{O}(\#s) \lor \neg\langle \exists f \mid 1\text{–}1(f) \ \&\ \mathbf{domain}(f) = \#s \ \&\ \mathbf{range}(f) = s\rangle$
Loc_def $\Rightarrow$ $\quad s_1 = s$
Loc_def $\Rightarrow$ $\quad f = \{[x, \mathrm{enum}(x, s_1)] : x \in \mathrm{enum\_Ord}(s_1)\}$

-- Our general fcn_symbol theory tells us that this has enum_Ord(s) as range, and it is easily seen, using the definition of enum_Ord that it has domain s.

APPLY $\langle x_\Theta : y, y_\Theta : zz\rangle$ fcn_symbol$(f(x) \mapsto \mathrm{enum}(x,s_1), g \mapsto f, s \mapsto \mathrm{enum_Ord}(s_1)) \Rightarrow$
$\quad \mathrm{Svm}(f) \ \&\ \mathbf{domain}(f) = \mathrm{enum\_Ord}(s_1) \ \&\ \mathbf{range}(f) = \{\mathrm{enum}(x,s_1) : x \in \mathrm{enum\_Ord}(s_1)\} \ \&\ \big(y, zz \in \mathrm{enum\_Ord}(s_1) \ \&\ \mathrm{enum}(y,s_1) = \mathrm{enum}(zz,s_1) \ \&\ y \neq zz\big) \lor 1\text{–}1(f)$
EQUAL $\Rightarrow$ $\quad \mathrm{Svm}(f) \ \&\ \mathbf{domain}(f) = \mathrm{enum\_Ord}(s) \ \&\ \mathbf{range}(f) = \{\mathrm{enum}(x,s) : x \in \mathrm{enum\_Ord}(s)\} \ \&\ Stat3:\ \big(y, zz \in \mathrm{enum\_Ord}(s) \ \&\ \mathrm{enum}(y,s) = \mathrm{enum}(zz,s) \ \&\ y \neq zz\big) \lor 1\text{–}1(f)$
ELEM $\Rightarrow$ $\quad \mathrm{Svm}(f) \ \&\ \mathbf{domain}(f) = \mathrm{enum\_Ord}(s) \ \&\ \mathbf{range}(f) = s$

-- Next suppose that f is not 1-1. Then, by Stat4 3, there would exist two distinct elements y and zz of $\mathsf{enum\_Ord(s)}$ such that $\mathsf{enum(y,s) = enum(zz,s)}$, which is impossible by the definition of $\mathsf{enum\_Ord}$.

Suppose $\Rightarrow$ $\neg 1\text{–}1(\mathsf{f})$
ELEM $\Rightarrow$ $\mathsf{y, zz \in enum\_Ord(s)} \;\&\; \mathsf{enum(y,s) = enum(zz,s)} \;\&\; \mathsf{y \neq zz}$
$\langle \mathsf{s} \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat5:\ \langle \forall \mathsf{y \in enum\_Ord(s), z \in enum\_Ord(s)} \mid \mathsf{y \neq z} \rightarrow \mathsf{enum(y,s) \neq enum(z,s)} \rangle$
$\langle \mathsf{y, zz} \rangle \hookrightarrow Stat5 \Rightarrow$ $\mathsf{y, zz \in enum\_Ord(s)} \;\&\; \mathsf{y \neq zz} \rightarrow \mathsf{enum(y,s) \neq enum(zz,s)}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $1\text{–}1(\mathsf{f})$

-- We know at this point that there is a 1-1 mapping f between s and some ordinal; hence there is a least ordinal which can be mapped to f in 1-1 fashion.

Use_def(next) $\Rightarrow$ $\mathsf{next(enum\_Ord(s)) = enum\_Ord(s) \cup \{enum\_Ord\}(s)}$
Suppose $\Rightarrow$ $Stat6:\ \mathsf{enum\_Ord(s)} \notin \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{f}) = \mathsf{s} \rangle \}$
$\langle \mathsf{enum\_Ord(s)} \rangle \hookrightarrow Stat6 \Rightarrow$ $\neg \big(\mathsf{enum\_Ord(s) \in next(enum\_Ord(s))} \;\&\; \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathsf{enum\_Ord(s)} \;\&\; \mathbf{range}(\mathsf{f}) = \mathsf{s} \rangle \big)$
ELEM $\Rightarrow$ $Stat7:\ \neg \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathsf{enum\_Ord(s)} \;\&\; \mathbf{range}(\mathsf{f}) = \mathsf{s} \rangle$
$\langle \mathsf{f} \rangle \hookrightarrow Stat7 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat8:\ \mathsf{enum\_Ord(s)} \in \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{f}) = \mathsf{s} \rangle \}$
$\langle Stat8 \rangle$ ELEM $\Rightarrow$ $\{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \} \neq \emptyset$
$\langle \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \} \rangle \hookrightarrow T0 \Rightarrow$ $Stat9:$
$\mathbf{arb}(\{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \}) \in \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \}$
$\mathbf{arb}(\{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \}) \cap \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) =$

-- By definition, this least ordinal is the cardinality #s of s, and so $\#\mathsf{s}$ is an ordinal, and is in 1-1 correspondence with s. Therefore our theorem can only be false if some smaller ordinal o is also in 1-1 correspondence with s.

Use_def(#) $\Rightarrow$ $\#\mathsf{s} = \mathbf{arb}(\{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \})$
$\langle Stat9, * \rangle$ ELEM $\Rightarrow$ $Stat10:$
$\#\mathsf{s} \in \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \} \;\&$
$\#\mathsf{s} \cap \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \} = \emptyset$
$\langle \mathsf{x}_1 \rangle \hookrightarrow Stat10 \Rightarrow$ $\#\mathsf{s} = \mathsf{x}_1 \;\&\; \#\mathsf{s} \in \mathsf{next(enum\_Ord(s))} \;\&\; \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x}_1 \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle$
EQUAL $\Rightarrow$ $\langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \#\mathsf{s} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle$
$\langle \mathsf{next(enum\_Ord(s))}, \#\mathsf{s} \rangle \hookrightarrow T11 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat11:\ \langle \exists \mathsf{o} \in \#\mathsf{s}, \mathsf{g} \mid 1\text{–}1(\mathsf{g}) \;\&\; \mathbf{range}(\mathsf{g}) = \mathsf{s} \;\&\; \mathbf{domain}(\mathsf{g}) = \mathsf{o} \rangle$
$\langle \mathsf{o} \rangle \hookrightarrow Stat11 \Rightarrow$ $Stat12:\ \mathsf{o} \in \#\mathsf{s} \;\&\; \langle \exists \mathsf{g} \mid 1\text{–}1(\mathsf{g}) \;\&\; \mathbf{domain}(\mathsf{g}) = \mathsf{o} \;\&\; \mathbf{range}(\mathsf{g}) = \mathsf{s} \rangle$

-- But by definition $\#\mathsf{s}$ is the least ordinal in 1-1 correspondence with s, a final contradiction which completes the proof of the present theorem.

Use_def(#) $\Rightarrow$ $Stat13:$
$\#\mathsf{s} \in \{\mathsf{x} : \mathsf{x \in next(enum\_Ord(s))} \mid \langle \exists \mathsf{ff} \mid 1\text{–}1(\mathsf{ff}) \;\&\; \mathbf{domain}(\mathsf{ff}) = \mathsf{x} \;\&\; \mathbf{range}(\mathsf{ff}) = \mathsf{s} \rangle \} \;\&$

$\#s \cap \{x : x \in \text{next}(\text{enum\_Ord}(s)) \mid \langle \exists ff \mid 1\text{–}1(ff) \,\&\, \mathbf{domain}(ff) = x \,\&\, \mathbf{range}(ff) = s\rangle\} = \emptyset$

ELEM $\Rightarrow$ $\mathit{Stat14}: o \notin \{x : x \in \text{next}(\text{enum\_Ord}(s)) \mid \langle \exists ff \mid 1\text{–}1(ff) \,\&\, \mathbf{domain}(ff) = x \,\&\, \mathbf{range}(ff) = s\rangle\}$

$\langle o\rangle \hookrightarrow \mathit{Stat14} \Rightarrow$ $o \notin \text{next}(\text{enum\_Ord}(s))$

$\langle x_2\rangle \hookrightarrow \mathit{Stat13} \Rightarrow$ $\#s \in \text{next}(\text{enum\_Ord}(s))$

$\langle \#s\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Using theorem 121 it is easy to prove that no set s can be a member of its own cardinality.

Theorem 150 (122) $S \notin \#S$. PROOF:

Suppose_not(s) $\Rightarrow$ $s \in \#s$

$\langle s\rangle \hookrightarrow \mathit{T121} \Rightarrow$ $\mathit{Stat1}: \neg\langle \exists o \in \#s, g \mid 1\text{–}1(g) \,\&\, \mathbf{range}(g) = s \,\&\, \mathbf{domain}(g) = o\rangle$

$\langle s\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}: \neg\langle \exists g \mid 1\text{–}1(g) \,\&\, \mathbf{range}(g) = s \,\&\, \mathbf{domain}(g) = s\rangle$

$\langle s\rangle \hookrightarrow \mathit{T94} \Rightarrow$ $1\text{–}1(\iota_s) \,\&\, \mathbf{domain}(\iota_s) = s \,\&\, \mathbf{range}(\iota_s) = s$

$\langle \iota_s\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem states a special property of the choice operator **arb** which is sometimes useful.

-- 'arb' is monotone decreasing for non - empty sets of ordinals

Theorem 151 (123) $\mathcal{O}(R) \,\&\, R \supseteq S \,\&\, S \supseteq T \rightarrow \mathbf{arb}(S) \in \mathbf{arb}(T) \vee \mathbf{arb}(S) = \mathbf{arb}(T) \vee T = \emptyset$. PROOF:

Suppose_not(o, s, t) $\Rightarrow$ $\mathcal{O}(o) \,\&\, o \supseteq s \,\&\, s \supseteq t \,\&\, \mathbf{arb}(s) \notin \mathbf{arb}(t) \,\&\, \mathbf{arb}(s) \neq \mathbf{arb}(t) \,\&\, t \neq \emptyset$

-- For consider a nonempty subset t of a set s of ordinals. Since **arb**(t) is a member of t and hence of s, it cannot be a member of **arb**(s). Thus, since both **arb**(t) and **arb**(s) must be ordinals, our claim follows immediately from Theorem 28.

$\langle s\rangle \hookrightarrow \mathit{T0} \Rightarrow$ $(\mathbf{arb}(s) \in s \,\&\, \mathbf{arb}(s) \cap s = \emptyset) \vee (s = \emptyset \,\&\, \mathbf{arb}(s) = \emptyset)$

$\langle t\rangle \hookrightarrow \mathit{T0} \Rightarrow$ $\mathbf{arb}(t) \in t$

ELEM $\Rightarrow$ $\mathbf{arb}(s) \in s \,\&\, \mathbf{arb}(s) \cap s = \emptyset$

$\langle o, \mathbf{arb}(s)\rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(\mathbf{arb}(s))$

ELEM $\Rightarrow$ $\mathbf{arb}(t) \in o$

$\langle o, \mathbf{arb}(t)\rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(\mathbf{arb}(t))$

$\langle \mathbf{arb}(s), \mathbf{arb}(t)\rangle \hookrightarrow \mathit{T28} \Rightarrow$ $\mathbf{arb}(t) \in \mathbf{arb}(s) \cap s$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem, closely related to the preceding, tells us that **arb** selects the minimum element of any set of ordinals.

Theorem 152 (124) $\mathsf{S} \neq \emptyset \,\&\, \langle \forall \mathsf{x} \in \mathsf{S} \,|\, \mathcal{O}(\mathsf{x}) \rangle \rightarrow \langle \forall \mathsf{x} \in \mathsf{S} \,|\, \mathsf{x} \supseteq \mathbf{arb}(\mathsf{S}) \rangle$. PROOF:

Suppose_not(s) $\Rightarrow$ $\mathsf{s} \neq \emptyset \,\&\, \mathit{Stat1}: \langle \forall \mathsf{x} \in \mathsf{s} \,|\, \mathcal{O}(\mathsf{x}) \rangle \,\&\, \mathit{Stat2}: \neg \langle \forall \mathsf{x} \in \mathsf{s} \,|\, \mathsf{x} \supseteq \mathbf{arb}(\mathsf{s}) \rangle$

-- For otherwise there is a non-null set s of ordinals with a member x not including **arb**(s). But by the axiom of choice, x cannot be a member of **arb**(s), and so by Theorem 28 s must be a member, and hence a subset, of x, contradicting the definition of x, and so proving our theorem.

$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathsf{x} \in \mathsf{s} \,\&\, \mathsf{x} \not\supseteq \mathbf{arb}(\mathsf{s})$
$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathcal{O}(\mathsf{x})$
$\langle \mathsf{s} \rangle \hookrightarrow \mathit{T0} \Rightarrow$ $\mathbf{arb}(\mathsf{s}) \in \mathsf{s} \,\&\, \mathsf{x} \notin \mathbf{arb}(\mathsf{s})$
$\langle \mathbf{arb}(\mathsf{s}) \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathcal{O}(\mathbf{arb}(\mathsf{s}))$
$\langle \mathbf{arb}(\mathsf{s}), \mathsf{x} \rangle \hookrightarrow \mathit{T28} \Rightarrow$ $\mathbf{arb}(\mathsf{s}) \in \mathsf{x}$
$\langle \mathsf{x}, \mathbf{arb}(\mathsf{s}) \rangle \hookrightarrow \mathit{T31} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next aim is to prove that the cardinality of any subset t of a set s is no more than the cardinality of s, This will be established by showing that the enumerating ordinal of t is no larger than that of s. We prepare for this by showing that if t is a subset of an ordinal s, then the function enum(x, t) is nondecreasing in the variable x.

-- Lemma for following theorem

Theorem 153 (125) $\mathcal{O}(\mathsf{S}) \,\&\, T \subseteq \mathsf{S} \,\&\, \mathsf{X} \in \mathsf{S} \,\&\, \mathsf{Y} \in \mathsf{X} \rightarrow \mathsf{enum}(\mathsf{Y}, T) \in \mathsf{enum}(\mathsf{X}, T) \vee \mathsf{enum}(\mathsf{X}, T) \supseteq T$. PROOF:

Suppose_not(s, t, x, y_2) $\Rightarrow$ $\mathit{Stat1}: \mathcal{O}(\mathsf{s}) \,\&\, \mathsf{t} \subseteq \mathsf{s} \,\&\, \mathsf{x} \in \mathsf{s} \,\&\, \mathsf{y}_2 \in \mathsf{x} \,\&\, \mathsf{enum}(\mathsf{y}_2, \mathsf{t}) \notin \mathsf{enum}(\mathsf{x}, \mathsf{t}) \,\&\, \mathsf{enum}(\mathsf{x}, \mathsf{t}) \not\supseteq \mathsf{t}$

-- For suppose the contrary. Then enum(x, t) cannot be t, so by definition of enum it must be a member of t, hence of s, hence an ordinal. Likewise x must be an ordinal, and so y_2 must also be an ordinal. Moreover $\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\}$ must be nonempty.

Use_def(enum) $\Rightarrow$ $\mathsf{enum}(\mathsf{x}, \mathsf{t}) = \textbf{if } \mathsf{t} \subseteq \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\} \textbf{ then } \mathsf{t} \textbf{ else } \mathbf{arb}(\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\}) \textbf{ fi}$
ELEM $\Rightarrow$ $\mathit{Stat44}: \mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\} \neq \emptyset \,\&\, \mathsf{enum}(\mathsf{x}, \mathsf{t}) = \mathbf{arb}(\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\})$
$\langle \mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\} \rangle \hookrightarrow \mathit{T0} \Rightarrow$ $\mathsf{enum}(\mathsf{x}, \mathsf{t}) \in \mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\}$
ELEM $\Rightarrow$ $\mathsf{enum}(\mathsf{x}, \mathsf{t}) \in \mathsf{s}$
$\langle \mathsf{s}, \mathsf{enum}(\mathsf{x}, \mathsf{t}) \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(\mathsf{enum}(\mathsf{x}, \mathsf{t}))$
$\langle \mathsf{s}, \mathsf{x} \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(\mathsf{x})$
$\langle \mathsf{x}, \mathsf{y}_2 \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(\mathsf{y}_2)$

-- Thus y_2 must be a subset of x, and so it follows by the principle of set monotonicity that $\{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{y}_2\} \subseteq \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{x}\}$. But then $\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{y}_2\}$ must be nonempty, and so, by definition of enum, $\mathsf{enum}(\mathsf{y}_2, \mathsf{t}) = \mathbf{arb}(\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{y}_2\})$

$\langle x, y_2\rangle \hookrightarrow T12 \Rightarrow \quad y_2 \subseteq x$
ELEM $\Rightarrow \quad \mathcal{O}(\text{enum}(x,t))\ \&\ \mathcal{O}(x)\ \&\ \mathcal{O}(y_2)\ \&\ y_2 \subseteq x$
Set_monot $\Rightarrow \quad Stat55: \{\text{enum}(u,t) : u \in y_2\} \subseteq \{\text{enum}(u,t) : u \in x\}$

-- ?? Use_def (enum) ⇒ Stat66: enum (y2, t) = if t incin {enum (u, t): u in y2} then t else arb (t-{enum (u, t): u in y2}) end if

Use_def (enum) $\Rightarrow \quad \text{enum}(y_2,t) = \textbf{if } t \subseteq \{\text{enum}(u,t) : u \in y_2\} \textbf{ then } t \textbf{ else arb}(t \setminus \{\text{enum}(u,t) : u \in y_2\}) \textbf{ fi}$
ELEM $\Rightarrow \quad Stat66: \text{enum}(y_2,t) = \textbf{if } t \subseteq \{\text{enum}(u,t) : u \in y_2\} \textbf{ then } t \textbf{ else arb}(t \setminus \{\text{enum}(u,t) : u \in y_2\}) \textbf{ fi}$
$\langle Stat44, Stat55\rangle$ ELEM $\Rightarrow \quad Stat77: t \setminus \{\text{enum}(u,t) : u \in y_2\} \supseteq t \setminus \{\text{enum}(u,t) : u \in x\}\ \&\ t \setminus \{\text{enum}(u,t) : u \in y_2\} \neq \emptyset$
$\langle Stat66\rangle$ ELEM $\Rightarrow \quad Stat88: t \setminus \{\text{enum}(u,t) : u \in y_2\} \neq \emptyset\ \&\ \text{enum}(y_2,t) = \textbf{arb}(t \setminus \{\text{enum}(u,t) : u \in y_2\})$

-- Thus, since we have seen that $t \setminus \{\text{enum}(u,t) : u \in y_2\} \supseteq t \setminus \{\text{enum}(u,t) : u \in x\}$, it follows by Theorem 123 that $\text{enum}(y_2,t)$ is no larger than $\text{enum}(x,t)$, so it must equal $\text{enum}(x,t)$. But since y_2 is in x this is easily seen to be impossible, and so we have a contradiction which proves our theorem.

$\langle s, t \setminus \{\text{enum}(u,t) : u \in y_2\}, t \setminus \{\text{enum}(u,t) : u \in x\}\rangle \hookrightarrow T123(\langle Stat1, Stat77, Stat44\rangle) \Rightarrow \quad Stat99: \textbf{arb}(t \setminus \{\text{enum}(u,t) : u \in y_2\}) \in \textbf{arb}(t \setminus \{\text{enum}(u,t) : u \in x\}) \vee$
$\textbf{arb}(t \setminus \{\text{enum}(u,t) : u \in y_2\}) = \textbf{arb}(t \setminus \{\text{enum}(u,t) : u \in x\})$
$\langle Stat99, Stat44, Stat88\rangle$ ELEM $\Rightarrow \quad Stat2: \text{enum}(y_2,t) \in \text{enum}(x,t) \vee \text{enum}(y_2,t) = \text{enum}(x,t)$
$\langle Stat1, Stat2, *\rangle$ ELEM $\Rightarrow \quad Stat12: \text{enum}(y_2,t) = \text{enum}(x,t)$
$\langle t\rangle \hookrightarrow T41 \Rightarrow \quad Stat3: \langle \exists u \mid (\mathcal{O}(u)\ \&\ t = \{\text{enum}(y_2,t) : y_2 \in u\})\ \&\ \langle \forall y \in u, zz \in u \mid y \neq zz \rightarrow \text{enum}(y,t) \neq \text{enum}(zz,t)\rangle\rangle$
$\langle u\rangle \hookrightarrow Stat3 \Rightarrow \quad \mathcal{O}(u)\ \&\ t = \{\text{enum}(y,t) : y \in u\}\ \&\ Stat4: \langle \forall y \in u, zz \in u \mid y \neq zz \rightarrow \text{enum}(y,t) \neq \text{enum}(zz,t)\rangle$
Suppose $\Rightarrow \quad y_2 \notin u$
$\langle u, y_2\rangle \hookrightarrow T32 \Rightarrow \quad u \subseteq y_2$
Set_monot $\Rightarrow \quad \{\text{enum}(v,t) : v \in u\} \subseteq \{\text{enum}(v,t) : v \in y_2\}$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat10: y_2 \in u$
Suppose $\Rightarrow \quad x \notin u$
$\langle u, x\rangle \hookrightarrow T32 \Rightarrow \quad u \subseteq x$
Set_monot $\Rightarrow \quad \{\text{enum}(v,t) : v \in u\} \subseteq \{\text{enum}(v,t) : v \in x\}$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat11: x \in u$
$\langle x, y_2\rangle \hookrightarrow Stat4(\langle Stat1, Stat10, Stat11, Stat12\rangle) \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- We continue the sequence of steps which will lead us to a proof that the cardinality of a subset of a set s is no greater than $\#s$. Our plan is to prove this first for ordinals and then to generalize to arbitrary sets, using the fact that any set is in 1-1 correspondence with an ordinal. The following lemma states a related fact about the standard enumeration of ordinals.

-- Subsets of an ordinal enumerate at least as rapidly as the ordinal

Theorem 154 (126) $\mathcal{O}(\mathsf{S})\ \&\ T \subseteq \mathsf{S}\ \&\ \mathsf{X} \in \mathsf{S} \rightarrow \mathsf{enum}(\mathsf{X}, T) \supseteq \mathsf{X} \vee \mathsf{enum}(\mathsf{X}, T) \supseteq T$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s}, \mathsf{t}, \mathsf{b}) \Rightarrow \quad \mathcal{O}(\mathsf{s})\ \&\ \mathsf{t} \subseteq \mathsf{s}\ \&\ \mathsf{b} \in \mathsf{s}\ \&\ \neg\big(\mathsf{enum}(\mathsf{b}, \mathsf{t}) \supseteq \mathsf{b} \vee \mathsf{enum}(\mathsf{b}, \mathsf{t}) \supseteq \mathsf{t}\big)$

-- For suppose the contrary, so that there is an ordinal s with a subset t for which there exists a $\mathsf{b} \in \mathsf{s}$ such that $\mathsf{enum}(\mathsf{b}, \mathsf{t})$ fails to include b and is not t. By the principle of transfinite induction, there exists a least such element of s, which we call a.

$\mathsf{APPLY}\ \langle \mathsf{mt}_\Theta : \mathsf{a}\rangle\ \mathsf{transfinite\_induction}\Big(\mathsf{n} \mapsto \mathsf{b}, \mathsf{P}(\mathsf{x}) \mapsto \big(\mathsf{x} \in \mathsf{s}\ \&\ \neg(\mathsf{enum}(\mathsf{x}, \mathsf{t}) \supseteq \mathsf{x} \vee \mathsf{enum}(\mathsf{x}, \mathsf{t}) \supseteq \mathsf{t})\big)\Big) \Rightarrow$

$\mathit{Stat1}: \ \langle \forall \mathsf{x} \mid \big(\mathsf{a} \in \mathsf{s}\ \&\ \neg(\mathsf{enum}(\mathsf{a}, \mathsf{t}) \supseteq \mathsf{a} \vee \mathsf{enum}(\mathsf{a}, \mathsf{t}) \supseteq \mathsf{t})\big)\ \&\ \Big(\mathsf{x} \in \mathsf{a} \rightarrow \neg\big(\mathsf{x} \in \mathsf{s}\ \&\ \neg(\mathsf{enum}(\mathsf{x}, \mathsf{t}) \supseteq \mathsf{x} \vee \mathsf{enum}(\mathsf{x}, \mathsf{t}) \supseteq \mathsf{t})\big)\Big)\rangle$

$\langle \mathsf{a}_0 \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{a} \in \mathsf{s}\ \&\ \mathsf{enum}(\mathsf{a}, \mathsf{t}) \not\supseteq \mathsf{a}\ \&\ \mathsf{enum}(\mathsf{a}, \mathsf{t}) \not\supseteq \mathsf{t}$

-- It follows by definition of enum that t cannot be a subset of $\{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{a}\}$, and that $\mathsf{enum}(\mathsf{a}, \mathsf{t})$ is a member of $\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{y}, \mathsf{t}) : \mathsf{y} \in \mathsf{a}\}$

$\mathsf{Use\_def}(\mathsf{enum}) \Rightarrow \quad \mathsf{enum}(\mathsf{a}, \mathsf{t}) = \textbf{if}\ \mathsf{t} \subseteq \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{a}\}\ \textbf{then}\ \mathsf{t}\ \textbf{else}\ \mathbf{arb}(\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{u}, \mathsf{t}) : \mathsf{u} \in \mathsf{a}\})\ \textbf{fi}$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{t} \setminus \{\mathsf{enum}(\mathsf{y}, \mathsf{t}) : \mathsf{y} \in \mathsf{a}\} \neq \emptyset$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{enum}(\mathsf{a}, \mathsf{t}) = \mathbf{arb}(\mathsf{t} \setminus \{\mathsf{enum}(\mathsf{y}, \mathsf{t}) : \mathsf{y} \in \mathsf{a}\})$

$\langle \mathsf{t} \setminus \{\mathsf{enum}(\mathsf{y}, \mathsf{t}) : \mathsf{y} \in \mathsf{a}\} \rangle \hookrightarrow \mathit{T0} \Rightarrow \quad \mathsf{enum}(\mathsf{a}, \mathsf{t}) \in \mathsf{t} \setminus \{\mathsf{enum}(\mathsf{y}, \mathsf{t}) : \mathsf{y} \in \mathsf{a}\}$

-- Since a is a member of s, it must be an ordinal and a subset of s. Since $\mathsf{enum}(\mathsf{a}, \mathsf{t})$ belongs to t it belongs to s, and so it is also an ordinal. Since $\mathsf{enum}(\mathsf{a}, \mathsf{t})$ is not larger than a, it must be a member of a, so Theorem 125 tells us that $\mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big)$ is a member, and hence a subset, of $\mathsf{enum}(\mathsf{a}, \mathsf{t})$.

$\langle \mathsf{s}, \mathsf{a} \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(\mathsf{a})$

$\langle \mathsf{s}, \mathsf{a} \rangle \hookrightarrow \mathit{T31} \Rightarrow \quad \mathsf{a} \subseteq \mathsf{s}$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{enum}(\mathsf{a}, \mathsf{t}) \in \mathsf{s}$

$\langle \mathsf{s}, \mathsf{enum}(\mathsf{a}, \mathsf{t}) \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t})\big)$

$\langle \mathsf{a}, \mathsf{enum}(\mathsf{a}, \mathsf{t}) \rangle \hookrightarrow \mathit{T32} \Rightarrow \quad \mathsf{enum}(\mathsf{a}, \mathsf{t}) \in \mathsf{a}$

$\langle \mathsf{s}, \mathsf{t}, \mathsf{a}, \mathsf{enum}(\mathsf{a}, \mathsf{t}) \rangle \hookrightarrow \mathit{T125} \Rightarrow \quad \mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big) \in \mathsf{enum}(\mathsf{a}, \mathsf{t})$

-- Since $\mathsf{enum}(\mathsf{a}, \mathsf{t})$ is a member of t, $\mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big)$ cannot include t since if it did we would have the membership cycle $\mathsf{enum}(\mathsf{a}, \mathsf{t}) \in \mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big) \in \mathsf{enum}(\mathsf{a}, \mathsf{t})$. Applying Stat2 1 we therefore find that $\mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big) \supseteq \mathsf{enum}(\mathsf{a}, \mathsf{t})$, and hence the same impossible membership cycle, a contradiction which proves our theorem.

$\langle \mathsf{enum}(\mathsf{a}, \mathsf{t}) \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{enum}(\mathsf{a}, \mathsf{t}) \notin \mathsf{s} \vee \mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big) \supseteq \mathsf{enum}(\mathsf{a}, \mathsf{t}) \vee \mathsf{enum}\big(\mathsf{enum}(\mathsf{a}, \mathsf{t}), \mathsf{t}\big) \supseteq \mathsf{t}$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- It follows from the preceding theorem that $\{\mathsf{enum}(x,t) : x \in s\} \supseteq t$ for every subset t of an ordinal s

Theorem 155 (127) $\mathcal{O}(S) \,\&\, T \subseteq S \to \{\mathsf{enum}(x, T) : x \in S\} \supseteq T$. PROOF:

$\text{Suppose\_not}(s,t) \Rightarrow \quad \mathcal{O}(s) \,\&\, t \subseteq s \,\&\, \{\mathsf{enum}(x,t) : x \in S\} \not\supseteq t$

-- For suppose the contrary. Since by definition s is both a member and a subset of $\mathsf{next}(s)$, and since $\mathsf{next}(s)$ is an ordinal by Theorem 29, it follows by the previous theorem that $\mathsf{enum}(s,t) \supseteq t$

$\langle s\rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}(\mathsf{next}(s))$
$\text{Use\_def}(\mathsf{next}) \Rightarrow \quad \mathsf{next}(s) = s \cup \{s\}$
$\text{ELEM} \Rightarrow \quad s \in \mathsf{next}(s) \,\&\, t \subseteq \mathsf{next}(s)$
$\langle \mathsf{next}(s), t, s\rangle \hookrightarrow T126 \Rightarrow \quad \mathsf{enum}(s,t) \supseteq t$

-- But since by assumption $t \setminus \{\mathsf{enum}(y,t) : y \in s\}$ is not empty, it follows by definition of enum and by the axiom of choice that $\mathsf{enum}(s,t)$ is a member of t, and hence a member of itself, a contradiction which proves our theorem.

$\text{Use\_def}(\mathsf{enum}) \Rightarrow \quad \mathsf{enum}(s,t) = \textbf{if } t \subseteq \{\mathsf{enum}(y,t) : y \in s\} \textbf{ then } t \textbf{ else arb}(t \setminus \{\mathsf{enum}(y,t) : y \in s\}) \textbf{ fi}$
$\text{ELEM} \Rightarrow \quad t \setminus \{\mathsf{enum}(y,t) : y \in s\} \neq \emptyset$
$\langle t \setminus \{\mathsf{enum}(y,t) : y \in s\}\rangle \hookrightarrow T0 \Rightarrow \quad \mathsf{enum}(s,t) \in t$
$\text{ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- Next we show that if t is a subset of an ordinal s, there is an ordinal x no larger than s which 'enum' puts into 1-1 correspondence with t.

Theorem 156 (128) $\mathcal{O}(S) \,\&\, T \subseteq S \to \langle \exists x \subseteq S \mid \mathcal{O}(x) \,\&\, T = \{\mathsf{enum}(y, T) : y \in x\} \,\&\, \langle \forall y \in x, z \in x \mid y \neq z \longrightarrow \mathsf{enum}(y, T) \neq \mathsf{enum}(z, T)\rangle\rangle$. PROOF:

$\text{Suppose\_not}(s,t) \Rightarrow \quad \mathcal{O}(s) \,\&\, t \subseteq s \,\&\, \mathit{Stat1} : \neg\langle \exists x \subseteq s \mid \mathcal{O}(x) \,\&\, t = \{\mathsf{enum}(y,t) : y \in x\} \,\&\, \langle \forall y \in x, z \in x \mid y \neq z \longrightarrow \mathsf{enum}(y,t) \neq \mathsf{enum}(z,t)\rangle\rangle$

-- Suppose the contrary: that the map $\mathsf{enum}(.,t)$ does not put t in 1-1 correspondence with the set $\{\mathsf{enum}(y,t) : y \in x\}$ for any ordinal no larger than s. Since Theorem 127 tells us that $\{\mathsf{enum}(x,t) : x \in s\} \supseteq t$, we can use the principle of transfinite induction to find a smallest ordinal u for which $\{\mathsf{enum}(x,t) : x \in u\}$ includes t.

$\langle s,t\rangle \hookrightarrow T127 \Rightarrow \quad \{\mathsf{enum}(u,t) : u \in s\} \supseteq t$
$\text{ELEM} \Rightarrow \quad s \subseteq s$
$\text{APPLY } \langle \mathsf{mt}_\ominus : u\rangle \text{ transfinite\_induction}(n \mapsto s, P(x) \mapsto \mathcal{O}(x) \,\&\, x \subseteq s \,\&\, \{\mathsf{enum}(v,t) : v \in x\} \supseteq t) \Rightarrow$

$Stat2: \ \langle \forall x \,|\, \mathcal{O}(u) \;\&\; u \subseteq s \;\&\; \{enum(v,t) : v \in u\} \supseteq t \;\&\; \big(x \in u \rightarrow \neg(\mathcal{O}(x) \;\&\; x \subseteq s \;\&\; \{enum(w,t) : w \in x\} \supseteq t)\big)\rangle$

$\langle x_0 \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathcal{O}(u) \;\&\; u \subseteq s \;\&\; \{enum(v,t) : v \in u\} \supseteq t$

-- We now aim to derive a contradiction. By Stat3, enum cannot map u onto t in 1-1 fashion. Thus it either (Case 1) does not map u in 1-1 fashion, or (Case 2) maps u onto a set different from t. First suppose that we are in Case 2, so that there exist distinct y and zz in u such that $enum(y,t) = enum(zz,t)$

$\langle u \rangle \hookrightarrow Stat1 \Rightarrow \quad \neg\big(t = \{enum(y,t) : y \in u\} \;\&\; \langle \forall y \in u, z \in u \,|\, y \neq z \rightarrow enum(y,t) \neq enum(z,t)\rangle\big)$

Suppose $\Rightarrow \quad Stat4: \ \neg\langle \forall y \in u, z \in u \,|\, y \neq z \rightarrow enum(y,t) \neq enum(z,t)\rangle$

$\langle y, zz \rangle \hookrightarrow Stat4 \Rightarrow \quad y, zz \in u \;\&\; y \neq zz \;\&\; enum(y,t) = enum(zz,t)$

-- Since u is an ordinal, its members y and zz must also be ordinals, and must be subsets of u.

$\langle u, y \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(y)$

$\langle u, zz \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(zz)$

$\langle u, y \rangle \hookrightarrow T12 \Rightarrow \quad y \subseteq u$

$\langle u, zz \rangle \hookrightarrow T12 \Rightarrow \quad zz \subseteq u$

-- Since y is an ordinal, a member of u, and a subset of s, Stat3 3 tells us that $\{enum(v,t) : v \in y\}$ cannot include t. Also, since $enum(y,t) = enum(zz,t)$, it follows by Theorem 39 that t must belong either to $\{enum(v,t) : v \in y\}$ or $\{enum(v,t) : v \in zz\}$. Suppose the former, so that there is an ordinal $a \in y$ such that $t = enum(a,t)$. Plainly, a is a subset of y.

$\langle y \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat5: \ \{enum(v,t) : v \in y\} \not\supseteq t$

$\langle y, zz, t \rangle \hookrightarrow T39 \Rightarrow \quad t \in \{enum(v,t) : v \in y\} \vee t \in \{enum(v,t) : v \in zz\}$

Suppose $\Rightarrow \quad Stat6: \ t \in \{enum(v,t) : v \in y\}$

$\langle a \rangle \hookrightarrow Stat6 \Rightarrow \quad t = enum(a,t) \;\&\; a \in y$

$\langle y, a \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(a)$

$\langle y, a \rangle \hookrightarrow T31 \Rightarrow \quad a \subseteq y$

-- Since tv cannot be a member of t, it follows by definition of enum that t must be a subset of $\{enum(v,t) : v \in a\}$, and therefore of the larger set $\{enum(v,t) : v \in y\}$. But it was shown above that $\{enum(v,t) : v \in y\}$ cannot include t. This contradiction excludes the possibility that t is a member of $\{enum(v,t) : v \in y\}$, and so implies that t is a member of $\{enum(v,t) : v \in zz\}$.

Use_def(enum) $\Rightarrow \quad enum(a,t) = \textbf{if } t \subseteq \{enum(v,t) : v \in a\} \textbf{ then } t \textbf{ else arb}(t \setminus \{enum(v,t) : v \in a\}) \textbf{ fi}$

ELEM $\Rightarrow \quad t \subseteq \{enum(v,t) : v \in a\}$

Set_monot $\Rightarrow$ $\{\mathsf{enum}(v,t) : v \in a\} \subseteq \{\mathsf{enum}(v,t) : v \in y\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat7} : t \in \{\mathsf{enum}(v,t) : v \in zz\}$

-- However, an exactly similar argument refutes this possibility, implying that $t \neq \{\mathsf{enum}(y,t) : y \in u\}$. Also, as shown above, $\{\mathsf{enum}(v,t) : v \in u\} \supseteq t$, from which it is obvious that there is a $c \in \{\mathsf{enum}(y,t) : y \in u\}$, hence of the form $\mathsf{enum}(d,t)$ with $d \in u$, such that $c \notin t$.

$\langle b \rangle \hookrightarrow \mathit{Stat7} \Rightarrow$ $t = \mathsf{enum}(b,t) \;\&\; b \in zz$
$\langle zz, b \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(b)$
$\langle zz, b \rangle \hookrightarrow \mathit{T31} \Rightarrow$ $b \subseteq zz$
Use_def(enum) $\Rightarrow$ $\mathsf{enum}(b,t) = \textbf{if } t \subseteq \{\mathsf{enum}(v,t) : v \in b\} \textbf{ then } t \textbf{ else arb}(t \setminus \{\mathsf{enum}(v,t) : v \in b\}) \textbf{ fi}$
ELEM $\Rightarrow$ $t \subseteq \{\mathsf{enum}(v,t) : v \in b\}$
Set_monot $\Rightarrow$ $\{\mathsf{enum}(v,t) : v \in b\} \subseteq \{\mathsf{enum}(v,t) : v \in zz\}$
Set_monot $\Rightarrow$ $\{\mathsf{enum}(v,t) : v \in zz\} \subseteq \{\mathsf{enum}(v,t) : v \in u\}$
$\langle \mathit{Stat7} \rangle$ ELEM $\Rightarrow$ $t \in \{\mathsf{enum}(v,t) : v \in u\}$
$\langle \mathit{Stat7} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat8} : t \neq \{\mathsf{enum}(y,t) : y \in u\} \;\&\; t \subseteq \{\mathsf{enum}(y,t) : y \in u\}$
$\langle \mathit{Stat8} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat9} : t \not\supseteq \{\mathsf{enum}(y,t) : y \in u\}$
$\langle c \rangle \hookrightarrow \mathit{Stat9} \Rightarrow$ $\mathit{Stat10} : c \in \{\mathsf{enum}(y,t) : y \in u\} \;\&\; c \notin t$
$\langle d \rangle \hookrightarrow \mathit{Stat10} \Rightarrow$ $c = \mathsf{enum}(d,t) \;\&\; d \in u \;\&\; c \notin t$

-- Since u is an ordinal, its member d must also be an ordinal and a subset of d. Thus, by the minimality of u (Stat3 3), $\{\mathsf{enum}(v,t) : v \in d\}$ cannot include t, which, by definition of 'enum', tells us that c = enum (d, t) must be a member of t, contradicting what has just been proved. This excludes the case (Case 2) that we have had under consideration, and so leaves open only the possibility that $t \neq \{\mathsf{enum}(y,t) : y \in u\}$; but, since we have shown above that the second of these sets includes the first, if follows that the first does not include the second.

$\langle u, d \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(d)$
$\langle u, d \rangle \hookrightarrow \mathit{T12} \Rightarrow$ $d \subseteq u$
$\langle d \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathit{Stat10a} : \{\mathsf{enum}(v,t) : v \in d\} \not\supseteq t$
Use_def(enum) $\Rightarrow$ $\mathsf{enum}(d,t) = \textbf{if } t \subseteq \{\mathsf{enum}(y,t) : y \in d\} \textbf{ then } t \textbf{ else arb}(t \setminus \{\mathsf{enum}(y,t) : y \in d\}) \textbf{ fi}$
ELEM $\Rightarrow$ $t \setminus \{\mathsf{enum}(y,t) : y \in d\} \neq \emptyset$
$\langle t \setminus \{\mathsf{enum}(y,t) : y \in d\} \rangle \hookrightarrow \mathit{T0}(\langle \mathit{Stat10a} \rangle) \Rightarrow$ $\mathsf{enum}(d,t) \in t$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat11} : t \not\supseteq \{\mathsf{enum}(w,t) : w \in u\}$

-- Hence there is a $v \in u$ such that $\mathsf{enum}(v,t) \notin t$, so it follows by definition of 'enum' that $\mathsf{enum}(v,t) \notin t$, implying that $\{\mathsf{enum}(w) : w \in v\} \supseteq t$. Since the minimality of u rules this out, we have a contradiction which completes the proof of the present theorem.

$\langle \mathrm{dd} \rangle \hookrightarrow Stat11 \Rightarrow \quad Stat12: \mathrm{dd} \in \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{u}\} \ \& \ \mathrm{dd} \notin \mathrm{t}$
$\langle \mathrm{v} \rangle \hookrightarrow Stat12 \Rightarrow \quad \mathrm{v} \in \mathrm{u} \ \& \ \mathrm{enum}(\mathrm{v},\mathrm{t}) \notin \mathrm{t}$
$\langle \mathrm{u},\mathrm{v} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathrm{v})$
$\langle \mathrm{u},\mathrm{v} \rangle \hookrightarrow T31 \Rightarrow \quad \mathrm{v} \subseteq \mathrm{u}$
ELEM $\Rightarrow \quad \mathrm{v} \subseteq \mathrm{s}$
Use_def(enum) $\Rightarrow \quad \mathrm{enum}(\mathrm{v},\mathrm{t}) = \textbf{if } \mathrm{t} \subseteq \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{v}\} \textbf{ then } \mathrm{t} \textbf{ else arb}(\mathrm{t} \setminus \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{v}\}) \textbf{ fi}$
Suppose $\Rightarrow \quad \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{v}\} \not\supseteq \mathrm{t}$
ELEM $\Rightarrow \quad \mathrm{t} \setminus \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{v}\} \neq \emptyset$
$\langle \mathrm{t} \setminus \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{v}\} \rangle \hookrightarrow T0 \Rightarrow \quad \mathrm{false};$ Discharge $\Rightarrow \quad \{\mathrm{enum}(\mathrm{w},\mathrm{t}) : \mathrm{w} \in \mathrm{v}\} \supseteq \mathrm{t}$
$\langle \mathrm{v} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathrm{false};$ Discharge $\Rightarrow$ QED

-- In proving Theorem 130 below we will want the following direct consequence of Theorem 128, which tells us that any subset of an ordinal s is in 1-1 correspondence with an ordinal no greater than s.

Theorem 157 (129) $\mathcal{O}(\mathrm{S}) \ \& \ T \subseteq \mathrm{S} \rightarrow \langle \exists \mathrm{f} \mid 1\text{–}1(\mathrm{f}) \ \& \ \mathcal{O}(\mathbf{domain}(\mathrm{f})) \ \& \ \mathbf{domain}(\mathrm{f}) \subseteq \mathrm{S} \ \& \ \mathbf{range}(\mathrm{f}) = T \rangle$. PROOF:

Suppose_not(s, t) $\Rightarrow \quad \mathcal{O}(\mathrm{s}) \ \& \ \mathrm{t} \subseteq \mathrm{s} \ \& \ Stat1: \neg\langle \exists \mathrm{f} \mid 1\text{–}1(\mathrm{f}) \ \& \ \mathcal{O}(\mathbf{domain}(\mathrm{f})) \ \& \ \mathbf{domain}(\mathrm{f}) \subseteq \mathrm{s} \ \& \ \mathbf{range}(\mathrm{f}) = \mathrm{t} \rangle$

-- For suppose the contrary, and let t be a subset of an ordinal s for which there is no 1-1 correspondence with an ordinal no larger than s. By theorem 128 there is an ordinal xx no larger than s for which the map $\mathrm{y} \mapsto \mathrm{enum}(\mathrm{y},\mathrm{t})$ is a 1-1 mapping of xx onto t, contradicting the statement we have just made and so proving our theorem.

$\langle \mathrm{s},\mathrm{t} \rangle \hookrightarrow T128 \Rightarrow \quad Stat2: \langle \exists \mathrm{x} \subseteq \mathrm{s} \mid \mathcal{O}(\mathrm{x}) \ \& \ \mathrm{t} = \{\mathrm{enum}(\mathrm{y},\mathrm{t}) : \mathrm{y} \in \mathrm{x}\} \ \& \ \langle \forall \mathrm{y} \in \mathrm{x}, \mathrm{zz} \in \mathrm{x} \mid \mathrm{y} \neq \mathrm{zz} \rightarrow \mathrm{enum}(\mathrm{y},\mathrm{t}) \neq \mathrm{enum}(\mathrm{zz},\mathrm{t}) \rangle \rangle$
$\langle \mathrm{x}_2 \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathcal{O}(\mathrm{x}_2) \ \& \ \mathrm{x}_2 \subseteq \mathrm{s} \ \& \ \mathrm{t} = \{\mathrm{enum}(\mathrm{y},\mathrm{t}) : \mathrm{y} \in \mathrm{x}_2\} \ \& \ Stat3: \langle \forall \mathrm{y} \in \mathrm{x}_2, \mathrm{zz} \in \mathrm{x}_2 \mid \mathrm{y} \neq \mathrm{zz} \rightarrow \mathrm{enum}(\mathrm{y},\mathrm{t}) \neq \mathrm{enum}(\mathrm{zz},\mathrm{t}) \rangle$
Loc_def $\Rightarrow \quad \mathrm{f} = \{[\mathrm{x}, \mathrm{enum}(\mathrm{x},\mathrm{t})] : \mathrm{x} \in \mathrm{x}_2\}$
APPLY $\langle \mathrm{x}_\Theta : \mathrm{x}, \mathrm{y}_\Theta : \mathrm{y} \rangle$ fcn_symbol$(\mathrm{f}(\mathrm{x}) \mapsto \mathrm{enum}(\mathrm{x},\mathrm{t}), \mathrm{g} \mapsto \mathrm{f}, \mathrm{s} \mapsto \mathrm{x}_2) \Rightarrow$
$\quad \mathrm{Svm}(\mathrm{f}) \ \& \ \mathbf{domain}(\mathrm{f}) = \mathrm{x}_2 \ \& \ \mathbf{range}(\mathrm{f}) = \{\mathrm{enum}(\mathrm{x},\mathrm{t}) : \mathrm{x} \in \mathrm{x}_2\} \ \& \ (\mathrm{x},\mathrm{y} \in \mathrm{x}_2 \ \& \ \mathrm{enum}(\mathrm{x},\mathrm{t}) = \mathrm{enum}(\mathrm{y},\mathrm{t}) \ \& \ \mathrm{x} \neq \mathrm{y}) \lor 1\text{–}1(\mathrm{f})$
$\langle \mathrm{x},\mathrm{y} \rangle \hookrightarrow Stat3 \Rightarrow \quad 1\text{–}1(\mathrm{f})$
EQUAL $\Rightarrow \quad \mathcal{O}(\mathbf{domain}(\mathrm{f}))$
$\langle \mathrm{f} \rangle \hookrightarrow Stat1 \Rightarrow \quad \mathrm{false};$ Discharge $\Rightarrow$ QED

-- A straightforward consequence of Theorem 129, which is stated in the following theorem, is that #s is a cardinal in 1-1 correspondence with s (rather than merely in the single-valued correspondence which the definition of cardinality requires.) This result has several easy but important corollaries which we then digress to prove.

-- Cardinality theorem

Theorem 158 (130) $\mathsf{Card}(\#S)\ \&\ \mathcal{O}(\#S)\ \&\ \langle\exists f \mid 1\text{--}1(f)\ \&\ \mathbf{range}(f) = S\ \&\ \mathbf{domain}(f) = \#S\rangle$. PROOF:

Suppose_not(s) $\Rightarrow$ $\neg\mathsf{Card}(\#s) \vee \neg\mathcal{O}(\#s) \vee \neg\langle\exists f \mid 1\text{--}1(f)\ \&\ \mathbf{range}(f) = s\ \&\ \mathbf{domain}(f) = \#s\rangle$

-- For suppose the contrary, i. e. either $\#s$ is not a cardinal or is not in 1-1 correspondence with s. Since by Theorem 121 $\#s$ is an ordinal in 1-1 correspondence with s, it follows that $\#s$ must not be a cardinal. Thus, by definition of 'cardinal', there must exist a member c of $\#s$ and a 1-1 map of c onto $\#s$.

$\langle s\rangle \hookrightarrow T121 \Rightarrow$ $\mathcal{O}(\#s)\ \&\ Stat1:$
$\langle\exists f \mid 1\text{--}1(f)\ \&\ \mathbf{range}(f) = s\ \&\ \mathbf{domain}(f) = \#s\rangle\ \&\ Stat2:\ \neg\langle\exists o \in \#s, g \mid 1\text{--}1(g)\ \&\ \mathbf{range}(g) = s\ \&\ \mathbf{domain}(g) = o\rangle$
ELEM $\Rightarrow$ $\neg\mathsf{Card}(\#s)$
Use_def(Card) $\Rightarrow$ $\neg\big(\mathcal{O}(\#s)\ \&\ \langle\forall y \in \#s, ff \mid \mathbf{domain}(ff) \neq y \vee \mathbf{range}(ff) \neq \#s \vee \neg\mathsf{Svm}(ff)\rangle\big)$
ELEM $\Rightarrow$ $Stat3:\ \neg\langle\forall y \in \#s, ff \mid \mathbf{domain}(ff) \neq y \vee \mathbf{range}(ff) \neq \#s \vee \neg\mathsf{Svm}(ff)\rangle$
$\langle c, g\rangle \hookrightarrow Stat3 \Rightarrow$ $c \in \#s\ \&\ \mathbf{domain}(g) = c\ \&\ \mathbf{range}(g) = \#s\ \&\ \mathsf{Svm}(g)$
$\langle \#s, c\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(c)$

-- By Theorem 113, g has a 1-1 partial inverse h, which maps $\#s$ to a subset of c, so the inverse of h is a 1-1 mapping of this subset ssc onto $\#s$.

$\langle g\rangle \hookrightarrow T113 \Rightarrow$ $Stat4:\ \langle\exists h \mid \mathbf{domain}(h) = \mathbf{range}(g)\ \&\ \mathbf{range}(h) \subseteq \mathbf{domain}(g)\ \&\ 1\text{--}1(h)\ \&\ \langle\forall x \in \mathbf{range}(g) \mid g\upharpoonright(h\upharpoonright x) = x\rangle\rangle$
$\langle h\rangle \hookrightarrow Stat4 \Rightarrow$ $\mathbf{domain}(h) = \#s\ \&\ \mathbf{range}(h) \subseteq c\ \&\ 1\text{--}1(h)$
$\langle h\rangle \hookrightarrow T91 \Rightarrow$ $\mathbf{range}(h^{\leftarrow}) = \#s\ \&\ \mathbf{domain}(h^{\leftarrow}) \subseteq c\ \&\ 1\text{--}1(h^{\leftarrow})$

-- However, Theorem 129 tells us that there is a 1-1 map f of an ordinal contained in c onto ssc. By Theorem 31, this ordinal must either be c or a member of c, and so must be a member of $\#s$ in any case.

$\langle c, \mathbf{domain}(h^{\leftarrow})\rangle \hookrightarrow T129 \Rightarrow$ $Stat5:\ \langle\exists f \mid 1\text{--}1(f)\ \&\ \mathcal{O}(\mathbf{domain}(f))\ \&\ \mathbf{domain}(f) \subseteq c\ \&\ \mathbf{range}(f) = \mathbf{domain}(h^{\leftarrow})\rangle$
$\langle f\rangle \hookrightarrow Stat5 \Rightarrow$ $1\text{--}1(f)\ \&\ \mathcal{O}(\mathbf{domain}(f))\ \&\ \mathbf{domain}(f) \subseteq c\ \&\ \mathbf{range}(f) = \mathbf{domain}(h^{\leftarrow})$
$\langle c, \mathbf{domain}(f)\rangle \hookrightarrow T31 \Rightarrow$ $\mathbf{domain}(f) \in c \vee \mathbf{domain}(f) = c$
$\langle \#s, c\rangle \hookrightarrow T31 \Rightarrow$ $c \subseteq \#s$
ELEM $\Rightarrow$ $Stat6:\ \mathbf{domain}(f) \in \#s$

-- But now the map product $h^{\leftarrow}\bullet f$ is a 1-1 map of a member of $\#s$ onto $\#s$, so if we let ff be a 1-1 map of $\#s$ to s, $ff\bullet(h^{\leftarrow}\bullet f)$ is a 1-1 map of a member of $\#s$ onto s.

$\langle f, h^{\leftarrow}\rangle \hookrightarrow T86 \Rightarrow$ $\mathbf{domain}(h^{\leftarrow}\bullet f) = \mathbf{domain}(f)\ \&\ \mathbf{range}(h^{\leftarrow}\bullet f) = \#s$
$\langle h^{\leftarrow}, f\rangle \hookrightarrow T108 \Rightarrow$ $1\text{--}1(h^{\leftarrow}\bullet f)$
$\langle ff\rangle \hookrightarrow Stat1 \Rightarrow$ $1\text{--}1(ff)\ \&\ \mathbf{range}(ff) = s\ \&\ \mathbf{domain}(ff) = \#s$
$\langle h^{\leftarrow}\bullet f, ff\rangle \hookrightarrow T86 \Rightarrow$ $\mathbf{domain}(ff\bullet(h^{\leftarrow}\bullet f)) = \mathbf{domain}(f)\ \&\ \mathbf{range}(ff\bullet(h^{\leftarrow}\bullet f)) = s$

$\langle \mathsf{ff}, \mathsf{h}^{\leftarrow} \bullet \mathsf{f} \rangle \hookrightarrow T108 \Rightarrow \quad 1\text{–}1(\mathsf{ff} \bullet (\mathsf{h}^{\leftarrow} \bullet \mathsf{f}))$

-- By Stat7 7, this is impossible, a contradiction which proves our theorem.

$\langle \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat8: \ \mathbf{domain}(\mathsf{f}) \in \#\mathsf{s} \rightarrow \neg \langle \exists \mathsf{g} \mid 1\text{–}1(\mathsf{g}) \ \& \ \mathbf{range}(\mathsf{g}) = \mathsf{s} \ \& \ \mathbf{domain}(\mathsf{g}) = \mathbf{domain}(\mathsf{f}) \rangle$
$\langle \mathsf{ff} \bullet (\mathsf{h}^{\leftarrow} \bullet \mathsf{f}) \rangle \hookrightarrow Stat8 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Theorem 130 implies that any two sets which are in 1-1 correspondence have the same cardinality.

Theorem 159 (131) $1\text{–}1(\mathsf{F}) \rightarrow \#\mathbf{range}(\mathsf{F}) = \#\mathbf{domain}(\mathsf{F})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{h}) \Rightarrow \quad 1\text{–}1(\mathsf{h}) \ \& \ \#\mathbf{range}(\mathsf{h}) \neq \#\mathbf{domain}(\mathsf{h})$

-- We proceed by contradiction, and so suppose that there exists a 1-1 map whose range and domain have different cardinalities. Since each of these two is are in 1-1 correspondence with its cardinality, $\#\mathbf{range}(\mathsf{h})$ and $\#\mathbf{domain}(\mathsf{h})$ are in 1-1 correspondence with each other.

$\langle \mathbf{range}(\mathsf{h}) \rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#\mathbf{range}(\mathsf{h})) \ \& \ Stat1: \ \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathbf{range}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathbf{range}(\mathsf{h}) \rangle$
$\langle \mathsf{f} \rangle \hookrightarrow Stat1 \Rightarrow \quad 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathbf{range}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathbf{range}(\mathsf{h})$
$\langle \mathbf{domain}(\mathsf{h}) \rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#\mathbf{domain}(\mathsf{h})) \ \& \ Stat2: \ \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathbf{domain}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathbf{domain}(\mathsf{h}) \rangle$
$\langle \mathsf{g} \rangle \hookrightarrow Stat2 \Rightarrow \quad 1\text{–}1(\mathsf{g}) \ \& \ \mathbf{range}(\mathsf{g}) = \mathbf{domain}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{g}) = \#\mathbf{domain}(\mathsf{h})$
$\langle \mathsf{f} \rangle \hookrightarrow T91 \Rightarrow \quad 1\text{–}1(\mathsf{f}^{\leftarrow}) \ \& \ \mathbf{range}(\mathsf{f}^{\leftarrow}) = \#\mathbf{range}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{f}^{\leftarrow}) = \mathbf{range}(\mathsf{h})$
$\langle \mathsf{h}, \mathsf{g} \rangle \hookrightarrow T108 \Rightarrow \quad 1\text{–}1(\mathsf{h} \bullet \mathsf{g})$
$\langle \mathsf{g}, \mathsf{h} \rangle \hookrightarrow T86 \Rightarrow \quad \mathbf{range}(\mathsf{h} \bullet \mathsf{g}) = \mathbf{range}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{h} \bullet \mathsf{g}) = \#\mathbf{domain}(\mathsf{h})$
$\langle \mathsf{f}^{\leftarrow}, \mathsf{h} \bullet \mathsf{g} \rangle \hookrightarrow T108 \Rightarrow \quad 1\text{–}1(\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}))$
$\langle \mathsf{h} \bullet \mathsf{g}, \mathsf{f}^{\leftarrow} \rangle \hookrightarrow T86 \Rightarrow \quad \mathbf{range}(\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g})) = \#\mathbf{range}(\mathsf{h}) \ \& \ \mathbf{domain}(\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g})) = \#\mathbf{domain}(\mathsf{h})$
$\langle \mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}) \rangle \hookrightarrow T91 \Rightarrow \quad 1\text{–}1\big((\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}))^{\leftarrow}\big) \ \& \ \mathbf{range}\big((\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}))^{\leftarrow}\big) = \#\mathbf{domain}(\mathsf{h}) \ \& \ \mathbf{domain}\big((\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}))^{\leftarrow}\big) = \#\mathbf{range}(\mathsf{h})$

-- But by Theorem 28 one of the two distinct ordinals $\#\mathbf{range}(\mathsf{h})$ and $\#\mathbf{domain}(\mathsf{h})$, must be a member of the other, even though both are cardinals, so neither can be in 1-1 correspondence with anything smaller. This contradiction proves our theorem.

$\langle \#\mathbf{range}(\mathsf{h}), \#\mathbf{domain}(\mathsf{h}) \rangle \hookrightarrow T28 \Rightarrow \quad \#\mathbf{range}(\mathsf{h}) \in \#\mathbf{domain}(\mathsf{h}) \vee \#\mathbf{domain}(\mathsf{h}) \in \#\mathbf{range}(\mathsf{h})$
$\langle \#\mathbf{range}(\mathsf{h}) \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#\mathbf{range}(\mathsf{h}))$
$\langle \#\mathbf{domain}(\mathsf{h}) \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#\mathbf{domain}(\mathsf{h}))$
$\mathsf{Suppose} \Rightarrow \quad \#\mathbf{range}(\mathsf{h}) \in \#\mathbf{domain}(\mathsf{h})$
$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad Stat3: \ \langle \forall \mathsf{y} \in \#\mathbf{domain}(\mathsf{h}), \mathsf{f} \mid \mathbf{domain}(\mathsf{f}) \neq \mathsf{y} \vee \mathbf{range}(\mathsf{f}) \neq \#\mathbf{domain}(\mathsf{h}) \vee \neg \mathsf{Svm}(\mathsf{f}) \rangle$
$\langle \#\mathbf{range}(\mathsf{h}), (\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}))^{\leftarrow} \rangle \hookrightarrow Stat3 \Rightarrow \quad \neg \mathsf{Svm}\Big((\mathsf{f}^{\leftarrow} \bullet (\mathsf{h} \bullet \mathsf{g}))^{\leftarrow}\Big)$

$\mathsf{Use\_def(1\text{–}1)} \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $\#\mathbf{domain}(h) \in \#\mathbf{range}(h)$
$\mathsf{Use\_def(Card)} \Rightarrow$ $Stat4: \langle \forall y \in \#\mathbf{range}(h), f \,|\, \mathbf{domain}(f) \neq y \vee \mathbf{range}(f) \neq \#\mathbf{range}(h) \vee \neg \mathsf{Svm}(f) \rangle$
$\langle \#\mathbf{domain}(h), f^{\leftarrow} \bullet (h \bullet g) \rangle \hookrightarrow Stat4 \Rightarrow$ $\neg \mathsf{Svm}(f^{\leftarrow} \bullet (h \bullet g))$
$\mathsf{Use\_def(1\text{–}1)} \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- The following corollary rounds out Theorem 131 by proving that two sets can be in 1-1 correspondence if and only if they have the same cardinality. It is this result that captures the essence of the notion of cardinality.

Theorem 160 (132) $\langle \exists F \,|\, 1\text{–}1(F) \;\&\; \mathbf{range}(F) = S \;\&\; \mathbf{domain}(F) = T \rangle \leftrightarrow \#S = \#T$. PROOF:

$\mathsf{Suppose\_not}(s,t) \Rightarrow$ $(\neg \langle \exists f \,|\, 1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = t \rangle \;\&\; \#s = \#t) \vee (\langle \exists f \,|\, 1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = t \rangle \;\&\; \#s \neq \#t)$

-- We proceed by contradiction. Since Theorem 131 implies that s and t cannot have different cardinalities if they are in 1-1 correspondence, it must be that they have the same cardinality but are not in 1-1 correspondence.

$\mathsf{Suppose} \Rightarrow$ $Stat1: \langle \exists f \,|\, 1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = t \rangle \;\&\; \#s \neq \#t$
$\langle f \rangle \hookrightarrow Stat1 \Rightarrow$ $1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = t$
$\langle f \rangle \hookrightarrow T131 \Rightarrow$ $\#\mathbf{range}(f) = \#\mathbf{domain}(f)$
$\mathsf{EQUAL} \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $Stat2: \neg \langle \exists f \,|\, 1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = t \rangle \;\&\; \#s = \#t$

-- But by Theorem 130, s and t are in 1-1 correspondence with their respective cardinalities #s and #t, by mappings g and h respectively.

$\langle s \rangle \hookrightarrow T130 \Rightarrow$ $Stat3: \langle \exists f \,|\, 1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = \#s \rangle$
$\langle t \rangle \hookrightarrow T130 \Rightarrow$ $Stat4: \langle \exists f \,|\, 1\text{–}1(f) \;\&\; \mathbf{range}(f) = t \;\&\; \mathbf{domain}(f) = \#t \rangle$
$\langle g \rangle \hookrightarrow Stat3 \Rightarrow$ $1\text{–}1(g) \;\&\; \mathbf{range}(g) = s \;\&\; \mathbf{domain}(g) = \#s$
$\langle h \rangle \hookrightarrow Stat4 \Rightarrow$ $1\text{–}1(h) \;\&\; \mathbf{range}(h) = t \;\&\; \mathbf{domain}(h) = \#t$

-- Hence the map $g \bullet h^{\leftarrow}$, whose domain and range are t and s respectively, is 1-1, contradicting Stat5 5 and thereby proving our theorem.

$\langle h \rangle \hookrightarrow T91 \Rightarrow$ $1\text{–}1(h^{\leftarrow}) \;\&\; \mathbf{range}(h^{\leftarrow}) = \#t \;\&\; \mathbf{domain}(h^{\leftarrow}) = t$
$\langle g, h^{\leftarrow} \rangle \hookrightarrow T108 \Rightarrow$ $1\text{–}1(g \bullet h^{\leftarrow})$
$\langle h^{\leftarrow}, g \rangle \hookrightarrow T86 \Rightarrow$ $\mathbf{range}(g \bullet h^{\leftarrow}) = \mathbf{range}(g) \;\&\; \mathbf{domain}(g \bullet h^{\leftarrow}) = t$
$\langle g \bullet h^{\leftarrow} \rangle \hookrightarrow Stat2 \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- It follows from Theorem 132 that the enumerating ordinal of a set s, like any other set t in 1-1 correspondence with s, has the same cardinality as s.

-- The enumerating ordinal of a set has the same cardinality as the set

Theorem 161 (133) $\langle \exists o \,|\, \mathcal{O}(o) \,\&\, S = \{\mathsf{enum}(x, S) : x \in o\} \,\&\, \#o = \#S\rangle$. PROOF:

$\mathsf{Suppose\_not}(s_1) \Rightarrow$ $Stat1: \neg\langle \exists o \,|\, \mathcal{O}(o) \,\&\, s_1 = \{\mathsf{enum}(x, s_1) : x \in o\} \,\&\, \#o = \#s_1\rangle$

-- Suppose the contrary, so that no ordinal which enum puts in 1-1 correspondence with s has the same cardinality as s_1. Theorem 41 tells us that there is some ordinal o which enum puts in 1-1 correspondence with s_1, i. e. enum defines a 1-1 function f whose domain is o and whose range is s_1. Thus o and s_1 have the same cardinality, and so o is a counterexample to Stat2 2, a contradiction which proves our theorem.

$\langle s_1\rangle \hookrightarrow T41 \Rightarrow$ $Stat3: \langle \exists o \,|\, (\mathcal{O}(o) \,\&\, s_1 = \{\mathsf{enum}(x, s_1) : x \in o\}) \,\&\, \langle \forall y \in o, z \in o \,|\, y \neq z \rightarrow \mathsf{enum}(y, s_1) \neq \mathsf{enum}(z, s_1)\rangle\rangle$
$\langle o\rangle \hookrightarrow Stat3 \Rightarrow$ $\mathcal{O}(o) \,\&\, s_1 = \{\mathsf{enum}(x, s_1) : x \in o\} \,\&\, Stat4: \langle \forall y \in o, z \in o \,|\, y \neq z \rightarrow \mathsf{enum}(y, s_1) \neq \mathsf{enum}(z, s_1)\rangle$
$\mathsf{Loc\_def} \Rightarrow$ $f = \{[x, \mathsf{enum}(x, s_1)] : x \in o\}$
$\mathsf{APPLY}\ \langle x_\Theta : a, y_\Theta : b\rangle\ \mathsf{fcn\_symbol}(f(x) \mapsto \mathsf{enum}(x, s_1), g \mapsto f, s \mapsto o) \Rightarrow$
$\mathsf{Svm}(f) \,\&\, \mathbf{domain}(f) = o \,\&\, \mathbf{range}(f) = \{\mathsf{enum}(x, s_1) : x \in o\} \,\&\, (a, b \in o \,\&\, \mathsf{enum}(a, s_1) = \mathsf{enum}(b, s_1) \,\&\, a \neq b) \vee 1\text{–}1(f)$
$\mathsf{Suppose} \Rightarrow$ $Stat5: a, b \in o \,\&\, \mathsf{enum}(a, s_1) = \mathsf{enum}(b, s_1) \,\&\, a \neq b$
$\langle a, b\rangle \hookrightarrow Stat4 \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $1\text{–}1(f)$
$\mathsf{ELEM} \Rightarrow$ $1\text{–}1(f) \,\&\, \mathbf{domain}(f) = o \,\&\, \mathbf{range}(f) = s_1$
$\langle s_1, o\rangle \hookrightarrow T132 \Rightarrow$ $Stat6: \langle \exists f \,|\, 1\text{–}1(f) \,\&\, \mathbf{range}(f) = s_1 \,\&\, \mathbf{domain}(f) = o\rangle \rightarrow \#s_1 = \#o$
$\langle f\rangle \hookrightarrow Stat6 \Rightarrow$ $\#o = \#s_1$
$\langle o\rangle \hookrightarrow Stat1 \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- Theorem 131 also has the two following corollaries, which state facts basic to the arithmetic theory of infinite cardinals. The first of these is the associative law for cardinal arithmetic, which follows as an elementary consequence of Theorem 118.

-- Associative Law for Cardinals

Theorem 162 (134) $\#(A \times B \times C) = \#(A \times (B \times C))$. PROOF:

$\mathsf{Suppose\_not}(a, b, c) \Rightarrow$ $\#(a \times b \times c) \neq \#(a \times (b \times c))$
$\mathsf{Loc\_def} \Rightarrow$ $f = \{[[[x, y], zz], [x, [y, zz]]] : x \in a, y \in b, zz \in c\}$
$\langle f, a, b, c\rangle \hookrightarrow T118 \Rightarrow$ $1\text{–}1(f) \,\&\, \mathbf{domain}(f) = a \times b \times c \,\&\, \mathbf{range}(f) = a \times (b \times c)$
$\mathsf{EQUAL} \Rightarrow$ $\#\mathbf{domain}(f) \neq \#\mathbf{range}(f)$
$\langle f\rangle \hookrightarrow T131 \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- The following proof of the associative law for cardinal arithmetic is equally elementary, this time as a consequence of Theorem 119.

-- Commutative Law for Cardinals

Theorem 163 (135) $\#(\mathsf{A}\times\mathsf{B}) = \#(\mathsf{B}\times\mathsf{A})$. PROOF:

$\mathsf{Suppose\_not}(a,b) \Rightarrow \quad \#(a\times b) \neq \#(b\times a)$
$\mathsf{Loc\_def} \Rightarrow \quad f = \{[[x,y],[y,x]] : x\in a, y\in b\}$
$\langle f,a,b\rangle \hookrightarrow T119 \Rightarrow \quad 1\text{–}1(f) \,\&\, \mathbf{domain}(f) = a\times b \,\&\, \mathbf{range}(f) = b\times a$
$\mathsf{EQUAL} \Rightarrow \quad \#\mathbf{domain}(f) = \#(a\times b) \,\&\, \#\mathbf{range}(f) = \#(b\times a)$
$\langle f\rangle \hookrightarrow T131 \Rightarrow \quad \#(a\times b) = \#(b\times a)$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following utility lemma simply notes that only the null set can have cardinality $\emptyset$.

Theorem 164 (136) $\#\mathsf{S} = \emptyset \leftrightarrow \mathsf{S} = \emptyset$. PROOF:

$\mathsf{Suppose\_not}(s) \Rightarrow \quad \neg(\#s = \emptyset \leftrightarrow s = \emptyset)$

-- For since any set s is in 1-1 correspondence with its cardinality, our result follows immediately from Theorem 78

$\langle s\rangle \hookrightarrow T121 \Rightarrow \quad Stat1 : \langle \exists f \mid 1\text{–}1(f) \,\&\, \mathbf{range}(f) = s \,\&\, \mathbf{domain}(f) = \#s\rangle$
$\langle f\rangle \hookrightarrow Stat1 \Rightarrow \quad 1\text{–}1(f) \,\&\, \mathbf{range}(f) = s \,\&\, \mathbf{domain}(f) = \#s$
$\langle f\rangle \hookrightarrow T78 \Rightarrow \quad \mathbf{range}(f) = \emptyset \leftrightarrow \mathbf{domain}(f) = \emptyset$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- It is also useful to state the following variant of the preceding lemma, which notes that every nonzero cardinal is larger than $\emptyset$

Theorem 165 (137) $\emptyset \in \#\mathsf{S} \leftrightarrow \mathsf{S} \neq \emptyset$. PROOF:

$\mathsf{Suppose\_not}(s) \Rightarrow \quad (\emptyset \in \#s \,\&\, s = \emptyset) \vee (\emptyset \notin \#s \,\&\, s \neq \emptyset)$
$\langle s\rangle \hookrightarrow T136 \Rightarrow \quad \emptyset \notin \#s \,\&\, \#s \neq \emptyset$
$T121 \Rightarrow \quad \mathcal{O}(\#s)$
$\mathsf{Suppose} \Rightarrow \quad \neg\mathcal{O}(\emptyset)$
$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad \neg\langle \forall x\in\emptyset \mid x \subseteq \emptyset\rangle \vee \neg\langle \forall x\in\emptyset, y\in\emptyset \mid x\in y \vee y\in x \vee x = y\rangle$
$\mathsf{Suppose} \Rightarrow \quad Stat1 : \neg\langle \forall x\in\emptyset \mid x\subseteq\emptyset\rangle$
$\langle c\rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat2 : \neg\langle \forall x\in\emptyset, y\in\emptyset \mid x\in y \vee y\in x \vee x = y\rangle$
$\langle d\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathcal{O}(\emptyset)$
$\langle \#s, \emptyset\rangle \hookrightarrow T28 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that a set s is a cardinal if and only if it is its own cardinal.

Theorem 166 (138) $\mathsf{Card}(\mathsf{S}) \leftrightarrow \mathsf{S} = \#\mathsf{S}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s}) \Rightarrow \quad \big(\mathsf{Card}(\mathsf{s})\ \&\ \mathsf{s} \neq \#\mathsf{s}\big) \vee \big(\neg\mathsf{Card}(\mathsf{s})\ \&\ \mathsf{s} = \#\mathsf{s}\big)$

-- We proceed by contradiction. Since Theorem 130 tells us that $\#\mathsf{s}$ is a cardinal with which s is in 1-1 correspondence, only the first of the two cases displayed above need be considered. In this case s and $\#\mathsf{s}$ are both cardinals, hence ordinals, so one must be a member of the other.

$\langle\mathsf{s}\rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#\mathsf{s})\ \&\ \mathit{Stat1}: \ \big\langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{s}\ \&\ \mathbf{domain}(\mathsf{f}) = \#\mathsf{s}\big\rangle$
$\mathsf{Suppose} \Rightarrow \quad \neg\mathsf{Card}(\mathsf{s})\ \&\ \mathsf{s} = \#\mathsf{s}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{s} \neq \#\mathsf{s}$
$\langle\mathsf{f}\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{s}\ \&\ \mathbf{domain}(\mathsf{f}) = \#\mathsf{s}$
$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad \mathcal{O}(\mathsf{s})\ \&\ \mathit{Stat2}: \ \big\langle \forall \mathsf{y} \in \mathsf{s}, \mathsf{f} \mid \mathbf{domain}(\mathsf{f}) \neq \mathsf{y} \vee \mathbf{range}(\mathsf{f}) \neq \mathsf{s} \vee \neg\mathsf{Svm}(\mathsf{f})\big\rangle$
$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad \mathcal{O}(\#\mathsf{s})\ \&\ \mathit{Stat3}: \ \big\langle \forall \mathsf{y} \in \#\mathsf{s}, \mathsf{f} \mid \mathbf{domain}(\mathsf{f}) \neq \mathsf{y} \vee \mathbf{range}(\mathsf{f}) \neq \#\mathsf{s} \vee \neg\mathsf{Svm}(\mathsf{f})\big\rangle$
$\langle\mathsf{s}, \#\mathsf{s}\rangle \hookrightarrow T28 \Rightarrow \quad \mathsf{s} \in \#\mathsf{s} \vee \#\mathsf{s} \in \mathsf{s}$

-- But Stat4 4 (resp. Stat4 5) tells us that $\#\mathsf{s}$ cannot be a member of s (resp. s cannot be a member of $\#\mathsf{s}$), so we have a contradiction which proves our theorem.

$\mathsf{Suppose} \Rightarrow \quad \#\mathsf{s} \in \mathsf{s}$
$\langle\#\mathsf{s}, \mathsf{f}\rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \neg\mathsf{Svm}(\mathsf{f})$
$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{s} \in \#\mathsf{s}$
$\langle\mathsf{f}\rangle \hookrightarrow T91 \Rightarrow \quad 1\text{–}1(\mathsf{f}^{\leftarrow})\ \&\ \mathbf{range}(\mathsf{f}^{\leftarrow}) = \#\mathsf{s}\ \&\ \mathbf{domain}(\mathsf{f}^{\leftarrow}) = \mathsf{s}$
$\langle\mathsf{s}, \mathsf{f}^{\leftarrow}\rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \neg\mathsf{Svm}(\mathsf{f}^{\leftarrow})$
$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- In the following corollary we note that if c is a cardinal and is in 1-1 correspondence with a set s, then c is $\#\mathsf{s}$.

-- Uniqueness of Cardinality

Theorem 167 (139) $\mathsf{Card}(\mathsf{C})\ \&\ \big\langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{S}\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{C}\big\rangle \rightarrow \mathsf{C} = \#\mathsf{S}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{c}, \mathsf{s}) \Rightarrow \quad \mathsf{Card}(\mathsf{c})\ \&\ \mathit{Stat1}: \ \big\langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{S}\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{C}\big\rangle\ \&\ \mathsf{c} \neq \#\mathsf{s}$
$\langle\mathsf{f}\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{Card}(\mathsf{c})\ \&\ 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{s}\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{c}\ \&\ \mathsf{c} \neq \#\mathsf{s}$

-- Suppose the contrary, i. e. that there is a cardinal c in 1-1 correspondence with s which is different from $\#\mathsf{s}$.

$\langle\mathsf{f}\rangle \hookrightarrow T131 \Rightarrow \quad \#\mathbf{range}(\mathsf{f}) = \#\mathbf{domain}(\mathsf{f})$
$\mathsf{EQUAL} \Rightarrow \quad \#\mathsf{c} \neq \mathsf{c}$

$\langle c \rangle \hookrightarrow T138 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next two results are also simple corollaries of the foregoing: the cardinality operator "#" is idempotent, and any two distinct cardinals are ordered by membership.

Theorem 168 (140) $\#S = \#\#S$. PROOF:

Suppose_not(s) $\Rightarrow$ $\#s \neq \#\#s$
$\langle s \rangle \hookrightarrow T130 \Rightarrow$ $\mathrm{Card}(\#s)$
$\langle \#s \rangle \hookrightarrow T138 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 169 (141) $\#S \in \#T \vee \#S = \#T \vee \#T \in \#S$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\neg(\#s \in \#t \vee \#s = \#t \vee \#t \in \#s)$
$\langle s \rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#s)$
$\langle t \rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#t)$
$\langle \#s, \#t \rangle \hookrightarrow T28 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we note that the ordering of cardinals by membership is transitive, simply because cardinals must be ordinals and our claim holds for ordinals by Theorem 31.

Theorem 170 (142) $\#S \in \#T \,\&\, \#T \in \#R \rightarrow \#S \in \#R$. PROOF:

Suppose_not(s, t, r) $\Rightarrow$ $\#s \in \#t \,\&\, \#t \in \#r \,\&\, \#s \notin \#r$
$\langle r \rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#r)$
$\langle \#r, \#t \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(\#t)$
$\langle \#r, \#t \rangle \hookrightarrow T31 \Rightarrow$ $\mathcal{O}(\#r) \,\&\, \mathcal{O}(\#t) \rightarrow (\#t \subseteq \#r \leftrightarrow \#t \in \#r \vee \#t = \#r)$
ELEM $\Rightarrow$ $\#s \notin \#r$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We can use the preceding results to prove that subsets of an ordinal have a cardinality that is no larger than the ordinal. This is just a bit short of the more general result (which follows as Theorem 144) that subsets of a set s can never have a cardinality greater than #s.

-- Subsets of an ordinal have a cardinality that is no larger than the ordinal

Theorem 171 (143) $\mathcal{O}(S) \,\&\, T \subseteq S \rightarrow \#T \subseteq S$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s},\mathsf{t}) \Rightarrow \quad \mathcal{O}(\mathsf{s})\ \&\ \mathsf{t} \subseteq \mathsf{s}\ \&\ \#\mathsf{t} \not\subseteq \mathsf{s}$

-- Proceeding by contradiction, let the ordinal s and its subset t be a counterexample. By the preceding theorem, there is an ordinal u no larger than s which enum puts into 1-1 correspondence with t.

$\langle \mathsf{s},\mathsf{t}\rangle \hookrightarrow T128 \Rightarrow \quad Stat1:\ \langle \exists \mathsf{x} \subseteq \mathsf{s} \mid \mathcal{O}(\mathsf{x})\ \&\ \mathsf{t} = \{\mathsf{enum}(\mathsf{y},\mathsf{t}) : \mathsf{y} \in \mathsf{x}\}\ \&\ \langle \forall \mathsf{y} \in \mathsf{x}, \mathsf{z} \in \mathsf{x} \mid \mathsf{y} \neq \mathsf{z} \rightarrow \mathsf{enum}(\mathsf{y},\mathsf{t}) \neq \mathsf{enum}(\mathsf{z},\mathsf{t})\rangle\rangle$
$\langle \mathsf{u}\rangle \hookrightarrow Stat1 \Rightarrow \quad Stat10:\ \mathsf{u} \subseteq \mathsf{s}\ \&\ \mathcal{O}(\mathsf{u})\ \&\ \mathsf{t} = \{\mathsf{enum}(\mathsf{y},\mathsf{t}) : \mathsf{y} \in \mathsf{u}\}\ \&\ Stat2:\ \langle \forall \mathsf{y} \in \mathsf{u}, \mathsf{zz} \in \mathsf{u} \mid \mathsf{y} \neq \mathsf{zz} \rightarrow \mathsf{enum}(\mathsf{y},\mathsf{t}) \neq \mathsf{enum}(\mathsf{zz},\mathsf{t})\rangle$
$\mathsf{Loc\_def} \Rightarrow \quad \mathsf{g} = \{[\mathsf{x}, \mathsf{enum}(\mathsf{x},\mathsf{t})] : \mathsf{x} \in \mathsf{u}\}$
$\mathsf{APPLY}\ \langle \mathsf{x}_\Theta : \mathsf{x}, \mathsf{y}_\Theta : \mathsf{y}\rangle\ \mathsf{fcn\_symbol}(\mathsf{f}(\mathsf{x}) \mapsto \mathsf{enum}(\mathsf{x},\mathsf{t}), \mathsf{g} \mapsto \mathsf{g}, \mathsf{s} \mapsto \mathsf{u}) \Rightarrow$
$\quad \mathsf{Svm}(\mathsf{g})\ \&\ \mathbf{range}(\mathsf{g}) = \{\mathsf{enum}(\mathsf{x},\mathsf{t}) : \mathsf{x} \in \mathsf{u}\}\ \&\ \mathbf{domain}(\mathsf{g}) = \mathsf{u}\ \&\ (\mathsf{x},\mathsf{y} \in \mathsf{u}\ \&\ \mathsf{enum}(\mathsf{x},\mathsf{t}) = \mathsf{enum}(\mathsf{y},\mathsf{t})\ \&\ \mathsf{x} \neq \mathsf{y}) \vee 1\text{–}1(\mathsf{g})$
$\langle \mathsf{x},\mathsf{y}\rangle \hookrightarrow Stat2 \Rightarrow \quad 1\text{–}1(\mathsf{g})$

-- Theorem 131 now tells us that $\#\mathsf{t} = \#\mathsf{u}$. But it is easy to see that $\mathsf{u} \supseteq \#\mathsf{u}$ for every ordinal u.

$\langle \mathsf{g}\rangle \hookrightarrow T131 \Rightarrow \quad \#\mathbf{range}(\mathsf{g}) = \#\mathbf{domain}(\mathsf{g})$
$\mathsf{EQUAL} \Rightarrow \quad \#\mathsf{t} = \#\mathsf{u}$
$\mathsf{Suppose} \Rightarrow \quad Stat3:\ \mathsf{u} \not\supseteq \#\mathsf{u}$
$\langle \mathsf{u}\rangle \hookrightarrow T130 \Rightarrow \quad Stat11:\ \mathsf{Card}(\#\mathsf{u})\ \&\ \mathcal{O}(\#\mathsf{u})\ \&\ Stat4:\ \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{u}\ \&\ \mathbf{domain}(\mathsf{f}) = \#\mathsf{u}\rangle$
$\langle \mathsf{f}\rangle \hookrightarrow Stat4 \Rightarrow \quad Stat21:\ 1\text{–}1(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{u}\ \&\ \mathbf{domain}(\mathsf{f}) = \#\mathsf{u}$
$\langle \mathsf{f}\rangle \hookrightarrow T91(\langle Stat21\rangle) \Rightarrow \quad 1\text{–}1(\mathsf{f}^{\leftarrow})\ \&\ \mathbf{domain}(\mathsf{f}^{\leftarrow}) = \mathsf{u}\ \&\ \mathbf{range}(\mathsf{f}^{\leftarrow}) = \#\mathsf{u}$
$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad Stat14:\ \mathsf{Svm}(\mathsf{f}^{\leftarrow})$
$\langle \mathsf{u}, \#\mathsf{u}\rangle \hookrightarrow T26(\langle Stat3, Stat11, Stat10\rangle) \Rightarrow \quad Stat12:\ \#\mathsf{u} \supseteq \mathsf{u}$
$\langle \#\mathsf{u}, \mathsf{u}\rangle \hookrightarrow T31(\langle Stat10, Stat11, Stat12, Stat3\rangle) \Rightarrow \quad \mathsf{u} \in \#\mathsf{u}$
$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad Stat5:\ \langle \forall \mathsf{y} \in \#\mathsf{u}, \mathsf{f} \mid \mathbf{domain}(\mathsf{f}) \neq \mathsf{y} \vee \mathbf{range}(\mathsf{f}) \neq \#\mathsf{u} \vee \neg \mathsf{Svm}(\mathsf{f})\rangle$
$\langle \mathsf{u}, \mathsf{f}^{\leftarrow}\rangle \hookrightarrow Stat5 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{u} \supseteq \#\mathsf{u}$

-- But now, since $\mathsf{s} \supseteq \mathsf{u} \supseteq \#\mathsf{u} = \#\mathsf{t}$, we must have $\mathsf{s} \supseteq \#\mathsf{t}$, contradicting our original assumption and thus proving our theorem.

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- The following theorem, which the preceding theorem anticipates, states the basic fact that the cardinality of a subset t of s can be no larger than the cardinality of s.

-- Subset cardinality theorem

Theorem 172 (144) $T \subseteq \mathsf{S} \rightarrow \#T \subseteq \#\mathsf{S}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{t},\mathsf{s}) \Rightarrow \quad Stat1:\ \mathsf{t} \subseteq \mathsf{s}\ \&\ \#\mathsf{t} \not\subseteq \#\mathsf{s}$

-- For suppose that t is a subset of s but has a larger cardinality. Then s is in 1-1 correspondence with #s by some map f, and so the inverse of f maps t to a subset of #s in 1-1, and hence cardinality-preserving, fashion.

$\langle s \rangle \hookrightarrow T130 \Rightarrow$ *Stat2*: $\mathsf{Card}(\#s)$ & $\langle \exists f \mid 1\text{–}1(f)$ & $\mathbf{range}(f) = s$ & $\mathbf{domain}(f) = \#s \rangle$
$\langle f \rangle \hookrightarrow Stat2 \Rightarrow$ $1\text{–}1(f)$ & $\mathbf{range}(f) = s$ & $\mathbf{domain}(f) = \#s$
$\langle f \rangle \hookrightarrow T91 \Rightarrow$ $1\text{–}1(f^{\leftarrow})$ & $f = f^{\leftarrow\leftarrow}$ & $\mathbf{range}(f^{\leftarrow}) = \mathbf{domain}(f)$ & $\mathbf{domain}(f^{\leftarrow}) = \mathbf{range}(f)$
$\langle f^{\leftarrow}, t \rangle \hookrightarrow T53 \Rightarrow$ $1\text{–}1(f^{\leftarrow}{}_{|t})$
$\langle f^{\leftarrow}{}_{|t} \rangle \hookrightarrow T131 \Rightarrow$ *Stat3*: $\#\mathbf{domain}(f^{\leftarrow}{}_{|t}) = \#\mathbf{range}(f^{\leftarrow}{}_{|t})$
$\langle f^{\leftarrow}, t \rangle \hookrightarrow T72 \Rightarrow$ $\mathbf{range}(f^{\leftarrow}{}_{|t}) \subseteq \#s$

-- Since the cardinal #s is an ordinal, it follows by Theorem 143 that $\#\mathbf{range}(f^{\leftarrow}{}_{|t})$ is not larger than #s. But since $f^{\leftarrow}{}_{|t}$ is 1-1 and has domain t, $\#\mathbf{range}(f^{\leftarrow}{}_{|t}) = \#t$; so $\#t \subseteq \#s$, a contradiction which proves our theorem.

$\langle s \rangle \hookrightarrow T121 \Rightarrow$ $\mathcal{O}(\#s)$
$\langle \#s, \mathbf{range}(f^{\leftarrow}{}_{|t}) \rangle \hookrightarrow T143 \Rightarrow$ *Stat4*: $\#\mathbf{range}(f^{\leftarrow}{}_{|t}) \subseteq \#s$
$\langle f^{\leftarrow}, t \rangle \hookrightarrow T84 \Rightarrow$ $\mathbf{domain}(f^{\leftarrow}{}_{|t}) = t$
EQUAL $\Rightarrow$ *Stat5*: $\#\mathbf{domain}(f^{\leftarrow}{}_{|t}) = \#t$
$\langle Stat1, Stat3, Stat4, Stat5 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following somewhat more general result tells us that the range of a single-valued map is never more numerous than its domain.

Theorem 173 (145) $\mathsf{Svm}(F) \rightarrow \#\mathbf{range}(F) \subseteq \#\mathbf{domain}(F)$. PROOF:

Suppose_not(f) $\Rightarrow$ $\mathsf{Svm}(f)$ & $\#\mathbf{range}(f) \not\subseteq \#\mathbf{domain}(f)$
$\langle f \rangle \hookrightarrow T113 \Rightarrow$ *Stat1*: $\langle \exists h \mid (\mathbf{domain}(h) = \mathbf{range}(f)$ & $\mathbf{range}(h) \subseteq \mathbf{domain}(f)$ & $1\text{–}1(h))$ & $\langle \forall x \in \mathbf{range}(f) \mid f\upharpoonright(h\upharpoonright x) = x \rangle \rangle$
$\langle h \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathbf{domain}(h) = \mathbf{range}(f)$ & $\mathbf{range}(h) \subseteq \mathbf{domain}(f)$ & $1\text{–}1(h)$
$\langle \mathbf{range}(h), \mathbf{domain}(f) \rangle \hookrightarrow T144 \Rightarrow$ $\#\mathbf{range}(h) \subseteq \#\mathbf{domain}(f)$
EQUAL $\Rightarrow$ $\#\mathbf{domain}(h) = \#\mathbf{range}(f)$
$\langle h \rangle \hookrightarrow T131 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the domain and range of any map f (and indeed of any set f) is no greater than f.

Theorem 174 (146) $\#\mathbf{domain}(F) \subseteq \#F$. PROOF:

Suppose_not(f) $\Rightarrow$ $\#\mathbf{domain}(f) \not\subseteq \#f$

-- For **domain**(f) is a single-valued image of f by the map $\{[x, x^{[1]}] : x \in f\}$, and so the present theorem is a corollary of Theorem 145.

Loc_def ⇒ $g = \{[x, x^{[1]}] : x \in f\}$
APPLY $\langle\,\rangle$ fcn_symbol $(f(x) \mapsto x^{[1]}, g \mapsto g, s \mapsto f)$ ⇒
$Svm(g)$ & $\mathbf{domain}(g) = f$ & $\mathbf{range}(g) = \{x^{[1]} : x \in f\}$
Use_def(**domain**) ⇒ $\mathbf{range}(g) = \mathbf{domain}(f)$
$\langle g \rangle \hookrightarrow T145$ ⇒ $\#\mathbf{range}(g) \subseteq \#\mathbf{domain}(g)$
EQUAL ⇒ $\#\mathbf{domain}(f) \subseteq \#f$
ELEM ⇒ false; Discharge ⇒ QED

Theorem 175 (147) $\#\mathbf{range}(F) \subseteq \#F$. PROOF:

Suppose_not(f) ⇒ $\#\mathbf{range}(f) \nsubseteq \#f$

-- For **range**(f) is a single-valued image of f by the map $\{[x, x^{[2]}] : x \in f\}$, and so the present theorem is a corollary of Theorem 145.

Loc_def ⇒ $g = \{[x, x^{[2]}] : x \in f\}$
APPLY $\langle\,\rangle$ fcn_symbol $(f(x) \mapsto x^{[2]}, g \mapsto g, s \mapsto f)$ ⇒
$Svm(g)$ & $\mathbf{domain}(g) = f$ & $\mathbf{range}(g) = \{x^{[2]} : x \in f\}$
Use_def(**range**) ⇒ $\mathbf{range}(g) = \mathbf{range}(f)$
$\langle g \rangle \hookrightarrow T145$ ⇒ $\#\mathbf{range}(g) \subseteq \#\mathbf{domain}(g)$
EQUAL ⇒ $\#\mathbf{range}(f) \subseteq \#f$
ELEM ⇒ false; Discharge ⇒ QED

-- If the map f is single-valued, the inequality given by Theorem 146 can be sharpened to an equality:

Theorem 176 (148) $Svm(F) \rightarrow \#\mathbf{domain}(F) = \#F$. PROOF:

Suppose_not(f) ⇒ $Svm(f)$ & $\#\mathbf{domain}(f) \neq \#f$

-- For in this case **domain**(f) is easily seen to be a 1-1 image of f by the map $\{[x, x^{[1]}] : x \in f\}$, and so the present theorem is a corollary of theorem 131.

Loc_def ⇒ $g = \{[u, u^{[1]}] : u \in f\}$
APPLY $\langle x_\Theta : x, y_\Theta : y \rangle$ fcn_symbol $(f(x) \mapsto x^{[1]}, g \mapsto g, s \mapsto f)$ ⇒
$Svm(g)$ & $\mathbf{domain}(g) = f$ & $\mathbf{range}(g) = \{x^{[1]} : x \in f\}$ & $(x, y \in f$ & $x^{[1]} = y^{[1]}$ & $x \neq y) \vee 1\text{–}1(g)$

Use_def(**domain**) $\Rightarrow$ $\mathbf{range}(g) = \mathbf{domain}(f)$
Suppose $\Rightarrow$ $\neg 1\text{–}1(g)$
ELEM $\Rightarrow$ $x, y \in f \;\&\; x^{[1]} = y^{[1]} \;\&\; x \neq y$
Use_def(Svm) $\Rightarrow$ *Stat1* : $\langle \forall x \in f, y \in f \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\langle x, y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ $1\text{–}1(g)$
$\langle g \rangle \hookrightarrow$ *T131* $\Rightarrow$ $\#\mathbf{domain}(g) = \#\mathbf{range}(g)$
EQUAL $\Rightarrow$ $\#\mathbf{domain}(f) = \#f$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that the restriction of a single-valued map f to any subset a of **domain**(f) has the same cardinality as a.

Theorem 177 (10000*a*) $\mathrm{Svm}(F) \;\&\; \mathbf{domain}(F) \supseteq A \rightarrow \#F_{|A} = \#A$. PROOF:

Suppose_not(f, a) $\Rightarrow$ $\mathrm{Svm}(f) \;\&\; \mathbf{domain}(f) \supseteq a \;\&\; \#f_{|a} \neq \#a$
$\langle f, a \rangle \hookrightarrow$ *T84* $\Rightarrow$ $\mathbf{domain}(f_{|a}) = a$
EQUAL $\Rightarrow$ $\#\mathbf{domain}(f_{|a}) = \#a$
$\langle f, a \rangle \hookrightarrow$ *T52* $\Rightarrow$ $\mathrm{Svm}(f_{|a})$
$\langle f_{|a} \rangle \hookrightarrow$ *T148* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The preceding theorems allow us to add an additional result to our utility fcn_symbol theory. This simply states the fact that the cardinality of a single-valued map is the cardinality of its domain, and is at least as large as the cardinality of its range.

ENTER_THEORY fcn_symbol
-- Add an additional result to the fcn_symbol theory

Theorem 178 (fcn_symbol$_1$) $\#\{[xx, f(xx)] : xx \in s\} = \#s \;\&\; \#\{f(xx) : xx \in s\} \subseteq \#s$. PROOF:

Suppose_not(s) $\Rightarrow$ $\neg(\#\{[x, f(x)] : x \in s\} = \#s \;\&\; \#\{f(x) : x \in s\} \subseteq \#s)$
Assump $\Rightarrow$ $g = \{[x, f(x)] : x \in s\}$
Tfcn_symbol $\cdot$ 1 $\Rightarrow$ $\mathbf{domain}(g) = s$
Tfcn_symbol $\cdot$ 5 $\Rightarrow$ $\mathbf{range}(g) = \{f(x) : x \in s\}$
Tfcn_symbol $\cdot$ 7 $\Rightarrow$ $\mathrm{Svm}(g)$
$\langle g \rangle \hookrightarrow$ *T148* $\Rightarrow$ $\#\mathbf{domain}(g) = \#g$
EQUAL $\Rightarrow$ $\#\{[x, f(x)] : x \in s\} = \#s$
$\langle g \rangle \hookrightarrow$ *T147* $\Rightarrow$ $\#\mathbf{range}(g) \subseteq \#g$
EQUAL $\Rightarrow$ $\#\{f(x) : x \in s\} \subseteq \#\{[x, f(x)] : x \in s\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- Return to the top - level theory

-- Next we show that if t is not null, the condition #s ⊇ #t is equivalent to the existence of a single-valued map of s onto t.

Theorem 179 (149) $\#S \supseteq \#T \leftrightarrow T = \emptyset \lor \langle \exists f \mid \mathrm{Svm}(f) \mathbin{\&} \mathbf{domain}(f) = S \mathbin{\&} \mathbf{range}(f) = T \rangle$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\big(\#s \supseteq \#t \mathbin{\&} t \neq \emptyset \mathbin{\&} \neg\langle \exists f \mid \mathrm{Svm}(f) \mathbin{\&} \mathbf{domain}(f) = s \mathbin{\&} \mathbf{range}(f) = t\rangle\big) \lor$
$\#s \not\supseteq \#t \mathbin{\&} t = \emptyset \lor \langle \exists f \mid \mathrm{Svm}(f) \mathbin{\&} \mathbf{domain}(f) = s \mathbin{\&} \mathbf{range}(f) = t\rangle$

-- Proceed by contradiction, and so suppose that either (Case 1) #s is at least as large as #t but there exists no single-valued map of s onto t, or that (Case 2) there exists such a map but that #t is larger than #s. Consider Case 1 first. By Theorem 121, there are 1-1 maps h and g which respectively send #s and #t to s and t.

Suppose $\Rightarrow$ $\#s \supseteq \#t \mathbin{\&} t \neq \emptyset \mathbin{\&}$ *Stat1* : $\neg\langle \exists ff \mid \mathrm{Svm}(ff) \mathbin{\&} \mathbf{domain}(ff) = s \mathbin{\&} \mathbf{range}(ff) = t\rangle$
$\langle s \rangle \hookrightarrow T121 \Rightarrow$ *Stat2* : $\langle \exists f \mid 1\text{–}1(f) \mathbin{\&} \mathbf{range}(f) = s \mathbin{\&} \mathbf{domain}(f) = \#s\rangle$
$\langle h \rangle \hookrightarrow Stat2 \Rightarrow$ $1\text{–}1(h) \mathbin{\&} \mathbf{range}(h) = s \mathbin{\&} \mathbf{domain}(h) = \#s$
$\langle t \rangle \hookrightarrow T121 \Rightarrow$ *Stat3* : $\langle \exists f \mid 1\text{–}1(f) \mathbin{\&} \mathbf{range}(f) = t \mathbin{\&} \mathbf{domain}(f) = \#t\rangle$
$\langle g \rangle \hookrightarrow Stat3 \Rightarrow$ $1\text{–}1(g) \mathbin{\&} \mathbf{range}(g) = t \mathbin{\&} \mathbf{domain}(g) = \#t$

-- Moreover, we can easily define a single-valued map of #s onto its subset #t. Simply note that ∅ is an element of #t, and map all elements of #s which belong to #t into themselves, and all other elements of #s onto ∅.

$\langle t \rangle \hookrightarrow T137 \Rightarrow$ $\emptyset \in \#t$
Loc_def $\Rightarrow$ $f = \{[x, \mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi}] : x \in \#s\}$
APPLY $\langle\ \rangle$ fcn_symbol$\big(f(x) \mapsto \mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi}, g \mapsto f, s \mapsto \#s\big) \Rightarrow$
$\mathrm{Svm}(f) \mathbin{\&} \mathbf{domain}(f) = \#s \mathbin{\&} \mathbf{range}(f) = \{\mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi} : x \in \#s\}$
Suppose $\Rightarrow$ *Stat4* : $\mathbf{range}(f) \neq \#t$
$\langle c \rangle \hookrightarrow Stat4 \Rightarrow$ $(c \in \{\mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi} : x \in \#s\} \mathbin{\&} c \notin \#t) \lor (c \notin \{\mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi} : x \in \#s\} \mathbin{\&} c \in \#t)$
Suppose $\Rightarrow$ *Stat5* : $c \notin \{\mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi} : x \in \#s\} \mathbin{\&} c \in \#t$
$\langle c \rangle \hookrightarrow Stat5 \Rightarrow$ $c \neq \mathbf{if}\ c \in \#t\ \mathbf{then}\ c\ \mathbf{else}\ \emptyset\ \mathbf{fi} \lor c \notin \#s$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6* : $c \in \{\mathbf{if}\ x \in \#t\ \mathbf{then}\ x\ \mathbf{else}\ \emptyset\ \mathbf{fi} : x \in \#s\} \mathbin{\&} c \notin \#t$
$\langle d \rangle \hookrightarrow Stat6 \Rightarrow$ $c = \mathbf{if}\ d \in \#t\ \mathbf{then}\ d\ \mathbf{else}\ \emptyset\ \mathbf{fi}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathrm{Svm}(f) \mathbin{\&} \mathbf{domain}(f) = \#s \mathbin{\&} \mathbf{range}(f) = \#t$

-- Theorem 91 now tells us that h has a 1-1 inverse which maps s onto #s. Hence g•(f•h←) is a single-valued map of s onto t, showing that Case 1 is impossible.

$\langle h \rangle \hookrightarrow T91 \Rightarrow$ $1\text{–}1(h^{\leftarrow})$ & $\mathbf{range}(h^{\leftarrow}) = \#s$ & $\mathbf{domain}(h^{\leftarrow}) = s$
Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(h)$ & $\mathsf{Svm}(g)$ & $\mathsf{Svm}(h^{\leftarrow})$
$\langle f, h^{\leftarrow} \rangle \hookrightarrow T103 \Rightarrow$ $\mathsf{Svm}(f \bullet h^{\leftarrow})$
$\langle h^{\leftarrow}, f \rangle \hookrightarrow T86 \Rightarrow$ $\mathbf{domain}(f \bullet h^{\leftarrow}) = s$ & $\mathbf{range}(f \bullet h^{\leftarrow}) = \#t$
$\langle g, f \bullet h^{\leftarrow} \rangle \hookrightarrow T103 \Rightarrow$ $\mathsf{Svm}\left(g \bullet (f \bullet h^{\leftarrow})\right)$
$\langle f \bullet h^{\leftarrow}, g \rangle \hookrightarrow T86 \Rightarrow$ $\mathbf{domain}(g \bullet (f \bullet h^{\leftarrow})) = s$ & $\mathbf{range}(g \bullet (f \bullet h^{\leftarrow})) = t$
$\langle g \bullet (f \bullet h^{\leftarrow}) \rangle \hookrightarrow Stat1 \Rightarrow$ false; -

-- So only Case2 remains to be considered. But in this case there is a single-valued map hh of s onto t, and so theorems 133, 144, and 145 lead to an immediate contradiction, proving our theorem.

Discharge $\Rightarrow$ $\#s \not\supseteq \#t$ & $t = \emptyset \vee \langle \exists f \mid \mathsf{Svm}(f)$ & $\mathbf{domain}(f) = s$ & $\mathbf{range}(f) = t \rangle$
ELEM $\Rightarrow$ $\#t \neq \emptyset$
$\langle t \rangle \hookrightarrow T136 \Rightarrow$ $t \neq \emptyset$
ELEM $\Rightarrow$ $Stat7:$ $\langle \exists f \mid \mathsf{Svm}(f)$ & $\mathbf{domain}(f) = s$ & $\mathbf{range}(f) = t \rangle$
$\langle hh \rangle \hookrightarrow Stat7 \Rightarrow$ $\mathsf{Svm}(hh)$ & $\mathbf{domain}(hh) = s$ & $\mathbf{range}(hh) = t$
$\langle hh \rangle \hookrightarrow T148 \Rightarrow$ $\#hh = \#\mathbf{domain}(hh)$
EQUAL $\Rightarrow$ $\#hh = \#s$
$\langle hh \rangle \hookrightarrow T147 \Rightarrow$ $\#hh \supseteq \#\mathbf{range}(hh)$
EQUAL $\Rightarrow$ $\#hh \supseteq \#t$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The inverse image of a set under a map is the image of the set under the inverse of the map. This notion, which appears in many arguments, has the following direct definition.

-- Inverse Map Image

DEF 14*d*. $X \urcorner Y \quad =_{\text{Def}} \quad \{p^{[1]} : p \in X \mid p^{[2]} \in Y\}$

-- We can use the foregoing defintion to prove various elementary properties of the inverse image, beginning with the following, which in effect tells us that the inverse image operation is additive.

Theorem 180 (150) $\mathbf{domain}(G) = G \urcorner R \cup G \urcorner \mathbf{range}(G) \backslash R$. PROOF:

Suppose_not(g, r) $\Rightarrow$ $\mathbf{domain}(g) \neq g \urcorner r \cup g \urcorner \mathbf{range}(g) \backslash r$

-- Forif not, expansion of the definitions involved brigns us to the inequality between setformers seen below.

Use_def (**domain**) $\Rightarrow$ $\{p^{[1]} : p \in g\} \neq g \upharpoonleft r \cup g \upharpoonleft \mathbf{range}(g) \backslash r$

Use_def ($\upharpoonleft$) $\Rightarrow$ $\{p^{[1]} : p \in g\} \neq \{p^{[1]} : p \in g \mid p^{[2]} \in r\} \cup \{p^{[1]} : p \in g \mid p^{[2]} \in \mathbf{range}(g) \backslash r\}$

-- But it is easily seen, using set monotonicity that the left side of this last inequality includes the right, and so the right side must fail to include the left. Hence here is a point q in the left-hand set but in neither of those on the right.

Set_monot $\Rightarrow$ $\{p^{[1]} : p \in g\} \supseteq \{p^{[1]} : p \in g \mid p^{[2]} \in r\}$

Set_monot $\Rightarrow$ $\{p^{[1]} : p \in g \mid p^{[2]} \in \mathbf{range}(g) \backslash r\} \subseteq \{p^{[1]} : p \in g\}$

ELEM $\Rightarrow$ *Stat1* : $\{p^{[1]} : p \in g\} \not\subseteq \{p^{[1]} : p \in g \mid p^{[2]} \in r\} \cup \{p^{[1]} : p \in g \mid p^{[2]} \in \mathbf{range}(g) \backslash r\}$

$\langle q \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $q \in \{p^{[1]} : p \in g\}$ & *Stat3* : $q \notin \{p^{[1]} : p \in g \mid p^{[2]} \in r\}$ & *Stat4* : $q \notin \{p^{[1]} : p \in g \mid p^{[2]} \in \mathbf{range}(g) \backslash r\}$

-- It follows immediately that $q = p^{[1]}$ for some $p \in g$, but that $q \notin \mathbf{range}(g)$, an impossibility which completes our proof.

$\langle p \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $q = p^{[1]}$ & $p \in g$

$\langle p \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $p^{[2]} \notin r$

$\langle p \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $p^{[2]} \notin \mathbf{range}(g)$

Use_def (**range**) $\Rightarrow$ *Stat5* : $p^{[2]} \notin \{p^{[2]} : p \in g\}$

$\langle p \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the inverse images of two disjoint sets by a single-valued map are disjoint.

Theorem 181 (151) $S \cap R = \emptyset$ & $\mathrm{Svm}(G) \rightarrow G \upharpoonleft S \cap G \upharpoonleft R = \emptyset$. PROOF:

Suppose_not (s, r, g) $\Rightarrow$ $s \cap r = \emptyset$ & $\mathrm{Svm}(g)$ & $g \upharpoonleft s \cap g \upharpoonleft r \neq \emptyset$

-- For if not there must exist points p and q, in s and r respectively, such that $p^{[1]} = q^{[1]}$.

Use_def ($\upharpoonleft$) $\Rightarrow$ *Stat1* : $\{p^{[1]} : p \in g \mid p^{[2]} \in s\} \cap \{p^{[1]} : p \in g \mid p^{[2]} \in r\} \neq \emptyset$

$\langle c \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $c \in \{p^{[1]} : p \in g \mid p^{[2]} \in s\}$ & *Stat3* : $c \in \{p^{[1]} : p \in g \mid p^{[2]} \in r\}$

$\langle p \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $c = p^{[1]}$ & $p \in g$ & $p^{[2]} \in s$

$\langle q \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $c = q^{[1]}$ & $q \in g$ & $q^{[2]} \in r$

Use_def (Svm) $\Rightarrow$ *Stat4* : $\langle \forall x \in g, y \in g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$

-- But then $p = q$, so that $p^{[2]} = q^{[2]}$ must belong to both s and r, a contradiction which completes our proof.

$\langle p, q\rangle \hookrightarrow Stat4 \Rightarrow \quad p = q$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following entirely elementary lemma tells us that the inverse image by a map g of any element in the range of g is a nonempty set.

Theorem 182 (152) $Y \in \mathbf{range}(G) \rightarrow G \upharpoonleft \{Y\} \neq \emptyset$. PROOF:

Suppose_not$(y, r) \Rightarrow \quad y \in \mathbf{range}(g) \;\&\; g \upharpoonleft \{y\} = \emptyset$
Use_def$(\mathbf{range}) \Rightarrow \quad Stat1: \; y \in \{p^{[2]} : p \in g\}$
Use_def$(\upharpoonleft) \Rightarrow \quad Stat2: \; \{p^{[1]} : p \in g \mid p^{[2]} \in \{y\}\} = \emptyset$
$\langle p\rangle \hookrightarrow Stat1 \Rightarrow \quad y = p^{[2]} \;\&\; p \in g$
$\langle p\rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following easy lemma tells us that the "INV_IM" operator instroduced above is the same as 'range of inverse', i. e. $f \upharpoonleft r \neq \mathbf{range}(f^{\leftarrow}{}_{|r})$.

Theorem 183 (153) $G \upharpoonleft R = \mathbf{range}(G^{\leftarrow}{}_{|R})$. PROOF:

Suppose_not$(f, r) \Rightarrow \quad f \upharpoonleft r \neq \mathbf{range}(f^{\leftarrow}{}_{|r})$

-- For the proof, we have only to expand our assertion using the defintions of the operators involved and simplify the resulting setformers.

Use_def$(\mathbf{range}) \Rightarrow \quad f \upharpoonleft r \neq \{q^{[2]} : q \in f^{\leftarrow}{}_{|r}\}$
Use_def$(|) \Rightarrow \quad f \upharpoonleft r \neq \{q^{[2]} : q \in \{s \in f^{\leftarrow} \mid s^{[1]} \in r\}\}$
Use_def$(^{\leftarrow}) \Rightarrow \quad f \upharpoonleft r \neq \{q^{[2]} : q \in \{s \in \{[t^{[2]}, t^{[1]}] : t \in f\} \mid s^{[1]} \in r\}\}$
SIMPLF $\Rightarrow \quad \{s^{[2]} : s \in \{[t^{[2]}, t^{[1]}] : t \in f\} \mid s^{[1]} \in r\} = \{q^{[2]} : q \in \{s \in \{[t^{[2]}, t^{[1]}] : t \in f\} \mid s^{[1]} \in r\}\}$
SIMPLF $\Rightarrow \quad \left\{[t^{[2]}, t^{[1]}]^{[2]} : t \in f \mid [t^{[2]}, t^{[1]}]^{[1]} \in r\right\} = \{s^{[2]} : s \in \{[t^{[2]}, t^{[1]}] : t \in f\} \mid s^{[1]} \in r\}$
ELEM $\Rightarrow \quad f \upharpoonleft r \neq \left\{[t^{[2]}, t^{[1]}]^{[2]} : t \in f \mid [t^{[2]}, t^{[1]}]^{[1]} \in r\right\}$
Use_def$(\upharpoonleft) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that for single-valued maps, the inverse image $G \upharpoonleft R$ can be expressed in an evident way as the set of images of the individual members of R under G.

Theorem 184 (154) $\mathsf{Svm}(G^{\leftarrow}) \rightarrow G \upharpoonleft R = \{G^{\leftarrow} \restriction w : w \in R \cap \mathbf{range}(G)\}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{f},\mathsf{r}) \Rightarrow \quad \mathsf{Svm}(\mathsf{f}^{\leftarrow})\ \&\ \mathsf{f} \,\text{↰}\, \mathsf{r} \neq \{\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathsf{r} \cap \mathbf{range}(\mathsf{f})\}$

-- Suppose that f and r furnish a counterexample to our assertion. Since Theorems 152 and 65 allow us to write $\mathsf{f} \,\text{↰}\, \mathsf{r}$ as $\left\{\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\right\}$, the setformer inequality seen below would follow, and so there would exist an element c in one of these sets but not the other.

$\langle \mathsf{f},\mathsf{r}\rangle \hookrightarrow T153 \Rightarrow \quad \mathbf{range}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}}) \neq \{\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathsf{r} \cap \mathbf{range}(\mathsf{f})\}$

$\langle \mathsf{f}^{\leftarrow},\mathsf{r}\rangle \hookrightarrow T43 \Rightarrow \quad Stat1:\ \mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \subseteq \mathsf{f}^{\leftarrow}$

$\langle \mathsf{f}^{\leftarrow},\mathsf{r}\rangle \hookrightarrow T52 \Rightarrow \quad \mathsf{Svm}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})$

$\langle \mathsf{f}^{\leftarrow}{}_{|\mathsf{r}}\rangle \hookrightarrow T66 \Rightarrow \quad Stat2:\ \{\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathsf{r} \cap \mathbf{range}(\mathsf{f})\} \neq \left\{\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\right\}$

$\langle \mathsf{f}^{\leftarrow},\mathsf{r}\rangle \hookrightarrow T84 \Rightarrow \quad \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}}) = \mathbf{domain}(\mathsf{f}^{\leftarrow}) \cap \mathsf{r}$

$\langle \mathsf{f}\rangle \hookrightarrow T89 \Rightarrow \quad \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}}) = \mathsf{r} \cap \mathbf{range}(\mathsf{f})$

$\langle \mathsf{c}\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{c} \in \{\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathsf{r} \cap \mathbf{range}(\mathsf{f})\} \leftrightarrow \mathsf{c} \notin \left\{\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\right\}$

-- If c is in the first set but not the second, it would follow since $\mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}}) = \mathsf{r} \cap \mathbf{range}(\mathsf{f})$ that there was some y in $\mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})$ for which $\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} \neq \mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y}$, which is impossible. Hence c must belong to the second of our two sets but not the first.

$\mathsf{Suppose} \Rightarrow \quad Stat3:\ \mathsf{c} \in \{\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathsf{r} \cap \mathbf{range}(\mathsf{f})\}\ \&\ \mathsf{c} \notin \left\{\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\right\}$

$\langle \mathsf{y},\mathsf{y}\rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{y} \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\ \&\ \mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} \neq \mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y}$

$\langle \mathsf{f}^{\leftarrow},\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}},\mathsf{y}\rangle \hookrightarrow T106 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat4:\ \mathsf{c} \in \left\{\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\right\}\ \&\ \mathsf{c} \notin \{\mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y} :\ \mathsf{y} \in \mathsf{r} \cap \mathbf{range}(\mathsf{f})\}$

-- However an equally elementary contradiction results in this case also, completing our proof.

$\langle \mathsf{y}',\mathsf{y}'\rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{y}' \in \mathbf{domain}(\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}})\ \&\ \mathsf{f}^{\leftarrow} \upharpoonright \mathsf{y}' \neq \mathsf{f}^{\leftarrow}{}_{|\mathsf{r}} \upharpoonright \mathsf{y}'$

$\langle \mathsf{f}^{\leftarrow},\mathsf{f}^{\leftarrow}{}_{|\mathsf{r}},\mathsf{y}'\rangle \hookrightarrow T106 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- It is equally easy to show that for 1-1 maps g the inverse image under g of a singleton set $\{\mathsf{y}\}$ is the image of the point y under the inverse map $\mathsf{g}^{\leftarrow}$.

Theorem 185 (155) $1\text{–}1(\mathsf{G})\ \&\ \mathsf{Y} \in \mathbf{range}(\mathsf{G}) \rightarrow \mathsf{G} \,\text{↰}\, \{\mathsf{Y}\} = \{\mathsf{G}^{\leftarrow} \upharpoonright \mathsf{Y}\}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{g},\mathsf{y}) \Rightarrow \quad Stat1:\ 1\text{–}1(\mathsf{g})\ \&\ \mathsf{y} \in \mathbf{range}(\mathsf{g})\ \&\ \mathsf{g} \,\text{↰}\, \{\mathsf{y}\} \neq \{\mathsf{g}^{\leftarrow} \upharpoonright \mathsf{y}\}$

-- Indeed, the negative of our assertion reduces to the set theoretic inequality seen below, and since the first hand side of this inequality is plainly included in its left hand side, there would have to exist an element u in the first of these sets but not the second.

$\langle g \rangle \hookrightarrow T91 \Rightarrow \quad 1\text{–}1(g^{\leftarrow})$
$\text{Use\_def}(1\text{–}1) \Rightarrow \quad Stat2: \ \mathsf{Svm}(g^{\leftarrow})$
$\langle g, \{y\} \rangle \hookrightarrow T154([Stat1, Stat2]) \Rightarrow \quad \{g^{\leftarrow} \upharpoonright v : v \in \{y\} \cap \mathbf{range}(g)\} \neq \{g^{\leftarrow} \upharpoonright y\}$
$\text{Suppose} \Rightarrow \quad \{g^{\leftarrow} \upharpoonright v : v \in \{y\} \cap \mathbf{range}(g)\} \not\supseteq \{g^{\leftarrow} \upharpoonright y\}$
$\text{ELEM} \Rightarrow \quad Stat3: \ g^{\leftarrow} \upharpoonright y \notin \{g^{\leftarrow} \upharpoonright v : v \in \{y\} \cap \mathbf{range}(g)\}$
$\langle y \rangle \hookrightarrow Stat3 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat4: \ \{g^{\leftarrow} \upharpoonright v : v \in \{y\} \cap \mathbf{range}(g)\} \not\subseteq \{g^{\leftarrow} \upharpoonright y\}$
$\langle u \rangle \hookrightarrow Stat4 \Rightarrow \quad Stat5: \ u \in \{g^{\leftarrow} \upharpoonright v : v \in \{y\} \cap \mathbf{range}(g)\} \ \& \ u \neq g^{\leftarrow} \upharpoonright y$

-- But this supposition immediately leads to a contradiction wihc proves our theorem.

$\langle v \rangle \hookrightarrow Stat5 \Rightarrow \quad u = g^{\leftarrow} \upharpoonright v \ \& \ v = y \ \& \ u \neq g^{\leftarrow} \upharpoonright y$
$\text{EQUAL} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- Next we show that the range of any map g on the inverse image of a set r by g is the intersection of r with the range of g.

Theorem 186 (156) $\mathsf{Svm}(G) \rightarrow \mathbf{range}(G_{|G^{\urcorner}R}) = \mathbf{range}(G) \cap R$. PROOF:

$\text{Suppose\_not}(g, r) \Rightarrow \quad \mathsf{Svm}(g) \ \& \ Stat1: \ \mathbf{range}(g_{|g^{\urcorner}r}) \neq \mathbf{range}(g) \cap r$

$\langle c \rangle \hookrightarrow Stat1 \Rightarrow \quad c \in \mathbf{range}(g_{|g^{\urcorner}r}) \leftrightarrow c \notin \mathbf{range}(g) \vee c \notin r$
$\text{Use\_def}(|) \Rightarrow \quad c \in \mathbf{range}(\{q : q \in g \mid q^{[1]} \in g \urcorner r\}) \leftrightarrow c \notin \mathbf{range}(g) \vee c \notin r$
$\text{Use\_def}(\mathbf{range}) \Rightarrow \quad c \in \{p^{[2]} : p \in \{q : q \in g \mid q^{[1]} \in g \urcorner r\}\} \leftrightarrow c \notin \{p^{[2]} : p \in g\} \vee c \notin r$
$\text{SIMPLF} \Rightarrow \quad c \in \{q^{[2]} : q \in g \mid q^{[1]} \in g \urcorner r\} \leftrightarrow c \notin \{p^{[2]} : p \in g\} \vee c \notin r$
$\text{Set\_monot} \Rightarrow \quad \{q^{[2]} : q \in g \mid q^{[1]} \in g \urcorner r\} \subseteq \{p^{[2]} : p \in g\}$
$\text{ELEM} \Rightarrow \quad (c \in \{q^{[2]} : q \in g \mid q^{[1]} \in g \urcorner r\} \ \& \ c \notin r) \vee (c \notin \{q^{[2]} : q \in g \mid q^{[1]} \in g \urcorner r\} \ \& \ c \in \{p^{[2]} : p \in g\} \ \& \ c \in r)$
$\text{Suppose} \Rightarrow \quad Stat2: \ c \in \{q^{[2]} : q \in g \mid q^{[1]} \in g \urcorner r\} \ \& \ c \notin r$
$\langle q \rangle \hookrightarrow Stat2 \Rightarrow \quad c = q^{[2]} \ \& \ q \in g \ \& \ q^{[1]} \in g \urcorner r$
$\text{Use\_def}(\urcorner) \Rightarrow \quad Stat3: \ q^{[1]} \in \{p^{[1]} : p \in g \mid p^{[2]} \in r\}$
$\langle p \rangle \hookrightarrow Stat3 \Rightarrow \quad q^{[1]} = p^{[1]} \ \& \ p \in g \ \& \ p^{[2]} \in r$
$\text{Use\_def}(\mathsf{Svm}) \Rightarrow \quad Stat4: \ \langle \forall x \in g, y \in g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
$\langle q, p \rangle \hookrightarrow Stat4 \Rightarrow \quad q = p$

-- But then $c = q^{[2]} = p^{[2]}$, leading to a contradiction which eliminates this case.

$\text{ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat5: \ c \in \{p^{[2]} : p \in g\} \ \& \ c \notin \{q^{[2]} : q \in g \mid q^{[1]} \in g \urcorner r\} \ \& \ c \in r$
$\langle p', p' \rangle \hookrightarrow Stat5 \Rightarrow \quad c = p'^{[2]} \ \& \ p' \in g \ \& \ p'^{[1]} \notin g \urcorner r$
$\text{Use\_def}(\urcorner) \Rightarrow \quad Stat6: \ p'^{[1]} \notin \{t^{[1]} : t \in g \mid t^{[2]} \in r\}$

$\langle p' \rangle \hookrightarrow Stat6 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that for any single_valued map g and set a, the range of g on the inverse image of a under g is **range**(g) ∩ a.

Theorem 187 (157) $\mathsf{Svm}(G) \rightarrow \mathbf{range}(G_{|G \urcorner A}) = \mathbf{range}(G) \cap A$. PROOF:

Suppose_not(g, a) $\Rightarrow$ $\mathsf{Svm}(g) \;\&\; \mathbf{range}(g_{|g \urcorner a}) \neq \mathbf{range}(g) \cap a$

-- For supposing the contrary and using the definitions of the operators involved we are led to the set-theoretic inequality seen below.

Use_def(**range**) $\Rightarrow$ $\{p^{[2]} : p \in g_{|g \urcorner a}\} \neq \{p^{[2]} : p \in g\} \cap a$

Use_def(|) $\Rightarrow$ $\{p^{[2]} : p \in \{q : q \in g \,|\, q^{[1]} \in g \urcorner a\}\} \neq \{p^{[2]} : p \in g\} \cap a$

SIMPLF $\Rightarrow$ $\{p^{[2]} : p \in g \,|\, p^{[1]} \in g \urcorner a\} \neq \{p^{[2]} : p \in g\} \cap a$

Use_def(↰) $\Rightarrow$ $Stat1: \{p^{[2]} : p \in g \,|\, p^{[1]} \in \{q^{[1]} : q \in g \,|\, q^{[2]} \in a\}\} \neq \{p^{[2]} : p \in g\} \cap a$

$\langle c \rangle \hookrightarrow Stat1 \Rightarrow$ $c \in \{p^{[2]} : p \in g \,|\, p^{[1]} \in \{q^{[1]} : q \in g \,|\, q^{[2]} \in a\}\} \leftrightarrow c \notin \{p^{[2]} : p \in g\} \vee c \notin a$

Suppose $\Rightarrow$ $Stat2: c \in \{p^{[2]} : p \in g \,|\, p^{[1]} \in \{q^{[1]} : q \in g \,|\, q^{[2]} \in a\}\} \;\&\; c \notin \{p^{[2]} : p \in g\} \vee c \notin a$

$\langle p \rangle \hookrightarrow Stat2 \Rightarrow$ $c = p^{[2]} \;\&\; p \in g \;\&\; Stat3: p^{[1]} \in \{q^{[1]} : q \in g \,|\, q^{[2]} \in a\} \;\&\; c \notin \{p^{[2]} : p \in g\} \vee c \notin a$

$\langle q, p \rangle \hookrightarrow Stat3 \Rightarrow$ $c = p^{[2]} \;\&\; p \in g \;\&\; p^{[1]} = q^{[1]} \;\&\; q \in g \;\&\; q^{[2]} \in a \;\&\; c \notin a$

Use_def(Svm) $\Rightarrow$ $Stat4: \langle \forall x \in g, y \in g \,|\, x^{[1]} = y^{[1]} \rightarrow x = y \rangle$

$\langle p, q \rangle \hookrightarrow Stat4 \Rightarrow$ $p = q$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following simple lemma tells us that for single-valued maps g and sets a, b, the inverse image of b under the restriction of g to the inverse image of a is simply the intersection $g \urcorner a \cap g \urcorner b$.

Theorem 188 (158) $\mathsf{Svm}(G) \rightarrow G_{|G \urcorner A} \urcorner B = G \urcorner A \cap G \urcorner B$. PROOF:

Suppose_not(g, a, b) $\Rightarrow$ $\mathsf{Svm}(g) \;\&\; g_{|g \urcorner a} \urcorner b \neq g \urcorner a \cap g \urcorner b$

-- For supposing the contrary, and using the definitions of "*INV_IM*" and "*ON*", we are led to the setformer inequality seen below.

Use_def(↰) $\Rightarrow$ $\{w^{[1]} : w \in g_{|g \urcorner a} \,|\, w^{[2]} \in b\} \neq \{w^{[1]} : w \in g \,|\, w^{[2]} \in a\} \cap \{w^{[1]} : w \in g \,|\, w^{[2]} \in b\}$

Use_def(|) $\Rightarrow$ $\{w^{[1]} : w \in g_{|g \urcorner a} \,|\, w^{[2]} \in b\} = \{w^{[1]} : w \in \{u : u \in g \,|\, u^{[1]} \in g \urcorner a\} \,|\, w^{[2]} \in b\}$

SIMPLF $\langle\ \rangle \Rightarrow$ $\{w^{[1]} : w \in \{u : u \in g \,|\, u^{[1]} \in g \urcorner a\} \,|\, w^{[2]} \in b\} = \{w^{[1]} : w \in g \,|\, w^{[1]} \in g \urcorner a \;\&\; w^{[2]} \in b\}$

ELEM $\Rightarrow$ $\{w^{[1]} : w \in g \mid w^{[1]} \in g \,\text{↰}\, a \;\&\; w^{[2]} \in b\} \neq \{w^{[1]} : w \in g \mid w^{[2]} \in a\} \cap \{w^{[1]} : w \in g \mid w^{[2]} \in b\}$

-- One more use of the definitions of "INV_IM" reduces this last inequality to the following still more elementary form:

Use_def(↰) $\Rightarrow$ $\{w^{[1]} : w \in g \mid w^{[1]} \in \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; w^{[2]} \in b\} \neq$
$\{w^{[1]} : w \in g \mid w^{[2]} \in a\} \cap \{w^{[1]} : w \in g \mid w^{[2]} \in b\}$

-- But it is easy to see that the condition appearing in the left-hand setformer in this last inequality can be replaced by the condition $w^{[2]} \in a \;\&\; w^{[2]} \in b$.

Use_def(Svm) $\Rightarrow$ $Stat1 : \langle \forall x \in g, y \in g \mid x^{[1]} = y^{[1]} \rightarrow x = y \rangle$
Suppose $\Rightarrow$ $Stat2 : \{w^{[1]} : w \in g \mid w^{[1]} \in \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; w^{[2]} \in b\} \neq \{w^{[1]} : w \in g \mid w^{[2]} \in a \;\&\; w^{[2]} \in b\}$
$\langle c \rangle \hookrightarrow Stat2 \Rightarrow$
$c \in g \;\&$
$c^{[1]} \neq c^{[1]} \vee ((c^{[1]} \in \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; c^{[2]} \in b) \;\&\; \neg(c^{[2]} \in a \;\&\; c^{[2]} \in b)) \vee (\neg(c^{[1]} \in \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; c^{[2]} \in b) \;\&\; c^{[2]} \in a \;\&\; c^{[2]} \in b)$
Suppose $\Rightarrow$ $Stat3 : (c^{[1]} \in \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; c^{[2]} \in b) \;\&\; \neg(c^{[2]} \in a \;\&\; c^{[2]} \in b)$
$\langle v \rangle \hookrightarrow Stat3 \Rightarrow$ $v \in g \;\&\; c^{[1]} = v^{[1]} \;\&\; v^{[2]} \in a$
$\langle v, c \rangle \hookrightarrow Stat1 \Rightarrow$ $c = v$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg(c^{[1]} \in \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; c^{[2]} \in b) \;\&\; c^{[2]} \in a \;\&\; c^{[2]} \in b$
ELEM $\Rightarrow$ $Stat4 : c^{[1]} \notin \{u^{[1]} : u \in g \mid u^{[2]} \in a\} \;\&\; c^{[2]} \in b$
$\langle c \rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ $\{w^{[1]} : w \in g \mid w^{[2]} \in a \;\&\; w^{[2]} \in b\} \neq \{w^{[1]} : w \in g \mid w^{[2]} \in a\} \cap \{w^{[1]} : w \in g \mid w^{[2]} \in b\}$

-- But it is easy to see that this last inequality is impossible. Indeed, by set monotonicity the left-hand side of this inequality is included in the right, so that the inequality would imply the existence of a point u such that $u^{[2]} \in a$, $u^{[2]} \in b$, whiel the conjuction of these two conditions was false, and evident impossibility which completes the proof of our theorem.

Set_monot $\langle\,\rangle \Rightarrow$ $\{w^{[1]} : w \in g \mid w^{[2]} \in a \;\&\; w^{[2]} \in b\} \subseteq \{w^{[1]} : w \in g \mid w^{[2]} \in a\}$
Set_monot $\langle\,\rangle \Rightarrow$ $\{w^{[1]} : w \in g \mid w^{[2]} \in a \;\&\; w^{[2]} \in b\} \subseteq \{w^{[1]} : w \in g \mid w^{[2]} \in b\}$
ELEM $\Rightarrow$ $Stat5 : \{w^{[1]} : w \in g \mid w^{[2]} \in a \;\&\; w^{[2]} \in b\} \not\supseteq \{w^{[1]} : w \in g \mid w^{[2]} \in a\} \cap \{w^{[1]} : w \in g \mid w^{[2]} \in b\}$
$\langle d \rangle \hookrightarrow Stat5 \Rightarrow$ $d \in \{w^{[1]} : w \in g \mid w^{[2]} \in a\} \cap \{w^{[1]} : w \in g \mid w^{[2]} \in b\} \;\&\; Stat6 : d \notin \{w^{[1]} : w \in g \mid w^{[2]} \in a \;\&\; w^{[2]} \in b\}$
ELEM $\Rightarrow$ $Stat7 : d \in \{w^{[1]} : w \in g \mid w^{[2]} \in a\} \;\&\; Stat8 : d \in \{w^{[1]} : w \in g \mid w^{[2]} \in b\}$
$\langle u_1 \rangle \hookrightarrow Stat7 \Rightarrow$ $u_1 \in g \;\&\; d = {u_1}^{[1]} \;\&\; {u_1}^{[2]} \in a$
$\langle u_2 \rangle \hookrightarrow Stat8 \Rightarrow$ $u_2 \in g \;\&\; d = {u_2}^{[1]} \;\&\; {u_2}^{[2]} \in b$
$\langle u_1, u_2 \rangle \hookrightarrow Stat1 \Rightarrow$ $u_1 = u_2$
EQUAL $\Rightarrow$ ${u_1}^{[2]} \in b$
$\langle u_1 \rangle \hookrightarrow Stat6 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It follows trivially using the principle of set monotonicity that the expression $g \upharpoonleft a$ is monotone increasing in its second argument.

Theorem 189 (159) $B \subseteq A \rightarrow G \upharpoonleft B \subseteq G \upharpoonleft A$. PROOF:

Suppose_not$(b, a, g) \Rightarrow$ $b \subseteq a \,\&\, g \upharpoonleft b \not\subseteq g \upharpoonleft a$
Set_monot $\Rightarrow$ $\{p^{[1]} : p \in g \mid p^{[2]} \in b\} \subseteq \{p^{[1]} : p \in g \mid p^{[2]} \in a\}$
Use_def$(\upharpoonleft) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following simple lemma tells us that successive restriction of a map f, first to a set a and then to b, produces the same result as restriction of f to the intersection set $a \cap b$

Theorem 190 (160) $(F_{|A})_{|B} = F_{|A \cap B}$. PROOF:

Suppose_not$(f, a, b) \Rightarrow$ $(f_{|a})_{|b} \neq f_{|a \cap b}$

-- For a counterexample f, a, b would imply the existence of a point p which was in one of the two sets in the following quality but not the other, and evident impossibility which proves our assertion.

Use_def$(|) \Rightarrow$ $\{p : p \in f_{|a} \mid p^{[1]} \in b\} \neq \{p : p \in f \mid p^{[1]} \in a \cap b\}$
Use_def$(|) \Rightarrow$ $\{p : p \in f_{|a} \mid p^{[1]} \in b\} = \{p : p \in \{q : q \in f \mid q^{[1]} \in a\} \mid p^{[1]} \in b\}$
SIMPLF $\Rightarrow$ *Stat1* : $\{p : p \in f \mid p^{[1]} \in a \,\&\, p^{[1]} \in b\} \neq \{p : p \in f \mid p^{[1]} \in a \cap b\}$
Set_monot $\Rightarrow$ $\{p : p \in f \mid p^{[1]} \in a \,\&\, p^{[1]} \in b\} = \{p : p \in f \mid p^{[1]} \in a \cap b\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

6 Finiteness

-- Our arguments till now have concerned ordinals and cardinals irrespective of whether they are finite or infinite. Now we introduce the concept of finiteness and prove its basic properties, in preparation for introduction of the set of integers and derivation of the basic properties of integers.

-- Finiteness

DEF 18. Finite(X) $\leftrightarrow_{\text{Def}}$ $\neg\langle\exists f \mid 1\text{–}1(f) \,\&\, \mathbf{domain}(f) = X \,\&\, \mathbf{range}(f) \subseteq X \,\&\, X \neq \mathbf{range}(f)\rangle$

-- We begin our work with the finiteness concept by proving the elementary but basic fact that the null set is a finite cardinal.

-- 0 is a finite cardinal

Theorem 191 (161) $\mathcal{O}(\emptyset)\ \&\ \mathsf{Finite}(\emptyset)\ \&\ \mathsf{Card}(\emptyset)$. PROOF:

Suppose_not $\Rightarrow$ $\neg\big(\mathcal{O}(\emptyset)\ \&\ \mathsf{Finite}(\emptyset)\ \&\ \mathsf{Card}(\emptyset)\big)$

-- For the fact that $\emptyset$ is an ordinal is an immediate consequence of the definition of $\mathcal{O}$,

Suppose $\Rightarrow$ $\neg\mathcal{O}(\emptyset)$
Use_def$(\mathcal{O})$ $\Rightarrow$ $\neg\langle\forall x\in\emptyset\,|\,x\subseteq\emptyset\rangle \vee \neg\langle\forall x\in\emptyset, y\in\emptyset\,|\,x\in y\vee y\in x\vee x=y\rangle$
Suppose $\Rightarrow$ *Stat1* : $\neg\langle\forall x\in\emptyset\,|\,x\subseteq\emptyset\rangle$
$\langle c\rangle\hookrightarrow$*Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat2* : $\neg\langle\forall x\in\emptyset, y\in\emptyset\,|\,x\in y\vee y\in x\vee x=y\rangle$
$\langle d\rangle\hookrightarrow$*Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathcal{O}(\emptyset)$

-- Similarly, the fact that $\emptyset$ is a cardinal and is finite is an immediate consequence of the definitions involved.

Suppose $\Rightarrow$ $\neg\mathsf{Finite}(\emptyset)$
Use_def(Finite) $\Rightarrow$ *Stat3* : $\langle\exists f\,|\,1\text{–}1(f)\ \&\ \mathbf{domain}(f)=\emptyset\ \&\ \mathbf{range}(f)\subseteq\emptyset\ \&\ \emptyset\neq\mathbf{range}(f)\rangle$
$\langle f\rangle\hookrightarrow$*Stat3* $\Rightarrow$ $\mathbf{range}(f)\subseteq\emptyset\ \&\ \emptyset\neq\mathbf{range}(f)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Finite}(\emptyset)$
Suppose $\Rightarrow$ $\neg\mathsf{Card}(\emptyset)$
Use_def(Card) $\Rightarrow$ $\neg\mathcal{O}(\emptyset)\vee\neg\langle\forall y\in\emptyset, f\,|\,\mathbf{domain}(f)\neq y\vee\mathbf{range}(f)\neq\emptyset\vee\neg\mathsf{Svm}(f)\rangle$
ELEM $\Rightarrow$ *Stat4* : $\neg\langle\forall y\in\emptyset, f\,|\,\mathbf{domain}(f)\neq y\vee\mathbf{range}(f)\neq\emptyset\vee\neg\mathsf{Svm}(f)\rangle$
$\langle a\rangle\hookrightarrow$*Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem states the important but elementary fact that subsets of a finite set are finite.
-- A subset of a finite set is finite

Theorem 192 (162) $\mathsf{Finite}(S)\ \&\ S\supseteq T\rightarrow\mathsf{Finite}(T)$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\mathsf{Finite}(s)\ \&\ s\supseteq t\ \&\ \neg\mathsf{Finite}(t)$

-- For suppose that there existed a finite set s having an infinite subset t. Then by definition there would be a 1-1 map f of t into a proper subset of itself. This can be extended to a 1-1 map of s into a proper subset of itself simply by setting the extension to the identity map on s\t. But since s is finite by assumption, this is impossible.

Use_def(Finite) $\Rightarrow$ *Stat1* : $\neg\langle\exists g\,|\,1\text{–}1(g)\ \&\ \mathbf{domain}(g)=s\ \&\ \mathbf{range}(g)\subseteq s\ \&\ s\neq\mathbf{range}(g)\rangle$
Use_def(Finite) $\Rightarrow$ *Stat2* : $\langle\exists h\,|\,1\text{–}1(h)\ \&\ \mathbf{domain}(h)=t\ \&\ \mathbf{range}(h)\subseteq t\ \&\ t\neq\mathbf{range}(h)\rangle$

$\langle h\rangle \hookrightarrow Stat2 \Rightarrow$ $1\text{–}1(h)$ & $\mathbf{domain}(h) = t$ & $\mathbf{range}(h) \subseteq t$ & $t \neq \mathbf{range}(h)$
$\langle s\backslash t\rangle \hookrightarrow T94 \Rightarrow$ $1\text{–}1\left(\iota_{s\backslash t}\right)$ & $\mathbf{domain}(\iota_{s\backslash t}) = s\backslash t$ & $\mathbf{range}(\iota_{s\backslash t}) = s\backslash t$
$\langle h, \iota_{s\backslash t}\rangle \hookrightarrow T80 \Rightarrow$ $1\text{–}1\left(h \cup \iota_{s\backslash t}\right)$
$\langle h, \iota_{s\backslash t}\rangle \hookrightarrow T70 \Rightarrow$ $\mathbf{domain}(h \cup \iota_{s\backslash t}) = \mathbf{domain}(h) \cup (s\backslash t)$
$\langle h, \iota_{s\backslash t}\rangle \hookrightarrow T71 \Rightarrow$ $\mathbf{range}(h \cup \iota_{s\backslash t}) = \mathbf{range}(h) \cup (s\backslash t)$
$\langle h \cup \iota_{s\backslash t}\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that if the domain of a 1-1 map is finite, so is its range. This result is then easily generalized to single valued maps (Theorem 165 below).

Theorem 193 (163) $1\text{–}1(F) \rightarrow \mathsf{Finite}(\mathbf{domain}(F)) \longrightarrow \mathsf{Finite}(\mathbf{range}(F))$. PROOF:

Suppose_not$(f_2) \Rightarrow$ $1\text{–}1(f_2)$ & $\mathsf{Finite}(\mathbf{domain}(f_2))$ & $\neg\mathsf{Finite}(\mathbf{range}(f_2))$

-- If we suppose the contrary, then by definition there must exist a g mapping $\mathbf{range}(f_2)$ into a proper subset of itself. But then $f_2{}^{\leftarrow}\bullet(g\bullet f_2)$ is a 1-1 mapping of $\mathbf{domain}(f_2)$ into itself.

Use_def(Finite) $\Rightarrow$ $Stat1: \neg\langle \exists g \mid 1\text{–}1(g)$ & $\mathbf{domain}(g) = \mathbf{domain}(f_2)$ & $\mathbf{range}(g) \subseteq \mathbf{domain}(f_2)$ & $\mathbf{range}(g) \neq \mathbf{domain}(f_2)\rangle$
Use_def(Finite) $\Rightarrow$ $Stat2: \langle \exists g \mid 1\text{–}1(g)$ & $\mathbf{domain}(g) = \mathbf{range}(f_2)$ & $\mathbf{range}(g) \subseteq \mathbf{range}(f_2)$ & $\mathbf{range}(g) \neq \mathbf{range}(f_2)\rangle$
$\langle g\rangle \hookrightarrow Stat2 \Rightarrow$ $1\text{–}1(g)$ & $\mathbf{domain}(g) = \mathbf{range}(f_2)$ & $\mathbf{range}(g) \subseteq \mathbf{range}(f_2)$ & $\mathbf{range}(g) \neq \mathbf{range}(f_2)$
$\langle f_2\rangle \hookrightarrow T91 \Rightarrow$ $1\text{–}1(f_2{}^{\leftarrow})$ & $\mathbf{domain}(f_2) = \mathbf{range}(f_2{}^{\leftarrow})$ & $\mathbf{range}(f_2) = \mathbf{domain}(f_2{}^{\leftarrow})$
$\langle g, f_2\rangle \hookrightarrow T108 \Rightarrow$ $1\text{–}1(g\bullet f_2)$
$\langle f_2{}^{\leftarrow}, g\bullet f_2\rangle \hookrightarrow T108 \Rightarrow$ $1\text{–}1(f_2{}^{\leftarrow}\bullet(g\bullet f_2))$
Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(f_2)$ & $\mathsf{Svm}(g)$ & $\mathsf{Svm}(f_2{}^{\leftarrow})$
$\langle f_2, g\rangle \hookrightarrow T86 \Rightarrow$ $\mathbf{domain}(g\bullet f_2) = \mathbf{domain}(f_2)$ & $\mathbf{range}(g\bullet f_2) = \mathbf{range}(g)$
$\langle g\bullet f_2, f_2{}^{\leftarrow}\rangle \hookrightarrow T87 \Rightarrow$ $\mathbf{domain}(f_2{}^{\leftarrow}\bullet(g\bullet f_2)) = \mathbf{domain}(f_2)$ & $\mathbf{range}(f_2{}^{\leftarrow}\bullet(g\bullet f_2)) \subseteq \mathbf{domain}(f_2)$

-- and it is easily seen that $\mathbf{range}(f_2{}^{\leftarrow}\bullet(g\bullet f_2))$ must be a proper subset of $\mathbf{domain}(f_2)$, contradicting the finiteness of $\mathbf{domain}(f_2)$, and so proving the present theorem.

$\langle g\bullet f_2, f_2{}^{\leftarrow}\rangle \hookrightarrow T85 \Rightarrow$ $\mathbf{range}(f_2{}^{\leftarrow}\bullet(g\bullet f_2)) = \mathbf{range}(f_2{}^{\leftarrow}{}_{|\mathbf{range}(g\bullet f_2)})$
EQUAL $\Rightarrow$ $\mathbf{range}(f_2{}^{\leftarrow}\bullet(g\bullet f_2)) = \mathbf{range}(f_2{}^{\leftarrow}{}_{|\mathbf{range}(g)})$
$\langle f_2{}^{\leftarrow}, \mathbf{range}(g)\rangle \hookrightarrow T88 \Rightarrow$ $\mathbf{range}(f_2{}^{\leftarrow}) \neq \mathbf{range}(f_2{}^{\leftarrow}{}_{|\mathbf{range}(g)})$
$\langle f_2{}^{\leftarrow}\rangle \hookrightarrow T62 \Rightarrow$ $f_2{}^{\leftarrow}{}_{|\mathbf{domain}(f_2{}^{\leftarrow})} = f_2{}^{\leftarrow}$
EQUAL $\Rightarrow$ $\mathbf{range}(f_2{}^{\leftarrow}{}_{|\mathbf{domain}(f_2{}^{\leftarrow})}) \neq \mathbf{range}(f_2{}^{\leftarrow}{}_{|\mathbf{range}(g)})$
ELEM $\Rightarrow$ $\mathbf{range}(f_2{}^{\leftarrow}\bullet(g\bullet f_2)) \neq \mathbf{domain}(f_2)$
$\langle f_2{}^{\leftarrow}\bullet(g\bullet f_2)\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- If s is a 1-1 map, the implication given in the preceding theorem can be strengthened to an equivalence:

Theorem 194 (164) $1\text{–}1(\mathsf{F}) \rightarrow \big(\mathsf{Finite}(\mathbf{domain}(\mathsf{F})) \leftrightarrow \mathsf{Finite}(\mathbf{range}(\mathsf{F}))\big)$. PROOF:

Suppose_not(f) $\Rightarrow$ $1\text{–}1(\mathsf{f}) \;\&\; \neg\big(\mathsf{Finite}(\mathbf{domain}(\mathsf{f})) \leftrightarrow \mathsf{Finite}(\mathbf{range}(\mathsf{f}))\big)$

-- For in this case Theorem 163 applies to both f and its inverse, giving us a pair of implications, and so yielding the asserted equivalence.

$\big\langle \mathsf{f} \big\rangle \hookrightarrow T163 \Rightarrow$ $\mathsf{Finite}(\mathbf{domain}(\mathsf{f})) \rightarrow \mathsf{Finite}(\mathbf{range}(\mathsf{f}))$
$\big\langle \mathsf{f} \big\rangle \hookrightarrow T91 \Rightarrow$ $1\text{–}1(\mathsf{f}^{\leftarrow}) \;\&\; \mathbf{domain}(\mathsf{f}^{\leftarrow}) = \mathbf{range}(\mathsf{f}) \;\&\; \mathbf{range}(\mathsf{f}^{\leftarrow}) = \mathbf{domain}(\mathsf{f})$
$\big\langle \mathsf{f}^{\leftarrow} \big\rangle \hookrightarrow T163 \Rightarrow$ $\mathsf{Finite}(\mathbf{domain}(\mathsf{f}^{\leftarrow})) \rightarrow \mathsf{Finite}(\mathbf{range}(\mathsf{f}^{\leftarrow}))$
EQUAL $\Rightarrow$ $\mathsf{Finite}(\mathbf{range}(\mathsf{f})) \rightarrow \mathsf{Finite}(\mathbf{domain}(\mathsf{f}))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We can also extend Theorem 163 from 1-1 maps to single-valued maps in general:

-- A single_valued map with finite domain has a finite range

Theorem 195 (165) $\mathsf{Svm}(\mathsf{F}) \;\&\; \mathsf{Finite}(\mathbf{domain}(\mathsf{F})) \rightarrow \mathsf{Finite}(\mathbf{range}(\mathsf{F}))$. PROOF:

Suppose_not(f) $\Rightarrow$ $\mathsf{Svm}(\mathsf{f}) \;\&\; \mathsf{Finite}(\mathbf{domain}(\mathsf{f})) \;\&\; \neg\mathsf{Finite}(\mathbf{range}(\mathsf{f}))$

-- The result follows easily from Theorem 164 if we use Theorem 113, which tells us that there is a 1-1 map, partially inverse to f, which maps $\mathbf{range}(\mathsf{f})$ into a subset of $\mathbf{domain}(\mathsf{f})$.

$\big\langle \mathsf{f} \big\rangle \hookrightarrow T113 \Rightarrow$ $Stat1: \big\langle \exists \mathsf{h} \mid \mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{f}) \;\&\; \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f}) \;\&\; 1\text{–}1(\mathsf{h}) \;\&\; \big\langle \forall \mathsf{x} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{f}{\upharpoonright}(\mathsf{h}{\upharpoonright}\mathsf{X}) = \mathsf{X}\big\rangle\big\rangle$
$\big\langle \mathsf{invm} \big\rangle \hookrightarrow Stat1 \Rightarrow$ $\mathbf{domain}(\mathsf{invm}) = \mathbf{range}(\mathsf{f}) \;\&\; \mathbf{range}(\mathsf{invm}) \subseteq \mathbf{domain}(\mathsf{f}) \;\&\; 1\text{–}1(\mathsf{invm})$
EQUAL $\Rightarrow$ $\neg\mathsf{Finite}(\mathbf{domain}(\mathsf{invm}))$
$\big\langle \mathsf{invm} \big\rangle \hookrightarrow T164 \Rightarrow$ $\neg\mathsf{Finite}(\mathbf{range}(\mathsf{invm}))$
$\big\langle \mathbf{domain}(\mathsf{f}), \mathbf{range}(\mathsf{invm}) \big\rangle \hookrightarrow T162 \Rightarrow$ $\neg\mathsf{Finite}(\mathbf{domain}(\mathsf{f}))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following corollary to Theorem 164 states that a set is finite if and only if its cardinality is finite. The proof merely applies Theorem 164 twice, once to a 1-1 map from $\#\mathsf{s}$ to s, and once to the inverse of this map.

Theorem 196 (166) $\mathsf{Finite}(\mathsf{S}) \leftrightarrow \mathsf{Finite}(\#\mathsf{S})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s}) \Rightarrow \quad \neg\big(\mathsf{Finite}(\mathsf{s}) \leftrightarrow \mathsf{Finite}(\#\mathsf{s})\big)$
$\big\langle \mathsf{s} \big\rangle \hookrightarrow T130 \Rightarrow \quad Stat1: \ \mathsf{Card}(\#\mathsf{s}) \ \& \ \big\langle \exists \mathsf{f} \,|\, 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathsf{s} \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathsf{s} \big\rangle$
$\big\langle \mathsf{f} \big\rangle \hookrightarrow Stat1 \Rightarrow \quad 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) = \mathsf{s} \ \& \ \mathbf{range}(\mathsf{f}) = \#\mathsf{s}$
$\big\langle \mathsf{f} \big\rangle \hookrightarrow T164 \Rightarrow \quad \mathsf{Finite}(\mathsf{s}) \rightarrow \mathsf{Finite}(\#\mathsf{s})$
$\big\langle \mathsf{f} \big\rangle \hookrightarrow T91 \Rightarrow \quad 1\text{–}1(\mathsf{f}^{\leftarrow}) \ \& \ \mathbf{range}(\mathsf{f}^{\leftarrow}) = \#\mathsf{s} \ \& \ \mathbf{domain}(\mathsf{f}^{\leftarrow}) = \mathsf{s}$
$\big\langle \mathsf{f} \big\rangle \hookrightarrow T164 \Rightarrow \quad \mathsf{Finite}(\#\mathsf{s}) \rightarrow \mathsf{Finite}(\mathsf{s})$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Even if t is a proper subset of a general set s we can only assert that #t is no greater than #s. But if s is finite, then, as the following theorem shows, #t must be less than s.

-- Proper subsets of a finite set have fewer elements

Theorem 197 (167) $\mathsf{Finite}(\mathsf{S}) \ \& \ T \subseteq \mathsf{S} \ \& \ T \neq \mathsf{S} \rightarrow \# T \in \#\mathsf{S}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s},\mathsf{t}) \Rightarrow \quad \mathsf{Finite}(\mathsf{s}) \ \& \ \mathsf{t} \subseteq \mathsf{s} \ \& \ \mathsf{t} \neq \mathsf{s} \ \& \ \#\mathsf{t} \notin \#\mathsf{s}$

-- For, proceeding by contradiction, suppose that t is a proper subset of the finite set s and #t ∉ #s. By Theorem 130, there exist 1-1 maps f and g of #s and #t to s and t respectively, and then $\mathsf{g}^{\leftarrow}$ is a 1-1 map of #s to s.

$\big\langle \mathsf{t} \big\rangle \hookrightarrow T130 \Rightarrow \quad Stat1: \ \mathsf{Card}(\#\mathsf{t}) \ \& \ \mathcal{O}(\#\mathsf{t}) \ \& \ \big\langle \exists \mathsf{f} \,|\, 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathsf{t} \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathsf{t} \big\rangle$
$\big\langle \mathsf{f} \big\rangle \hookrightarrow Stat1 \Rightarrow \quad 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathsf{t} \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathsf{t}$
$\big\langle \mathsf{s} \big\rangle \hookrightarrow T130 \Rightarrow \quad Stat2: \ \mathsf{Card}(\#\mathsf{s}) \ \& \ \mathcal{O}(\#\mathsf{s}) \ \& \ \big\langle \exists \mathsf{f} \,|\, 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{range}(\mathsf{f}) = \mathsf{s} \ \& \ \mathbf{domain}(\mathsf{f}) = \#\mathsf{s} \big\rangle$
$\big\langle \mathsf{g} \big\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{Card}(\#\mathsf{s}) \ \& \ 1\text{–}1(\mathsf{g}) \ \& \ \mathbf{domain}(\mathsf{g}) = \#\mathsf{s} \ \& \ \mathbf{range}(\mathsf{g}) = \mathsf{s}$
$\big\langle \mathsf{g} \big\rangle \hookrightarrow T91 \Rightarrow \quad 1\text{–}1(\mathsf{g}^{\leftarrow}) \ \& \ \mathbf{domain}(\mathsf{g}^{\leftarrow}) = \mathsf{s} \ \& \ \mathbf{range}(\mathsf{g}^{\leftarrow}) = \#\mathsf{s}$

-- Since #t ∉ #s and both are ordinals, it follows by Theorems 83 and 16 that #t = #s, and so $\mathsf{f} \bullet \mathsf{g}^{\leftarrow}$ is a 1-1 map of s onto t. Thus by the definition of finiteness t cannot be a proper subset of s, a contradiction proving our theorem.

$\big\langle \mathsf{t},\mathsf{s} \big\rangle \hookrightarrow T144 \Rightarrow \quad \#\mathsf{s} \supseteq \#\mathsf{t}$
$\big\langle \#\mathsf{s},\#\mathsf{t} \big\rangle \hookrightarrow T31 \Rightarrow \quad \#\mathsf{t} = \#\mathsf{s}$
$\big\langle \mathsf{g}^{\leftarrow},\mathsf{f} \big\rangle \hookrightarrow T86 \Rightarrow \quad \mathbf{domain}(\mathsf{f} \bullet \mathsf{g}^{\leftarrow}) = \mathsf{s} \ \& \ \mathbf{range}(\mathsf{f} \bullet \mathsf{g}^{\leftarrow}) = \mathsf{t}$
$\big\langle \mathsf{f},\mathsf{g}^{\leftarrow} \big\rangle \hookrightarrow T108 \Rightarrow \quad 1\text{–}1(\mathsf{f} \bullet \mathsf{g}^{\leftarrow})$
$\mathsf{Use\_def}(\mathsf{Finite}) \Rightarrow \quad Stat3: \ \neg\big\langle \exists \mathsf{f} \,|\, 1\text{–}1(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) = \mathsf{s} \ \& \ \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \ \& \ \mathbf{range}(\mathsf{f}) \neq \mathsf{s} \big\rangle$
$\big\langle \mathsf{f} \bullet \mathsf{g}^{\leftarrow} \big\rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Another property of finite sets (which could be used to define them) is stated in the following theorem: a finite set is not the image of any proper subset of itself by a single-valued map.

Theorem 198 (168) $\mathsf{Finite}(\mathsf{S}) \leftrightarrow \neg\langle\exists \mathsf{f} \mid \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) = \mathsf{S} \,\&\, \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{S} \,\&\, \mathsf{S} \neq \mathbf{domain}(\mathsf{f})\rangle$. PROOF:

Suppose_not(s) $\Rightarrow$ $\neg\big(\mathsf{Finite}(\mathsf{s}) \leftrightarrow \neg\langle\exists \mathsf{f} \mid \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) = \mathsf{s} \,\&\, \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{domain}(\mathsf{f})\rangle\big)$

-- First suppose the contrary, and first consider the case in which s is finite but there exists a single-valued map f of a proper subset s of onto s. By Theorem 113, this has a 1-1 partial inverse mapping s to a proper subset of s, which is impossible by definition of finiteness. Hence we need only consider the case in which s is not finite, but there exists no single-valued map f of a proper subset of s onto s.

Suppose $\Rightarrow$ $\mathsf{Finite}(\mathsf{s}) \,\&\, \mathit{Stat1}: \langle\exists \mathsf{f} \mid \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) = \mathsf{s} \,\&\, \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{domain}(\mathsf{f})\rangle$
$\langle\mathsf{f}\rangle\hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) = \mathsf{s} \,\&\, \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{domain}(\mathsf{f})$
$\langle\mathsf{f}\rangle\hookrightarrow T113 \Rightarrow$ $\mathit{Stat2}: \langle\exists \mathsf{h} \mid (\mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f}) \,\&\, 1\text{–}1(\mathsf{h})) \,\&\, \langle\forall \mathsf{x} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{f}\upharpoonright(\mathsf{h}\upharpoonright\mathsf{x}) = \mathsf{x}\rangle\rangle$
$\langle\mathsf{h}\rangle\hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathbf{domain}(\mathsf{h}) = \mathsf{s} \,\&\, \mathbf{range}(\mathsf{h}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{range}(\mathsf{h}) \,\&\, 1\text{–}1(\mathsf{h})$
Use_def(Finite) $\Rightarrow$ $\mathit{Stat3}: \neg\langle\exists \mathsf{h} \mid 1\text{–}1(\mathsf{h}) \,\&\, \mathbf{domain}(\mathsf{h}) = \mathsf{s} \,\&\, \mathbf{range}(\mathsf{h}) \subseteq \mathsf{s} \,\&\, \mathbf{range}(\mathsf{h}) \neq \mathsf{s}\rangle$
$\langle\mathsf{h}\rangle\hookrightarrow \mathit{Stat3} \Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{Finite}(\mathsf{s}) \,\&\, \mathit{Stat4}: \neg\langle\exists \mathsf{f} \mid \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) = \mathsf{s} \,\&\, \mathbf{domain}(\mathsf{f}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{domain}(\mathsf{f})\rangle$

-- Since by definition of finiteness there exists a 1-1 map g of a proper subset of s onto s, and since the inverse of g is also a 1-1 map, we have a contradiction in this case also, and so our theorem is proved.

Use_def(Finite) $\Rightarrow$ $\mathit{Stat5}: \langle\exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \,\&\, \mathbf{domain}(\mathsf{f}) = \mathsf{s} \,\&\, \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{range}(\mathsf{f})\rangle$
$\langle\mathsf{g}\rangle\hookrightarrow \mathit{Stat5} \Rightarrow$ $1\text{–}1(\mathsf{g}) \,\&\, \mathbf{domain}(\mathsf{g}) = \mathsf{s} \,\&\, \mathbf{range}(\mathsf{g}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{range}(\mathsf{g})$
$\langle\mathsf{g}\rangle\hookrightarrow T91 \Rightarrow$ $1\text{–}1(\mathsf{g}^{\leftarrow}) \,\&\, \mathbf{range}(\mathsf{g}^{\leftarrow}) = \mathsf{s} \,\&\, \mathbf{domain}(\mathsf{g}^{\leftarrow}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{domain}(\mathsf{g}^{\leftarrow})$
Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(\mathsf{g}^{\leftarrow})$
$\langle\mathsf{g}^{\leftarrow}\rangle\hookrightarrow \mathit{Stat4} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Since members of an ordinal s are also subsets of s, it follows immediately from Theorem 162 that any member of a finite ordinal is finite.

Theorem 199 (169) $\mathcal{O}(\mathsf{S}) \,\&\, \mathsf{Finite}(\mathsf{S}) \,\&\, T \in \mathsf{S} \rightarrow \mathsf{Finite}(T)$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\mathcal{O}(\mathsf{s}) \,\&\, \mathsf{Finite}(\mathsf{s}) \,\&\, \mathsf{t} \in \mathsf{s} \,\&\, \neg\mathsf{Finite}(\mathsf{t})$
$\langle\mathsf{s},\mathsf{t}\rangle\hookrightarrow T12 \Rightarrow$ $\mathsf{t} \subseteq \mathsf{s}$
$\langle\mathsf{s},\mathsf{t}\rangle\hookrightarrow T162 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- A further corollary of Theorem 168 is that any infinite ordinal is larger than any finite ordinal:

-- Any infinite ordinal is larger than any finite ordinal

Theorem 200 (170) $\mathcal{O}(S) \;\&\; \mathcal{O}(T) \;\&\; \neg\mathsf{Finite}(S) \;\&\; \mathsf{Finite}(T) \rightarrow T \in S$. PROOF:

$\mathsf{Suppose\_not}(s, t) \Rightarrow \quad \mathcal{O}(s) \;\&\; \mathcal{O}(t) \;\&\; \neg\mathsf{Finite}(s) \;\&\; \mathsf{Finite}(t) \;\&\; t \notin s$
$\langle s, t\rangle \hookrightarrow T28 \Rightarrow \quad s \in t \lor t \in s \lor s = t$
$\mathsf{EQUAL} \Rightarrow \quad s \neq t$
$\mathsf{ELEM} \Rightarrow \quad s \in t$
$\langle t, s\rangle \hookrightarrow T169 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- To exploit the fact that sets which are in 1-1 correspondence have the same cardinality, we sometimes need to make use of elementary constructions of such maps. The following lemma captures one such case: elements of a set s can always be interchanged by some 1-1 map.

-- Interchange Lemma

Theorem 201 (171) $X, Y \in S \rightarrow \langle \exists f \mid 1\text{–}1(f) \;\&\; \mathbf{range}(f) = S \;\&\; \mathbf{domain}(f) = S \;\&\; f{\upharpoonright}X = Y \;\&\; f{\upharpoonright}Y = X\rangle$. PROOF:

$\mathsf{Suppose\_not}(a, s, b) \Rightarrow \quad Stat1:\; a, b \in s \;\&\; Stat2:\; \neg\langle \exists f \mid 1\text{–}1(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) = s \;\&\; f{\upharpoonright}a = b \;\&\; f{\upharpoonright}b = a\rangle$

-- For the desired map f can be defined by $f{\upharpoonright}x = \mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi}$,

$\mathsf{Loc\_def} \Rightarrow \quad f = \{[x, \mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi}] :\; x \in s\}$

-- and it is easily seen that the range of this map is s.

$\mathsf{APPLY}\ \langle x_\Theta : c_2, y_\Theta : d_2\rangle\ \mathsf{fcn\_symbol}(f(x) \mapsto \mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi}, g \mapsto f, s \mapsto s) \Rightarrow$
$\quad Stat3a:\; \langle \forall x \mid x \in s \rightarrow f{\upharpoonright}x = \mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi}\rangle \;\&\; (c_2, d_2 \in s \;\&\; \mathbf{if}\ c_2 = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ c_2 = b\ \mathbf{then}\ a\ \mathbf{else}\ c_2\ \mathbf{fi\ fi} = \mathbf{if}\ d_2 = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ d_2 =$
$\mathsf{APPLY}\ \langle\,\rangle\ \mathsf{fcn\_symbol}(f(x) \mapsto \mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi}, g \mapsto f, s \mapsto s) \Rightarrow$
$\quad Stat3:\; \mathsf{Svm}(f) \;\&\; \mathbf{domain}(f) = s \;\&\; \mathbf{range}(f) = \{\mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi} :\; x \in s\}$
$\mathsf{Suppose} \Rightarrow \quad Stat4:\; s \neq \{\mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi} :\; x \in s\}$
$\langle c\rangle \hookrightarrow Stat4 \Rightarrow \quad Stat5:\; c \in s \leftrightarrow c \notin \{\mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi} :\; x \in s\}$
$\mathsf{Suppose} \Rightarrow \quad Stat6:\; c \in s \;\&\; Stat7:\; c \notin \{\mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi} :\; x \in s\}$
$\mathsf{Suppose} \Rightarrow \quad c = b$
$\langle a\rangle \hookrightarrow Stat7 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad c \neq b$
$\mathsf{Suppose} \Rightarrow \quad c = a$
$\langle b\rangle \hookrightarrow Stat7 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad c \neq a$
$\langle c\rangle \hookrightarrow Stat7 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad c \notin s \;\&\; Stat8:\; c \in \{\mathbf{if}\ x = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ x = b\ \mathbf{then}\ a\ \mathbf{else}\ x\ \mathbf{fi\ fi} :\; x \in s\}$
$\langle d\rangle \hookrightarrow Stat8 \Rightarrow \quad Stat8a:\; c = \mathbf{if}\ d = a\ \mathbf{then}\ b\ \mathbf{else\ if}\ d = b\ \mathbf{then}\ a\ \mathbf{else}\ d\ \mathbf{fi\ fi} \;\&\; d \in s \;\&\; c \notin s$
$\langle Stat1, Stat8a\rangle\ \mathsf{ELEM} \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad Stat9:\; \mathbf{range}(f) = s$

-- The fact that f is 1-1 is also elementary, so f has all the properties which our theorem asserts.

$TELEM \Rightarrow$ $\quad \textbf{if } c_2 = a \textbf{ then } b \textbf{ else if } c_2 = b \textbf{ then } a \textbf{ else } c_2 \textbf{ fi fi} = \textbf{if } d_2 = a \textbf{ then } b \textbf{ else if } d_2 = b \textbf{ then } a \textbf{ else } d_2 \textbf{ fi fi} \rightarrow c_2 = d_2$

$\langle Stat3a\rangle$ ELEM $\Rightarrow$ $\quad Stat11: \ 1\text{–}1(f)$

$\langle b\rangle \hookrightarrow Stat3a(\langle Stat1\rangle) \Rightarrow$ $\quad Stat13: \ f\restriction b = a$

$\langle a\rangle \hookrightarrow Stat3a(\langle Stat1\rangle) \Rightarrow$ $\quad Stat12: \ f\restriction a = b$

$\langle f\rangle \hookrightarrow Stat2(\langle Stat9, Stat3, Stat11, Stat12, Stat13\rangle) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

-- The following utility lemma gives us expressions for the map restriction of any single-valued map, and for the range and domain of this restriction. Using Theorem 171, we can easily prove that the successor set of any finite set is also finite.

Theorem 202 (172) $\quad \text{Finite}(S) \leftrightarrow \text{Finite}(S \cup \{X\})$. PROOF:

Suppose_not(s, a) $\Rightarrow$ $\quad (\text{Finite}(s) \,\&\, \neg\text{Finite}(s \cup \{a\})) \vee (\neg\text{Finite}(s) \,\&\, \text{Finite}(s \cup \{a\}))$

-- Since any subset of a finite set is finite, our theorem can only be false if s is finite and $s \cup \{a\}$ is not, in which case a is not in s and there must exist a 1-1 map g of $s \cup \{a\}$ into a subset of $s \cup \{a\}$ which omits some element c of $s \cup \{a\}$.

Suppose $\Rightarrow$ $\quad \neg\text{Finite}(s) \,\&\, \text{Finite}(s \cup \{a\})$

ELEM $\Rightarrow$ $\quad s \cup \{a\} \supseteq s$

$\langle s \cup \{a\}, s\rangle \hookrightarrow T162 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad \text{Finite}(s) \,\&\, \neg\text{Finite}(s \cup \{a\})$

Suppose $\Rightarrow$ $\quad s = s \cup \{a\}$

EQUAL $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad a \notin s$

Use_def(Finite) $\Rightarrow$ $\quad Stat1:$

$\langle \exists f \mid 1\text{–}1(f) \,\&\, \textbf{domain}(f) = s \cup \{a\} \,\&\, \textbf{range}(f) \subseteq s \cup \{a\} \,\&\, s \cup \{a\} \neq \textbf{range}(f)\rangle \,\&\, Stat2: \ \neg\langle \exists f \mid 1\text{–}1(f) \,\&\, \textbf{domain}(f) = s \,\&\, \textbf{range}(f) \subseteq s \,\&\, s \neq \textbf{range}(f)\rangle$

$\langle g\rangle \hookrightarrow Stat1 \Rightarrow$ $\quad 1\text{–}1(g) \,\&\, \textbf{domain}(g) = s \cup \{a\} \,\&\, Stat3: \ s \cup \{a\} \neq \textbf{range}(g) \,\&\, \textbf{range}(g) \subseteq s \cup \{a\}$

$\langle c\rangle \hookrightarrow Stat3 \Rightarrow$ $\quad c \in s \cup \{a\} \,\&\, c \notin \textbf{range}(g)$

ELEM $\Rightarrow$ $\quad a \notin s \,\&\, a, c \in \textbf{domain}(g) \,\&\, c \notin \textbf{range}(g)$

-- But by Theorem 171 there is a 1-1 map f of $s \cup \{a\}$ onto itself which interchanges a and c. The product map f•g is therefore a 1-1 map f of $s \cup \{a\}$ into itself whose range omits a.

$\langle a, \textbf{domain}(g), c\rangle \hookrightarrow T171 \Rightarrow$ $\quad Stat4: \ \langle \exists f \mid 1\text{–}1(f) \,\&\, \textbf{range}(f) = \textbf{domain}(g) \,\&\, \textbf{domain}(f) = \textbf{domain}(g) \,\&\, f\restriction a = c \,\&\, f\restriction c = a\rangle$

$\langle f\rangle \hookrightarrow Stat4 \Rightarrow$ $\quad 1\text{–}1(f) \,\&\, \textbf{range}(f) = \textbf{domain}(g) \,\&\, \textbf{domain}(f) = \textbf{domain}(g) \,\&\, f\restriction a = c \,\&\, f\restriction c = a$

$\langle g, f\rangle \hookrightarrow T85 \Rightarrow$ $\quad \textbf{range}(f\bullet g) = \textbf{range}(f_{|\textbf{range}(g)}) \,\&\, \textbf{domain}(f\bullet g) = s \cup \{a\}$

Use_def(1–1) $\Rightarrow$ $\quad \text{Svm}(f)$

Suppose $\Rightarrow$ $\quad a \in \textbf{range}(f_{|\textbf{range}(g)})$

$\langle f, \textbf{range}(g)\rangle \hookrightarrow T101 \Rightarrow$ $\quad Stat5: \ a \in \{f\restriction x : \ x \in \textbf{domain}(f) \mid x \in \textbf{range}(g)\}$

$\langle e\rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad a = f\restriction e \;\&\; e \in \mathbf{domain}(f) \;\&\; e \in \mathbf{range}(g)$
$\mathsf{ELEM} \Rightarrow \quad c \in \mathbf{domain}(f) \;\&\; a = f\restriction c$
$\langle f, e, c\rangle \hookrightarrow \mathit{T102} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad a \notin \mathbf{range}(f\bullet g)$

-- Hence the restriction of f to s is a 1-1 mapping of s into itself.

$\langle f, \mathbf{range}(g)\rangle \hookrightarrow \mathit{T72} \Rightarrow \quad \mathbf{range}(f_{|\mathbf{range}(g)}) \subseteq s \cup \{a\}$
$\mathsf{ELEM} \Rightarrow \quad \mathbf{range}(f\bullet g) \subseteq s$
$\langle f, g\rangle \hookrightarrow \mathit{T108} \Rightarrow \quad 1\text{–}1(f\bullet g)$
$\langle f\bullet g, s\rangle \hookrightarrow \mathit{T53} \Rightarrow \quad 1\text{–}1\Big((f\bullet g)_{|s}\Big)$
$\langle f\bullet g, s\rangle \hookrightarrow \mathit{T84} \Rightarrow \quad \mathbf{domain}((f\bullet g)_{|s}) = s$
$\langle f\bullet g, s\rangle \hookrightarrow \mathit{T72} \Rightarrow \quad \mathbf{range}((f\bullet g)_{|s}) \subseteq s$

-- But it is easily seen that $f\bullet g\restriction a$ must be an element of s,

$\mathsf{ELEM} \Rightarrow \quad a \in \mathbf{domain}(f\bullet g)$
$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad \mathsf{Svm}(f\bullet g)$
$\langle a, f\bullet g\rangle \hookrightarrow \mathit{T64} \Rightarrow \quad f\bullet g\restriction a \in s$

-- and that $f\bullet g\restriction a$ is not a member of $\mathbf{range}(f\bullet g_{|s})$

$\langle f\bullet g\rangle \hookrightarrow \mathit{T65} \Rightarrow \quad f\bullet g = \{[x, f\bullet g\restriction x] : x \in \mathbf{domain}(f\bullet g)\}$
$\mathsf{EQUAL} \Rightarrow \quad f\bullet g = \{[x, f\bullet g\restriction x] : x \in s \cup \{a\}\}$
$\mathsf{Suppose} \Rightarrow \quad f\bullet g\restriction a \in \mathbf{range}((f\bullet g)_{|s})$
$\mathsf{Use\_def}(|) \Rightarrow \quad f\bullet g\restriction a \in \mathbf{range}(\{x \in f\bullet g \mid x^{[1]} \in s\})$
$\mathsf{EQUAL} \Rightarrow \quad f\bullet g\restriction a \in \mathbf{range}(\{x \in \{[x, f\bullet g\restriction x] : x \in s \cup \{a\}\} \mid x^{[1]} \in s\})$
$\mathsf{SIMPLF} \Rightarrow \quad f\bullet g\restriction a \in \mathbf{range}\Big(\Big\{[x, f\bullet g\restriction x] : x \in s \cup \{a\} \mid [x, f\bullet g\restriction x]^{[1]} \in s\Big\}\Big)$
$\mathsf{Use\_def}(\mathbf{range}) \Rightarrow \quad f\bullet g\restriction a \in \Big\{y^{[2]} : y \in \Big\{[x, f\bullet g\restriction x] : x \in s \cup \{a\} \mid [x, f\bullet g\restriction x]^{[1]} \in s\Big\}\Big\}$
$\mathsf{SIMPLF} \Rightarrow \quad \mathit{Stat6}: \; f\bullet g\restriction a \in \Big\{[x, f\bullet g\restriction x]^{[2]} : x \in s \cup \{a\} \mid [x, f\bullet g\restriction x]^{[1]} \in s\Big\}$
$\langle b\rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad b \in s \cup \{a\} \;\&\; [b, f\bullet g\restriction b]^{[1]} \in s \;\&\; f\bullet g\restriction a = [b, f\bullet g\restriction b]^{[2]}$
$\mathsf{ELEM} \Rightarrow \quad f\bullet g\restriction a = f\bullet g\restriction b \;\&\; b \in s$
$\langle f\bullet g, a\rangle \hookrightarrow \mathit{T93} \Rightarrow \quad (f\bullet g)^{\leftarrow}\restriction(f\bullet g\restriction a) = a$
$\langle f\bullet g, b\rangle \hookrightarrow \mathit{T93} \Rightarrow \quad (f\bullet g)^{\leftarrow}\restriction(f\bullet g\restriction b) = b$
$\mathsf{EQUAL} \Rightarrow \quad a = b$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathbf{range}((f\bullet g)_{|s}) \neq s$

-- Therefore $\mathsf{f}\bullet\mathsf{g}_{|\mathsf{s}}$ is a 1-1 mapping of s into a proper subset of itself, violating our assumption that s is finite, and thereby proving the present theorem.

$\mathsf{Use\_def(Finite)} \Rightarrow \quad Stat7: \neg\langle\exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \,\&\, \mathbf{domain}(\mathsf{f}) = \mathsf{s} \,\&\, \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \,\&\, \mathsf{s} \neq \mathbf{range}(\mathsf{f})\rangle$

$\langle(\mathsf{f}\bullet\mathsf{g})_{|\mathsf{s}}\rangle \hookrightarrow Stat7 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Theorem 172 has the following obvious corollary.

Theorem 203 (173) $\mathsf{Finite}(\mathsf{S}) \rightarrow \mathsf{Finite}(\mathsf{next}(\mathsf{S}))$. PROOF:

$\mathsf{Suppose\_not(s)} \Rightarrow \quad \mathsf{Finite}(\mathsf{s}) \,\&\, \neg\mathsf{Finite}(\mathsf{next}(\mathsf{s}))$

$\mathsf{Use\_def(next)} \Rightarrow \quad \mathsf{next}(\mathsf{s}) = \mathsf{s} \cup \{\mathsf{s}\}$

$\langle \mathsf{s}, \mathsf{s}\rangle \hookrightarrow T172 \Rightarrow \quad \mathsf{Finite}(\mathsf{s} \cup \{\mathsf{s}\})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following equally obvious corollaries of Theorem 172 are also useful. The fist simply states that any singleton is finite.

Theorem 204 (174) $\mathsf{Finite}(\{\mathsf{S}\})$. PROOF:

$\mathsf{Suppose\_not} \Rightarrow \quad \neg\mathsf{Finite}(\{\mathsf{s}\})$

$T161 \Rightarrow \quad \mathsf{Finite}(\emptyset)$

$\langle \emptyset, \mathsf{s}\rangle \hookrightarrow T172 \Rightarrow \quad \mathsf{Finite}(\emptyset \cup \{\mathsf{s}\})$

$\mathsf{ELEM} \Rightarrow \quad \emptyset \cup \{\mathsf{s}\} = \{\mathsf{s}\}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next elementary result states that any unordered pair is also a finite set.

Theorem 205 (175) $\mathsf{Finite}(\{\mathsf{S}, T\})$. PROOF:

$\mathsf{Suppose\_not} \Rightarrow \quad \neg\mathsf{Finite}(\{\mathsf{s}, \mathsf{t}\})$

$T174 \Rightarrow \quad \mathsf{Finite}(\{\mathsf{s}\})$

$\langle \{\mathsf{s}\}, \mathsf{t}\rangle \hookrightarrow T172 \Rightarrow \quad \mathsf{Finite}(\{\mathsf{s}\} \cup \{\mathsf{t}\})$

$\mathsf{ELEM} \Rightarrow \quad \{\mathsf{s}\} \cup \{\mathsf{t}\} = \{\mathsf{s}, \mathsf{t}\}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that the set whose existence is asserted by the Axiom of Infinity must actually be infinite.

Theorem 206 (176) $\neg\mathsf{Finite}(\mathsf{s\_inf})$. PROOF:

Suppose_not ⇒ $\mathsf{Finite}(\mathsf{s\_inf})$
Loc_def ⇒ $g = \{[x,\{x\}] : x \in \mathsf{s\_inf}\}$

-- For by the axiomatic assumption defining s_inf, the single-valued map $\{[x,\{x\}] : x \in \mathsf{s\_inf}\}$ sends s_inf into itself.

APPLY $\langle x_\Theta : a_1, y_\Theta : a_2\rangle$ fcn_symbol$(f(x) \mapsto \{x\}, g \mapsto g, s \mapsto \mathsf{s_inf})$ ⇒
$\mathsf{Svm}(g) \,\&\, \mathbf{domain}(g) = \mathsf{s\_inf} \,\&\, \mathbf{range}(g) = \{\{x\} : x \in \mathsf{s\_inf}\} \,\&\, (a_1, a_2 \in \mathsf{s\_inf} \,\&\, \{a_1\} = \{a_2\} \,\&\, a_1 \neq a_2) \vee 1\text{–}1(g)$
Suppose ⇒ *Stat1* : $\mathbf{range}(g) \not\subseteq \mathsf{s\_inf}$
$\langle c\rangle \hookrightarrow$ *Stat1* ⇒ *Stat2* : $c \in \{\{x\} : x \in \mathsf{s\_inf}\} \,\&\, c \notin \mathsf{s\_inf}$
$\langle d\rangle \hookrightarrow$ *Stat2* ⇒ $d \in \mathsf{s\_inf} \,\&\, c = \{d\} \,\&\, c \notin \mathsf{s\_inf}$
T00 ⇒ $\mathsf{s\_inf} \neq \emptyset \,\&\,$ *Stat3* : $\langle \forall x \in \mathsf{s\_inf} \mid \{x\} \in \mathsf{s\_inf}\rangle$
$\langle d\rangle \hookrightarrow$ *Stat3* ⇒ $d \in \mathsf{s\_inf} \to \{d\} \in \mathsf{s\_inf}$
ELEM ⇒ false; Discharge ⇒ $\mathbf{range}(g) \subseteq \mathsf{s\_inf}$

-- But it is easily seen using the axiom of choice that **arb**(s_inf) cannot be in the range of the map g. Hence g is a 1-1 map which maps s_inf into a proper subset of itself, contradicting the definition of finiteness, and thereby proving our theorem

T00 ⇒ $\mathsf{s\_inf} \neq \emptyset$
$\langle \mathsf{s\_inf}\rangle \hookrightarrow$ *T0* ⇒ $\mathbf{arb}(\mathsf{s\_inf}) \in \mathsf{s\_inf} \,\&\, \mathbf{arb}(\mathsf{s\_inf}) \cap \mathsf{s\_inf} = \emptyset$
Suppose ⇒ $\mathbf{range}(g) = \mathsf{s\_inf}$
ELEM ⇒ *Stat4* : $\mathbf{arb}(\mathsf{s\_inf}) \in \{\{x\} : x \in \mathsf{s\_inf}\}$
$\langle e\rangle \hookrightarrow$ *Stat4* ⇒ $e \in \mathsf{s\_inf} \,\&\, \mathbf{arb}(\mathsf{s\_inf}) = \{e\}$
ELEM ⇒ false; Discharge ⇒ $\mathbf{range}(g) \neq \mathsf{s\_inf}$
Suppose ⇒ $\neg 1\text{–}1(g)$
ELEM ⇒ false; Discharge ⇒ $1\text{–}1(g)$
Use_def(Finite) ⇒ *Stat5* : $\neg\langle \exists f \mid 1\text{–}1(f) \,\&\, \mathbf{domain}(f) = \mathsf{s\_inf} \,\&\, \mathbf{range}(f) \subseteq \mathsf{s\_inf} \,\&\, \mathsf{s\_inf} \neq \mathbf{range}(f)\rangle$
$\langle g\rangle \hookrightarrow$ *Stat5* ⇒ false; Discharge ⇒ QED

-- It follows as a corollary of the preceding theorem that #s_inf is an infinite cardinal.

-- Infinite cardinality theorem

Theorem 207 (177) $\neg\mathsf{Finite}(\#\mathsf{s\_inf})$. PROOF:

Suppose_not ⇒ $\mathsf{Finite}(\#\mathsf{s\_inf})$
$\langle \mathsf{s\_inf}\rangle \hookrightarrow$ *T166* ⇒ $\mathsf{Finite}(\#\mathsf{s\_inf}) \leftrightarrow \mathsf{Finite}(\mathsf{s\_inf})$
T176 ⇒ $\neg\mathsf{Finite}(\mathsf{s\_inf})$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The next theorem tells us that for finite sets there is no difference between ordinals and cardinals, since all finite ordinals are cardinals.

-- All finite ordinals are cardinals

Theorem 208 (178) $\mathcal{O}(\mathsf{X})\ \&\ \mathsf{Finite}(\mathsf{X}) \rightarrow \mathsf{Card}(\mathsf{X})$. PROOF:

Suppose_not(x) $\Rightarrow$ $\mathcal{O}(\mathsf{x})\ \&\ \mathsf{Finite}(\mathsf{x})\ \&\ \neg\mathsf{Card}(\mathsf{x})$

-- For if x is a finite ordinal which is not a cardinal, then by definition there must exist a single-valued map f of a member y of x onto x. Plainly y must be an ordinal and a proper subset of x.

Use_def(Card) $\Rightarrow$ *Stat1* : $\neg\langle\forall \mathsf{y} \in \mathsf{x}, \mathsf{f} \mid \mathbf{domain}(\mathsf{f}) \neq \mathsf{y} \vee \mathbf{range}(\mathsf{f}) \neq \mathsf{x} \vee \neg\mathsf{Svm}(\mathsf{f})\rangle$
$\langle \mathsf{y}, \mathsf{f}\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathbf{domain}(\mathsf{f}) = \mathsf{y}\ \&\ \mathsf{y} \in \mathsf{x}\ \&\ \mathbf{range}(\mathsf{f}) = \mathsf{x}\ \&\ \mathsf{Svm}(\mathsf{f})$
$\langle \mathsf{x}, \mathsf{y}\rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathsf{y})$
$\langle \mathsf{x}, \mathsf{y}\rangle \hookrightarrow$ *T31* $\Rightarrow$ $\mathsf{y} \subseteq \mathsf{x}$
ELEM $\Rightarrow$ $\mathsf{y} \neq \mathsf{x}$

-- By Theorem 113, f has a 1-1 partial inverse h, whose domain is **range**(f) and whose range is a subset of **domain**(f). Since x is finite, this must map x onto all of itself, contradicting the fact that $\mathsf{y} = \mathbf{domain}(\mathsf{f})$ is a proper subset of x. This contradiction proves our theorem.

$\langle \mathsf{f}\rangle \hookrightarrow$ *T113* $\Rightarrow$ *Stat2* : $\langle\exists \mathsf{h} \mid (\mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f})\ \&\ 1\text{–}1(\mathsf{h}))\ \&\ \langle\forall \mathsf{x} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{f}\restriction(\mathsf{h}\restriction\mathsf{x}) = \mathsf{x}\rangle\rangle$
$\langle \mathsf{h}\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathbf{domain}(\mathsf{h}) = \mathbf{range}(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathbf{domain}(\mathsf{f})\ \&\ 1\text{–}1(\mathsf{h})$
Use_def(Finite) $\Rightarrow$ *Stat3* : $\neg\langle\exists \mathsf{p} \mid 1\text{–}1(\mathsf{p})\ \&\ \mathbf{domain}(\mathsf{p}) = \mathsf{x}\ \&\ \mathbf{range}(\mathsf{p}) \subseteq \mathsf{x}\ \&\ \mathsf{x} \neq \mathbf{range}(\mathsf{p})\rangle$
$\langle \mathsf{h}\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\neg(1\text{–}1(\mathsf{h})\ \&\ \mathbf{domain}(\mathsf{h}) = \mathsf{x}\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathsf{x}\ \&\ \mathsf{x} \neq \mathbf{range}(\mathsf{h}))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

7 The set of all Integers, basic arithmetic of integers and cardinals

-- Now we can take a decisive step into the realm of traditional mathematics by defining the set of positive integers as the smallest infinite cardinal. These are the 'unsigned' integers, including $\emptyset$. The proofs of their properties with which we now continue prepare for subsequent introduction of the signed integers, from these the rational and real numbers, and finally the complex numbers.

-- The set of integers

DEF 18a. $\mathbb{N} =_{\text{Def}}$ $\mathbf{arb}(\{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\})$

-- The definition just given is justified by the following theorem, which tells us that $\mathbb{N}$ is in fact an infinite ordinal, whose members are exactly the finite ordinals.

Theorem 209 (179) $\mathcal{O}(\mathbb{N})$ & $\neg\mathsf{Finite}(\mathbb{N})$ & $\big(\mathsf{Card}(\mathsf{X})$ & $\mathsf{Finite}(\mathsf{X}) \leftrightarrow \mathsf{X} \in \mathbb{N}\big)$. PROOF:

Suppose_not(x) $\Rightarrow$ $\neg\mathcal{O}(\mathbb{N}) \vee \mathsf{Finite}(\mathbb{N}) \vee \neg\big(\mathsf{Card}(\mathsf{x})$ & $\mathsf{Finite}(\mathsf{x}) \leftrightarrow \mathsf{x} \in \mathbb{N}\big)$

-- First we show that there exists an infinite ordinal, which will imply that there is a smallest infinite ordinal. This is done using the axiom of infinity: the cardinal of the infinite set which this axiom gives us must be infinite.

T177 $\Rightarrow$ $\neg\mathsf{Finite}(\#\mathsf{s\_inf})$
$\langle\mathsf{s\_inf}\rangle \hookrightarrow$ *T130* $\Rightarrow$ $\mathcal{O}(\#\mathsf{s\_inf})$
Use_def(next) $\Rightarrow$ $\mathsf{next}(\#\mathsf{s\_inf}) = \#\mathsf{s\_inf} \cup \{\#\mathsf{s\_inf}\}$
ELEM $\Rightarrow$ $\#\mathsf{s\_inf} \in \mathsf{next}(\#\mathsf{s\_inf})$
Suppose $\Rightarrow$ $\{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\} = \emptyset$
ELEM $\Rightarrow$ *Stat1* : $\#\mathsf{s\_inf} \notin \{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\}$
$\langle\#\mathsf{s\_inf}\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\#\mathsf{s\_inf} \notin \mathsf{next}(\#\mathsf{s\_inf}) \vee \mathsf{Finite}(\#\mathsf{s\_inf})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat2* : $\{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\} \neq \emptyset$

-- Since we have just shown that there is some infinite ordinal, the axiom of choice tells us that $\mathbb{N}$ must be an infinite ordinal.

$\big\langle \{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\} \big\rangle \hookrightarrow$ *T0* $\Rightarrow$ $\mathbf{arb}(\{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\}) \in \{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\}$ &
$\mathbf{arb}(\{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\}) \cap \{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\} = \emptyset$
Use_def($\mathbb{N}$) $\Rightarrow$ *Stat3* : $\mathbb{N} \in \{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\}$ & $\mathbb{N} \cap \{\mathsf{x} : \mathsf{x} \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(\mathsf{x})\} = \emptyset$
$\langle\mathbb{N}\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\mathbb{N} \in \mathsf{next}(\#\mathsf{s\_inf})$ & $\neg\mathsf{Finite}(\mathbb{N})$
$\langle\#\mathsf{s\_inf}\rangle \hookrightarrow$ *T29* $\Rightarrow$ $\mathcal{O}\big(\mathsf{next}(\#\mathsf{s\_inf})\big)$
$\big\langle\mathsf{next}(\#\mathsf{s\_inf}), \mathbb{N}\big\rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathbb{N})$

-- It follows from what has now been proved that only the third clause of our theorem can be false. Hence there exists an x for which $\mathsf{Card}(\mathsf{x})$ & $\mathsf{Finite}(\mathsf{x})$ and $\mathsf{x} \in \mathbb{N}$ are inequivalent. This inequivalence falls into two possible cases.

ELEM $\Rightarrow$ $\big(\mathsf{Card}(\mathsf{x})$ & $\mathsf{Finite}(\mathsf{x})$ & $\mathsf{x} \notin \mathbb{N}\big) \vee \big(\neg\mathsf{Card}(\mathsf{x}) \vee \neg\mathsf{Finite}(\mathsf{x})$ & $\mathsf{x} \in \mathbb{N}\big)$

-- The first of these cases is impossible, so the second must hold.

Suppose $\Rightarrow$ $\mathsf{Card}(x)$ & $\mathsf{Finite}(x)$ & $x \notin \mathbb{N}$
Use_def(Card) $\Rightarrow$ $\mathcal{O}(x)$ & $\mathsf{Finite}(x)$ & $x \notin \mathbb{N}$
$\langle \mathbb{N}, x \rangle \hookrightarrow T170 \Rightarrow$ $\mathcal{O}(x)$ & $\mathcal{O}(\mathbb{N})$ & $\neg\mathsf{Finite}(\mathbb{N})$ & $\mathsf{Finite}(x) \rightarrow x \in \mathbb{N}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{Card}(x) \vee \neg\mathsf{Finite}(x)$ & $x \in \mathbb{N}$

-- But since $\mathbb{N}$ has been defined as the smallest infinite ordinal, each member of $\mathbb{N}$ is a finite ordinal, and hence a cardinal by Theorem 178.

Suppose $\Rightarrow$ $\neg\mathsf{Finite}(x)$
$\langle \mathsf{next}(\#\mathsf{s\_inf}), \mathbb{N} \rangle \hookrightarrow T31 \Rightarrow$ $x \in \mathsf{next}(\#\mathsf{s\_inf})$ & $\neg\mathsf{Finite}(x)$
Suppose $\Rightarrow$ *Stat4* : $x \notin \{u : u \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(u)\}$
$\langle x \rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ $x \in \{u : u \in \mathsf{next}(\#\mathsf{s\_inf}) \mid \neg\mathsf{Finite}(u)\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Finite}(x)$
$\langle \mathbb{N}, x \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(x)$
$\langle x \rangle \hookrightarrow T178 \Rightarrow$ $\mathcal{O}(x)$ & $\mathsf{Finite}(x) \rightarrow \mathsf{Card}(x)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It follows trivially from Theorem 179 that every element of Z is its own cardinality.

Theorem 210 (180) $X \in \mathbb{N} \rightarrow X = \#X$ & $\mathsf{Card}(X)$ & $\mathcal{O}(X)$ & $\mathsf{Finite}(X)$. PROOF:

-- Simply because every element of Z is finite and a cardinal, hence an ordinal.

Suppose_not(m) $\Rightarrow$ *Stat1* : $m \in \mathbb{N}$ & $m \neq \#m \vee \neg\mathsf{Card}(m) \vee \neg\mathcal{O}(m) \vee \neg\mathsf{Finite}(m)$
$\langle m \rangle \hookrightarrow T179 \Rightarrow$ $\mathsf{Card}(m)$ & $\mathsf{Finite}(m)$
$\langle m \rangle \hookrightarrow T138 \Rightarrow$ $m = \#m$
Use_def(Card) $\Rightarrow$ $\mathcal{O}(m)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The standard set-theoretic (von Neumann) definitions of the first few positive integers $1, 2, 3, \ldots$ are as follows: Def 18b: [Standard definitions of the integers, (1 is next (0)) & (2 is next (1)) & (3 is next (2)) & ...] one := next (0)

DEF 18*b*. $1 =_{\mathrm{Def}} \mathsf{next}(\emptyset)$
DEF 18*b*. $2 =_{\mathrm{Def}} \mathsf{next}(1)$
DEF 18*b*. $3 =_{\mathrm{Def}} \mathsf{next}(2)$
DEF 18*b*. $4 =_{\mathrm{Def}} \mathsf{next}(3)$

-- We show next that the set of integers is not merely an ordinal, but is indeed a cardinal.

-- The set of integers is a Cardinal

Theorem 211 (181) $\mathsf{Card}(\mathbb{N})$. PROOF:

Suppose_not $\Rightarrow$ $\neg\mathsf{Card}(\mathbb{N})$

-- For in the contrary case there would exist an element y of $\mathbb{N}$ and a 1-1 map f of y onto $\mathbb{N}$.

Use_def(Card) $\Rightarrow$ $\neg\mathcal{O}(\mathbb{N}) \vee \neg\langle\forall y \in \mathbb{N}, f \,|\, \mathbf{domain}(f) \neq y \vee \mathbf{range}(f) \neq \mathbb{N} \vee \neg\mathsf{Svm}(f)\rangle$
$\langle \mathsf{junk}\rangle \hookrightarrow T179 \Rightarrow$ $\mathcal{O}(\mathbb{N})$
ELEM $\Rightarrow$ *Stat1* : $\neg\langle\forall y \in \mathbb{N}, f \,|\, \mathbf{domain}(f) \neq y \vee \mathbf{range}(f) \neq \mathbb{N} \vee \neg\mathsf{Svm}(f)\rangle$

-- But then Theorem 165 would tell us that $\mathbb{N}$ is finite, a contradiction proving our theorem.

$\langle y, f\rangle \hookrightarrow Stat1 \Rightarrow$ $y \in \mathbb{N} \,\&\, \mathbf{domain}(f) = y \,\&\, \mathbf{range}(f) = \mathbb{N} \,\&\, \mathsf{Svm}(f)$
$\langle f\rangle \hookrightarrow T165 \Rightarrow$ $\mathsf{Finite}(\mathbf{domain}(f)) \rightarrow \mathsf{Finite}(\mathbf{range}(f))$
$\langle y\rangle \hookrightarrow T179 \Rightarrow$ $\mathsf{Finite}(y) \,\&\, \neg\mathsf{Finite}(\mathbb{N})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next aim is to define the first few integers and establish their elementary properties. This is done in the two following theorems.

Theorem 212 (182) $\mathcal{O}(\emptyset) \,\&\, \emptyset, 1, 2, 3 \in \mathbb{N} \,\&\, \mathsf{Card}(\emptyset) \,\&\, \mathsf{Card}(1) \,\&\, \mathsf{Card}(2) \,\&\, \mathsf{Card}(3)$. PROOF:

Suppose_not $\Rightarrow$ $\neg\big(\mathcal{O}(\emptyset) \,\&\, \emptyset, 1, 2, 3 \in \mathbb{N} \,\&\, \mathsf{Card}(\emptyset) \,\&\, \mathsf{Card}(1) \,\&\, \mathsf{Card}(2) \,\&\, \mathsf{Card}(3)\big)$

-- All these statements are trivial corollaries of the fact that 0 is a cardinal, and of theorems 147, 159, and 164.

$T161 \Rightarrow$ $\mathsf{Finite}(\emptyset) \,\&\, \mathsf{Card}(\emptyset)$
$\langle\emptyset\rangle \hookrightarrow T179 \Rightarrow$ $\emptyset \in \mathbb{N}$
Use_def(Card) $\Rightarrow$ $\mathcal{O}(\emptyset)$
Use_def(1) $\Rightarrow$ $1 = \mathsf{next}(\emptyset)$
$\langle\emptyset\rangle \hookrightarrow T29 \Rightarrow$ $\mathcal{O}\big(\mathsf{next}(\emptyset)\big)$
EQUAL $\Rightarrow$ $\mathcal{O}(1)$
$\langle\emptyset\rangle \hookrightarrow T173 \Rightarrow$ $\mathsf{Finite}\big(\mathsf{next}(\emptyset)\big)$
EQUAL $\Rightarrow$ $\mathsf{Finite}(1)$
$\langle 1\rangle \hookrightarrow T178 \Rightarrow$ $\mathsf{Card}(1)$
$\langle 1\rangle \hookrightarrow T179 \Rightarrow$ $1 \in \mathbb{N}$

Use_def(2) ⇒ $2 = \mathsf{next}(1)$
$\langle 1 \rangle \hookrightarrow T29$ ⇒ $\mathcal{O}\big(\mathsf{next}(1)\big)$
EQUAL ⇒ $\mathcal{O}(2)$
$\langle 1 \rangle \hookrightarrow T173$ ⇒ $\mathsf{Finite}\big(\mathsf{next}(1)\big)$
EQUAL ⇒ $\mathsf{Finite}(2)$
$\langle 2 \rangle \hookrightarrow T178$ ⇒ $\mathsf{Card}(2)$
$\langle 2 \rangle \hookrightarrow T179$ ⇒ $2 \in \mathbb{N}$
Use_def(3) ⇒ $3 = \mathsf{next}(2)$
$\langle 2 \rangle \hookrightarrow T29$ ⇒ $\mathcal{O}\big(\mathsf{next}(2)\big)$
EQUAL ⇒ $\mathcal{O}(3)$
$\langle 2 \rangle \hookrightarrow T173$ ⇒ $\mathsf{Finite}\big(\mathsf{next}(2)\big)$
EQUAL ⇒ $\mathsf{Finite}(3)$
$\langle 3 \rangle \hookrightarrow T178$ ⇒ $\mathsf{Card}(3)$
$\langle 3 \rangle \hookrightarrow T179$ ⇒ $3 \in \mathbb{N}$
ELEM ⇒ false; Discharge ⇒ QED

-- The following corollary to Theorem 182 merely adds the elementary fact that the first 3 integers are all different.

Theorem 213 (183) $\emptyset, 1, 2, 3 \in \mathbb{N} \;\&\; 1 \neq \emptyset \;\&\; 2 \neq \emptyset \;\&\; 3 \neq \emptyset \;\&\; 1 \neq 2 \;\&\; 1 \neq 3 \;\&\; 2 \neq 3$. PROOF:

Suppose_not ⇒ $\neg(\emptyset, 1, 2, 3 \in \mathbb{N} \;\&\; 1 \neq \emptyset \;\&\; 2 \neq \emptyset \;\&\; 3 \neq \emptyset \;\&\; 1 \neq 2 \;\&\; 1 \neq 3 \;\&\; 2 \neq 3)$
$T182$ ⇒ $\emptyset, 1, 2, 3 \in \mathbb{N}$
Use_def(1) ⇒ $1 = \mathsf{next}(\emptyset)$
Use_def(2) ⇒ $2 = \mathsf{next}(1)$
Use_def(3) ⇒ $3 = \mathsf{next}(2)$
Use_def(next) ⇒ $\emptyset \in 1 \;\&\; 1 \in 2 \;\&\; 2 \in 3$
ELEM ⇒ false; Discharge ⇒ QED

-- Next, in preparation for our account of integer arithmetic, we define the main arithmetic operators, not merely for integers, but for all cardinals, whether finite or infinite.

-- Cardinal sum
DEF 19. $\mathsf{X} + \mathsf{Y} =_{\text{Def}} \#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{X}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{Y}\})$
-- Cardinal product
DEF 20. $\mathsf{X} * \mathsf{Y} =_{\text{Def}} \#(\mathsf{X} \times \mathsf{Y})$
DEF 21. $\mathcal{P}\mathsf{X} =_{\text{Def}} \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{X}\}$
-- Cardinal Difference
DEF 22. $\mathsf{X} - \mathsf{Y} =_{\text{Def}} \#(\mathsf{X} \backslash \mathsf{Y})$

-- The quotient m **div** n is defined as the largest integer k such that $\mathsf{k} * \mathsf{n}$ is no larger than m, and m **mod** n is defined as the remainder $\mathsf{m} - \mathsf{m}\ \mathbf{div}\ \mathsf{n} * \mathsf{n}$.

-- Integer Quotient ; Note that x / 0 is defined as Z for x in Z

Def 23. $\mathsf{X}\ \mathbf{div}\ \mathsf{Y} \quad =_{\text{Def}} \quad \bigcup\{\mathsf{k} \in \mathbb{N} \mid \mathsf{k} * \mathsf{Y} \subseteq \mathsf{X}\}$

-- Integer Remainder

Def 24. $\mathsf{X}\ \mathbf{mod}\ \mathsf{Y} \quad =_{\text{Def}} \quad \mathsf{X} - \mathsf{X}\ \mathbf{div}\ \mathsf{Y} * \mathsf{Y}$

-- The fact that the power set of the null set is the singleton whose sole member is the null set is an elementary consequence of the definition of 'pow'.

Theorem 214 (184) $\mathcal{P}\emptyset = \{\emptyset\}$. Proof:

Suppose_not $\Rightarrow$ $\mathit{Stat1}: \ \mathcal{P}\emptyset \neq \{\emptyset\}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $(\mathsf{c} \in \mathcal{P}\emptyset \ \&\ \mathsf{c} \neq \emptyset) \vee (\mathsf{c} \notin \mathcal{P}\emptyset \ \&\ \mathsf{c} = \emptyset)$
Use_def $(\mathcal{P}) \Rightarrow$ $(\mathsf{c} \in \{\mathsf{x} : \mathsf{x} \subseteq \emptyset\} \ \&\ \mathsf{c} \neq \emptyset) \vee (\mathsf{c} \notin \{\mathsf{x} : \mathsf{x} \subseteq \emptyset\} \ \&\ \mathsf{c} = \emptyset)$
Suppose $\Rightarrow$ $\mathit{Stat2}: \ \mathsf{c} \notin \{\mathsf{x} : \mathsf{x} \subseteq \emptyset\} \ \&\ \mathsf{c} = \emptyset$
$\langle \emptyset \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat3}: \ \mathsf{c} \in \{\mathsf{x} : \mathsf{x} \subseteq \emptyset\} \ \&\ \mathsf{c} \neq \emptyset$
$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ false; Discharge $\Rightarrow$ Qed

-- = = = = = = = = = = = = = = = = = Additional laws concerning union-set =

Theorem 215 (185) $\bigcup\emptyset = \emptyset \ \&\ \left(\mathsf{M} \neq \emptyset \to \bigcup\mathsf{M} = \mathbf{arb}(\mathsf{M}) \cup \bigcup(\mathsf{M} \setminus \{\mathbf{arb}(\mathsf{M})\})\right)$. Proof:

-- For, if not, there would be a counterexample m to the second part of the statement, since the first part of the statement trivially follows from the very definition of the union-set operator.

Suppose_not $(\mathsf{m}) \Rightarrow$ $\bigcup\emptyset \neq \emptyset \vee \left(\mathsf{m} \neq \emptyset \ \&\ \bigcup\mathsf{m} \neq \mathbf{arb}(\mathsf{m}) \cup \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})\right)$
Suppose $\Rightarrow$ $\bigcup\emptyset \neq \emptyset$
Use_def $(\bigcup) \Rightarrow$ $\mathit{Stat1}: \ \{\mathsf{x} : \mathsf{y} \in \emptyset, \mathsf{x} \in \mathsf{y}\} \neq \emptyset$
$\langle \mathsf{c}, \mathsf{y} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}: \ \mathsf{y} \in \emptyset$
$\langle \mathit{Stat2} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat3}: \ \mathsf{m} \neq \emptyset \ \&\ \bigcup\mathsf{m} \neq \mathbf{arb}(\mathsf{m}) \cup \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$

-- Every non-null set m, and in particular our counterexample, can be decomposed as disjoint union $\mathsf{m} = \{\mathbf{arb}(\mathsf{m})\} \cup (\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$.

$\langle \mathit{Stat3} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat4}: \ \mathsf{m} = \{\mathbf{arb}(\mathsf{m})\} \cup (\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$

-- However, the union-set of the union of two disjoint sets equals the union of their respective union-sets; in our case, when one of the two sets is singleton and hence its member equals its union-set, this tells us that $\bigcup m = \mathbf{arb}(m) \cup \bigcup(m \setminus \{\mathbf{arb}(m)\})$ and hence leads to a contradiction.

Use_def$(\bigcup) \Rightarrow$ *Stat5*: $\{x : y \in m, x \in y\} \neq \mathbf{arb}(m) \cup \{x : y \in m \setminus \{\mathbf{arb}(m)\}, x \in y\}$
$\langle b \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ *Stat6*: $b \notin \{x : y \in m, x \in y\} \leftrightarrow b \in \mathbf{arb}(m) \vee b \in \{x : y \in m \setminus \{\mathbf{arb}(m)\}, x \in y\}$
Suppose $\Rightarrow$ *Stat7*: $b \in \{x : y \in m, x \in y\}\ \&\ b \notin \mathbf{arb}(m)\ \&\ b \notin \{x : y \in m \setminus \{\mathbf{arb}(m)\}, x \in y\}$
$\langle u, x \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ *Stat8*: $u \in m\ \&\ b \in u$
$\langle$ *Stat7*, *Stat8* $\rangle$ ELEM $\Rightarrow$ $u \in m \setminus \{\mathbf{arb}(m)\}$
$\langle w, v, u, b \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ *Stat9*: $b \notin u$
$\langle$ *Stat8*, *Stat9* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat10*: $b \notin \{x : y \in m, x \in y\}\ \&\ b \in \mathbf{arb}(m) \vee b \in \{x : y \in m \setminus \{\mathbf{arb}(m)\}, x \in y\}$
Suppose $\Rightarrow$ *Stat11*: $b \in \mathbf{arb}(m)$
$\langle \mathbf{arb}(m), b \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ *Stat12*: $\neg(\mathbf{arb}(m) \in m\ \&\ b \in \mathbf{arb}(m))$
$\langle$ *Stat4*, *Stat11*, *Stat12* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat13*: $b \in \{x : y \in m \setminus \{\mathbf{arb}(m)\}, x \in y\}$
$\langle ww, xx \rangle \hookrightarrow$ *Stat13* $\Rightarrow$ $ww \in m \setminus \{\mathbf{arb}(m)\}\ \&\ b \in ww$
$\langle ww, b \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following elementary lemma just tells us that the two sets entering into the definition of cardinal addition are always disjoint.

Theorem 216 (186) $\{[x, \emptyset] : x \in N\} \cap \{[x, 1] : x \in M\} = \emptyset$. PROOF:

Suppose_not$(n, m) \Rightarrow$ *Stat1*: $\{[x, \emptyset] : x \in n\} \cap \{[x, 1] : x \in m\} \neq \emptyset$
$\langle e \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2*: $e \in \{[x, \emptyset] : x \in n\}\ \&\ e \in \{[x, 1] : x \in m\}$
$\langle x, y \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $e = [x, \emptyset]\ \&\ e = [y, 1]$
T183 $\Rightarrow$ $1 \neq \emptyset$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the sum of two cardinalities $\#n$, $\#m$ can be computed using any pair of disjoint sets of which the first has cardinality $\#n$ and the second has cardinality $\#m$.

-- First Disjoint sum Lemma

Theorem 217 (187) $N \cap M = \emptyset\ \&\ K \cap J = \emptyset\ \&\ \#N = \#K\ \&\ \#M = \#J \rightarrow \#(N \cup M) = \#(K \cup J)$. PROOF:

Suppose_not$(n, m, k, j) \Rightarrow$ $(n \cap m = \emptyset\ \&\ k \cap j = \emptyset\ \&\ \#n = \#k\ \&\ \#m = \#j)\ \&\ \#(n \cup m) \neq \#(k \cup j)$
ELEM $\Rightarrow$ *Stat0*: $n \cap m = \emptyset\ \&\ k \cap j = \emptyset$

-- For supposing the contrary, and noting that there exist 1-1 maps f and g of n to j and m to k, we see immediately that $f \cup g$ is a 1-1 map of $m \cup n$ onto $j \cup k$, and so our claim follows using Theorem 80.

$\langle n, k\rangle \hookrightarrow T132 \Rightarrow$ $Stat1: \langle \exists f \mid 1\text{–}1(f) \,\&\, \mathbf{range}(f) = n \,\&\, \mathbf{domain}(f) = k\rangle$
$\langle f\rangle \hookrightarrow Stat1 \Rightarrow$ $1\text{–}1(f) \,\&\, \mathbf{range}(f) = n \,\&\, \mathbf{domain}(f) = k$
$\langle m, j\rangle \hookrightarrow T132 \Rightarrow$ $Stat2: \langle \exists f \mid 1\text{–}1(f) \,\&\, \mathbf{range}(f) = m \,\&\, \mathbf{domain}(f) = j\rangle$
$\langle g\rangle \hookrightarrow Stat2 \Rightarrow$ $1\text{–}1(g) \,\&\, \mathbf{range}(g) = m \,\&\, \mathbf{domain}(g) = j$
$\langle f, g\rangle \hookrightarrow T80(\langle Stat0\rangle) \Rightarrow$ $1\text{–}1(f \cup g)$
$\langle f, g\rangle \hookrightarrow T71 \Rightarrow$ $\mathbf{range}(f \cup g) = \mathbf{range}(f) \cup \mathbf{range}(g)$
$\langle f, g\rangle \hookrightarrow T70 \Rightarrow$ $\mathbf{domain}(f \cup g) = \mathbf{domain}(f) \cup \mathbf{domain}(g)$

-- Our result now follows from Theorem 131.

$\langle f \cup g\rangle \hookrightarrow T131 \Rightarrow$ $\#\mathbf{domain}(f \cup g) = \#\mathbf{range}(f \cup g)$
EQUAL $\Rightarrow$ $\#(\mathbf{domain}(f) \cup \mathbf{domain}(g)) = \#(\mathbf{range}(f) \cup \mathbf{range}(g))$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma, which simply notes a consequence of the elementary fact that $\{[x, a] : x \in n\}$ and n are in 1-1 correspondence, prepares for the proof of Theorem 190 below.

Theorem 218 (188) $\#\{[x, A] : x \in M\} = \#M$. PROOF:

-- Since $\{[x, [x, a]] : x \in m\}$ is clearly a 1-1 map, the present lemma is an obvious consequence of Theorem 131.

Suppose_not$(a, m) \Rightarrow$ $\#\{[x, a] : x \in m\} \neq \#m$
Loc_def $\Rightarrow$ $f = \{[x, [x, a]] : x \in m\}$
APPLY $\langle x_\Theta : x, y_\Theta : y\rangle$ fcn_symbol$(g \mapsto f, f(x) \mapsto [x, a], s \mapsto m) \Rightarrow$
$\mathbf{domain}(f) = m \,\&\, \mathbf{range}(f) = \{[x, a] : x \in m\} \,\&\, (x, y \in m \,\&\, [x, a] = [y, a] \,\&\, x \neq y) \vee 1\text{–}1(f)$
ELEM $\Rightarrow$ $1\text{–}1(f)$
$\langle f\rangle \hookrightarrow T131 \Rightarrow$ $\#\mathbf{domain}(f) = \#\mathbf{range}(f)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the numerical sum $n + m$ of any two disjoint sets is simply the number of elements in their union.

-- Disjoint sum Lemma

Theorem 219 (189) $N \cap M = \emptyset \rightarrow N + M = \#(N \cup M)$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m}) \Rightarrow \quad \mathsf{n} \cap \mathsf{m} = \emptyset \;\&\; \mathsf{n} + \mathsf{m} \neq \#(\mathsf{n} \cup \mathsf{m})$
$\mathsf{Use\_def}(+) \Rightarrow \quad \#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\}) \neq \#(\mathsf{n} \cup \mathsf{m})$
$\langle \emptyset, \mathsf{n} \rangle \hookrightarrow T188 \Rightarrow \quad \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} = \#\mathsf{n}$
$\langle 1, \mathsf{m} \rangle \hookrightarrow T188 \Rightarrow \quad \#\{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} = \#\mathsf{m}$
$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T186 \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} = \emptyset$
$\langle \mathsf{n}, \mathsf{m}, \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\}, \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} \rangle \hookrightarrow T187 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next theorem tells us that the sum of two cardinals i and j can be calculated using any two sets n and m whose cardinalities are n and m respectively.

Theorem 220 (190) $\mathsf{N} + \mathsf{M} = \#\mathsf{N} + \#\mathsf{M}$. PROOF:

-- Supposing that our theorem is false and expanding the definition of $+$ brings us to the cardinal inequality seen just below.

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m}) \Rightarrow \quad \mathsf{n} + \mathsf{m} \neq \#\mathsf{n} + \#\mathsf{m}$
$\mathsf{Use\_def}(+) \Rightarrow \quad \#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\}) \neq \#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\})$

-- But the pairs of sets appearing in this inequality are evidently disjoint, and have the respective cardinalities $\#\mathsf{n}$, $\#\mathsf{m}$, $\#\#\mathsf{n}$, $\#\#\mathsf{m}$.

$T183 \Rightarrow \quad Stat1 : \; 1 \neq \emptyset$
$\mathsf{Suppose} \Rightarrow \quad Stat2 : \; \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} \neq \emptyset$
$\langle \mathsf{c} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{c} \in \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\}$
$\mathsf{ELEM} \Rightarrow \quad Stat3 : \; \mathsf{c} \in \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \;\&\; \mathsf{c} \in \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\}$
$\langle \mathsf{d}, \mathsf{e} \rangle \hookrightarrow Stat3([]) \Rightarrow \quad Stat4 : \; \mathsf{d} \in \mathsf{n} \;\&\; \mathsf{c} = [\mathsf{d},\emptyset] \;\&\; \mathsf{e} \in \mathsf{m} \;\&\; \mathsf{c} = [\mathsf{e},1]$
$\langle Stat4, Stat1 \rangle \; \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} = \emptyset$
$\mathsf{Suppose} \Rightarrow \quad Stat5 : \; \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\} \neq \emptyset$
$\langle \mathsf{cc} \rangle \hookrightarrow Stat5 \Rightarrow \quad \mathsf{cc} \in \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\}$
$\mathsf{ELEM} \Rightarrow \quad Stat6 : \; \mathsf{cc} \in \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\} \;\&\; \mathsf{cc} \in \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\}$
$\langle \mathsf{dd}, \mathsf{ee} \rangle \hookrightarrow Stat6([]) \Rightarrow \quad Stat7 : \; \mathsf{dd} \in \#\mathsf{n} \;\&\; \mathsf{cc} = [\mathsf{dd},\emptyset] \;\&\; \mathsf{ee} \in \#\mathsf{m} \;\&\; \mathsf{cc} = [\mathsf{ee},1]$
$\langle Stat7, Stat1 \rangle \; \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\} = \emptyset$
$\langle \emptyset, \mathsf{n} \rangle \hookrightarrow T188 \Rightarrow \quad \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} = \#\mathsf{n}$
$\langle 1, \mathsf{m} \rangle \hookrightarrow T188 \Rightarrow \quad \#\{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} = \#\mathsf{m}$
$\langle \emptyset, \#\mathsf{n} \rangle \hookrightarrow T188 \Rightarrow \quad \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\} = \#\#\mathsf{n}$
$\langle 1, \#\mathsf{m} \rangle \hookrightarrow T188 \Rightarrow \quad \#\{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\} = \#\#\mathsf{m}$

-- Hence our assertion follows from theorems 137 and 172.

$\langle \mathsf{n} \rangle \hookrightarrow T140 \Rightarrow \quad \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{n}\} = \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{n}\}$
$\langle \mathsf{m} \rangle \hookrightarrow T140 \Rightarrow \quad \#\{[\mathsf{x},1] : \mathsf{x} \in \mathsf{m}\} = \#\{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{m}\}$

$\langle\, \{[x,\emptyset] : x \in n\}, \{[x,1] : x \in m\}, \{[x,\emptyset] : x \in \#n\}, \{[x,1] : x \in \#m\}\,\rangle \hookrightarrow T187 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is sometimes convenient to use theorem 186 in the following variant form:

-- Second disjoint sum Lemma

Theorem 221 (191) $N \cap M = \emptyset \rightarrow \#N + \#M = \#(N \cup M)$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n \cap m = \emptyset \;\&\; \#n + \#m \neq \#(n \cup m)$
$\langle n, m\rangle \hookrightarrow T190 \Rightarrow$ $n + m \neq \#(n \cup m)$
$\langle n, m\rangle \hookrightarrow T189 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The fact that cardinal multiplication of any n by 1 leaves n unchanged will be derived from the corresponding fact for Cartesian products, as stated in the following theorem.

Theorem 222 (192) $\#(\{C\} \times N) = \#N$. PROOF:

Suppose_not$(c, n) \Rightarrow$ $\#(\{c\} \times n) \neq \#n$

-- Make the contrary hypothesis, and use the definition of $\times$.

Use_def$(\times) \Rightarrow$ $\{c\} \times n = \{[x,y] : x \in \{c\}, y \in n\}$
SIMPLF $\Rightarrow$ $\{c\} \times n = \{[c,x] : x \in n\}$
EQUAL $\Rightarrow$ $\#(\{c\} \times n) = \#\{[c,x] : x \in n\}$

-- It is easily seen that {[x, [c, x]]: x in n} is a 1-1 map of n to {[c, x]: x in n}

Loc_def $\Rightarrow$ $f = \{[x,[c,x]] : x \in n\}$
APPLY $\langle x_\Theta : y, y_\Theta : zz\rangle$ fcn_symbol$(f(x) \mapsto [c,x], g \mapsto f, s \mapsto n) \Rightarrow$
$\mathrm{Svm}(f) \;\&\; \mathbf{domain}(f) = n \;\&\; \mathbf{range}(f) = \{[c,x] : x \in n\} \;\&\; (y, zz \in n \;\&\; [c,y] = [c,zz] \;\&\; y \neq zz) \vee 1\text{–}1(f)$
ELEM $\Rightarrow$ $1\text{–}1(f)$

-- Thus the present theorem follows immediately from Theorem 131.

$\langle f\rangle \hookrightarrow T131 \Rightarrow$ $\#\mathbf{domain}(f) = \#\mathbf{range}(f)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following minor variant of Theorem 192 has much the same proof.

Theorem 223 (193) $\#(\mathsf{N} \times \{\mathsf{C}\}) = \#\mathsf{N}$. PROOF:

$\mathsf{Suppose\_not}(n, c) \Rightarrow \quad \#(n \times \{c\}) \neq \#n$

-- Make the contrary hypothesis, and use the definition of $\times$.

$\mathsf{Use\_def}(\times) \Rightarrow \quad n \times \{c\} = \{[x, y] : x \in n, y \in \{c\}\}$
$\mathsf{SIMPLF} \Rightarrow \quad n \times \{c\} = \{[x, c] : x \in n\}$
$\mathsf{EQUAL} \Rightarrow \quad \#(n \times \{c\}) = \# \{[x, c] : x \in n\}$

-- It is easily seen that {[x, [x, c]]: x in n} is a 1-1 map of n to {[x, c]: x in n}

$\mathsf{Loc\_def} \Rightarrow \quad f = \{[x, [x, c]] : x \in n\}$
$\mathsf{APPLY}\ \langle x_\Theta : y, y_\Theta : zz\rangle\ \mathsf{fcn\_symbol}(f(x) \mapsto [x, c], g \mapsto f, s \mapsto n) \Rightarrow$
$\quad \mathsf{Svm}(f)\ \&\ \mathbf{domain}(f) = n\ \&\ \mathbf{range}(f) = \{[x, c] : x \in n\}\ \&\ (y, zz \in n\ \&\ [y, c] = [zz, c]\ \&\ y \neq zz) \vee 1\text{–}1(f)$
$\mathsf{ELEM} \Rightarrow \quad 1\text{–}1(f)$

-- Thus the present theorem follows immediately from Theorem 131.

$\langle f\rangle \hookrightarrow T131 \Rightarrow \quad \#\mathbf{domain}(f) = \#\mathbf{range}(f)$
$\mathsf{EQUAL} \Rightarrow \quad \#n = \# \{[x, c] : x \in n\}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Theorem 191 has the following corollary, which restates the definition of the arithmetic sum in a more 'algebraic' form.

Theorem 224 (194) $\mathsf{A} \neq \mathsf{B} \rightarrow \#\mathsf{N} + \#\mathsf{M} = \#(\mathsf{N} \times \{\mathsf{A}\} \cup \mathsf{M} \times \{\mathsf{B}\})$. PROOF:

$\mathsf{Suppose\_not}(a, b, n, m) \Rightarrow \quad a \neq b\ \&\ \#n + \#m \neq \#(n \times \{a\} \cup m \times \{b\})$
$\langle \{a\}, \{b\}, n, m\rangle \hookrightarrow T117 \Rightarrow \quad n \times \{a\} \cap (m \times \{b\}) = \emptyset$
$\langle n \times \{a\}, m \times \{b\}\rangle \hookrightarrow T191 \Rightarrow \quad \#(n \times \{a\} \cup m \times \{b\}) = \#(n \times \{a\}) + \#(m \times \{b\})$
$\langle n, a\rangle \hookrightarrow T193 \Rightarrow \quad \#(n \times \{a\}) = \#n$
$\langle m, b\rangle \hookrightarrow T193 \Rightarrow \quad \#(m \times \{b\}) = \#m$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following easy corollaries of Theorem 190 generalize it slightly. The proofs, which use Theorem 142, have an elementary algebraic flavor. We show first that the arithmetic sum of any two sets is the sum of the first and the cardinality of the second.

Theorem 225 (195) $\mathsf{N} + \mathsf{M} = \mathsf{N} + \#\mathsf{M}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m}) \Rightarrow \quad \mathsf{n} + \mathsf{m} \neq \mathsf{n} + \#\mathsf{m}$
$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T190 \Rightarrow \quad \mathsf{n} + \mathsf{m} = \#\mathsf{n} + \#\mathsf{m}$
$\langle \mathsf{n},\#\mathsf{m}\rangle \hookrightarrow T190 \Rightarrow \quad \mathsf{n} + \#\mathsf{m} = \#\mathsf{n} + \#\#\mathsf{m}$
$\langle \mathsf{m}\rangle \hookrightarrow T140 \Rightarrow \quad \#\#\mathsf{m} = \#\mathsf{m}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n} + \#\mathsf{m} = \#\mathsf{n} + \#\mathsf{m}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that the arithmetic sum of any two sets is the sum of the second and the cardinality of the first.

Theorem 226 (196) $\mathsf{N} + \mathsf{M} = \#\mathsf{N} + \mathsf{M}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m}) \Rightarrow \quad \mathsf{n} + \mathsf{m} \neq \#\mathsf{n} + \mathsf{m}$
$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T190 \Rightarrow \quad \mathsf{n} + \mathsf{m} = \#\mathsf{n} + \#\mathsf{m}$
$\langle \#\mathsf{n},\mathsf{m}\rangle \hookrightarrow T190 \Rightarrow \quad \#\mathsf{n} + \mathsf{m} = \#\#\mathsf{n} + \#\mathsf{m}$
$\langle \mathsf{m}\rangle \hookrightarrow T140 \Rightarrow \quad \#\#\mathsf{n} = \#\mathsf{n}$
$\mathsf{EQUAL} \Rightarrow \quad \#\mathsf{n} + \mathsf{m} = \#\mathsf{n} + \#\mathsf{m}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following 3-set variants of our earlier disjoint sum lemma are useful in proving the associativity of arithmetic addition. We first prove that the arithmetic sum of any three disjoint sets is the cardinality of their union.

Theorem 227 (197) $\mathsf{N} \cap \mathsf{M} = \emptyset \;\&\; \mathsf{N} \cap \mathsf{K} = \emptyset \;\&\; \mathsf{M} \cap \mathsf{K} = \emptyset \rightarrow \mathsf{N} + \mathsf{M} + \mathsf{K} = \#(\mathsf{N} \cup \mathsf{M} \cup \mathsf{K})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m},\mathsf{k}) \Rightarrow \quad \mathsf{n} \cap \mathsf{m} = \emptyset \;\&\; \mathsf{n} \cap \mathsf{k} = \emptyset \;\&\; \mathsf{m} \cap \mathsf{k} = \emptyset \;\&\; \mathsf{n} + \mathsf{m} + \mathsf{k} \neq \#(\mathsf{n} \cup \mathsf{m} \cup \mathsf{k})$
$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T189 \Rightarrow \quad \mathsf{n} + \mathsf{m} = \#(\mathsf{n} \cup \mathsf{m})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n} + \mathsf{m} + \mathsf{k} = \#(\mathsf{n} \cup \mathsf{m}) + \mathsf{k}$
$\langle \mathsf{n} \cup \mathsf{m},\mathsf{k}\rangle \hookrightarrow T196 \Rightarrow \quad \#(\mathsf{n} \cup \mathsf{m}) + \mathsf{k} = \mathsf{n} \cup \mathsf{m} + \mathsf{k}$
$\langle \mathsf{n} \cup \mathsf{m},\mathsf{k}\rangle \hookrightarrow T189 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following result also asserts that the arithmetic sum of any three disjoint sets is the cardinality of their union, but now with the arithmetic sum differently associated.

Theorem 228 (198) $\mathsf{N} \cap \mathsf{M} = \emptyset \;\&\; \mathsf{N} \cap \mathsf{K} = \emptyset \;\&\; \mathsf{M} \cap \mathsf{K} = \emptyset \rightarrow \mathsf{N} + (\mathsf{M} + \mathsf{K}) = \#\big(\mathsf{N} \cup (\mathsf{M} \cup \mathsf{K})\big)$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m},\mathsf{k}) \Rightarrow \quad \mathsf{n} \cap \mathsf{m} = \emptyset \;\&\; \mathsf{n} \cap \mathsf{k} = \emptyset \;\&\; \mathsf{m} \cap \mathsf{k} = \emptyset \;\&\; \mathsf{n} + (\mathsf{m} + \mathsf{k}) \neq \#\big(\mathsf{n} \cup (\mathsf{m} \cup \mathsf{k})\big)$
$\langle \mathsf{m},\mathsf{k}\rangle \hookrightarrow T189 \Rightarrow \quad \mathsf{m} + \mathsf{k} = \#(\mathsf{m} \cup \mathsf{k})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n} + (\mathsf{m} + \mathsf{k}) = \mathsf{n} + \#(\mathsf{m} \cup \mathsf{k})$

$\langle n, m \cup k \rangle \hookrightarrow T195 \Rightarrow \quad n + \#(m \cup k) = n + (m \cup k)$
$\langle n, m \cup k \rangle \hookrightarrow T189 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- Our next theorem tells us that the product of two cardinals $\#n$ and $\#m$ can also be calculated using any two sets n and m whose cardinalities are $\#n$ and $\#m$ respectively.

Theorem 229 (199) $N * M = \#N * \#M$. PROOF:

$\text{Suppose\_not}(n, m) \Rightarrow \quad n * m \neq \#n * \#m$

-- Supposing that our theorem is false and expanding the definition of $*$ brings us to the cardinal inequality seen just below.

$\text{Use\_def}(*) \Rightarrow \quad \#(n \times m) \neq \#(\#n \times \#m)$

-- Theorem 130 tells us that there always exist 1-1 maps of # n onto n and of # m onto m.

$\langle n \rangle \hookrightarrow T130 \Rightarrow \quad Stat1: \ \text{Card}(\#n) \ \& \ \langle \exists f \mid 1\text{–}1(f) \ \& \ \mathbf{range}(f) = n \ \& \ \mathbf{domain}(f) = \#n \rangle$
$\langle f \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat2: \ 1\text{–}1(f) \ \& \ \mathbf{domain}(f) = \#n \ \& \ \mathbf{range}(f) = n$
$\langle m \rangle \hookrightarrow T130 \Rightarrow \quad Stat3: \ \text{Card}(\#m) \ \& \ \langle \exists f \mid 1\text{–}1(f) \ \& \ \mathbf{range}(f) = m \ \& \ \mathbf{domain}(f) = \#m \rangle$
$\langle g \rangle \hookrightarrow Stat3 \Rightarrow \quad Stat4: \ 1\text{–}1(g) \ \& \ \mathbf{domain}(g) = \#m \ \& \ \mathbf{range}(g) = m$
$\text{Use\_def}(1\text{–}1) \Rightarrow \quad \text{Svm}(f)$
$\text{Use\_def}(1\text{–}1) \Rightarrow \quad \text{Svm}(g)$
$\text{Use\_def}(\text{Svm}) \Rightarrow \quad \text{Is\_map}(f)$
$\text{Use\_def}(\text{Svm}) \Rightarrow \quad \text{Is\_map}(g)$
$\text{Loc\_def} \Rightarrow \quad g_1 = g$
$\text{Loc\_def} \Rightarrow \quad h = \{[x, [f \restriction x^{[1]}, g_1 \restriction x^{[2]}]] : x \in \{[x, y] : x \in \#n, y \in \#m\}\}$

-- Now consider the map h defined by $\{[x, [f \restriction x^{[1]}, g \restriction x^{[2]}]] : x \in \#n \times \#m\}$. We will show that this is a 1-1 map of $\#n \times \#m$ onto $n \times m$.

$\text{APPLY} \ \langle x_\Theta : x, y_\Theta : y \rangle \ \text{fcn\_symbol}(f(x) \mapsto [f \restriction x^{[1]}, g_1 \restriction x^{[2]}], g \mapsto h, s \mapsto \{[x, y] : x \in \#n, y \in \#m\}) \Rightarrow$
$\quad \text{Svm}(h) \ \& \ \mathbf{range}(h) = \{[f \restriction x^{[1]}, g_1 \restriction x^{[2]}] : x \in \{[x, y] : x \in \#n, y \in \#m\}\} \ \& \ \mathbf{domain}(h) = \{[x, y] : x \in \#n, y \in \#m\} \ \& \ (x, y \in \{[x, y] : x \in \#n, y \in \#m\} \ \& \ [f \restriction x^{[1]}, g_1 \restriction x^{[2]}]$
$\text{EQUAL} \Rightarrow \quad Stat5:$
$\quad \text{Svm}(h) \ \& \ \mathbf{range}(h) = \{[f \restriction x^{[1]}, g \restriction x^{[2]}] : x \in \{[x, y] : x \in \#n, y \in \#m\}\} \ \&$
$\qquad \mathbf{domain}(h) = \{[x, y] : x \in \#n, y \in \#m\} \ \& \ (x, y \in \mathbf{domain}(h) \ \& \ [f \restriction x^{[1]}, g \restriction x^{[2]}] = [f \restriction y^{[1]}, g \restriction y^{[2]}] \ \& \ x \neq y) \vee 1\text{–}1(h)$
$\text{SIMPLF} \Rightarrow \quad \mathbf{range}(h) = \{[f \restriction [x, y]^{[1]}, g \restriction [x, y]^{[2]}] : x \in \#n, y \in \#m\}$
$\text{Use\_def}(\times) \Rightarrow \quad \#n \times \#m = \{[x, y] : x \in \#n, y \in \#m\}$

ELEM $\Rightarrow$ $\mathbf{domain}(\mathsf{h}) = \#\mathsf{n} \times \#\mathsf{m}$

-- Next we will show that h is 1-1. Indeed, the distinct x and y which appear in the final clause of the conjunction appearing as Stat6 6 above must have the forms $[\mathsf{x}_1, \mathsf{y}_1]$ and $[\mathsf{x}_2, \mathsf{y}_2]$ respectively, and so either $\mathsf{x}_1 \neq \mathsf{x}_2$ or $\mathsf{y}_1 \neq \mathsf{y}_2$.

Suppose $\Rightarrow$ $\neg 1\text{–}1(\mathsf{h})$
ELEM $\Rightarrow$ $\mathit{Stat7}:\ \mathsf{x} \neq \mathsf{y}$
ELEM $\Rightarrow$ $\mathit{Stat8}:\ \mathsf{x} \in \{[\mathsf{u}, \mathsf{y}] :\ \mathsf{u} \in \#\mathsf{n}, \mathsf{y} \in \#\mathsf{m}\}$
$\langle \mathsf{x}_1, \mathsf{y}_1 \rangle \hookrightarrow \mathit{Stat8} \Rightarrow$ $\mathit{Stat9}:\ \mathsf{x} = [\mathsf{x}_1, \mathsf{y}_1]\ \&\ \mathsf{x}_1 \in \#\mathsf{n}\ \&\ \mathsf{y}_1 \in \#\mathsf{m}$
ELEM $\Rightarrow$ $\mathit{Stat10}:\ \mathsf{y} \in \{[\mathsf{u}, \mathsf{v}] :\ \mathsf{u} \in \#\mathsf{n}, \mathsf{v} \in \#\mathsf{m}\}$
$\langle \mathsf{x}_2, \mathsf{y}_2 \rangle \hookrightarrow \mathit{Stat10} \Rightarrow$ $\mathit{Stat11}:\ \mathsf{y} = [\mathsf{x}_2, \mathsf{y}_2]\ \&\ \mathsf{x}_2 \in \#\mathsf{n}\ \&\ \mathsf{y}_2 \in \#\mathsf{m}$
$\langle \mathit{Stat9}, \mathit{Stat11}, * \rangle$ ELEM $\Rightarrow$ $\mathit{Stat12}:\ \mathsf{x} = [\mathsf{x}_1, \mathsf{y}_1]\ \&\ \mathsf{y} = [\mathsf{x}_2, \mathsf{y}_2]$
$\langle \mathit{Stat7}, \mathit{Stat12} \rangle$ ELEM $\Rightarrow$ $\mathsf{x}_1 \neq \mathsf{x}_2 \vee \mathsf{y}_1 \neq \mathsf{y}_2$

-- If $\mathsf{x}_1 \neq \mathsf{x}_2$, then since f is 1-1 it follows that $\mathsf{f}(\mathsf{x}_1) \neq \mathsf{f}(\mathsf{x}_2)$, contradicting $[\mathsf{f}\upharpoonright\mathsf{x}_1, \mathsf{g}\upharpoonright\mathsf{y}_1] = [\mathsf{f}\upharpoonright\mathsf{x}_2, \mathsf{g}\upharpoonright\mathsf{y}_2]$. Much the same argument applies if $\mathsf{y}_1 \neq \mathsf{y}_2$. Thus $[\mathsf{x}_1, \mathsf{y}_1] = [\mathsf{x}_2, \mathsf{y}_2]$, i. e. $\mathsf{x} = \mathsf{y}$, implying that h is 1-1.

EQUAL $\Rightarrow$ $\left[\mathsf{f}\upharpoonright\mathsf{x}^{[1]}, \mathsf{g}\upharpoonright\mathsf{x}^{[2]}\right] = \left[\mathsf{f}\upharpoonright[\mathsf{x}_1, \mathsf{y}_1]^{[1]}, \mathsf{g}\upharpoonright[\mathsf{x}_1, \mathsf{y}_1]^{[2]}\right]$
ELEM $\Rightarrow$ $[\mathsf{x}_1, \mathsf{y}_1]^{[1]} = \mathsf{x}_1\ \&\ [\mathsf{x}_1, \mathsf{y}_1]^{[2]} = \mathsf{y}_1$
EQUAL $\Rightarrow$ $\left[\mathsf{f}\upharpoonright\mathsf{x}^{[1]}, \mathsf{g}\upharpoonright\mathsf{x}^{[2]}\right] = [\mathsf{f}\upharpoonright\mathsf{x}_1, \mathsf{g}\upharpoonright\mathsf{y}_1]$
EQUAL $\Rightarrow$ $\left[\mathsf{f}\upharpoonright\mathsf{y}^{[1]}, \mathsf{g}\upharpoonright\mathsf{y}^{[2]}\right] = \left[\mathsf{f}\upharpoonright[\mathsf{x}_2, \mathsf{y}_2]^{[1]}, \mathsf{g}\upharpoonright[\mathsf{x}_2, \mathsf{y}_2]^{[2]}\right]$
ELEM $\Rightarrow$ $[\mathsf{x}_2, \mathsf{y}_2]^{[1]} = \mathsf{x}_2\ \&\ [\mathsf{x}_2, \mathsf{y}_2]^{[2]} = \mathsf{y}_2$
EQUAL $\Rightarrow$ $\left[\mathsf{f}\upharpoonright\mathsf{y}^{[1]}, \mathsf{g}\upharpoonright\mathsf{y}^{[2]}\right] = [\mathsf{f}\upharpoonright\mathsf{x}_2, \mathsf{g}\upharpoonright\mathsf{y}_2]$
Use_def(1–1) $\Rightarrow$ $\mathit{Stat13}:\ \langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$
Use_def(1–1) $\Rightarrow$ $\mathit{Stat14}:\ \langle \forall \mathsf{x} \in \mathsf{g}, \mathsf{y} \in \mathsf{g} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{f}, \mathsf{x}_1 \rangle \hookrightarrow \mathit{T69} \Rightarrow$ $[\mathsf{x}_1, \mathsf{f}\upharpoonright\mathsf{x}_1] \in \mathsf{f}$
$\langle \mathsf{f}, \mathsf{x}_2 \rangle \hookrightarrow \mathit{T69} \Rightarrow$ $[\mathsf{x}_2, \mathsf{f}\upharpoonright\mathsf{x}_2] \in \mathsf{f}$
$\langle [\mathsf{x}_1, \mathsf{f}\upharpoonright\mathsf{x}_1], [\mathsf{x}_2, \mathsf{f}\upharpoonright\mathsf{x}_2] \rangle \hookrightarrow \mathit{Stat13} \Rightarrow$ $[\mathsf{x}_1, \mathsf{f}\upharpoonright\mathsf{x}_1]^{[2]} = [\mathsf{x}_2, \mathsf{f}\upharpoonright\mathsf{x}_2]^{[2]} \rightarrow$
$[\mathsf{x}_1, \mathsf{f}\upharpoonright\mathsf{x}_1] = [\mathsf{x}_2, \mathsf{f}\upharpoonright\mathsf{x}_2]$
ELEM $\Rightarrow$ $\mathsf{x}_1 = \mathsf{x}_2$

-- Much the same argument applies if $\mathsf{y}_1 \neq \mathsf{y}_2$.

$\langle \mathsf{g}, \mathsf{y}_1 \rangle \hookrightarrow \mathit{T69} \Rightarrow$ $\mathit{Stat15}:\ [\mathsf{y}_1, \mathsf{g}\upharpoonright\mathsf{y}_1] \in \mathsf{g}$
$\langle \mathsf{g}, \mathsf{y}_2 \rangle \hookrightarrow \mathit{T69} \Rightarrow$ $[\mathsf{y}_2, \mathsf{g}\upharpoonright\mathsf{y}_2] \in \mathsf{g}$
$\langle [\mathsf{y}_1, \mathsf{g}\upharpoonright\mathsf{y}_1], [\mathsf{y}_2, \mathsf{g}\upharpoonright\mathsf{y}_2] \rangle \hookrightarrow \mathit{Stat14}(\langle \mathit{Stat15} \rangle) \Rightarrow$ $[\mathsf{y}_1, \mathsf{g}\upharpoonright\mathsf{y}_1]^{[2]} = [\mathsf{y}_2, \mathsf{g}\upharpoonright\mathsf{y}_2]^{[2]} \rightarrow$
$[\mathsf{y}_1, \mathsf{g}\upharpoonright\mathsf{y}_1] = [\mathsf{y}_2, \mathsf{g}\upharpoonright\mathsf{y}_2]$

ELEM $\Rightarrow$ $y_1 = y_2$
EQUAL $\Rightarrow$ $[x_1, y_1] = [x_2, y_2]$
EQUAL $\Rightarrow$ $x = y$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ 1–1(h)

-- Finally, we show that the range of h is $n \times m$. First suppose that **range**(h) is not included in $n \times m$. Then there exists a $c = [f\restriction cx_1, g\restriction cy_1]$, with $cx_1 \in \#n$ and $cy_1 \in \#m$, such that $c \notin n \times m$, which is impossible since f and g map $\#n$ and $\#m$ into n and m respectively.

Suppose $\Rightarrow$ $n \times m \neq \left\{\left[f\restriction[x,y]^{[1]}, g\restriction[x,y]^{[2]}\right] : x \in \#n, y \in \#m\right\}$
Use_def($\times$) $\Rightarrow$ $n \times m = \{[x,y] : x \in n, y \in m\}$
Suppose $\Rightarrow$ *Stat16* : $\{[x,y] : x \in n, y \in m\} \not\supseteq \left\{\left[f\restriction[x,y]^{[1]}, g\restriction[x,y]^{[2]}\right] : x \in \#n, y \in \#m\right\}$
$\langle c\rangle \hookrightarrow$ *Stat16* $\Rightarrow$ *Stat17* :
$c \in \left\{\left[f\restriction[x,y]^{[1]}, g\restriction[x,y]^{[2]}\right] : x \in \#n, y \in \#m\right\}$ & *Stat18* : $c \notin \{[x,y] : x \in n, y \in m\}$
$\langle cx_1, cy_1\rangle \hookrightarrow$ *Stat17* $\Rightarrow$ *Stat19* : $c = \left[f\restriction[cx_1, cy_1]^{[1]}, g\restriction[cx_1, cy_1]^{[2]}\right]$ & $cx_1 \in \#n$ & $cy_1 \in \#m$
$\langle$*Stat2, Stat4, Stat19*$\rangle$ ELEM $\Rightarrow$ $cx_1 \in$ **domain**(f) & $cy_1 \in$ **domain**(g)
$\langle cx_1, f\rangle \hookrightarrow$ *T64* $\Rightarrow$ $f\restriction cx_1 \in n$
$\langle cy_1, g\rangle \hookrightarrow$ *T64* $\Rightarrow$ $g\restriction cy_1 \in m$
ELEM $\Rightarrow$ $[cx_1, cy_1]^{[1]} = cx_1$
ELEM $\Rightarrow$ $[cx_1, cy_1]^{[2]} = cy_1$
EQUAL $\Rightarrow$ $c = [f\restriction cx_1, g\restriction cy_1]$
$\langle f\restriction cx_1, g\restriction cy_1\rangle \hookrightarrow$ *Stat18* $\Rightarrow$
$\neg(f\restriction cx_1 \in n \,\&\, g\restriction cy_1 \in m \,\&\, c = [f\restriction cx_1, g\restriction cy_1])$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat20* : $\{[x,y] : x \in n, y \in m\} \not\subseteq \left\{\left[f\restriction[x,y]^{[1]}, g\restriction[x,y]^{[2]}\right] : x \in \#n, y \in \#m\right\}$

-- On the other hand, if $n \times m$ is not included in **range**(h), there is an element $d = [dx_1, dy_1]$ in the first of these two sets but not in the second. but then there exist $ex_1 \in \#n$ and $ey_1 \in \#m$ such that $dx_1 = f\restriction ex_1$ and $dy_1 = g\restriction ey_1$, so d does belong to the second set.

$\langle d\rangle \hookrightarrow$ *Stat20* $\Rightarrow$ *Stat21* :
$d \in \{[x,y] : x \in n, y \in m\}$ & *Stat22* : $d \notin \left\{\left[f\restriction[x,y]^{[1]}, g\restriction[x,y]^{[2]}\right] : x \in \#n, y \in \#m\right\}$
$\langle dx_1, dy_1\rangle \hookrightarrow$ *Stat21* $\Rightarrow$ $dx_1 \in$ **range**(f) & $dy_1 \in$ **range**(g) & $d = [dx_1, dy_1]$
$\langle f\rangle \hookrightarrow$ *T65* $\Rightarrow$ $f = \{[x, f\restriction x] : x \in$ **domain**$(f)\}$
$\langle g\rangle \hookrightarrow$ *T65* $\Rightarrow$ $g = \{[x, g\restriction x] : x \in$ **domain**$(g)\}$
APPLY $\langle x_\Theta : x_3, y_\Theta : x_4\rangle$ fcn_symbol$(f(x) \mapsto g\restriction x, g \mapsto g, s \mapsto$ **domain**$(g))$ $\Rightarrow$ **range**$(g) =$

$\{g\upharpoonright x : x \in \mathbf{domain}(g)\}$

APPLY $\langle x_\Theta : y_3, y_\Theta : y_4\rangle$ fcn_symbol$(f(x) \mapsto f\upharpoonright x, g \mapsto f, s \mapsto \mathbf{domain}(f)) \Rightarrow$ $\mathbf{range}(f) = \{f\upharpoonright x : x \in \mathbf{domain}(f)\}$

ELEM $\Rightarrow$ *Stat23* : $dx_1 \in \{f\upharpoonright x : x \in \mathbf{domain}(f)\}$

ELEM $\Rightarrow$ *Stat24* : $dy_1 \in \{g\upharpoonright x : x \in \mathbf{domain}(g)\}$

$\langle ex_1\rangle \hookrightarrow$ *Stat23* $\Rightarrow$ *Stat25* : $ex_1 \in \#n \,\&\, dx_1 = f\upharpoonright ex_1$

$\langle ey_1\rangle \hookrightarrow$ *Stat24* $\Rightarrow$ $ey_1 \in \#m \,\&\, dy_1 = g\upharpoonright ey_1$

$\langle ex_1, ey_1\rangle \hookrightarrow$ *Stat22* $\Rightarrow$ $\neg(ex_1 \in \#n \,\&\, ey_1 \in \#m \,\&\, d = \left[f\upharpoonright[ex_1, ey_1]^{[1]}, g\upharpoonright[ex_1, ey_1]^{[2]}\right])$

$\langle$*Stat23*, $*\rangle$ ELEM $\Rightarrow$ $[ex_1, ey_1]^{[1]} = ex_1 \,\&\, [ex_1, ey_1]^{[2]} = ey_1$

$\langle$*Stat25*, $*\rangle$ ELEM $\Rightarrow$ $d \neq \left[f\upharpoonright[ex_1, ey_1]^{[1]}, g\upharpoonright[ex_1, ey_1]^{[2]}\right]$

EQUAL $\Rightarrow$ $d \neq [f\upharpoonright ex_1, g\upharpoonright ey_1]$

EQUAL $\Rightarrow$ $d = [f\upharpoonright ex_1, g\upharpoonright ey_1]$

Discharge $\Rightarrow$ $n \times m = \left\{\left[f\upharpoonright[x, y]^{[1]}, g\upharpoonright[x, y]^{[2]}\right] : x \in \#n, y \in \#m\right\}$

ELEM $\Rightarrow$ $n \times m = \mathbf{range}(h)$

$\langle h\rangle \hookrightarrow$ *T131* $\Rightarrow$ $\#\mathbf{range}(h) = \#\mathbf{domain}(h)$

EQUAL $\Rightarrow$ $\#(n \times m) = \#(\#n \times \#m)$

-- This final contradiction completes the proof of our theorem.

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following corollary of Theorem 199 is the 'product' analog of Theorem 195. Its simple proof is algebraic in flavor.

Theorem 230 (200) $N * M = N * \#M$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n * m \neq n * \#m$

$\langle n, m\rangle \hookrightarrow$ *T199* $\Rightarrow$ $n * m = \#n * \#m$

$\langle n, \#m\rangle \hookrightarrow$ *T199* $\Rightarrow$ $n * \#m = \#n * \#\#m$

$\langle m\rangle \hookrightarrow$ *T140* $\Rightarrow$ $\#\#m = \#m$

EQUAL $\Rightarrow$ $n * \#m = \#n * \#m$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- it is also useful to state the following variants of the same fact. The first of these tells us that the arithmetic product of two sets is the same as the arithmetic product of their cardinalities.

Theorem 231 (201) $\#(N \times M) = \#(\#N \times \#M)$. PROOF:

$\mathsf{Suppose\_not}(n, m) \Rightarrow \quad \#(n \times m) \neq \#(\#n \times \#m)$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#(n \times m) = n * m$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#(\#n \times \#m) = \#n * \#m$
$\langle n, m \rangle \hookrightarrow T199 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that the arithmetic product of two sets is the same as the arithmetic product of the first by the cardinality of the second.

Theorem 232 (202) $\#(N \times M) = \#(N \times \#M)$. PROOF:

$\mathsf{Suppose\_not}(n, m) \Rightarrow \quad \#(n \times m) \neq \#(n \times \#m)$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#(n \times m) = n * m$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#(n \times \#m) = n * \#m$
$\langle n, m \rangle \hookrightarrow T200 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The final result in this smalls series show that the arithmetic product of two sets is the same as the arithmetic product of the cardinality of the first by the second.

Theorem 233 (203) $\#(N \times M) = \#(\#N \times M)$. PROOF:

$\mathsf{Suppose\_not}(n, m) \Rightarrow \quad \#(n \times m) \neq \#(\#n \times m)$
$\langle n, m \rangle \hookrightarrow T135 \Rightarrow \quad \#(m \times n) \neq \#(\#n \times m)$
$\langle \#n, m \rangle \hookrightarrow T135 \Rightarrow \quad \#(m \times n) \neq \#(m \times \#n)$
$\langle m, n \rangle \hookrightarrow T202 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we note that a proper subset of a finite set n has a cardinality which is definitely smaller than that of n.

Theorem 234 (204) $\mathsf{Finite}(N) \,\&\, M \subseteq N \,\&\, M \neq N \rightarrow \#M \in \#N$. PROOF:

$\mathsf{Suppose\_not}(n, m) \Rightarrow \quad \mathsf{Finite}(n) \,\&\, m \subseteq n \,\&\, m \neq n \,\&\, \#m \notin \#n$

-- Suppose than n and m constitute a counterexample to our theorem. Since $m \subseteq n$, $\#m \subseteq \#n$ and since both $\#m$ and $\#n$ are ordinals it follows by Theorem 144 that $\#m = \#n$. Thus m and n are in 1-1 correspondence, which is impossible by the definition of finiteness.

$\langle n \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#n)$
$\langle m \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#m)$

$\langle \#n, \#m \rangle \hookrightarrow T32 \Rightarrow \quad \#m \supseteq \#n$
$\langle m, n \rangle \hookrightarrow T144 \Rightarrow \quad \#m = \#n$
$\langle m, n \rangle \hookrightarrow T132 \Rightarrow \quad Stat1: \langle \exists f \mid 1\text{–}1(f) \,\&\, \mathbf{range}(f) = m \,\&\, \mathbf{domain}(f) = n \rangle$
$\langle g \rangle \hookrightarrow Stat1 \Rightarrow \quad 1\text{–}1(g) \,\&\, \mathbf{domain}(g) = n \,\&\, \mathbf{range}(g) \subseteq n \,\&\, \mathbf{range}(g) \neq n$
Use_def(Finite) $\Rightarrow \quad Stat2: \neg\langle \exists f \mid 1\text{–}1(f) \,\&\, \mathbf{domain}(f) = n \,\&\, \mathbf{range}(f) \subseteq n \,\&\, \mathbf{range}(f) \neq n \rangle$
$\langle g \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Using the result just proved, we can give the following variant of the standard theory of transfinite induction.

THEORY finite_induction(n, P(x))
Finite(n) & P(n)
END finite_induction

ENTER_THEORY finite_induction

-- We show that if some finite set n has a property P, there must exist a subset of n which is finite, has property P, but has no strict subset also having property P. This refines the ordinary principle of induction, which would only tell us that n has an element having property P, but itself having no element also having property P.

DEF finite_induction$_0$. $\quad m_\Theta =_{\text{Def}} \mathbf{arb}(\{m : m \subseteq n \mid P(m) \,\&\, \langle \forall k \subseteq m \mid k \neq m \rightarrow \neg P(k) \rangle\})$

Theorem 235 (finite_induction$_1$) $m_\Theta \subseteq n \,\&\, P(m_\Theta) \,\&\, \langle \forall k \subseteq m_\Theta \mid k \neq m_\Theta \rightarrow \neg P(k) \rangle$. PROOF:

-- The proof works by applying standard transfinite induction to the cardinality $\#m$ of sets m having the property P(m). Since by our hypothesis # n is an integer for which there exists a set y such that $\#y = \#n \,\&\, y \subseteq n \,\&\, P(y)$, the standard principle of induction tells us that there must exist a smallest integer m with this property.

Suppose_not $\Rightarrow \quad Stat1: \neg\langle \exists m \mid m \subseteq n \,\&\, P(m) \,\&\, \langle \forall k \subseteq m \mid k \neq m \rightarrow \neg P(k) \rangle \rangle$
Assump $\Rightarrow$ Finite(n) & P(n)
$\langle n \rangle \hookrightarrow T166 \Rightarrow$ Finite(#n)
$\langle n \rangle \hookrightarrow T130 \Rightarrow$ Card(#n)
$\langle \#n \rangle \hookrightarrow T179 \Rightarrow \quad \#n \in \mathbb{N} \,\&\, P(n)$
ELEM $\Rightarrow \quad \#n = \#n \,\&\, n \subseteq n \,\&\, P(n)$
Suppose $\Rightarrow \quad Stat2: \neg\langle \exists y \mid \#y = \#n \,\&\, y \subseteq n \,\&\, P(y) \rangle$
$\langle n \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow \quad \#n \in \mathbb{N} \,\&\, \langle \exists y \mid \#y = \#n \,\&\, y \subseteq n \,\&\, P(y) \rangle$
Loc_def $\Rightarrow \quad n = nn$
EQUAL $\Rightarrow \quad \#n \in \mathbb{N} \,\&\, \langle \exists y \mid \#y = \#n \,\&\, y \subseteq nn \,\&\, P(y) \rangle$

APPLY $\langle \mathsf{mt}_\Theta : \mathsf{j}\rangle$ transfinite_induction$(\mathsf{n} \mapsto \#\mathsf{n}, \mathsf{P}(\mathsf{x}) \mapsto \mathsf{x} \in \mathbb{N} \,\&\, \langle \exists \mathsf{y} \mid \#\mathsf{y} = \mathsf{x} \,\&\, \mathsf{y} \subseteq \mathsf{nn} \,\&\, \mathsf{P}(\mathsf{y})\rangle) \Rightarrow$

$\langle \forall \mathsf{k} \mid (\mathsf{j} \in \mathbb{N} \,\&\, \langle \exists \mathsf{y} \mid \#\mathsf{y} = \mathsf{j} \,\&\, \mathsf{y} \subseteq \mathsf{nn} \,\&\, \mathsf{P}(\mathsf{y})\rangle) \,\&\, \Big(\mathsf{k} \in \mathsf{j} \rightarrow \neg(\mathsf{k} \in \mathbb{N} \,\&\, \langle \exists \mathsf{y} \mid \#\mathsf{y} = \mathsf{k} \,\&\, \mathsf{y} \subseteq \mathsf{nn} \,\&\, \mathsf{P}(\mathsf{y})\rangle)\Big)\rangle$

EQUAL $\Rightarrow$ $\mathit{Stat4}:\ \langle \forall \mathsf{k} \mid (\mathsf{j} \in \mathbb{N} \,\&\, \langle \exists \mathsf{y} \mid \#\mathsf{y} = \mathsf{j} \,\&\, \mathsf{y} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{y})\rangle) \,\&\, \Big(\mathsf{k} \in \mathsf{j} \rightarrow \neg(\mathsf{k} \in \mathbb{N} \,\&\, \langle \exists \mathsf{y} \mid \#\mathsf{y} = \mathsf{k} \,\&\, \mathsf{y} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{y})\rangle)\Big)\rangle$

$\langle \mathsf{a}\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $\mathsf{j} \in \mathbb{N} \,\&\, \mathit{Stat3}:\ \langle \exists \mathsf{y} \mid \#\mathsf{y} = \mathsf{j} \,\&\, \mathsf{y} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{y})\rangle$

$\langle \mathsf{m}\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $\mathit{Stat3a}:\ \#\mathsf{m} = \mathsf{j} \,\&\, \mathsf{m} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{m})$

Pred_monot $\Rightarrow$ Finite$(\#\mathsf{m})$

Pred_monot $\Rightarrow$ Finite(m)

$\langle \mathsf{m}\rangle \hookrightarrow \mathit{T130} \Rightarrow$ $\mathit{Stat4a}:\ \mathcal{O}(\#\mathsf{m})$

-- But now if the set m has any proper subset k such that P(k), then by Theorem 167 #k would be less than #m. Hence m has the minimality property demanded by the present theorem.

Suppose $\Rightarrow$ $\mathit{Stat5}:\ \neg\langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \rightarrow \neg\mathsf{P}(\mathsf{k})\rangle$

$\langle \mathsf{k}\rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ $\mathsf{k} \subseteq \mathsf{m} \,\&\, \mathsf{k} \neq \mathsf{m} \,\&\, \mathsf{P}(\mathsf{k})$

Set_monot $\Rightarrow$ $\#\mathsf{k} \subseteq \#\mathsf{m}$

Pred_monot $\Rightarrow$ Finite$(\#\mathsf{k})$

$\langle \mathsf{m}, \mathsf{k}\rangle \hookrightarrow \mathit{T167} \Rightarrow$ $\#\mathsf{k} \neq \#\mathsf{m}$

$\langle \mathsf{k}\rangle \hookrightarrow \mathit{T130} \Rightarrow$ Card$(\#\mathsf{k}) \,\&\, \mathcal{O}(\#\mathsf{k})$

$\langle \#\mathsf{m}, \#\mathsf{k}\rangle \hookrightarrow \mathit{T32}(\langle \mathit{Stat4a}\rangle) \Rightarrow$ $\#\mathsf{k} \in \#\mathsf{m}$

$\langle \#\mathsf{k}\rangle \hookrightarrow \mathit{T179} \Rightarrow$ $\#\mathsf{k} \in \mathbb{N}$

$\langle \#\mathsf{k}\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $\mathit{Stat6}:\ \neg\langle \exists \mathsf{y} \mid \#\mathsf{y} = \#\mathsf{k} \,\&\, \mathsf{y} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{y})\rangle$

$\langle \mathsf{k}\rangle \hookrightarrow \mathit{Stat6}(\langle \mathit{Stat3a}\rangle) \Rightarrow$ false; Discharge $\Rightarrow$ $\langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \rightarrow \neg\mathsf{P}(\mathsf{k})\rangle$

ELEM $\Rightarrow$ $\mathsf{m} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{m}) \,\&\, \langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \rightarrow \neg\mathsf{P}(\mathsf{k})\rangle$

$\langle \mathsf{m}\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ false; Discharge $\Rightarrow$ QED

$\langle \mathsf{m}\rangle \hookrightarrow$finite_induction $\cdot 1 \Rightarrow$ $\mathsf{m} \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{m}) \,\&\, \langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \rightarrow \neg\mathsf{P}(\mathsf{k})\rangle$

ENTER_THEORY Set_theory

DISPLAY finite_induction

THEORY finite_induction$(\mathsf{n}, \mathsf{P}(\mathsf{x}))$

Finite$(\mathsf{n}) \,\&\, \mathsf{P}(\mathsf{n})$

$\Rightarrow (\mathsf{m}_\Theta)$

$\mathsf{m}_\Theta \subseteq \mathsf{n} \,\&\, \mathsf{P}(\mathsf{m}_\Theta) \,\&\, \langle \forall \mathsf{k} \subseteq \mathsf{m}_\Theta \mid \mathsf{k} \neq \mathsf{m}_\Theta \rightarrow \neg\mathsf{P}(\mathsf{k})\rangle$

END finite_induction

-- We can use the variant form of induction just derived to prove that the union of two sets is finite if and only if both of the sets are finite.

Theorem 236 (205) $\mathsf{Finite}(\mathsf{N})\ \&\ \mathsf{Finite}(\mathsf{M}) \leftrightarrow \mathsf{Finite}(\mathsf{N}\cup\mathsf{M})$. Proof:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m}) \Rightarrow\quad \neg\big(\mathsf{Finite}(\mathsf{n})\ \&\ \mathsf{Finite}(\mathsf{m}) \leftrightarrow \mathsf{Finite}(\mathsf{n}\cup\mathsf{m})\big)$

-- For if $\mathsf{n}\cup\mathsf{m}$ is finite, so are its subsets m and n.

$\mathsf{Suppose} \Rightarrow\quad \mathsf{Finite}(\mathsf{n}\cup\mathsf{m})\ \&\ \neg\big(\mathsf{Finite}(\mathsf{n})\ \&\ \mathsf{Finite}(\mathsf{m})\big)$
$\mathsf{ELEM} \Rightarrow\quad \mathsf{n}\subseteq\mathsf{n}\cup\mathsf{m}\ \&\ \mathsf{m}\subseteq\mathsf{n}\cup\mathsf{m}$
$\big\langle\mathsf{n}\cup\mathsf{m},\mathsf{n}\big\rangle\hookrightarrow T162 \Rightarrow\quad \mathsf{Finite}(\mathsf{n})$
$\big\langle\mathsf{n}\cup\mathsf{m},\mathsf{m}\big\rangle\hookrightarrow T162 \Rightarrow\quad \mathsf{Finite}(\mathsf{m})$

-- Thus we only need to consider the possibility that n and m are finite, but $\mathsf{n}\cup\mathsf{m}$ is not. In this case, we can apply the theory of finite induction developed just above to show that there exists a finite set nn such that $\mathsf{nn}\cup\mathsf{m}_2$ is infinite, but $\mathsf{n}_2\cup\mathsf{m}$ is finite for every proper subset n_2 of nn.

$\mathsf{ELEM} \Rightarrow\quad \mathsf{Finite}(\mathsf{n})\ \&\ \mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{n}\cup\mathsf{m})$
$\mathsf{ELEM} \Rightarrow\quad \mathsf{Finite}(\mathsf{n})\ \&\ \big\langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{n}\cup\mathsf{m})\big\rangle$
$\mathsf{APPLY}\ \ \langle\mathsf{m}_\Theta :\ \mathsf{nn}\rangle\ \mathsf{finite\_induction}\Big(\mathsf{n}\mapsto\mathsf{n},\mathsf{P}(\mathsf{x})\mapsto\big(\mathsf{Finite}(\mathsf{x})\ \&\ \big\langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{x}\cup\mathsf{m})\big\rangle\big)\Big) \Rightarrow$
$\mathsf{Finite}(\mathsf{nn})\ \&\ Stat1:\ \big\langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{nn}\cup\mathsf{m})\big\rangle\ \&\ Stat2:\ \big\langle\forall\mathsf{n}_2\subseteq\mathsf{nn}\mid\mathsf{n}_2\neq\mathsf{nn}\rightarrow\neg\big(\mathsf{Finite}(\mathsf{n}_2)\ \&\ \big\langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{n}_2\cup\mathsf{m})\big\rangle\big)\big\rangle$
$\big\langle\mathsf{m}_2\big\rangle\hookrightarrow Stat1 \Rightarrow\quad \mathsf{Finite}(\mathsf{m}_2)\ \&\ \neg\mathsf{Finite}(\mathsf{nn}\cup\mathsf{m}_2)$

-- If nn is nonempty, then we can remove one element c from it, getting a proper subset $\mathsf{nn}\setminus\{\mathsf{c}\}$ for which $\mathsf{nn}\setminus\{\mathsf{c}\}\cup\mathsf{m}_2$ must therefore be finite. But then $\mathsf{nn}\cup\mathsf{m}_2$ is also finite by Theorem 172, a contradiction.

$\mathsf{Suppose} \Rightarrow\quad Stat3:\ \mathsf{nn}\neq\emptyset$
$\big\langle\mathsf{c}\big\rangle\hookrightarrow Stat3 \Rightarrow\quad \mathsf{c}\in\mathsf{nn}$
$\mathsf{ELEM} \Rightarrow\quad \mathsf{nn}\setminus\{\mathsf{c}\}\subseteq\mathsf{nn}\ \&\ \mathsf{nn}\setminus\{\mathsf{c}\}\neq\mathsf{nn}\ \&\ \mathsf{nn}=\mathsf{nn}\setminus\{\mathsf{c}\}\cup\{\mathsf{c}\}$
$\big\langle\mathsf{nn}\setminus\{\mathsf{c}\}\big\rangle\hookrightarrow Stat2 \Rightarrow\quad Stat4:\ \neg\big\langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{nn}\setminus\{\mathsf{c}\}\cup\mathsf{m})\big\rangle$
$\big\langle\mathsf{m}_2\big\rangle\hookrightarrow Stat4 \Rightarrow\quad \mathsf{Finite}(\mathsf{nn}\setminus\{\mathsf{c}\}\cup\mathsf{m}_2)$

-- It follows that nn must be empty. But this obviously contradicts the fact that m_2 is finite, and so concludes the proof of the present theorem.

$\big\langle\mathsf{nn}\setminus\{\mathsf{c}\}\cup\mathsf{m}_2,\mathsf{c}\big\rangle\hookrightarrow T172 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad \mathsf{nn}=\emptyset$
$\mathsf{ELEM} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ Qed

-- = = = = = = = = = = = = = = = = Additional laws concerning finiteness = The union-set of a finite set of finite sets is finite:

Theorem 237 (206) $\langle\forall x \in S \mid \mathsf{Finite}(x)\rangle \,\&\, \mathsf{Finite}(S) \rightarrow \mathsf{Finite}(\bigcup S)$. PROOF:

-- For if there were a counterexample s, then by the finite induction principle it would include an inclusion-minimal counterexample t.

Suppose_not(s) $\Rightarrow$ $\mathit{Stat1}: \langle\forall x \in s \mid \mathsf{Finite}(x)\rangle \,\&\, \mathsf{Finite}(s) \,\&\, \neg\mathsf{Finite}(\bigcup s)$
APPLY $\langle m_\Theta : m\rangle$ finite_induction$(n \mapsto s, P(y) \mapsto \neg\mathsf{Finite}(\bigcup y)) \Rightarrow$
$\mathit{Stat2}: m \subseteq s \,\&\, \neg\mathsf{Finite}(\bigcup m) \,\&\, \langle\forall k \subseteq m \mid k \neq m \longrightarrow \mathsf{Finite}(\bigcup k)\rangle$

-- Such a counterexample m cannot be $\emptyset$, because $\bigcup\emptyset = \emptyset$, which is finite. Consequently, the union set of m can be decomposed as the disjoint union $\bigcup m = \mathbf{arb}(m) \cup \bigcup(m \setminus \{\mathbf{arb}(m)\})$.

$\langle m\rangle \hookrightarrow \mathit{T185} \Rightarrow$ $\mathit{Stat3}: \bigcup\emptyset = \emptyset \,\&\, (m \neq \emptyset \rightarrow \bigcup m = \mathbf{arb}(m) \cup \bigcup(m \setminus \{\mathbf{arb}(m)\}))$
$\mathit{T161} \Rightarrow$ $\mathit{Stat4}: \mathsf{Finite}(\emptyset)$
Suppose $\Rightarrow$ $\mathit{Stat5}: m = \emptyset$
EQUAL $\langle \mathit{Stat5}, \mathit{Stat3}, \mathit{Stat4}\rangle \Rightarrow$ $\mathit{Stat6}: \mathsf{Finite}(\bigcup m)$
$\langle \mathit{Stat2}, \mathit{Stat6}\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat7}: m \neq \emptyset$

-- Since $\mathbf{arb}(m)$ belongs to m, it is finite; moreover, by the minimality of m, $\bigcup(m \setminus \{\mathbf{arb}(m)\})$ is also finite.

$\langle \mathit{Stat3}, \mathit{Stat7}, \mathit{Stat2}\rangle$ ELEM $\Rightarrow$ $\mathit{Stat8}: \bigcup m = \mathbf{arb}(m) \cup \bigcup(m \setminus \{\mathbf{arb}(m)\}) \,\&\, \mathbf{arb}(m) \in s \,\&\, m \setminus \{\mathbf{arb}(m)\} \subseteq m \,\&\, m \setminus \{\mathbf{arb}(m)\} \neq m$
$\langle \mathbf{arb}(m)\rangle \hookrightarrow \mathit{Stat1}(\langle \mathit{Stat8}\rangle) \Rightarrow$ $\mathit{Stat9}: \mathsf{Finite}(\mathbf{arb}(m))$
$\langle \mathit{Stat2}\rangle$ ELEM $\Rightarrow$ $\mathit{Stat10}: \langle\forall k \subseteq m \mid k \neq m \longrightarrow \mathsf{Finite}(\bigcup k)\rangle$
$\langle m \setminus \{\mathbf{arb}(m)\}\rangle \hookrightarrow \mathit{Stat10} \Rightarrow$ $\mathit{Stat11}: \mathsf{Finite}(\bigcup(m \setminus \{\mathbf{arb}(m)\}))$
$\langle \mathbf{arb}(m), \bigcup(m \setminus \{\mathbf{arb}(m)\})\rangle \hookrightarrow \mathit{T205}([\mathit{Stat9}, \mathit{Stat11}]) \Rightarrow$ $\mathit{Stat12}: \mathsf{Finite}(\mathbf{arb}(m) \cup \bigcup(m \setminus \{\mathbf{arb}(m)\}))$

-- This implies that $\bigcup m$ is finite, giving a contradiction which proves the desired conclusion.

$\langle \mathit{Stat8}, \mathit{Stat12}, \mathit{Stat2}\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- = Next we prove a variant Theorem 205 which states that the union of two sets is finite if and only if their arithmetic sum is finite.

Theorem 238 (207) $\mathsf{Finite}(N + M) \leftrightarrow \mathsf{Finite}(N \cup M)$. PROOF:

$\mathsf{Suppose\_not}(n, m) \Rightarrow \quad \mathit{Stat0}:\ \neg\big(\mathsf{Finite}(n + m) \leftrightarrow \mathsf{Finite}(n \cup m)\big)$

-- For suppose that n, m give a counterexample. By Theorems 152 and 190 our assertion reduces to the pair of conditions $\mathsf{Finite}(n) \leftrightarrow \mathsf{Finite}(\{[x, \emptyset] : x \in n\})$ and $\mathsf{Finite}(m) \leftrightarrow \mathsf{Finite}(\{[x, 1] : x \in m\})$.

$\mathsf{Use\_def}(+) \Rightarrow \quad \mathsf{Finite}(n + m) \leftrightarrow \mathsf{Finite}\big(\#(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\})\big)$

$\big\langle \{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\} \big\rangle \hookrightarrow \mathit{T166} \Rightarrow$

$\quad \mathsf{Finite}\big(\#(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\})\big) \leftrightarrow \mathsf{Finite}(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\})$

$\big\langle \{[x, \emptyset] : x \in n\}, \{[x, 1] : x \in m\} \big\rangle \hookrightarrow \mathit{T205} \Rightarrow$

$\quad \mathsf{Finite}(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\}) \leftrightarrow \mathsf{Finite}(\{[x, \emptyset] : x \in n\})\ \&\ \mathsf{Finite}(\{[x, 1] : x \in m\})$

$\langle n, m \rangle \hookrightarrow \mathit{T205} \Rightarrow \quad \mathsf{Finite}(n \cup m) \leftrightarrow \mathsf{Finite}(n)\ \&\ \mathsf{Finite}(m)$

$\langle \mathit{Stat0}, * \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat1}:\ \neg\Big(\big(\mathsf{Finite}(n) \leftrightarrow \mathsf{Finite}(\{[x, \emptyset] : x \in n\})\big)\ \&\ \big(\mathsf{Finite}(m) \leftrightarrow \mathsf{Finite}(\{[x, 1] : x \in m\})\big)\Big)$

$\mathsf{Loc\_def} \Rightarrow \quad f = \big\{\big[x, x^{[1]}\big] : x \in \{[v, \emptyset] : v \in n\}\big\}$

-- But n and $\{[x, \emptyset] : x \in n\}$ are plainly in 1-1 correspondence, and similarly for m.

$\mathsf{APPLY}\ \langle x_\Theta : a, y_\Theta : b \rangle\ \mathsf{fcn\_symbol}\big(f(x) \mapsto x^{[1]}, g \mapsto f, s \mapsto \{[x, \emptyset] : x \in n\}\big) \Rightarrow$

$\quad \mathit{Stat2}:\ \mathsf{Svm}(f)\ \&\ \mathbf{domain}(f) = \{[x, \emptyset] : x \in n\}\ \&\ \mathbf{range}(f) = \big\{x^{[1]} : x \in \{[v, \emptyset] : v \in n\}\big\}\ \&\ \mathit{Stat3}:\ (a \in \{[x, \emptyset] : x \in n\}\ \&\ b \in \{[y, \emptyset] : y \in n\}\ \&\ a^{[1]} = b^{[1]}\ \&\ a \neq b) \vee 1\text{–}$

$\langle aa, bb \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathit{Stat4}:\ (aa, bb \in n\ \&\ a = [aa, \emptyset]\ \&\ b = [bb, \emptyset]\ \&\ a^{[1]} = b^{[1]}\ \&\ a \neq b) \vee 1\text{–}1(f)$

$\langle \mathit{Stat4} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat4a}:\ 1\text{–}1(f)$

$\mathsf{Loc\_def} \Rightarrow \quad g = \big\{\big[x, x^{[1]}\big] : x \in \{[x, 1] : x \in m\}\big\}$

$\mathsf{APPLY}\ \langle x_\Theta : c, y_\Theta : d \rangle\ \mathsf{fcn\_symbol}\big(f(x) \mapsto x^{[1]}, g \mapsto g, s \mapsto \{[x, 1] : x \in m\}\big) \Rightarrow$

$\quad \mathit{Stat5}:\ \mathsf{Svm}(g)\ \&\ \mathbf{domain}(g) = \{[x, 1] : x \in m\}\ \&\ \mathbf{range}(g) = \big\{x^{[1]} : x \in \{[v, 1] : v \in m\}\big\}\ \&\ \mathit{Stat6}:\ (c \in \{[x, 1] : x \in m\}\ \&\ d \in \{[y, 1] : y \in m\}\ \&\ c^{[1]} = d^{[1]}\ \&\ c \neq d) \vee$

$\langle cc, dd \rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad \mathit{Stat7}:\ (c = [cc, 1]\ \&\ d = [dd, 1]\ \&\ c^{[1]} = d^{[1]}\ \&\ c \neq d) \vee 1\text{–}1(g)$

$\langle \mathit{Stat7} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat7a}:\ 1\text{–}1(g)$

$\mathsf{SIMPLF} \Rightarrow \quad \big\{x^{[1]} : x \in \{[v, \emptyset] : v \in n\}\big\} = \Big\{[v, \emptyset]^{[1]} : v \in n\Big\}$

$\mathsf{Set\_monot} \Rightarrow \quad \Big\{[v, \emptyset]^{[1]} : v \in n\Big\} = \{v : v \in n\}$

$\mathsf{SIMPLF} \Rightarrow \quad \{v : v \in n\} = n$

$\mathsf{ELEM} \Rightarrow \quad \mathit{Stat8}:\ \mathbf{range}(f) = n$

$\mathsf{SIMPLF} \Rightarrow \quad \big\{x^{[1]} : x \in \{[v, 1] : v \in m\}\big\} = \Big\{[v, 1]^{[1]} : v \in m\Big\}$

$\mathsf{Set\_monot} \Rightarrow \quad \Big\{[v, 1]^{[1]} : v \in m\Big\} = \{v : v \in m\}$

$\mathsf{SIMPLF} \Rightarrow \quad \{v : v \in m\} = m$

$\mathsf{ELEM} \Rightarrow \quad \mathbf{range}(g) = m$

$\langle f \rangle \hookrightarrow \mathit{T164}([\mathit{Stat4a}, \cap]) \Rightarrow \quad \mathit{Stat9}:\ \mathsf{Finite}(\mathbf{range}(f)) \leftrightarrow \mathsf{Finite}(\mathbf{domain}(f))$

-- From which our assertion is obvious.

EQUAL $\langle$ *Stat2, Stat8, Stat9* $\rangle \Rightarrow$ *Stat10* : $\mathsf{Finite}(n) \leftrightarrow \mathsf{Finite}(\{[x, \emptyset] : x \in n\})$
$\langle g \rangle \hookrightarrow T164(\langle Stat7a \rangle) \Rightarrow$ $\mathsf{Finite}(\mathbf{range}(g)) \leftrightarrow \mathsf{Finite}(\mathbf{domain}(g))$
EQUAL $\langle$ *Stat5* $\rangle \Rightarrow$ *Stat11* : $\mathsf{Finite}(m) \leftrightarrow \mathsf{Finite}(\{[x, 1] : x \in m\})$
$\langle$ *Stat1, Stat10, Stat11* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It follows as a corollary of the preceding theorem that the arithmetic sum of two sets is finite if and only if both of the sets are finite.

Theorem 239 (208) $\mathsf{Finite}(N) \,\&\, \mathsf{Finite}(M) \leftrightarrow \mathsf{Finite}(N + M)$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $\neg(\mathsf{Finite}(n) \,\&\, \mathsf{Finite}(m) \leftrightarrow \mathsf{Finite}(n + m))$
$\langle n, m \rangle \hookrightarrow T207 \Rightarrow$ $\mathsf{Finite}(n + m) \leftrightarrow \mathsf{Finite}(n \cup m)$
$\langle n, m \rangle \hookrightarrow T205 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The next two results, both trivial corollaries of Theorem 114, state that an arithmetic product is zero if either of its factors is zero.

Theorem 240 (209) $N * \emptyset = \emptyset$. PROOF:

Suppose_not$(n) \Rightarrow$ $n * \emptyset \neq \emptyset$
Use_def$(*) \Rightarrow$ $n * \emptyset = \#(n \times \emptyset)$
$\langle n \rangle \hookrightarrow T114 \Rightarrow$ $n \times \emptyset = \emptyset$
EQUAL $\Rightarrow$ $n * \emptyset = \#\emptyset$
$\langle \emptyset \rangle \hookrightarrow T136 \Rightarrow$ QED

Theorem 241 (210) $\emptyset * N = \emptyset$. PROOF:

Suppose_not$(n) \Rightarrow$ $\emptyset * n \neq \emptyset$
Use_def$(*) \Rightarrow$ $\emptyset * n = \#(\emptyset \times n)$
$\langle n \rangle \hookrightarrow T114 \Rightarrow$ $\emptyset \times n = \emptyset$
EQUAL $\Rightarrow$ $\emptyset * n = \#\emptyset$
$\langle \emptyset \rangle \hookrightarrow T136 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is also trivial to show that arithmetic addition of $\emptyset$ to any n leaves n unchanged.

Theorem 242 (211) $\#\mathsf{N}+\emptyset=\#\mathsf{N}$. PROOF:

Suppose_not(n) $\Rightarrow$ $\#\mathsf{n}+\emptyset\neq\#\mathsf{n}$
Use_def(+) $\Rightarrow$ $\#\mathsf{n}+\emptyset=\#(\{[\mathsf{x},\emptyset]:\mathsf{x}\in\#\mathsf{n}\}\cup\{[\mathsf{x},1]:\mathsf{x}\in\emptyset\})$
Set_monot $\Rightarrow$ $\{[\mathsf{x},1]:\mathsf{x}\in\emptyset\}=\{\mathsf{x}:\mathsf{x}\in\emptyset\}$
SIMPLF $\Rightarrow$ $\{[\mathsf{x},1]:\mathsf{x}\in\emptyset\}=\emptyset$
ELEM $\Rightarrow$ $\{[\mathsf{x},\emptyset]:\mathsf{x}\in\#\mathsf{n}\}\cup\{[\mathsf{x},1]:\mathsf{x}\in\emptyset\}=\{[\mathsf{x},\emptyset]:\mathsf{x}\in\#\mathsf{n}\}$
EQUAL $\Rightarrow$ $\#\mathsf{n}+\emptyset=\#\{[\mathsf{x},\emptyset]:\mathsf{x}\in\#\mathsf{n}\}$
$\langle\emptyset,\#\mathsf{n}\rangle\hookrightarrow T188\Rightarrow$ $\#\{[\mathsf{x},\emptyset]:\mathsf{x}\in\#\mathsf{n}\}=\#\#\mathsf{n}$
$\langle\mathsf{n}\rangle\hookrightarrow T140\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The two following results simply translate Theorem 192 into a statement concerning integer multiplication.

Theorem 243 (212) $1*\mathsf{N}=\#\mathsf{N}$. PROOF:

Suppose_not(n) $\Rightarrow$ $1*\mathsf{n}\neq\#\mathsf{n}$
Use_def($*$) $\Rightarrow$ $1*\mathsf{n}=\#(1\times\mathsf{n})$
Use_def(1) $\Rightarrow$ $1=\mathsf{next}(\emptyset)$
Use_def(next) $\Rightarrow$ $1=\emptyset\cup\{\emptyset\}$
ELEM $\Rightarrow$ $1=\{\emptyset\}$
EQUAL $\Rightarrow$ $\#\mathsf{n}\neq\#(\{\emptyset\}\times\mathsf{n})$
$\langle\emptyset,\mathsf{n}\rangle\hookrightarrow T192\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 244 (213) $\mathsf{N}*1=\#\mathsf{N}$. PROOF:

Suppose_not(n) $\Rightarrow$ $\mathsf{n}*1\neq\#\mathsf{n}$
Use_def($*$) $\Rightarrow$ $\mathsf{n}*1=\#(\mathsf{n}\times1)$
Use_def(1) $\Rightarrow$ $1=\mathsf{next}(\emptyset)$
Use_def(next) $\Rightarrow$ $1=\emptyset\cup\{\emptyset\}$
ELEM $\Rightarrow$ $1=\{\emptyset\}$
EQUAL $\Rightarrow$ $\#\mathsf{n}\neq\#(\mathsf{n}\times\{\emptyset\})$
$\langle\mathsf{n},\emptyset\rangle\hookrightarrow T193\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we state and prove a result valid not only in integer but in cardinal arithmetic: the product of n and m is no smaller than n if the cardinal m is non-zero.

Theorem 245 (214) $\mathsf{M}\neq\emptyset\rightarrow\#(\mathsf{N}\times\mathsf{M})\supseteq\#\mathsf{N}$. PROOF:

$\mathsf{Suppose\_not}(m,n) \Rightarrow \quad \mathit{Stat1}:\ m \neq \emptyset\ \&\ \#(n \times m) \not\supseteq \#n$

-- For suppose that m and n are a counterexample to our assertion, and let d belong to m. Consider the single-valued map f defined by f = {[x, car (x)]: x in {[y, d]: y in n}}, whose range is easily seen to be n, while its domain is plainly a subset of n PROD m.

$\langle d \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad d \in m\ \&\ \{d\} \subseteq m$
$\mathsf{Loc\_def} \Rightarrow \quad f = \left\{ \left[x, x^{[1]}\right] : x \in \{[y,d] : y \in n\}\right\}$
$\mathsf{APPLY}\ \langle\ \rangle\ \mathsf{fcn\_symbol}\left(f(x) \mapsto x^{[1]}, g \mapsto f, s \mapsto \{[y,d] : y \in n\}\right) \Rightarrow$
$\quad \mathsf{Svm}(f)\ \&\ \mathbf{domain}(f) = \{[y,d] : y \in n\}\ \&\ \mathbf{range}(f) = \left\{x^{[1]} : x \in \{[y,d] : y \in n\}\right\}$
$\mathsf{SIMPLF} \Rightarrow \quad \mathbf{range}(f) = \left\{[y,d]^{[1]} : y \in n\right\}$
$\mathsf{Set\_monot} \Rightarrow \quad \left\{[y,d]^{[1]} : y \in n\right\} = \{y : y \in n\}$
$\mathsf{ELEM} \Rightarrow \quad \mathbf{range}(f) = \{y : y \in n\}$
$\mathsf{SIMPLF} \Rightarrow \quad \mathbf{range}(f) = n$
$\mathsf{EQUAL} \Rightarrow \quad \mathit{Stat2}:\ \#\mathbf{range}(f) = \#n$
$\mathsf{SIMPLF} \Rightarrow \quad \{[y,d] : y \in n\} = \{[y,z] : y \in n, z \in \{d\}\}$
$\mathsf{Set\_monot} \Rightarrow \quad \{[y,z] : y \in n, z \in \{d\}\} \subseteq \{[y,z] : y \in n, z \in m\}$
$\mathsf{Use\_def}(\times) \Rightarrow \quad \mathbf{domain}(f) \subseteq n \times m$

-- It follows by theorems 85 and 83 that the cardinality of the domain of f is no more than that of $n \times m$, while the cardinality of $\mathbf{range}(f)$ is no more than that of $\mathbf{domain}(f)$.

$\langle \mathbf{domain}(f), n \times m \rangle \hookrightarrow \mathit{T144} \Rightarrow \quad \mathit{Stat3}:\ \#\mathbf{domain}(f) \subseteq \#(n \times m)$
$\langle f \rangle \hookrightarrow \mathit{T145} \Rightarrow \quad \mathit{Stat4}:\ \#\mathbf{range}(f) \subseteq \#\mathbf{domain}(f)$
$\langle \mathit{Stat1}, \mathit{Stat3}, \mathit{Stat4}, \mathit{Stat2}, * \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following elementary lemma prepares for the proof of the commutative law for cardinal addition.

Theorem 246 (215) $A \neq B \rightarrow N + M = \#(N \times \{A\} \cup M \times \{B\})$. PROOF:

$\mathsf{Suppose\_not}(a,b,n,m) \Rightarrow \quad a \neq b\ \&\ n + m \neq \#(n \times \{a\} \cup m \times \{b\})$

-- For supposing the contrary we can derive a contradiction using theorems 127, 145, 122a, and T192 in the following order:

$\langle n, m \rangle \hookrightarrow \mathit{T190} \Rightarrow \quad n + m = \#n + \#m$
$\langle \{a\}, \{b\}, n, m \rangle \hookrightarrow \mathit{T117} \Rightarrow \quad n \times \{a\} \cap (m \times \{b\}) = \emptyset$
$\langle n \times \{a\}, m \times \{b\} \rangle \hookrightarrow \mathit{T189} \Rightarrow \quad n \times \{a\} + m \times \{b\} = \#(n \times \{a\} \cup m \times \{b\})$
$\langle n \times \{a\}, m \times \{b\} \rangle \hookrightarrow \mathit{T190} \Rightarrow \quad n \times \{a\} + m \times \{b\} = \#(n \times \{a\}) + \#(m \times \{b\})$

ELEM $\Rightarrow$ $\#(\mathsf{n} \times \{\mathsf{a}\}) + \#(\mathsf{m} \times \{\mathsf{b}\}) = \#(\mathsf{n} \times \{\mathsf{a}\} \cup \mathsf{m} \times \{\mathsf{b}\})$
$\langle \mathsf{n}, \mathsf{a} \rangle \hookrightarrow$ *T193* $\Rightarrow$ $\#(\mathsf{n} \times \{\mathsf{a}\}) = \#\mathsf{n}$
$\langle \mathsf{m}, \mathsf{b} \rangle \hookrightarrow$ *T193* $\Rightarrow$ $\#(\mathsf{m} \times \{\mathsf{b}\}) = \#\mathsf{m}$
EQUAL $\Rightarrow$ $\#\mathsf{n} + \#\mathsf{m} = \#(\mathsf{n} \times \{\mathsf{a}\} \cup \mathsf{m} \times \{\mathsf{b}\})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove the commutative laws for cardinal arithmetic. For addition the proof results almost immediately from Theorem 215.

-- Commutative law for addition

Theorem 247 (216) $\mathsf{N} + \mathsf{M} = \mathsf{M} + \mathsf{N}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathsf{n} + \mathsf{m} \neq \mathsf{m} + \mathsf{n}$
T183 $\Rightarrow$ $1 \neq \emptyset$
$\langle \emptyset, 1, \mathsf{n}, \mathsf{m} \rangle \hookrightarrow$ *T215* $\Rightarrow$ $\mathsf{n} + \mathsf{m} = \#(\mathsf{n} \times \{\emptyset\} \cup \mathsf{m} \times \{1\})$
$\langle 1, \emptyset, \mathsf{m}, \mathsf{n} \rangle \hookrightarrow$ *T215* $\Rightarrow$ $\mathsf{m} + \mathsf{n} = \#(\mathsf{m} \times \{1\} \cup \mathsf{n} \times \{\emptyset\})$
ELEM $\Rightarrow$ $\mathsf{n} \times \{\emptyset\} \cup \mathsf{m} \times \{1\} = \mathsf{m} \times \{1\} \cup \mathsf{n} \times \{\emptyset\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The commutative law for cardinal multiplication is an equally elementary consequence of Theorem 215.

Theorem 248 (217) $\mathsf{N} * \mathsf{M} = \mathsf{M} * \mathsf{N}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathsf{n} * \mathsf{m} \neq \mathsf{m} * \mathsf{n}$
Use_def($*$) $\Rightarrow$ $\#(\mathsf{n} \times \mathsf{m}) \neq \#(\mathsf{m} \times \mathsf{n})$
$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow$ *T135* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma states several facts which can be regarded as modified 'distributive laws' for the Cartesian product.

Theorem 249 (218) $\mathsf{A} \times \mathsf{X} \cap (\mathsf{B} \times \mathsf{X}) = \mathsf{A} \cap \mathsf{B} \times \mathsf{X}$ & $\mathsf{A} \times \mathsf{X} \cup \mathsf{B} \times \mathsf{X} = (\mathsf{A} \cup \mathsf{B}) \times \mathsf{X}$ & $\mathsf{X} \times \mathsf{A} \cap (\mathsf{X} \times \mathsf{B}) = \mathsf{X} \times (\mathsf{A} \cap \mathsf{B})$ & $\mathsf{X} \times \mathsf{A} \cup \mathsf{X} \times \mathsf{B} = \mathsf{X} \times (\mathsf{A} \cup \mathsf{B})$. PROOF:

-- For suppose the contrary, and let a, b, and c be a counterexample to one of the four clauses of our theorem.

Suppose_not(a, c, b) $\Rightarrow$ $\mathsf{a} \times \mathsf{c} \cap (\mathsf{b} \times \mathsf{c}) \neq \mathsf{a} \cap \mathsf{b} \times \mathsf{c} \lor$
$\mathsf{a} \times \mathsf{c} \cup \mathsf{b} \times \mathsf{c} \neq (\mathsf{a} \cup \mathsf{b}) \times \mathsf{c} \lor \mathsf{c} \times \mathsf{a} \cap (\mathsf{c} \times \mathsf{b}) \neq \mathsf{c} \times (\mathsf{a} \cap \mathsf{b}) \lor \mathsf{c} \times \mathsf{a} \cup \mathsf{c} \times \mathsf{b} \neq \mathsf{c} \times (\mathsf{a} \cup \mathsf{b})$

-- Use of the definition of the cartesian product and of its evident additivity in both its arguments tells us that neither the second or the fourth clause of our theorem can be violated.

Suppose ⇒ $a \times c \cup b \times c \neq (a \cup b) \times c$

Use_def(×) ⇒ $\{[x,y] : x \in a, y \in c\} \cup \{[x,y] : x \in b, y \in c\} \neq \{[x,y] : x \in a \cup b, y \in c\}$

Set_monot ⇒ $\{[x,y] : x \in a, y \in c\} \cup \{[x,y] : x \in b, y \in c\} = \{[x,y] : x \in a \cup b, y \in c\}$

ELEM ⇒ false; Discharge ⇒ $a \times c \cup b \times c = (a \cup b) \times c$

Suppose ⇒ $c \times a \cup c \times b \neq c \times (a \cup b)$

Use_def(×) ⇒ $\{[x,y] : x \in c, y \in a\} \cup \{[x,y] : x \in c, y \in b\} \neq \{[x,y] : x \in c, y \in a \cup b\}$

Set_monot ⇒ $\{[x,y] : x \in c, y \in a\} \cup \{[x,y] : x \in c, y \in b\} = \{[x,y] : x \in c, y \in a \cup b\}$

ELEM ⇒ false; Discharge ⇒ $c \times a \cup c \times b = c \times (a \cup b)$

-- Next suppose that the second clause of the theorem is violated. Using the definition of Cartesian product, we see that there must exist an element d which belongs to one of the two sets $\{[x,y] : x \in a, y \in c\} \cap \{[x,y] : x \in b, y \in c\}$ and $\{[x,y] : x \in a \cap b, y \in c\}$ but not the other. But if d is in the first of these two sets but not the second we are led to the elementary contradiction seen below.

Suppose ⇒ $a \times c \cap (b \times c) \neq a \cap b \times c$

Use_def(×) ⇒ *Stat1* : $\{[x,y] : x \in a, y \in c\} \cap \{[x,y] : x \in b, y \in c\} \neq \{[x,y] : x \in a \cap b, y \in c\}$

$\langle d \rangle \hookrightarrow$ *Stat1* ⇒

$\big(\neg(d \in \{[x,y] : x \in a, y \in c\} \;\&\; d \in \{[x,y] : x \in b, y \in c\}) \;\&\; d \in \{[x,y] : x \in a \cap b, y \in c\}\big) \vee$
$d \in \{[x,y] : x \in a, y \in c\} \;\&\; d \in \{[x,y] : x \in b, y \in c\} \;\&\; d \notin \{[x,y] : x \in a \cap b, y \in c\}$

Suppose ⇒ *Stat2* :

$(d \in \{[x,y] : x \in a, y \in c\} \;\&\; d \in \{[x,y] : x \in b, y \in c\}) \;\&$ *Stat3* : $d \notin \{[x,y] : x \in a \cap b, y \in c\}$

$\langle a_1, c_1, b_1, c_2 \rangle \hookrightarrow$ *Stat2* ⇒ *Stat4* : $d = [a_1, c_1] \;\&\; a_1 \in a \;\&\; c_1 \in c \;\&\; d = [b_1, c_2] \;\&\; b_1 \in b \;\&\; c_2 \in c$

$\langle$ *Stat4* $\rangle$ ELEM ⇒ $a_1 \in a \cap b$

$\langle a_1, c_1 \rangle \hookrightarrow$ *Stat3* ⇒ $d \neq [a_1, c_1] \;\&\; a_1 \in a \cap b \;\&\; c_1 \in c$

-- Thus d must be in the second of the two sets displayed above, but not the first. However, this assertion is in evident contradiction with the monotone dependence of the cartesian product on its arguments. This shows that it is only the fourth clause of our theorem that could be false.

ELEM ⇒ false; Discharge ⇒ $\neg(d \in \{[x,y] : x \in a, y \in c\} \;\&\; d \in \{[x,y] : x \in b, y \in c\}) \;\&\; d \in \{[x,y] : x \in a \cap b, y \in c\}$

Set_monot ⇒ $\{[x,y] : x \in a \cap b, y \in c\} \subseteq \{[x,y] : x \in a, y \in c\}$

Set_monot ⇒ $\{[x,y] : x \in a \cap b, y \in c\} \subseteq \{[x,y] : x \in b, y \in c\}$

ELEM ⇒ false; Discharge ⇒ $c \times a \cap (c \times b) \neq c \times (a \cap b)$

-- But an argument almost identical to that just given can be used to rule out this last possibility. For, using the definition of Cartesian product, we see that there must exist an element d_2 which belongs to one of the two sets seen just below, but not the other.

Use_def($\times$) $\Rightarrow$ $\mathit{Stat5}:\ \{[x,y]:\ x\in c, y\in a\}\cap\{[x,y]:\ x\in c, y\in b\}\neq\{[x,y]:\ x\in c, y\in a\cap b\}$

$\langle d_2\rangle\hookrightarrow \mathit{Stat5}\Rightarrow$

$\big(\neg(d_2\in\{[x,y]:\ x\in c, y\in a\}\ \&\ d_2\in\{[x,y]:\ x\in c, y\in b\})\ \&\ d_2\in\{[x,y]:\ x\in c, y\in a\cap b\}\big)\vee$
$d_2\in\{[x,y]:\ x\in c, y\in a\}\ \&\ d_2\in\{[x,y]:\ x\in c, y\in b\}\ \&\ d_2\notin\{[x,y]:\ x\in c, y\in a\cap b\}$

-- If d_2 is in the first of these two sets but not the second we are led to the elementary contradiction seen below.

Suppose $\Rightarrow$ $\mathit{Stat6}:$

$d_2\in\{[x,y]:\ x\in c, y\in a\}\ \&\ d_2\in\{[x,y]:\ x\in c, y\in b\}\ \&\ \mathit{Stat7}:\ d_2\notin\{[x,y]:\ x\in c, y\in a\cap b\}$

$\langle a_{21}, c_{21}, b_{21}, c_{22}\rangle\hookrightarrow \mathit{Stat6}\Rightarrow$ $d_2=[a_{21},c_{21}]\ \&\ a_{21}\in c\ \&\ c_{21}\in a\ \&\ d_2=[b_{21},c_{22}]\ \&\ b_{21}\in c\ \&\ c_{22}\in b$

ELEM $\Rightarrow$ $c_{21}\in a\cap b$

$\langle a_{21}, c_{21}\rangle\hookrightarrow \mathit{Stat7}\Rightarrow$ $d_2\neq[a_{21},c_{21}]\ \&\ a_{21}\in c\ \&\ c_{21}\in a\cap b$

-- Thus d_2 must be in the second of the two sets seen just above, but not the first. However, this assertion is in contradiction with the monotone dependence of the cartesian product on its arguments. This shows that none of the clauses of our theorem can be false, which is what we wanted to prove.

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg(d_2\in\{[x,y]:\ x\in c, y\in a\}\ \&\ d_2\in\{[x,y]:\ x\in c, y\in b\})\ \&\ d_2\in\{[x,y]:\ x\in c, y\in a\cap b\}$

Set_monot $\Rightarrow$ $\{[x,y]:\ x\in c, y\in a\cap b\}\subseteq\{[x,y]:\ x\in c, y\in a\}$

Set_monot $\Rightarrow$ $\{[x,y]:\ x\in c, y\in a\cap b\}\subseteq\{[x,y]:\ x\in c, y\in b\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following 'monotonicity' consequence of Theorem 218 is often useful.

Theorem 250 (219) $A\subseteq B\ \&\ C\subseteq D\rightarrow A\times C\subseteq B\times D$. PROOF:

Suppose_not(a, b, c, d) $\Rightarrow$ $a\subseteq b\ \&\ c\subseteq d\ \&\ a\times c\nsubseteq b\times d$

-- For the proof we have only to use Theore 203 twice and the use the transitivity of inclusion.

$\langle c, a, d\rangle\hookrightarrow T218\Rightarrow$ $a\times c\cap(a\times d)=a\times(c\cap d)$

$\langle a, d, b\rangle\hookrightarrow T218\Rightarrow$ $a\times d\cap(b\times d)=a\cap b\times d$

ELEM $\Rightarrow$ $c\cap d=c\ \&\ a\cap b=a$

EQUAL $\Rightarrow$ $a\times c\cap(a\times d)=a\times c\ \&\ a\times d\cap(b\times d)=a\times d$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Sometimes one needs the following generalization of Theorem 218.

Theorem 251 (220) $\mathsf{A \times C \cap (B \times D) = A \cap B \times (C \cap D)}$. PROOF:

$\mathsf{Suppose\_not(a, c, b, d)} \Rightarrow \quad \mathsf{a \times c \cap (b \times d) \neq a \cap b \times (c \cap d)}$

-- For suppose that a, c, b, d form a counterexample to our assertion.

$\langle \mathsf{a \cap b, a, c \cap d, c} \rangle \hookrightarrow T219 \Rightarrow \quad \mathsf{a \cap b \times (c \cap d) \subseteq a \times c}$
$\langle \mathsf{a \cap b, b, c \cap d, d} \rangle \hookrightarrow T219 \Rightarrow \quad \mathsf{a \cap b \times (c \cap d) \subseteq b \times d}$
$\mathsf{ELEM} \Rightarrow \quad Stat1: \ \mathsf{a \cap b \times (c \cap d) \not\supseteq a \times c \cap (b \times d)}$
$\langle \mathsf{u} \rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{u \in a \times c \ \& \ u \in b \times d \ \& \ u \notin a \cap b \times (c \cap d)}$
$\mathsf{Use\_def(\times)} \Rightarrow \quad Stat2:$
$\mathsf{u \in \{[x, y] : x \in a, y \in c\} \ \& \ u \in \{[x, y] : x \in b, y \in d\}} \ \& \ Stat3: \ \mathsf{u \notin \{[x, y] : x \in a \cap b, y \in c \cap d\}}$
$\langle \mathsf{x_1, y_1, x_2, y_2} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{x_1 \in a \ \& \ y_1 \in c \ \& \ u = [x_1, y_1] \ \& \ x_2 \in b \ \& \ y_2 \in d \ \& \ u = [x_2, y_2]}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{x_1 \in b \ \& \ y_1 \in d}$
$\langle \mathsf{x_1, y_1} \rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we prove the associative law, first for cardinal addition, then for cardinal multiplication.

Theorem 252 (221) $\mathsf{N + (M + K) = (N + M) + K}$. PROOF:

$\mathsf{Suppose\_not(n, m, k)} \Rightarrow \quad \mathsf{n + (m + k) \neq n + m + k}$

-- For let n, m, and k be a counterexample to our assertion. It is clear that the sets $\mathsf{n \times \{\emptyset\}}$, $\mathsf{m \times \{1\}}$, and $\mathsf{k \times \{2\}}$ are all disjoint, so that by Theorem 197 the sums $\mathsf{m \times \{1\} + k \times \{2\} + n \times \{\emptyset\}}$ and $\mathsf{n \times \{\emptyset\} \cup m \times \{1\} \cup k \times \{2\}}$ can both be written as the cardinality of their union and so are equal.

$T183 \Rightarrow \quad Stat1: \ 1 \neq \emptyset \ \& \ 2 \neq \emptyset \ \& \ 1 \neq 2$
$\mathsf{ELEM} \Rightarrow \quad Stat2: \ \{1\} \cap \{\emptyset\} = \emptyset \ \& \ \{2\} \cap \{\emptyset\} = \emptyset \ \& \ \{1\} \cap \{2\} = \emptyset$
$\langle \{\emptyset\}, \{1\}, \mathsf{n, m} \rangle \hookrightarrow T117 \Rightarrow \quad \mathsf{n \times \{\emptyset\} \cap (m \times \{1\}) = \emptyset}$
$\langle \{1\}, \{2\}, \mathsf{m, k} \rangle \hookrightarrow T117 \Rightarrow \quad \mathsf{m \times \{1\} \cap (k \times \{2\}) = \emptyset}$
$\langle \{\emptyset\}, \{2\}, \mathsf{n, k} \rangle \hookrightarrow T117 \Rightarrow \quad \mathsf{n \times \{\emptyset\} \cap (k \times \{2\}) = \emptyset}$
$\langle \mathsf{n \times \{\emptyset\}, m \times \{1\}, k \times \{2\}} \rangle \hookrightarrow T197 \Rightarrow \quad \mathsf{n \times \{\emptyset\} + m \times \{1\} + k \times \{2\} =}$
$\mathsf{\#(n \times \{\emptyset\} \cup m \times \{1\} \cup k \times \{2\})}$
$\langle \mathsf{m \times \{1\}, k \times \{2\}, n \times \{\emptyset\}} \rangle \hookrightarrow T197 \Rightarrow \quad \mathsf{m \times \{1\} + k \times \{2\} + n \times \{\emptyset\} =}$
$\mathsf{\#(m \times \{1\} \cup k \times \{2\} \cup n \times \{\emptyset\})}$
$\langle Stat1, Stat2, * \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{m \times \{1\} \cup k \times \{2\} \cup n \times \{\emptyset\} = n \times \{\emptyset\} \cup m \times \{1\} \cup k \times \{2\}}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{\#(m \times \{1\} \cup k \times \{2\} \cup n \times \{\emptyset\}) = \#(n \times \{\emptyset\} \cup m \times \{1\} \cup k \times \{2\})}$
$\langle \mathsf{m \times \{1\} + k \times \{2\}, n \times \{\emptyset\}} \rangle \hookrightarrow T216 \Rightarrow \quad \mathsf{m \times \{1\} + k \times \{2\} + n \times \{\emptyset\} =}$
$\mathsf{n \times \{\emptyset\} + (m \times \{1\} + k \times \{2\})}$

ELEM $\Rightarrow$ $\quad$ *Stat3*: $(\mathsf{n} \times \{\emptyset\} + \mathsf{m} \times \{1\}) + \mathsf{k} \times \{2\} = \mathsf{n} \times \{\emptyset\} + (\mathsf{m} \times \{1\} + \mathsf{k} \times \{2\})$

-- The cardinalities of the sets $\mathsf{n} \times \{\emptyset\}$ etc. are clearly equal to $\#\mathsf{n}$, $\#\mathsf{m}$, and $\#\mathsf{k}$ respectively, and so the inner sums appearing in this last formula can be replaced by sums like $\mathsf{n} + \mathsf{m}$ etc.

$\langle \mathsf{n}, \emptyset \rangle \hookrightarrow T193 \Rightarrow$ $\quad \#(\mathsf{n} \times \{\emptyset\}) = \#\mathsf{n}$
$\langle \mathsf{m}, 1 \rangle \hookrightarrow T193 \Rightarrow$ $\quad \#(\mathsf{m} \times \{1\}) = \#\mathsf{m}$
$\langle \mathsf{k}, 2 \rangle \hookrightarrow T193 \Rightarrow$ $\quad \#(\mathsf{k} \times \{2\}) = \#\mathsf{k}$
$\langle \mathsf{n} \times \{\emptyset\}, \mathsf{m} \times \{1\} \rangle \hookrightarrow T190 \Rightarrow$ $\quad \mathsf{n} \times \{\emptyset\} + \mathsf{m} \times \{1\} = \#(\mathsf{n} \times \{\emptyset\}) + \#(\mathsf{m} \times \{1\})$
EQUAL $\Rightarrow$ $\quad \mathsf{n} \times \{\emptyset\} + \mathsf{m} \times \{1\} = \#\mathsf{n} + \#\mathsf{m}$
$\langle \mathsf{m} \times \{1\}, \mathsf{k} \times \{2\} \rangle \hookrightarrow T190 \Rightarrow$ $\quad \mathsf{m} \times \{1\} + \mathsf{k} \times \{2\} = \#(\mathsf{m} \times \{1\}) + \#(\mathsf{k} \times \{2\})$
EQUAL $\Rightarrow$ $\quad \mathsf{m} \times \{1\} + \mathsf{k} \times \{2\} = \#\mathsf{m} + \#\mathsf{k}$
$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T190 \Rightarrow$ $\quad \#\mathsf{n} + \#\mathsf{m} = \mathsf{n} + \mathsf{m}$
$\langle \mathsf{m}, \mathsf{k} \rangle \hookrightarrow T190 \Rightarrow$ $\quad \#\mathsf{m} + \#\mathsf{k} = \mathsf{m} + \mathsf{k}$
EQUAL $\Rightarrow$ $\quad \mathsf{n} + \mathsf{m} + \mathsf{k} \times \{2\} = \mathsf{n} \times \{\emptyset\} + (\mathsf{m} + \mathsf{k})$

-- However, we can replace the sets $\mathsf{k} \times \{2\}$ and $\mathsf{n} \times \{\emptyset\}$ appearing in this last formula by their cardinalities.

$\langle \mathsf{n} + \mathsf{m}, \mathsf{k} \times \{2\} \rangle \hookrightarrow T195 \Rightarrow$ $\quad \mathsf{n} + \mathsf{m} + \mathsf{k} \times \{2\} = \mathsf{n} + \mathsf{m} + \#(\mathsf{k} \times \{2\})$
$\langle \mathsf{n} \times \{\emptyset\}, \mathsf{m} + \mathsf{k} \rangle \hookrightarrow T196 \Rightarrow$ $\quad \mathsf{n} \times \{\emptyset\} + (\mathsf{m} + \mathsf{k}) = \#(\mathsf{n} \times \{\emptyset\}) + (\mathsf{m} + \mathsf{k})$
EQUAL $\Rightarrow$ $\quad \mathsf{n} + \mathsf{m} + \#\mathsf{k} = \#\mathsf{n} + (\mathsf{m} + \mathsf{k})$

-- And finally can use remove the two cardinality operators appearing in this last formula to obtain the assertion of the present theorem.

$\langle \mathsf{n} + \mathsf{m}, \mathsf{k} \rangle \hookrightarrow T195 \Rightarrow$ $\quad \mathsf{n} + \mathsf{m} + \mathsf{k} = \#\mathsf{n} + (\mathsf{m} + \mathsf{k})$
$\langle \mathsf{n}, \mathsf{m} + \mathsf{k} \rangle \hookrightarrow T196 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Our next two theorems respectively give the associative law for multiplication and the distributive law for multiplication over addition.

Theorem 253 (222) $\mathsf{N} * (\mathsf{M} * \mathsf{K}) = (\mathsf{N} * \mathsf{M}) * \mathsf{K}$. PROOF:

Suppose_not$(\mathsf{n}, \mathsf{m}, \mathsf{k})$ $\Rightarrow$ $\quad \mathsf{n} * (\mathsf{m} * \mathsf{k}) \neq \mathsf{n} * \mathsf{m} * \mathsf{k}$

-- For suppose that n, m, and k be a counterexample to the asserted associative law. Using the definition of $*$, we can easily see that $\#\ \mathsf{n} \times (\mathsf{m} \times \mathsf{k})$ and $\#(\mathsf{n} \times \mathsf{m} \times \mathsf{k})$ must then be different.

$\mathsf{Use\_def}(*) \Rightarrow \quad n * \#(m \times k) \neq \#(n \times m) * k$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#\big(n \times \#(m \times k)\big) \neq \#\big(\#(n \times m) \times k\big)$
$\big\langle n, m \times k\big\rangle \hookrightarrow T202 \Rightarrow \quad \#\big(n \times \#(m \times k)\big) = \#\big(n \times (m \times k)\big)$
$\big\langle n \times m, k\big\rangle \hookrightarrow T203 \Rightarrow \quad \#\big(\#(n \times m) \times k\big) = \#(n \times m \times k)$

-- But this clearly violates Theorem 134.

$\big\langle n, m, k\big\rangle \hookrightarrow T134 \Rightarrow \quad \#\big(n \times (m \times k)\big) = \#(n \times m \times k)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following is the 'right-hand' version of the distributive law for integer multiplication over addition.

Theorem 254 (223) $N * (M + K) = N * M + N * K$. PROOF:

$\mathsf{Suppose\_not}(n, m, k) \Rightarrow \quad n * (m + k) \neq n * m + n * k$

-- For supposing the contrary, we can use the definition of $+$ on the left of the resulting inequality and of $*$ on the right, and then simplify further, removing superfluous cardinality operators to get the final inequality seen just before the next comment below.

$T183 \Rightarrow \quad Stat1: \emptyset \neq 1$
$\big\langle m, k\big\rangle \hookrightarrow T190 \Rightarrow \quad m + k = \#m + \#k$
$\big\langle \emptyset, 1, m, k\big\rangle \hookrightarrow T194 \Rightarrow \quad \#m + \#k = \#(m \times \{\emptyset\} \cup k \times \{1\})$
$\mathsf{EQUAL} \Rightarrow \quad n * \#(m \times \{\emptyset\} \cup k \times \{1\}) \neq n * m + n * k$
$\mathsf{Use\_def}(*) \Rightarrow \quad n * \#(m \times \{\emptyset\} \cup k \times \{1\}) \neq \#(n \times m) + \#(n \times k)$
$\big\langle n, m \times \{\emptyset\} \cup k \times \{1\}\big\rangle \hookrightarrow T200 \Rightarrow \quad n * (m \times \{\emptyset\} \cup k \times \{1\}) \neq \#(n \times m) + \#(n \times k)$
$\big\langle \#(n \times m), n \times k\big\rangle \hookrightarrow T195 \Rightarrow \quad n * (m \times \{\emptyset\} \cup k \times \{1\}) \neq \#(n \times m) + n \times k$
$\big\langle \#(n \times m), n \times k\big\rangle \hookrightarrow T216 \Rightarrow \quad \#(n \times m) + n \times k = n \times k + \#(n \times m)$
$\big\langle n \times k, n \times m\big\rangle \hookrightarrow T195 \Rightarrow \quad \#(n \times m) + n \times k = n \times k + n \times m$
$\big\langle n \times m, n \times k\big\rangle \hookrightarrow T216 \Rightarrow \quad \#(n \times m) + n \times k = n \times m + n \times k$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#\big(n \times (m \times \{\emptyset\} \cup k \times \{1\})\big) \neq n \times m + n \times k$

-- The right-hand side of this last inequality can then be rewritten as follows using Theorem 194:

$\big\langle n \times m, n \times k\big\rangle \hookrightarrow T190 \Rightarrow \quad n \times m + n \times k = \#(n \times m) + \#(n \times k)$
$\big\langle \emptyset, 1, n \times m, n \times k\big\rangle \hookrightarrow T194 \Rightarrow \quad \#(n \times m) + \#(n \times k) = \#(n \times m \times \{\emptyset\} \cup n \times k \times \{1\})$
$\big\langle m \times \{\emptyset\}, n, k \times \{1\}\big\rangle \hookrightarrow T218 \Rightarrow \quad n \times (m \times \{\emptyset\} \cup k \times \{1\}) = n \times (m \times \{\emptyset\}) \cup n \times (k \times \{1\})$
$\mathsf{EQUAL} \Rightarrow \quad \#\big(n \times (m \times \{\emptyset\}) \cup n \times (k \times \{1\})\big) \neq \#(n \times m \times \{\emptyset\} \cup n \times k \times \{1\})$

-- But the pairs of terms in this last inequality are easily seen to be disjoint:

$\langle \{\emptyset\}, \{1\}, \mathsf{m}, \mathsf{k}\rangle \hookrightarrow T117([Stat1, Stat1]) \Rightarrow \quad \mathsf{m} \times \{\emptyset\} \cap (\mathsf{k} \times \{1\}) = \emptyset$
$\langle \mathsf{m} \times \{\emptyset\}, \mathsf{n}, \mathsf{k} \times \{1\}\rangle \hookrightarrow T218 \Rightarrow \quad \mathsf{n} \times (\mathsf{m} \times \{\emptyset\}) \cap (\mathsf{n} \times (\mathsf{k} \times \{1\})) = \mathsf{n} \times (\mathsf{m} \times \{\emptyset\} \cap (\mathsf{k} \times \{1\}))$
EQUAL $\Rightarrow \quad \mathsf{n} \times (\mathsf{m} \times \{\emptyset\}) \cap (\mathsf{n} \times (\mathsf{k} \times \{1\})) = \mathsf{n} \times \emptyset$
$\langle \mathsf{n}\rangle \hookrightarrow T114 \Rightarrow \quad \mathsf{n} \times (\mathsf{m} \times \{\emptyset\}) \cap (\mathsf{n} \times (\mathsf{k} \times \{1\})) = \emptyset$
$\langle \{\emptyset\}, \{1\}, \mathsf{n} \times \mathsf{m}, \mathsf{n} \times \mathsf{k}\rangle \hookrightarrow T117 \Rightarrow \quad \mathsf{n} \times \mathsf{m} \times \{\emptyset\} \cap (\mathsf{n} \times \mathsf{k} \times \{1\}) = \emptyset$

-- It now follows using Theorem 191 that the cardinalities of unions seen above can be rewritten as arithmetic sums:

$\langle \mathsf{n} \times (\mathsf{m} \times \{\emptyset\}), \mathsf{n} \times (\mathsf{k} \times \{1\})\rangle \hookrightarrow T191 \Rightarrow \quad \#(\mathsf{n} \times (\mathsf{m} \times \{\emptyset\}) \cup \mathsf{n} \times (\mathsf{k} \times \{1\})) =$
$\quad \#(\mathsf{n} \times (\mathsf{m} \times \{\emptyset\})) + \#(\mathsf{n} \times (\mathsf{k} \times \{1\}))$
$\langle \mathsf{n} \times \mathsf{m} \times \{\emptyset\}, \mathsf{n} \times \mathsf{k} \times \{1\}\rangle \hookrightarrow T191 \Rightarrow \quad \#(\mathsf{n} \times \mathsf{m} \times \{\emptyset\} \cup \mathsf{n} \times \mathsf{k} \times \{1\}) =$
$\quad \#(\mathsf{n} \times \mathsf{m} \times \{\emptyset\}) + \#(\mathsf{n} \times \mathsf{k} \times \{1\})$
ELEM $\Rightarrow \quad Stat9: \ \#(\mathsf{n} \times (\mathsf{m} \times \{\emptyset\})) + \#(\mathsf{n} \times (\mathsf{k} \times \{1\})) \neq \#(\mathsf{n} \times \mathsf{m} \times \{\emptyset\}) + \#(\mathsf{n} \times \mathsf{k} \times \{1\})$

-- But this last inequality is easily seen to be impossible:

$\langle \mathsf{n}, \mathsf{m}, \{\emptyset\}\rangle \hookrightarrow T134 \Rightarrow \quad \#(\mathsf{n} \times (\mathsf{m} \times \{\emptyset\})) = \#(\mathsf{n} \times \mathsf{m} \times \{\emptyset\})$
$\langle \mathsf{n}, \mathsf{k}, \{1\}\rangle \hookrightarrow T134 \Rightarrow \quad \#(\mathsf{n} \times (\mathsf{k} \times \{1\})) = \#(\mathsf{n} \times \mathsf{k} \times \{1\})$
EQUAL $\langle Stat9\rangle \Rightarrow$ false; Discharge $\Rightarrow$ Qed

-- Next we show that the product of two finite sets is finite. The proof uses the method of finite induction introduced above.

Theorem 255 (224) $\mathsf{Finite}(\mathsf{N}) \,\&\, \mathsf{Finite}(\mathsf{M}) \rightarrow \mathsf{Finite}(\mathsf{N} * \mathsf{M})$. Proof:

Suppose_not$(\mathsf{n}, \mathsf{m}) \Rightarrow \quad \mathsf{Finite}(\mathsf{n}) \,\&\, \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{n} * \mathsf{m})$

-- For suppose that there exist finite n and m such that $\mathsf{n} \times \mathsf{m}$ is infinite. By our theory of finite induction, there exists finite k and m' such that $\mathsf{k} \times \mathsf{m}'$ is infinite, but $\mathsf{j} \times \mathsf{m}_2$ is finite for every finite m and proper subset j of k.

Use_def$(*) \Rightarrow \quad \mathsf{Finite}(\mathsf{n}) \,\&\, \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\#(\mathsf{n} \times \mathsf{m}))$
$\langle \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathsf{n} \,\&\, \mathsf{y} \in \mathsf{m}\}\rangle \hookrightarrow T166 \Rightarrow \quad \mathsf{Finite}(\mathsf{n}) \,\&\, \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{n} \times \mathsf{m})$
Suppose $\Rightarrow \quad Stat1: \ \neg\langle \exists \mathsf{m} \mid \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{n} \times \mathsf{m})\rangle$
$\langle \mathsf{m}\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow \quad \mathsf{Finite}(\mathsf{n}) \,\&\, \langle \exists \mathsf{m} \mid \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{n} \times \mathsf{m})\rangle$
APPLY $\langle \mathsf{m}_\Theta : \mathsf{k}_1\rangle$ finite_induction$(\mathsf{n} \mapsto \mathsf{n}, \mathsf{p}(\mathsf{n}) \mapsto \mathsf{Finite}(\mathsf{n}) \,\&\, \langle \exists \mathsf{m} \mid \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{n} \times \mathsf{m})\rangle) \Rightarrow$
$\quad \mathsf{Finite}(\mathsf{k}_1) \,\&\, \langle \exists \mathsf{m} \mid \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{k}_1 \times \mathsf{m})\rangle \,\&\, \langle \forall \mathsf{j} \subseteq \mathsf{k}_1 \mid \mathsf{j} \neq \mathsf{k}_1 \rightarrow \neg(\mathsf{Finite}(\mathsf{j}) \,\&\, \langle \exists \mathsf{m} \mid \mathsf{Finite}(\mathsf{m}) \,\&\, \neg\mathsf{Finite}(\mathsf{j} \times \mathsf{m})\rangle)\rangle$

Loc_def ⇒ $\mathsf{k} = \mathsf{k}_1$
EQUAL ⇒ $\mathsf{Finite}(\mathsf{k})\ \&\ \mathit{Stat2}: \langle \exists \mathsf{m} \mid \mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{k}\times\mathsf{m})\rangle\ \&\ \mathit{Stat3}: \langle \forall \mathsf{j}\subseteq\mathsf{k}\mid \mathsf{j}\neq\mathsf{k}\rightarrow\neg\big(\mathsf{Finite}(\mathsf{j})\ \&\ \langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{j}\times\mathsf{m})\rangle\big)\rangle$
$\langle \mathsf{m}'\rangle \hookrightarrow \mathit{Stat2}$ ⇒ $\mathit{Stat4}: \mathsf{Finite}(\mathsf{m}')\ \&\ \neg\mathsf{Finite}(\mathsf{k}\times\mathsf{m}')$

-- Since $\emptyset \times \mathsf{m}'$ is $\emptyset$, k obviously cannot be empty, and so has a member c.

Suppose ⇒ $\mathsf{k} = \emptyset$
EQUAL ⇒ $\mathsf{k}\times\mathsf{m}' = \emptyset\times\mathsf{m}'$
$\langle \mathsf{m}'\rangle \hookrightarrow \mathit{T114}$ ⇒ $\mathsf{k}\times\mathsf{m}' = \emptyset$
EQUAL ⇒ $\neg\mathsf{Finite}(\emptyset)$
$\mathit{T161}$ ⇒ false; Discharge ⇒ $\mathit{Stat5}: \mathsf{k}\neq\emptyset$
$\langle \mathsf{c}\rangle \hookrightarrow \mathit{Stat5}$ ⇒ $\mathit{Stat6}: \mathsf{c}\in\mathsf{k}$

-- But then $\mathsf{k}\backslash\{\mathsf{c}\}$ is a proper subset of k, so $(\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}$ must be finite, and therefore $(\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}\cup\{\mathsf{c}\}\times\mathsf{m} = \mathsf{k}\times\mathsf{m}$ must also be finite, a contradiction which proves the present theorem.

$\langle \mathit{Stat6}\rangle$ ELEM ⇒ $\mathsf{k} = \mathsf{k}\backslash\{\mathsf{c}\}\cup\{\mathsf{c}\}\ \&\ \mathsf{k}\backslash\{\mathsf{c}\}\subseteq\mathsf{k}\ \&\ (\mathsf{k}\backslash\{\mathsf{c}\})\cap\{\mathsf{c}\} = \emptyset$
$\langle \mathsf{k}, \mathsf{k}\backslash\{\mathsf{c}\}\rangle \hookrightarrow \mathit{T162}$ ⇒ $\mathsf{Finite}(\mathsf{k}\backslash\{\mathsf{c}\})$
$\langle \mathsf{k}\backslash\{\mathsf{c}\}\rangle \hookrightarrow \mathit{Stat3}$ ⇒ $\mathit{Stat7}: \neg\langle\exists\mathsf{m}\mid\mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}\big((\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}\big)\rangle$
$\langle \mathsf{m}'\rangle \hookrightarrow \mathit{Stat7}$ ⇒ $\mathsf{Finite}\big((\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}'\big)$
$\langle \mathsf{k}\backslash\{\mathsf{c}\}, \mathsf{m}', \{\mathsf{c}\}\rangle \hookrightarrow \mathit{T218}$ ⇒ $(\mathsf{k}\backslash\{\mathsf{c}\}\cup\{\mathsf{c}\})\times\mathsf{m}' = (\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}'\cup\{\mathsf{c}\}\times\mathsf{m}'$
ELEM ⇒ $\mathsf{k}\backslash\{\mathsf{c}\}\cup\{\mathsf{c}\} = \mathsf{k}$
EQUAL ⇒ $\mathsf{k}\times\mathsf{m}' = (\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}'\cup\{\mathsf{c}\}\times\mathsf{m}'$
$\langle \mathsf{m}'\rangle \hookrightarrow \mathit{T166}$ ⇒ $\mathsf{Finite}(\#\mathsf{m}')$
$\langle \mathsf{c}, \mathsf{m}'\rangle \hookrightarrow \mathit{T192}$ ⇒ $\#(\{\mathsf{c}\}\times\mathsf{m}') = \#\mathsf{m}'$
EQUAL ⇒ $\mathsf{Finite}\big(\#(\{\mathsf{c}\}\times\mathsf{m}')\big)$
$\langle \{\mathsf{c}\}\times\mathsf{m}'\rangle \hookrightarrow \mathit{T166}$ ⇒ $\mathsf{Finite}(\{\mathsf{c}\}\times\mathsf{m}')$
$\langle (\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}', \{\mathsf{c}\}\times\mathsf{m}'\rangle \hookrightarrow \mathit{T205}$ ⇒ $\mathsf{Finite}\big((\mathsf{k}\backslash\{\mathsf{c}\})\times\mathsf{m}'\cup\{\mathsf{c}\}\times\mathsf{m}'\big)$
EQUAL ⇒ $\mathit{Stat8}: \mathsf{Finite}(\mathsf{k}\times\mathsf{m}')$
$\langle \mathit{Stat4}, \mathit{Stat8}\rangle$ ELEM ⇒ false; Discharge ⇒ QED

-- The following is a simple corollary of Theorem 224

Theorem 256 (225) $\mathsf{Finite}(\mathsf{N})\ \&\ \mathsf{Finite}(\mathsf{M})\rightarrow\mathsf{Finite}(\mathsf{N}\times\mathsf{M})$. PROOF:

Suppose_not(n,m) ⇒ $\mathsf{Finite}(\mathsf{n})\ \&\ \mathsf{Finite}(\mathsf{m})\ \&\ \neg\mathsf{Finite}(\mathsf{n}\times\mathsf{m})$
$\langle \mathsf{n}\times\mathsf{m}\rangle \hookrightarrow \mathit{T166}$ ⇒ $\neg\mathsf{Finite}\big(\#(\mathsf{n}\times\mathsf{m})\big)$
Use_def$(*)$ ⇒ $\neg\mathsf{Finite}\big(\#(\mathsf{n}*\mathsf{m})\big)$

$\langle n, m \rangle \hookrightarrow T224 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following result restates the preceding theorem as a statement about the arithmetic multiplication operator.

Theorem 257 (226) $(\text{Finite}(N) \,\&\, \text{Finite}(M)) \vee N = \emptyset \vee M = \emptyset \leftrightarrow \text{Finite}(N * M)$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $\neg\Big((\text{Finite}(n) \,\&\, \text{Finite}(m)) \vee n = \emptyset \vee m = \emptyset \leftrightarrow \text{Finite}(n * m)\Big)$

-- For let n, m be a counterexample to our assertion. It is easily seen that neither n nor m can be empty, so either both must be finite and $n * m$ infinite, or the revers.

$\langle m \rangle \hookrightarrow T210 \Rightarrow$ $\emptyset * m = \emptyset$
$\langle n \rangle \hookrightarrow T209 \Rightarrow$ $n * \emptyset = \emptyset$
$T161 \Rightarrow$ $\text{Finite}(\emptyset)$
Suppose $\Rightarrow$ $n = \emptyset$
EQUAL $\Rightarrow$ $\text{Finite}(n * m) \leftrightarrow \text{Finite}(\emptyset * M)$
EQUAL $\Rightarrow$ $\text{Finite}(n * m) \leftrightarrow \text{Finite}(\emptyset)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $n \neq \emptyset$
Suppose $\Rightarrow$ $m = \emptyset$
EQUAL $\Rightarrow$ $\text{Finite}(n * m) \leftrightarrow \text{Finite}(n * \emptyset)$
EQUAL $\Rightarrow$ $\text{Finite}(n * \emptyset) \leftrightarrow \text{Finite}(\emptyset)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $m \neq \emptyset$
ELEM $\Rightarrow$ $\neg(\text{Finite}(n) \,\&\, \text{Finite}(m) \leftrightarrow \text{Finite}(n * m))$

-- Since $\#(n \times m)$ is evidently no less than either n and m, the second case is ruled out, so we have only to consider the first case.

Suppose $\Rightarrow$ $Stat1:\ \text{Finite}(n * m) \,\&\, \neg(\text{Finite}(n) \,\&\, \text{Finite}(m))$
Use_def$(*) \Rightarrow$ $\text{Finite}\big(\#(n \times m)\big)$
$\langle m, n \rangle \hookrightarrow T214 \Rightarrow$ $\#(n \times m) \supseteq \#n$
$\langle \#(n \times m), \#n \rangle \hookrightarrow T162 \Rightarrow$ $\text{Finite}(\#n)$
$\langle n \rangle \hookrightarrow T166 \Rightarrow$ $Stat2:\ \text{Finite}(n)$
$\langle n, m \rangle \hookrightarrow T217 \Rightarrow$ $n * m = m * n$
EQUAL $\Rightarrow$ $\text{Finite}(m * n)$
Use_def$(*) \Rightarrow$ $\text{Finite}\big(\#(m \times n)\big)$
$\langle n, m \rangle \hookrightarrow T214 \Rightarrow$ $\#(m \times n) \supseteq \#m$
$\langle \#(m \times n), \#m \rangle \hookrightarrow T162 \Rightarrow$ $\text{Finite}(\#m)$
$\langle m \rangle \hookrightarrow T166 \Rightarrow$ $\text{Finite}(m)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Finite}(n)\ \&\ \mathsf{Finite}(m)\ \&\ \neg\mathsf{Finite}(n*m)$

-- But the preceding theorem rules out this case, so our proof is complete.

Use_def$(*)\Rightarrow$ $\neg\mathsf{Finite}\big(\#(n\times m)\big)$
$\langle n\times m\rangle\hookrightarrow T166\Rightarrow$ $\neg\mathsf{Finite}(n\times m)$
$\langle n,m\rangle\hookrightarrow T225\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We continue by proving that the power set of a finite set is also finite.

Theorem 258 (227) $\mathsf{Finite}(N)\leftrightarrow\mathsf{Finite}(\mathcal{P}N)$. PROOF:

Suppose_not$(n)\Rightarrow$ $\big(\mathsf{Finite}(n)\ \&\ \neg\mathsf{Finite}(\mathcal{P}n)\big)\vee\big(\neg\mathsf{Finite}(n)\ \&\ \mathsf{Finite}(\mathcal{P}n)\big)$

-- If the asserted equivalence is false, there must exist an either a finite n with an infinite power set, or an infinite n with a finite power set. Consider the second of these cases first. In this case $\{\mathbf{arb}(x):\ x\in\mathcal{P}n\backslash\{\emptyset\}\}$, which is the range of the function **arb** on the set n, must be finite. But since every singleton $\{y\}$, with $y\in n$, belongs to $\mathcal{P}n\backslash\{\emptyset\}$, it is plain that $\{\mathbf{arb}(x):\ x\in\mathcal{P}n\backslash\{\emptyset\}\}$ includes n, ruling out this case.

Suppose $\Rightarrow$ $\neg\mathsf{Finite}(n)\ \&\ \mathsf{Finite}(\mathcal{P}n)$
Loc_def $\Rightarrow$ $f=\{[x,\mathbf{arb}(x)]:\ x\in\mathcal{P}n\backslash\{\emptyset\}\}$
APPLY $\langle\ \rangle$ fcn_symbol$\big(f(x)\mapsto\mathbf{arb}(x),g\mapsto f,s\mapsto\mathcal{P}n\backslash\{\emptyset\}\big)\Rightarrow$
$\mathsf{Svm}(f)\ \&\ \mathbf{domain}(f)=\mathcal{P}n\backslash\{\emptyset\}\ \&\ \mathbf{range}(f)=\{\mathbf{arb}(x):\ x\in\mathcal{P}n\backslash\{\emptyset\}\}$
Suppose $\Rightarrow$ $Stat1:\ \mathbf{range}(f)\not\subseteq n$
$\langle c'\rangle\hookrightarrow Stat1\Rightarrow$ $c'\in\mathbf{range}(f)\ \&\ c'\notin n$
ELEM $\Rightarrow$ $Stat2:\ c'\in\{\mathbf{arb}(x):\ x\in\mathcal{P}n\backslash\{\emptyset\}\}$
$\langle d\rangle\hookrightarrow Stat2\Rightarrow$ $c'=\mathbf{arb}(d)\ \&\ d\in\mathcal{P}n\backslash\{\emptyset\}$
$\langle d\rangle\hookrightarrow T0\Rightarrow$ $c'\in d$
Use_def$(\mathcal{P})\Rightarrow$ $Stat3:\ d\in\{x:\ x\subseteq n\}$
$\langle d_1\rangle\hookrightarrow Stat3\Rightarrow$ $d\subseteq n$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{range}(f)\subseteq n$
Suppose $\Rightarrow$ $Stat4:\ n\not\subseteq\mathbf{range}(f)$
$\langle a\rangle\hookrightarrow Stat4\Rightarrow$ $a\notin\mathbf{range}(f)\ \&\ a\in n$
ELEM $\Rightarrow$ $Stat5:\ a\notin\{\mathbf{arb}(x):\ x\in\mathcal{P}n\backslash\{\emptyset\}\}$
$\langle\{a\}\rangle\hookrightarrow Stat5\Rightarrow$ $\neg(\{a\}\neq\emptyset\ \&\ a=\mathbf{arb}(\{a\})\ \&\ \{a\}\in\mathcal{P}n)$
ELEM $\Rightarrow$ $\{a\}\notin\mathcal{P}n$
Use_def$(\mathcal{P})\Rightarrow$ $Stat6:\ \{a\}\notin\{x:\ x\subseteq n\}$
$\langle\{a\}\rangle\hookrightarrow Stat6\Rightarrow$ $\{a\}\not\subseteq n$

$\mathsf{ELEM} \Rightarrow$ $\mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ $\mathbf{range}(\mathsf{f}) = \mathsf{n}$
$\mathsf{ELEM} \Rightarrow$ $\mathcal{P}\mathsf{n} \backslash \{\emptyset\} \subseteq \mathcal{P}\mathsf{n}$
$\langle \mathcal{P}\mathsf{n}, \mathcal{P}\mathsf{n} \backslash \{\emptyset\} \rangle \hookrightarrow T162 \Rightarrow$ $\mathsf{Finite}(\mathcal{P}\mathsf{n} \backslash \{\emptyset\})$
$\mathsf{EQUAL} \Rightarrow$ $\mathsf{Finite}(\mathbf{domain}(\mathsf{f}))$
$\langle \mathsf{f} \rangle \hookrightarrow T165 \Rightarrow$ $\mathsf{Finite}(\mathbf{range}(\mathsf{f}))$
$\mathsf{ELEM} \Rightarrow$ $\mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ $\mathsf{Finite}(\mathsf{n}) \;\&\; \neg\mathsf{Finite}(\mathcal{P}\mathsf{n})$

-- Thus it follows that if our theorem is false there must exist a finite n with an infinite power set, in which case the principle of finite induction tells us that there exists such a set m with no proper subset having the same property. Since $\{0\}$ is finite, m cannot be 0, and therefore it must have some member c.

APPLY $\langle \mathsf{m}_\Theta : \mathsf{m} \rangle$ $\mathsf{finite\_induction}(\mathsf{n} \mapsto \mathsf{n}, \mathsf{P}(\mathsf{x}) \mapsto \neg\mathsf{Finite}(\mathcal{P}\mathsf{x})) \Rightarrow$
$\mathsf{m} \subseteq \mathsf{n} \;\&\; \neg\mathsf{Finite}(\mathcal{P}\mathsf{m}) \;\&\; Stat7 : \langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \rightarrow \mathsf{Finite}(\mathcal{P}\mathsf{k}) \rangle$
$\mathsf{Suppose} \Rightarrow$ $\mathsf{m} = \emptyset$
$T184 \Rightarrow$ $\mathcal{P}\emptyset = \{\emptyset\}$
$\mathsf{EQUAL} \Rightarrow$ $\mathcal{P}\mathsf{m} = \{\emptyset\}$
$\mathsf{EQUAL} \Rightarrow$ $\neg\mathsf{Finite}(\{\emptyset\})$
$T161 \Rightarrow$ $\mathsf{Finite}(\emptyset)$
$\langle \emptyset, \emptyset \rangle \hookrightarrow T172 \Rightarrow$ $\mathsf{Finite}(\emptyset \cup \{\emptyset\})$
$\mathsf{ELEM} \Rightarrow$ $\emptyset \cup \{\emptyset\} = \{\emptyset\}$
$\mathsf{EQUAL} \Rightarrow$ $\mathsf{Finite}(\{\emptyset\})$
$\mathsf{ELEM} \Rightarrow$ $\mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ $Stat8 : \mathsf{m} \neq \emptyset$
$\langle \mathsf{c} \rangle \hookrightarrow Stat8 \Rightarrow$ $Stat9 : \mathsf{c} \in \mathsf{m}$

-- We can therefore decompose $\mathcal{P}\mathsf{m}$ into (i) the collection of all subsets of m which do contain c, and (ii) the collection of all subsets of m which do not contain c. It is easily seen that collection b is the power set of $\mathsf{m} \backslash \{\mathsf{c}\}$, and so, by the minimality of m, collection b must be finite, and therefore collection (i) must be infinite.

$\mathsf{Set\_monot} \Rightarrow$ $\{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{true}\} = \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \in \mathsf{x} \vee \mathsf{c} \notin \mathsf{x}\}$
$\mathsf{SIMPLF} \Rightarrow$ $\mathcal{P}\mathsf{m} = \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \in \mathsf{x} \vee \mathsf{c} \notin \mathsf{x}\}$
$\mathsf{Set\_monot} \Rightarrow$ $\{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \in \mathsf{x} \vee \mathsf{c} \notin \mathsf{x}\} = \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \in \mathsf{x}\} \cup \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\}$
$\mathsf{ELEM} \Rightarrow$ $\mathcal{P}\mathsf{m} = \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \in \mathsf{x}\} \cup \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\}$
$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ $\{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\} = \{\mathsf{x} : \mathsf{x} \in \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m}\} \mid \mathsf{c} \notin \mathsf{x}\}$
$\mathsf{SIMPLF} \Rightarrow$ $Stat10 : \{\mathsf{x} : \mathsf{x} \in \mathcal{P}\mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\} = \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\}$
$\mathsf{Suppose} \Rightarrow$ $Stat11 : \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\} \neq \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \backslash \{\mathsf{c}\}\}$
$\langle \mathsf{dq} \rangle \hookrightarrow Stat11 \Rightarrow$ $(\mathsf{dq} \in \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\} \;\&\; \mathsf{dq} \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \backslash \{\mathsf{c}\}\}) \vee (\mathsf{dq} \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\} \;\&\; \mathsf{dq} \in \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \backslash \{\mathsf{c}\}\})$
$\mathsf{Suppose} \Rightarrow$ $Stat12 : \mathsf{dq} \in \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \mid \mathsf{c} \notin \mathsf{x}\} \;\&\; Stat13 : \mathsf{dq} \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{m} \backslash \{\mathsf{c}\}\}$
$\langle \mathsf{a}' \rangle \hookrightarrow Stat12 \Rightarrow$ $\mathsf{dq} = \mathsf{a}' \;\&\; \mathsf{a}' \subseteq \mathsf{m} \;\&\; \mathsf{c} \notin \mathsf{a}'$
$\langle \mathsf{a}' \rangle \hookrightarrow Stat13 \Rightarrow$ $\neg(\mathsf{dq} = \mathsf{a}' \;\&\; \mathsf{a}' \subseteq \mathsf{m} \backslash \{\mathsf{c}\})$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat14:\ dq \notin \{x : x \subseteq m \mid c \notin x\}\ \&\ Stat15:\ dq \in \{x : x \subseteq m \setminus \{c\}\}$
$\langle b \rangle \hookrightarrow Stat15 \Rightarrow$ $dq = b\ \&\ b \subseteq m \setminus \{c\}$
$\langle b \rangle \hookrightarrow Stat14 \Rightarrow$ $\neg(dq \subseteq m\ \&\ c \notin dq)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat16:\ \{x : x \subseteq m \mid c \notin x\} = \{x : x \subseteq m \setminus \{c\}\}$
$\langle Stat16, Stat10 \rangle$ ELEM $\Rightarrow$ $\{x : x \in \mathcal{P}m \mid c \notin x\} = \{x : x \subseteq m \setminus \{c\}\}$
Use_def $(\mathcal{P}) \Rightarrow$ $\{x : x \in \mathcal{P}m \mid c \notin x\} = \mathcal{P}(m \setminus \{c\})$
ELEM $\Rightarrow$ $m \setminus \{c\} \subseteq m\ \&\ m \setminus \{c\} \neq m$
$\langle m \setminus \{c\} \rangle \hookrightarrow Stat7 \Rightarrow$ $\mathrm{Finite}(\mathcal{P}(m \setminus \{c\}))$
EQUAL $\Rightarrow$ $\mathrm{Finite}(\{x : x \subseteq m \setminus \{c\}\})\ \&\ \mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \notin x\})$
$\langle \{x : x \in \mathcal{P}m \mid c \in x\}, \{x : x \in \mathcal{P}m \mid c \notin x\} \rangle \hookrightarrow T205(\langle \cap \rangle) \Rightarrow$
$\mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \in x\})\ \&\ \mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \notin x\}) \leftrightarrow \mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \in x\} \cup \{x : x \in \mathcal{P}m \mid c \notin x\})$
EQUAL $\Rightarrow$ $\mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \in x\})\ \&\ \mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \notin x\}) \leftrightarrow \mathrm{Finite}(\mathcal{P}m)$
ELEM $\Rightarrow$ $\neg\mathrm{Finite}(\{x : x \in \mathcal{P}m \mid c \in x\})$
Use_def $(\mathcal{P}) \Rightarrow$ $\neg\mathrm{Finite}(\{x : x \in \{y : y \subseteq m\} \mid c \in x\})$
SIMPLF $\Rightarrow$ $\{x : x \in \{y : y \subseteq m\} \mid c \in x\} = \{y : y \subseteq m \mid c \in y\}$
EQUAL $\Rightarrow$ $\neg\mathrm{Finite}(\{y : y \subseteq m \mid c \in y\})$

-- But it is also easy to see that the single-valued map $ff(x) \mapsto x \cup \{c\}$ maps $\mathcal{P}(m \setminus \{c\})$ onto the collection (i) of sets, and the domain $\mathcal{P}(m \setminus \{c\})$ of this map, and hence its range, is plainly finite.

Loc_def $\Rightarrow$ $ff = \{[x, x \cup \{c\}] : x \in \{x : x \subseteq m \setminus \{c\}\}\}$
APPLY $\langle\ \rangle$ fcn_symbol $(f(x) \mapsto x \cup \{c\}, g \mapsto ff, s \mapsto \{x : x \subseteq m \setminus \{c\}\}) \Rightarrow$
$\mathrm{Svm}(ff)\ \&\ \mathbf{domain}(ff) = \{x : x \subseteq m \setminus \{c\}\}\ \&\ \mathbf{range}(ff) = \{x \cup \{c\} : x \in \{x : x \subseteq m \setminus \{c\}\}\}$
SIMPLF $\Rightarrow$ $\mathbf{range}(ff) = \{x \cup \{c\} : x \subseteq m \setminus \{c\}\}$
EQUAL $\Rightarrow$ $\mathrm{Finite}(\mathbf{domain}(ff))$
$\langle ff \rangle \hookrightarrow T165 \Rightarrow$ $\mathrm{Finite}(\mathbf{range}(ff))$
EQUAL $\Rightarrow$ $\mathrm{Finite}(\{x \cup \{c\} : x \subseteq m \setminus \{c\}\})$
Suppose $\Rightarrow$ $Stat17:\ \{x \cup \{c\} : x \subseteq m \setminus \{c\}\} \neq \{x : x \subseteq m \mid c \in x\}$
$\langle e \rangle \hookrightarrow Stat17(\langle \cap \rangle) \Rightarrow$ $Stat18:\ (e \in \{x \cup \{c\} : x \subseteq m \setminus \{c\}\}\ \&\ e \notin \{x : x \subseteq m \mid c \in x\}) \vee$
$e \notin \{x \cup \{c\} : x \subseteq m \setminus \{c\}\}\ \&\ e \in \{x : x \subseteq m \mid c \in x\}$
Suppose $\Rightarrow$ $Stat19:\ e \in \{x \cup \{c\} : x \subseteq m \setminus \{c\}\}\ \&\ Stat20:\ e \notin \{x : x \subseteq m \mid c \in x\}$
$\langle e_1 \rangle \hookrightarrow Stat19 \Rightarrow$ $Stat21:\ e = e_1 \cup \{c\}\ \&\ e_1 \subseteq m \setminus \{c\}$
$\langle e_1 \cup \{c\} \rangle \hookrightarrow Stat20 \Rightarrow$ $Stat22:\ \neg(e = e_1 \cup \{c\}\ \&\ e_1 \cup \{c\} \subseteq m\ \&\ c \in e_1 \cup \{c\})$
$\langle Stat21, Stat22, Stat9 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat23:\ \neg(e \in \{x \cup \{c\} : x \subseteq m \setminus \{c\}\}\ \&\ e \notin \{x : x \subseteq m \mid c \in x\})$
$\langle Stat23, Stat18, * \rangle$ ELEM $\Rightarrow$ $Stat24:\ e \notin \{x \cup \{c\} : x \subseteq m \setminus \{c\}\}\ \&\ Stat25:\ e \in \{x : x \subseteq m \mid c \in x\}$
$\langle e_2 \rangle \hookrightarrow Stat25 \Rightarrow$ $Stat26:\ e = e_2\ \&\ e_2 \subseteq m\ \&\ c \in e_2$
$\langle e_2 \setminus \{c\} \rangle \hookrightarrow Stat24 \Rightarrow$ $Stat27:\ \neg(e = e_2 \setminus \{c\} \cup \{c\}\ \&\ e_2 \setminus \{c\} \subseteq m \setminus \{c\})$
EQUAL $\langle Stat26, Stat27 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $\{x \cup \{c\} : x \subseteq m \setminus \{c\}\} = \{x : x \subseteq m \mid c \in x\}$

-- This contradiction proves our theorem.

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next proof, of Cantor's basic result that the cardinality of the power set of any set s is always larger than the cardinality of s, embodies a famous idea whose discovery encouraged development if the theory of infinite cardinality in its early days.

-- Cantor 's Theorem

Theorem 259 (228) $\#\mathsf{N} \in \#\mathcal{P}\mathsf{N}$. PROOF:

Suppose_not(n) $\Rightarrow$ $\#\mathsf{n} \notin \#\mathcal{P}\mathsf{n}$

-- For let n be a counterexample to our assertion. $\mathcal{P}\mathsf{n}$ in plainly not empty, and since $\#\mathsf{n}$ and $\#\mathcal{P}\mathsf{n}$ are both ordinals we must have $\#\mathsf{n} \supseteq \#\mathcal{P}\mathsf{n}$ by Theorem 32.

Suppose $\Rightarrow$ $\emptyset \notin \mathcal{P}\mathsf{n}$
Use_def($\mathcal{P}$) $\Rightarrow$ *Stat1* : $\emptyset \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{n}\}$
$\langle\emptyset\rangle\hookrightarrow$*Stat1* $\Rightarrow$ *Stat2* : $\emptyset \not\subseteq \mathsf{n}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathcal{P}\mathsf{n} \neq \emptyset$
$\langle\mathsf{n}\rangle\hookrightarrow$*T130* $\Rightarrow$ $\mathcal{O}(\#\mathsf{n})$
$\langle\mathcal{P}\mathsf{n}\rangle\hookrightarrow$*T130* $\Rightarrow$ $\mathcal{O}(\#\mathcal{P}\mathsf{n})$
$\langle\#\mathcal{P}\mathsf{n}, \#\mathsf{n}\rangle\hookrightarrow$*T32* $\Rightarrow$ $\#\mathsf{n} \supseteq \#\mathcal{P}\mathsf{n}$

-- Hence Theorem 149 tells us that there is a single valued map f of n onto $\mathcal{P}\mathsf{n}$. Consider the subset $\mathsf{s} = \{\mathsf{x} : \mathsf{x} \in \mathsf{n} \mid \mathsf{x} \notin \mathsf{f}\upharpoonright\mathsf{x}\}$ of n, which plainly belongs to $\mathcal{P}\mathsf{n}$.

$\langle\mathsf{n}, \mathcal{P}\mathsf{n}\rangle\hookrightarrow$*T149* $\Rightarrow$ *Stat3* : $\langle\exists \mathsf{f} \mid \mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{n}\ \&\ \mathbf{range}(\mathsf{f}) = \mathcal{P}\mathsf{n}\rangle$
$\langle\mathsf{f}\rangle\hookrightarrow$*Stat3* $\Rightarrow$ $\mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{n}\ \&\ \mathbf{range}(\mathsf{f}) = \mathcal{P}\mathsf{n}$
Loc_def $\Rightarrow$ $\mathsf{s} = \{\mathsf{x} : \mathsf{x} \in \mathsf{n} \mid \mathsf{x} \notin \mathsf{f}\upharpoonright\mathsf{x}\}$
Set_monot $\Rightarrow$ $\{\mathsf{x} : \mathsf{x} \in \mathsf{n} \mid \mathsf{x} \notin \mathsf{f}\upharpoonright\mathsf{x}\} \subseteq \{\mathsf{x} : \mathsf{x} \in \mathsf{n}\}$
ELEM $\Rightarrow$ $\mathsf{s} \subseteq \{\mathsf{x} : \mathsf{x} \in \mathsf{n}\}$
SIMPLF $\Rightarrow$ $\mathsf{s} \subseteq \mathsf{n}$
Suppose $\Rightarrow$ $\mathsf{s} \notin \mathcal{P}\mathsf{n}$
Use_def($\mathcal{P}$) $\Rightarrow$ *Stat4* : $\mathsf{s} \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{n}\}$
$\langle\mathsf{s}\rangle\hookrightarrow$*Stat4* $\Rightarrow$ $\mathsf{s} \not\subseteq \mathsf{n}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{s} \in \mathcal{P}\mathsf{n}$
ELEM $\Rightarrow$ $\mathsf{s} \in \mathbf{range}(\mathsf{f})$

-- It is clear since s belongs to the range of f that $\mathsf{s} = \mathsf{f}\upharpoonright\mathsf{c}$ for some $\mathsf{c} \in \mathsf{n}$.

$T65 \Rightarrow \quad \mathsf{f} = \{[\mathsf{x}, \mathsf{f}\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$
EQUAL $\Rightarrow \quad \mathsf{s} \in \mathbf{range}(\{[\mathsf{x}, \mathsf{f}\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\})$
Use_def(**range**) $\Rightarrow \quad \mathsf{s} \in \left\{\mathsf{y}^{[2]} : \mathsf{y} \in \{[\mathsf{x}, \mathsf{f}\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}\right\}$
SIMPLF $\Rightarrow \quad Stat5: \ \mathsf{s} \in \left\{[\mathsf{x}, \mathsf{f}\restriction\mathsf{x}]^{[2]} : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\right\}$
$\langle \mathsf{c} \rangle \hookrightarrow Stat5 \Rightarrow \quad \mathsf{c} \in \mathbf{domain}(\mathsf{f}) \ \& \ \mathsf{s} = [\mathsf{c}, \mathsf{f}\restriction\mathsf{c}]^{[2]}$
ELEM $\Rightarrow \quad \mathsf{c} \in \mathsf{n} \ \& \ \mathsf{s} = \mathsf{f}\restriction\mathsf{c}$

-- If $\mathsf{c} \in \mathsf{s}$, then it follows immediately that $\mathsf{c} \notin \mathsf{s}$.

Suppose $\Rightarrow \quad \mathsf{c} \in \mathsf{s}$
ELEM $\Rightarrow \quad Stat6: \ \mathsf{c} \in \{\mathsf{x} : \mathsf{x} \in \mathsf{n} \mid \mathsf{x} \notin \mathsf{f}\restriction\mathsf{x}\}$
$\langle \mathsf{d} \rangle \hookrightarrow Stat6 \Rightarrow \quad \mathsf{c} = \mathsf{d} \ \& \ \mathsf{c} \in \mathsf{n} \ \& \ \mathsf{c} \notin \mathsf{f}\restriction\mathsf{d}$
EQUAL $\Rightarrow \quad \mathsf{c} \notin \mathsf{f}\restriction\mathsf{c}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad \mathsf{c} \notin \mathsf{s}$

-- But in much the same way $\mathsf{c} \notin \mathsf{s}$ implies that $\mathsf{c} \in \mathsf{s}$, so we have a contradiction which proves Cantor's theorem.

ELEM $\Rightarrow \quad Stat7: \ \mathsf{c} \notin \{\mathsf{x} : \mathsf{x} \in \mathsf{n} \mid \mathsf{x} \notin \mathsf{f}\restriction\mathsf{x}\}$
$\langle \mathsf{c} \rangle \hookrightarrow Stat7 \Rightarrow \quad \neg(\mathsf{c} \in \mathsf{n} \ \& \ \mathsf{c} \notin \mathsf{f}\restriction\mathsf{c})$
ELEM $\Rightarrow \quad \neg(\mathsf{c} \in \mathsf{n} \ \& \ \mathsf{c} \notin \mathsf{s})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The two elementary lemmas which now follow state basic fact concerning arithmetic subtraction. We first show that a quantity subtracted from itself gives 0.

Theorem 260 (229) $\mathsf{N} - \mathsf{N} = \emptyset$. PROOF:

Suppose_not(n) $\Rightarrow \quad \mathsf{n} - \mathsf{n} \neq \emptyset$
Use_def($-$) $\Rightarrow \quad \mathsf{n} - \mathsf{n} = \#(\mathsf{n} \backslash \mathsf{n})$
ELEM $\Rightarrow \quad \mathsf{n} \backslash \mathsf{n} = \emptyset$
EQUAL $\Rightarrow \quad \mathsf{n} - \mathsf{n} = \#\emptyset$
$T161 \Rightarrow \quad \mathsf{Card}(\emptyset)$
$\langle \emptyset \rangle \hookrightarrow T138 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that a quantity subtracted from itself gives 0.

Theorem 261 (230) $\mathsf{N} - \emptyset = \#\mathsf{N}$. PROOF:

Suppose_not(n) ⇒ $n - \emptyset \neq \#n$
Use_def(−) ⇒ $\#(n \backslash \emptyset) \neq \#n$
ELEM ⇒ $n \backslash \emptyset = n$
EQUAL ⇒ false; Discharge ⇒ QED

-- Next we show that the cardinality of the sum of any two disjoint sets n, m is determined by the two separate cardinalities #n, #m.

Theorem 262 (231) $N \cap M = \emptyset \;\&\; N_2 \cap M_2 = \emptyset \;\&\; \#N = \#N_2 \;\&\; \#M = \#M_2 \rightarrow \#(N \cup M) = \#(N_2 \cup M_2)$. PROOF:

Suppose_not(n, m, n_2, m_2) ⇒ *Stat1* : $n \cap m = \emptyset \;\&\; n_2 \cap m_2 = \emptyset \;\&\; \#n = \#n_2 \;\&\; \#m = \#m_2 \;\&\; \#(n \cup m) \neq \#(n_2 \cup m_2)$

-- For our assertion results immediately by combining Theorems 174 and 175.

$\langle n, m \rangle \hookrightarrow$ *T189* ⇒ $\#(n \cup m) = n + m$
$\langle n_2, m_2 \rangle \hookrightarrow$ *T189* ⇒ $\#(n_2 \cup m_2) = n_2 + m_2$
$\langle n, m \rangle \hookrightarrow$ *T190* ⇒ $n + m = \#n + \#m$
$\langle n_2, m_2 \rangle \hookrightarrow$ *T190* ⇒ $n_2 + m_2 = \#n_2 + \#m_2$
EQUAL ⇒ false; Discharge ⇒ QED

-- Our next theorem states that when, in a subtraction, the subtrahend is an ordinal which exceeds the minuend, then the value of the difference is 0.

Theorem 263 (10020) $\mathcal{O}(Y) \;\&\; X \in Y \rightarrow X - Y = \emptyset$. PROOF:

Suppose_not(x, y) ⇒ $\mathcal{O}(y) \;\&\; x \in y \;\&\; x - y \neq \emptyset$
Use_def(−) ⇒ $\#(x \backslash y) \neq \emptyset$
$\langle x \backslash y \rangle \hookrightarrow$ *T136* ⇒ $x \backslash y \neq \emptyset$
$\langle y, x \rangle \hookrightarrow$ *T12* ⇒ false; Discharge ⇒ QED

-- The two following results state (and generalize) the fact that if an integer n is at least as large as an integer m, then n is the arithmetic sum of m and n − m. The first of these theorems generalizes this fact to arbitrary cardinals (and, indeed, sets). The second restates the same fact in a more narrowly arithmetic form.

-- Subtraction Lemma

Theorem 264 (232) $M \subseteq N \rightarrow \#N = \#M + (N - M)$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m},\mathsf{n}) \Rightarrow\quad \mathsf{m} \subseteq \mathsf{n} \,\&\, \#\mathsf{n} \neq \#\mathsf{m} + (\mathsf{n} - \mathsf{m})$

-- For our assertion results immediately from Theorem 189 and the definition of subtraction.

$\mathsf{ELEM} \Rightarrow\quad \mathsf{n} = \mathsf{m} \cup (\mathsf{n}\backslash\mathsf{m}) \,\&\, \mathsf{m} \cap (\mathsf{n}\backslash\mathsf{m}) = \emptyset$
$\langle \mathsf{m}, \mathsf{n}\backslash\mathsf{m}\rangle \hookrightarrow T189 \Rightarrow\quad \#\big(\mathsf{m} \cup (\mathsf{n}\backslash\mathsf{m})\big) = \mathsf{m} + (\mathsf{n}\backslash\mathsf{m})$
$\mathsf{EQUAL} \Rightarrow\quad \#\mathsf{n} = \mathsf{m} + (\mathsf{n}\backslash\mathsf{m})$
$\langle \mathsf{m}, \mathsf{n}\backslash\mathsf{m}\rangle \hookrightarrow T190 \Rightarrow\quad \#\mathsf{n} = \#\mathsf{m} + \#(\mathsf{n}\backslash\mathsf{m})$
$\mathsf{Use\_def}(-) \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

-- The following variant form of the preceding lemma is sometimes useful.

Theorem 265 (233) $\mathsf{M} \subseteq \mathsf{N} \to \#\mathsf{N} = \mathsf{N} - \mathsf{M} + \mathsf{M}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m},\mathsf{n}) \Rightarrow\quad \mathsf{m} \subseteq \mathsf{n} \,\&\, \#\mathsf{n} \neq \mathsf{n} - \mathsf{m} + \mathsf{m}$
$\langle \mathsf{m}, \mathsf{n}\rangle \hookrightarrow T232 \Rightarrow\quad \mathsf{n} - \mathsf{m} + \mathsf{m} \neq \#\mathsf{m} + (\mathsf{n} - \mathsf{m})$
$\langle \mathsf{m}, \mathsf{n} - \mathsf{m}\rangle \hookrightarrow T196 \Rightarrow\quad \mathsf{n} - \mathsf{m} + \mathsf{m} \neq \mathsf{m} + (\mathsf{n} - \mathsf{m})$
$\langle \mathsf{m}, \mathsf{n} - \mathsf{m}\rangle \hookrightarrow T216 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

-- If we confine our attention to cardinals $\#\mathsf{m}$ and $\#\mathsf{n}$, the preceding results tell us that if $\#\mathsf{m}$ is no larger than $\#\mathsf{n}$, we have $\#\mathsf{n} = \#\mathsf{m} + (\#\mathsf{n} - \#\mathsf{m})$.

-- Subtraction Lemma

Theorem 266 (234) $\#\mathsf{M} \in \#\mathsf{N} \lor \#\mathsf{M} = \#\mathsf{N} \to \#\mathsf{N} = \#\mathsf{M} + (\#\mathsf{N} - \#\mathsf{M})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m},\mathsf{n}) \Rightarrow\quad \#\mathsf{m} \in \#\mathsf{n} \lor \#\mathsf{m} = \#\mathsf{n} \,\&\, \#\mathsf{n} \neq \#\mathsf{m} + (\#\mathsf{n} - \#\mathsf{m})$

-- Our assertion follows trivially form the preceding, since the present hypotheses imply that $\#\mathsf{m} \subseteq \#\mathsf{n}$.

$\langle \mathsf{n}\rangle \hookrightarrow T130 \Rightarrow\quad \mathcal{O}(\#\mathsf{n})$
$\langle \mathsf{m}\rangle \hookrightarrow T130 \Rightarrow\quad \mathcal{O}(\#\mathsf{m})$
$\langle \#\mathsf{n}, \#\mathsf{m}\rangle \hookrightarrow T31 \Rightarrow\quad \#\mathsf{m} \subseteq \#\mathsf{n}$
$\langle \#\mathsf{m}, \#\mathsf{n}\rangle \hookrightarrow T232 \Rightarrow\quad \#\#\mathsf{n} = \#\#\mathsf{m} + (\#\mathsf{n} - \#\mathsf{m})$
$\langle \mathsf{N}\rangle \hookrightarrow T140 \Rightarrow\quad \#\#\mathsf{n} = \#\mathsf{n}$
$\langle \mathsf{M}\rangle \hookrightarrow T140 \Rightarrow\quad \#\#\mathsf{m} = \#\mathsf{m}$
$\mathsf{EQUAL} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

-- Next we show that the union set of a set s is the set-theoretic 'upper bound' of all its elements, i. e. the smallest set which includes all these elements.

-- Union set as an upper bound

Theorem 267 (235) $\langle \forall x \in S \mid x \subseteq \bigcup S \rangle \,\&\, (\langle \forall x \in S \mid x \subseteq T \rangle \rightarrow \bigcup S \subseteq T)$. PROOF:

$\mathsf{Suppose\_not}(s,t) \Rightarrow \quad \neg\langle \forall x \in s \mid x \subseteq \bigcup s \rangle \vee (\langle \forall x \in s \mid x \subseteq t \rangle \,\&\, \bigcup s \not\subseteq t)$

-- For if not, one of the two clauses of our theorem must be false. By definition of $\bigcup$, this cannot be the first clause, so it must be the second.

$\mathsf{Suppose} \Rightarrow \quad Stat1: \neg\langle \forall x \in s \mid x \subseteq \bigcup s \rangle$
$\langle x \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat2: x \in s \,\&\, x \not\subseteq \bigcup s$
$\langle c \rangle \hookrightarrow Stat2 \Rightarrow \quad c \in x \,\&\, c \notin \bigcup s$
$\mathsf{Use\_def}(\bigcup) \Rightarrow \quad Stat3: c \notin \{z : y \in s, z \in y\}$
$\langle x, c \rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat4: \langle \forall x \in s \mid x \subseteq t \rangle \,\&\, Stat5: \bigcup s \not\subseteq t$

-- But a second use of the definition of $\bigcup$ shows that this case is also impossible, proving our theorem.

$\langle d \rangle \hookrightarrow Stat5 \Rightarrow \quad d \in \bigcup s \,\&\, d \notin t$
$\mathsf{Use\_def}(\bigcup) \Rightarrow \quad Stat6: d \in \{z : y \in s, z \in y\}$
$\langle b, a \rangle \hookrightarrow Stat6 \Rightarrow \quad d = a \,\&\, b \in s \,\&\, a \in b$
$\langle b \rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- If the set appearing in the preceding theorem is a collection of ordinals, then its union set $\bigcup s$ must be an ordinal. Indeed $\bigcup s$ is the smallest ordinal not smaller than any of the elements of s (but this fact, an easy consequence of Theorem 235, is not derived in what follows.)

-- The union of a set of ordinals is an ordinal

Theorem 268 (236) $\langle \forall x \in S \mid \mathcal{O}(x) \rangle \rightarrow \mathcal{O}(\bigcup S)$. PROOF:

$\mathsf{Suppose\_not}(s) \Rightarrow \quad Stat1: \langle \forall x \in s \mid \mathcal{O}(x) \rangle \,\&\, \neg\mathcal{O}(\bigcup s)$

-- For suppose that s is a set of ordinals whose union set is not an ordinal. Then $\bigcup s$ either has an element x not included in $\bigcup s$, or a pair of distinct elements not related by membership. First consider the first of these two possibilities.

$\mathsf{Use\_def}(\mathcal{O}) \Rightarrow \quad \neg(\langle \forall x \in \bigcup s \mid x \subseteq \bigcup s \rangle \,\&\, \langle \forall x \in \bigcup s, y \in \bigcup s \mid x \in y \vee y \in x \vee x = y \rangle)$
$\mathsf{Use\_def}(\bigcup) \Rightarrow$

$\neg\langle\forall x \in \{zz : yy \in s, zz \in yy\} \mid x \subseteq \{zz : yy \in s, zz \in yy\}\rangle \vee$
$\neg\langle\forall x \in \{zz : yy \in s, zz \in yy\}, y \in \{zz : yy \in s, zz \in yy\} \mid x \in y \vee y \in x \vee x = y\rangle$

Suppose $\Rightarrow$ $\mathit{Stat2}: \neg\langle\forall x \in \{zz : yy \in s, zz \in yy\} \mid x \subseteq \{zz : yy \in s, zz \in yy\}\rangle$

-- In this case there would have to be a, c satisfying $a \in s$, $x \in a$, $c \in s$ such that $c \notin \{zz : yy \in s, zz \in yy\}$, an evident impossibility which rules out this case.

$\langle x\rangle\hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathit{Stat3}: x \in \{zz : yy \in s, zz \in yy\} \,\&\, x \not\subseteq \{zz : yy \in s, zz \in yy\}$
$\langle a, b\rangle\hookrightarrow \mathit{Stat3} \Rightarrow$ $x = b \,\&\, a \in s \,\&\, b \in a \,\&\, \mathit{Stat4}: x \not\subseteq \{zz : yy \in s, zz \in yy\}$
$\langle c\rangle\hookrightarrow \mathit{Stat4} \Rightarrow$ $c \in b \,\&\, \mathit{Stat5}: c \notin \{zz : yy \in s, zz \in yy\}$
$\langle a\rangle\hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathcal{O}(a)$
$\langle a, b\rangle\hookrightarrow \mathit{T12} \Rightarrow$ $c \in a$
$\langle a, c\rangle\hookrightarrow \mathit{Stat5} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat6}: \neg\langle\forall x \in \{zz : yy \in s, zz \in yy\}, y \in \{zz : yy \in s, zz \in yy\} \mid x \in y \vee y \in x \vee x = y\rangle$

-- But if $\bigcup s$ a pair of distinct elements u, v not related by membership, u would be a member of some ordinal au in s, and v would be a member of some ordinal av in s. Since one of these ordinals would necessarily include the other this is impossible, so out theorem is proved.

$\langle u, v\rangle\hookrightarrow \mathit{Stat6} \Rightarrow$ $\mathit{Stat7}: u, v \in \{zz : yy \in s, zz \in yy\} \,\&\, \neg(u \in v \vee v \in u \vee u = v)$
$\langle au, bu, av, bv\rangle\hookrightarrow \mathit{Stat7} \Rightarrow$ $au \in s \,\&\, u \in au \,\&\, av \in s \,\&\, v \in av \,\&\, \neg(u \in v \vee v \in u \vee u = v)$
$\langle au\rangle\hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathcal{O}(au)$
$\langle av\rangle\hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathcal{O}(av)$
$\langle au, u\rangle\hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(u)$
$\langle av, v\rangle\hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(v)$
$\langle u, v\rangle\hookrightarrow \mathit{T28} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that the arithmetic quotient of any n by a nonzero m is no larger than n.

Theorem 269 (237) $M \neq \emptyset \to N \; \textbf{\textit{div}} \; M \subseteq N$. PROOF:

Suppose_not$(m, n) \Rightarrow$ $\mathit{Stat1}: m \neq \emptyset \,\&\, n \; \mathbf{div} \; m \not\subseteq n$

-- For suppose that m, n is a counterexample to our assertion. Then by definition of division there must exist $k \in \mathbb{N}$ and $c \in k$, with $c \notin n$ and $k * m \subseteq n$.

Use_def(**div**) $\Rightarrow$ $\mathit{Stat2}: \bigcup\{k \in \mathbb{N} \mid k * m \subseteq n\} \not\subseteq n$
$\langle c\rangle\hookrightarrow \mathit{Stat2} \Rightarrow$ $c \notin n \,\&\, c \in \bigcup\{k \in \mathbb{N} \mid k * m \subseteq n\}$
Use_def$(\bigcup) \Rightarrow$ $c \in \{x : y \in \{k \in \mathbb{N} \mid k * m \subseteq n\}, x \in y\}$
SIMPLF $\Rightarrow$ $\mathit{Stat3}: c \in \{x : k \in \mathbb{N}, x \in k \mid k * m \subseteq n\}$

$\langle k, x \rangle \hookrightarrow Stat3 \Rightarrow \quad k \in \mathbb{N} \,\&\, c \in k \,\&\, k * m \subseteq n$

-- Let a be a member of m, so that $\#(k \times m) \supseteq \#(k \times \{a\})$ and therefore $k * m \supseteq k$. It follows that $n \supseteq k$, and so $c \in n$, a contradiction which proves our theorem.

$\langle a \rangle \hookrightarrow Stat1 \Rightarrow \quad a \in m$
$\langle k, k, \{a\}, m \rangle \hookrightarrow T219 \Rightarrow \quad k \times m \supseteq k \times \{a\}$
$\langle k \times \{a\}, k \times m \rangle \hookrightarrow T144 \Rightarrow \quad \#(k \times m) \supseteq \#(k \times \{a\})$
$\langle k, a \rangle \hookrightarrow T193 \Rightarrow \quad \#(k \times m) \supseteq \#k$
$\text{Use\_def}(*) \Rightarrow \quad k * m \supseteq \#k$
$\langle k \rangle \hookrightarrow T180 \Rightarrow \quad k * m \supseteq k$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma tells us that if n is and integer and m is nonzero, the arithmetic quotient of n by m is an integer.

Theorem 270 (238) $M \neq \emptyset \,\&\, N \in \mathbb{N} \rightarrow N \textbf{ div } M \in \mathbb{N} \,\&\, N \textbf{ div } M \subseteq N$. PROOF:

$\text{Suppose\_not}(m, n) \Rightarrow \quad n \in \mathbb{N} \,\&\, m \neq \emptyset \,\&\, n \textbf{ div } m \notin \mathbb{N} \vee n \textbf{ div } m \nsubseteq n$

-- For otherwise, since it is clear that every element of $\{k \in \mathbb{N} \mid k * m \subseteq n\}$ is an ordinal, It follows by definition that $n \textbf{ div } m$ is an ordinal.

$\text{Suppose} \Rightarrow \quad Stat1: \neg\langle \forall x \in \{k \in \mathbb{N} \mid k * m \subseteq n\} \mid \mathcal{O}(x) \rangle$
$\langle x_2 \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat2: x_2 \in \{k \in \mathbb{N} \mid k * m \subseteq n\} \,\&\, \neg\mathcal{O}(x_2)$
$\langle \rangle \hookrightarrow Stat2 \Rightarrow \quad x_2 \in \mathbb{N} \,\&\, \neg\mathcal{O}(x_2)$
$\langle x_2 \rangle \hookrightarrow T180 \Rightarrow$ false; Discharge $\Rightarrow \quad \langle \forall x \in \{k \in \mathbb{N} \mid k * m \subseteq n\} \mid \mathcal{O}(x) \rangle$
$\langle \{k \in \mathbb{N} \mid k * m \subseteq n\} \rangle \hookrightarrow T236 \Rightarrow \quad \mathcal{O}(\bigcup\{k \in \mathbb{N} \mid k * m \subseteq n\})$
$\text{Use\_def}(\textbf{div}) \Rightarrow \quad n \textbf{ div } m = \bigcup\{k \in \mathbb{N} \mid k * m \subseteq n\}$
$\langle junk \rangle \hookrightarrow T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
EQUAL $\Rightarrow \quad \mathcal{O}(n \textbf{ div } m)$

-- Since the second clause of our theorem is true by Theorem 237, $n \textbf{ div } m$ cannot be an integer, and since it is an ordinal, it cannot be finite. But since n is finite this is impossible, so our theorem is proved.

$\langle m, n \rangle \hookrightarrow T237 \Rightarrow \quad n \textbf{ div } m \subseteq n \,\&\, n \textbf{ div } m \notin \mathbb{N}$
$\langle n \rangle \hookrightarrow T179 \Rightarrow \quad \text{Finite}(n)$
$\langle m, n \rangle \hookrightarrow T237 \Rightarrow \quad n \textbf{ div } m \subseteq n$
$\langle n, n \textbf{ div } m \rangle \hookrightarrow T162 \Rightarrow \quad \text{Finite}(n \textbf{ div } m)$
$\langle n \textbf{ div } m \rangle \hookrightarrow T178 \Rightarrow \quad \text{Card}(n \textbf{ div } m)$

$\langle n \ \mathbf{div} \ m \rangle \hookrightarrow T179 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next result asserts that the set of integers is closed under arithmetic addition, multiplication, and subtraction.

Theorem 271 (239) $N, M \in \mathbb{N} \rightarrow N + M, N * M, N - M \in \mathbb{N}$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n, m \in \mathbb{N} \ \& \ \neg n + m, n * m, n - m \in \mathbb{N}$

-- We simply use the definitions of these operators, which make it immediately clear that the sum, product, and difference are all cardinals, and so must be integers if they are finite. But this has been proved earlier for the sum and product, and is obvious for the difference.

Use_def$(+) \Rightarrow$ $n + m = \#(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\})$
Use_def$(*) \Rightarrow$ $n * m = \#(n \times m)$
Use_def$(-) \Rightarrow$ $n - m = \#(n \backslash m)$
$\langle n \rangle \hookrightarrow T179 \Rightarrow$ Finite(n)
$\langle m \rangle \hookrightarrow T179 \Rightarrow$ Finite(m)
$\langle n, m \rangle \hookrightarrow T208 \Rightarrow$ Finite$(n + m)$
$\langle n, m \rangle \hookrightarrow T224 \Rightarrow$ Finite$(n * m)$
ELEM $\Rightarrow$ $n \backslash m \subseteq n$
$\langle n, n \backslash m \rangle \hookrightarrow T162 \Rightarrow$ Finite$(n \backslash m)$
$\langle n \backslash m \rangle \hookrightarrow T166 \Rightarrow$ Finite$\big(\#(n \backslash m)\big)$
EQUAL $\Rightarrow$ Finite$(n - m)$
$\big\langle \{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\} \big\rangle \hookrightarrow T130 \Rightarrow$
Card$\big(\#(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in m\})\big)$
EQUAL $\Rightarrow$ Card$(n + m)$
$\langle n \times m \rangle \hookrightarrow T130 \Rightarrow$ Card$\big(\#(n \times m)\big)$
EQUAL $\Rightarrow$ Card$(n * m)$
$\langle n \backslash m \rangle \hookrightarrow T130 \Rightarrow$ Card$\big(\#(n \backslash m)\big)$
EQUAL $\Rightarrow$ Card$(n - m)$
$\langle n + m \rangle \hookrightarrow T179 \Rightarrow$ $n + m \in \mathbb{N}$
$\langle n * m \rangle \hookrightarrow T179 \Rightarrow$ $n * m \in \mathbb{N}$
$\langle n - m \rangle \hookrightarrow T179 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the sum $m + n$ of two integers is strictly larger than n if $m \neq \emptyset$.

-- Strict monotonicity of addition

Theorem 272 (240) $\mathsf{M},\mathsf{N} \in \mathbb{N} \,\&\, \mathsf{N} \neq \emptyset \rightarrow \mathsf{M} \in \mathsf{M} + \mathsf{N}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m},\mathsf{n}) \Rightarrow \quad \mathit{Stat1}: \; \mathsf{m},\mathsf{n} \in \mathbb{N} \,\&\, \mathit{Stat2}: \; \mathsf{n} \neq \emptyset \,\&\, \mathsf{m} \notin \mathsf{m} + \mathsf{n}$

-- For suppose that m, n is a counterexample to our assertion. Since the set $\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\}$ whose cardinality defines the sum is finite, It follows by Theorem 167 that we have only to prove that its second term is not included in its first. But these two terms are clearly disjoint, and the second must be nonempty since n is nonempty. So our assertion is clear.

$\mathsf{Use\_def}(+) \Rightarrow \quad \mathsf{m} + \mathsf{n} = \#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\})$
$\langle \mathsf{m},\mathsf{n} \rangle \hookrightarrow T186 \Rightarrow \quad \mathit{Stat3}: \; \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cap \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\} = \emptyset$
$\langle \mathsf{e} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{e} \in \mathsf{n}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat4}: \; [\mathsf{e},1] \notin \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\}$
$\langle \mathsf{e} \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat5}: \; \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\} \neq \emptyset$
$\langle \mathit{Stat5}, \mathit{Stat3} \rangle \; \mathsf{ELEM} \Rightarrow$
$\quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \subseteq \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\} \,\&$
$\qquad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \neq \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\}$
$\langle \mathsf{m},\mathsf{n} \rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{m} + \mathsf{n} \in \mathbb{N}$
$\langle \mathsf{m}+\mathsf{n} \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(\mathsf{m}+\mathsf{n})$
$\mathsf{Use\_def}(+) \Rightarrow \quad \mathsf{m} + \mathsf{n} = \#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Finite}\big(\#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\})\big)$
$\langle \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\} \rangle \hookrightarrow T166 \Rightarrow$
$\quad \mathsf{Finite}(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\})$
$\langle \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \mathsf{n}\}, \{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \rangle \hookrightarrow T167 \Rightarrow \quad \mathit{Stat6}:$
$\quad \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} \in \mathsf{m} + \mathsf{n}$
$\langle \emptyset, \mathsf{m} \rangle \hookrightarrow T188 \Rightarrow \quad \mathit{Stat7}: \; \#\{[\mathsf{x},\emptyset] : \mathsf{x} \in \mathsf{m}\} = \#\mathsf{m}$
$\langle \mathsf{m} \rangle \hookrightarrow T180 \Rightarrow \quad \mathit{Stat8}: \; \#\mathsf{m} = \mathsf{m}$
$\langle \mathit{Stat1}, \mathit{Stat6}, \mathit{Stat7}, \mathit{Stat8}, * \rangle \; \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next elementary theorem asserts that only the empty set has cardinality $\emptyset$.

Theorem 273 (241) $\#\mathsf{N} = \emptyset \rightarrow \mathsf{N} = \emptyset$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}) \Rightarrow \quad \#\mathsf{n} = \emptyset \,\&\, \mathsf{n} \neq \emptyset$

-- For if $\#\ \mathrm{n} = 0$, then theorem 133 tell us that n is the range of a map with empty domain, an impossibility.

$\langle \mathsf{n} \rangle \hookrightarrow T121 \Rightarrow \quad \mathit{Stat1}: \; \langle \exists \mathsf{f} \mid 1\text{–}1(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) = \mathsf{n} \,\&\, \mathbf{domain}(\mathsf{f}) = \#\mathsf{n} \rangle$
$\langle \mathsf{f} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathbf{domain}(\mathsf{f}) = \emptyset \,\&\, \mathbf{range}(\mathsf{f}) \neq \emptyset$

$\langle f \rangle \hookrightarrow T78 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem asserts that for integers addition is not merely monotonic, but even strictly monotonic, in its second argument.

-- Strict monotonicity of addition

Theorem 274 (242) $M, N \in \mathbb{N} \,\&\, K \in N \rightarrow M + K \in M + N$. PROOF:

Suppose_not$(m, n, k) \Rightarrow$ *Stat1*: $m, n \in \mathbb{N} \,\&\, k \in n \,\&\, m + k \notin m + n$

-- For suppose that n, m, k constitute a counterexample to our theorem. Since the set $\mathbb{N}$ of integers is an ordinal, its members n and k are both ordinals and subsets of $\mathbb{N}$. n is plainly different from k, and equal to its own cardinality

$\langle \text{junk} \rangle \hookrightarrow T179 \Rightarrow$ $\mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, n \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(n)$
Use_def$(\mathcal{O}) \Rightarrow$ *Stat2*: $\langle \forall n \in \mathbb{N} \mid n \subseteq \mathbb{N} \rangle \,\&\, \langle \forall k \in n \mid k \subseteq n \rangle$
$\langle n, k \rangle \hookrightarrow Stat2 \Rightarrow$ *Stat3*: $n \subseteq \mathbb{N} \,\&\, k \subseteq n$
$\langle n \rangle \hookrightarrow T180 \Rightarrow$ *Stat4*: $n = \#n$
ELEM $\Rightarrow$ $k \neq n$
ELEM $\Rightarrow$ *Stat5*: $\#n \subseteq \mathbb{N}$

-- Since k is a subset of n, the set-theoretic facts stated just below are immediate. It follows from this that $\#(n \backslash k)$ is an integer no larger than $\#n$. Moreover k is a cardinal, and therefore is its own cardinality.

ELEM $\Rightarrow$ $n = n \backslash k \cup k \,\&\, (n \backslash k) \cap k = \emptyset \,\&\, n \supseteq n \backslash k$
$\langle k \rangle \hookrightarrow T180 \Rightarrow$ $k = \#k$
$\langle n \backslash k, n \rangle \hookrightarrow T144 \Rightarrow$ $\#n \supseteq \#(n \backslash k)$
$\langle n \backslash k \rangle \hookrightarrow T130 \Rightarrow$ $\text{Card}(\#(n \backslash k)) \,\&\, \mathcal{O}(\#(n \backslash k))$
$\langle n \rangle \hookrightarrow T130 \Rightarrow$ $\text{Card}(\#n) \,\&\, \mathcal{O}(\#n)$
$\langle \#n, \#(n \backslash k) \rangle \hookrightarrow T31 \Rightarrow$ *Stat6*: $\#(n \backslash k) \in \#n \vee \#(n \backslash k) = \#n$
$\langle Stat6, Stat5, Stat1, Stat4, * \rangle$ ELEM $\Rightarrow$ $\#(n \backslash k) \in \mathbb{N}$

-- It follows easily using Theorem 191 that n = (# (n-k) PLUS k), and so by what has been stated above, and a little algebra, it follows that (m PLUS k) notin ((m PLUS k) PLUS # (n-k)). But the set n-k is nonempty, and so by Theorem 240 the second of these sets is strictly larger than the first, a contradiction which proves our assertion.

$\langle n \backslash k, k \rangle \hookrightarrow T191 \Rightarrow$ $\#(n \backslash k) + \#k = \#(n \backslash k \cup k)$
ELEM $\Rightarrow$ $n \backslash k \cup k = n$

EQUAL $\Rightarrow$ $\#\mathsf{n} = \#(\mathsf{n}\backslash\mathsf{k}) + \#\mathsf{k}$
$\langle$*Stat3, Stat4, Stat1, ∗*$\rangle$ ELEM $\Rightarrow$ $\mathsf{k} \in \mathbb{N}$
EQUAL $\Rightarrow$ *Stat7* : $\#\mathsf{n} = \#(\mathsf{n}\backslash\mathsf{k}) + \mathsf{k}$
$\langle$*Stat4, Stat7, ∗*$\rangle$ ELEM $\Rightarrow$ $\mathsf{n} = \#(\mathsf{n}\backslash\mathsf{k}) + \mathsf{k}$
EQUAL $\Rightarrow$ $\mathsf{m} + \mathsf{k} \notin \mathsf{m} + \big(\#(\mathsf{n}\backslash\mathsf{k}) + \mathsf{k}\big)$
ALGEBRA $\Rightarrow$ $\mathsf{m} + \big(\#(\mathsf{n}\backslash\mathsf{k}) + \mathsf{k}\big) = \mathsf{m} + \mathsf{k} + \#(\mathsf{n}\backslash\mathsf{k})\ \&\ \mathsf{m} + \mathsf{k} \in \mathbb{N}$
ELEM $\Rightarrow$ $\mathsf{m} + \mathsf{k} \notin \mathsf{m} + \mathsf{k} + \#(\mathsf{n}\backslash\mathsf{k})$
$\langle \mathsf{n}\backslash\mathsf{k}\rangle \hookrightarrow$ *T241* $\Rightarrow$ $\#(\mathsf{n}\backslash\mathsf{k}) \neq \emptyset$
$\langle \mathsf{m} + \mathsf{k}, \#(\mathsf{n}\backslash\mathsf{k})\rangle \hookrightarrow$ *T240* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that for integers (though not for more general cardinals) cancellation of the common second argument of an equality between sums is allowed.

-- Cancellation

Theorem 275 (243) $\mathsf{M}, \mathsf{N}, \mathsf{K} \in \mathbb{N}\ \&\ \mathsf{M} + \mathsf{K} = \mathsf{N} + \mathsf{K} \rightarrow \mathsf{M} = \mathsf{N}$. PROOF:

Suppose_not$(\mathsf{m}, \mathsf{n}, \mathsf{k})$ $\Rightarrow$ $\mathsf{m}, \mathsf{n}, \mathsf{k} \in \mathbb{N}\ \&\ \mathsf{m} + \mathsf{k} = \mathsf{n} + \mathsf{k}\ \&\ \mathsf{n} \neq \mathsf{m}$

-- Suppose that n, m, k are a counterexample to our assertion. Since the integers m and n are obviously ordinals, they are either equal, or one of them is smaller than the other.

$\langle \mathsf{n}\rangle \hookrightarrow$ *T180* $\Rightarrow$ $\mathcal{O}(\mathsf{n})$
$\langle \mathsf{m}\rangle \hookrightarrow$ *T180* $\Rightarrow$ $\mathcal{O}(\mathsf{m})$
$\langle \mathsf{n}, \mathsf{m}\rangle \hookrightarrow$ *T28* $\Rightarrow$ $\mathsf{n} \in \mathsf{m} \vee \mathsf{n} = \mathsf{m} \vee \mathsf{m} \in \mathsf{n}$

-- But since addition is strictly monotone in its second argument by Theorem 242, and commutative by Theorem 216, it is impossible that either m or n should be smaller than the other, so our result follows.

$\langle \mathsf{m}, \mathsf{k}\rangle \hookrightarrow$ *T216* $\Rightarrow$ $\mathsf{m} + \mathsf{k} = \mathsf{k} + \mathsf{m}$
$\langle \mathsf{n}, \mathsf{k}\rangle \hookrightarrow$ *T216* $\Rightarrow$ $\mathsf{n} + \mathsf{k} = \mathsf{k} + \mathsf{n}$
Suppose $\Rightarrow$ $\mathsf{n} \in \mathsf{m}$
$\langle \mathsf{k}, \mathsf{m}, \mathsf{n}\rangle \hookrightarrow$ *T242* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{n} = \mathsf{m} \vee \mathsf{m} \in \mathsf{n}$
Suppose $\Rightarrow$ $\mathsf{m} \in \mathsf{n}$
$\langle \mathsf{k}, \mathsf{n}, \mathsf{m}\rangle \hookrightarrow$ *T242* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem asserts that cardinal addition is monotone increasing (but not necessarily strictly monotone) in its first argument.

-- Monotonicity of Addition

Theorem 276 (244) $M \subseteq N \rightarrow M + K \subseteq N + K$. PROOF:

Suppose_not$(m, n, k) \Rightarrow \quad m \subseteq n \;\&\; m + k \not\subseteq n + k$

-- Indeed, the set monotonicity principle and the monotonicity of cardinality (Theorem 144) together rule out the existence of a counterexample n, m, k to our assertion.

Use_def$(+) \Rightarrow \quad \#(\{[x, \emptyset] : x \in m\} \cup \{[x, 1] : x \in k\}) \not\subseteq \#(\{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in k\})$
Set_monot $\Rightarrow \quad \{[x, \emptyset] : x \in m\} \subseteq \{[x, \emptyset] : x \in n\}$
ELEM $\Rightarrow \quad \{[x, \emptyset] : x \in m\} \cup \{[x, 1] : x \in k\} \subseteq \{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in k\}$
$\langle \{[x, \emptyset] : x \in m\} \cup \{[x, 1] : x \in k\}, \{[x, \emptyset] : x \in n\} \cup \{[x, 1] : x \in k\} \rangle \hookrightarrow T144 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following easy result asserts that cardinal multiplication is monotone increasing (but not necessarily strictly monotone) in its first argument.

-- Monotonicity of Multiplication

Theorem 277 (245) $M \subseteq N \rightarrow M * K \subseteq N * K$. PROOF:

Suppose_not$(m, n, k) \Rightarrow \quad m \subseteq n \;\&\; m * k \not\subseteq n * k$

-- For the set monotonicity principle and the monotonicity of cardinality (Theorem 144) together rule out the existence of a counterexample n, m, k to our assertion.

Use_def$(*) \Rightarrow \quad \#(m \times k) \not\subseteq \#(n \times k)$
Use_def$(\times) \Rightarrow \quad \#\{[x, y] : x \in m, y \in k\} \not\subseteq \#\{[x, y] : x \in n, y \in k\}$
Set_monot $\Rightarrow \quad \{[x, y] : x \in m, y \in k\} \subseteq \{[x, y] : x \in n, y \in k\}$
$\langle \{[x, y] : x \in m, y \in k\}, \{[x, y] : x \in n, y \in k\} \rangle \hookrightarrow T144 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following corollary gives the strict version of the preceding result, in case the sets involved are integers.

-- Strict monotonicity of integer multiplication

Theorem 278 (246) $M \in N \;\&\; N, K \in \mathbb{N} \;\&\; K \neq \emptyset \rightarrow M * K \in N * K$. PROOF:

Suppose_not$(m, n, k) \Rightarrow \quad (m \in n \;\&\; n, k \in \mathbb{N} \;\&\; k \neq \emptyset) \;\&\; m * k \notin n * k$

-- First we can write $n \times k$ as the disjoint union $m \times k \cup (n \backslash m) \times k$, and since the second term in this union is noempty, $m \times k$ is a proper subset of $n \times k$. Since all these sets are clearly finite, it follows by Theorem 167 that $\#(m \times k)$ is smaller than $\#(m \times k)$. By definition of "*TIMES*", this is our assertion.

$\langle n \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(n) \,\&\, \mathrm{Finite}(n)$
$\langle k \rangle \hookrightarrow T180 \Rightarrow \quad \mathrm{Finite}(k)$
$\langle n, m \rangle \hookrightarrow T12 \Rightarrow \quad m \subseteq n$
ELEM $\Rightarrow \quad n = m \cup (n \backslash m) \,\&\, n \backslash m \neq \emptyset \,\&\, m \cap (n \backslash m) = \emptyset$
Use_def($*$) $\Rightarrow \quad \#(m \times k) \notin \#(n \times k)$
$\langle m, k, n \backslash m \rangle \hookrightarrow T218 \Rightarrow \quad (m \cup (n \backslash m)) \times k = m \times k \cup (n \backslash m) \times k \,\&$
$m \times k \cap ((n \backslash m) \times k) = m \cap (n \backslash m) \times k$
EQUAL $\Rightarrow \quad \mathit{Stat1}: \; n \times k = m \times k \cup (n \backslash m) \times k \,\&\, m \times k \cap ((n \backslash m) \times k) = \emptyset \times k$
$\langle k \rangle \hookrightarrow T114 \Rightarrow \quad \mathit{Stat2}: \; m \times k \cap ((n \backslash m) \times k) = \emptyset$
$\langle k, n \backslash m \rangle \hookrightarrow T214 \Rightarrow \quad \#((n \backslash m) \times k) \supseteq \#(n \backslash m)$
$\langle n \backslash m \rangle \hookrightarrow T136 \Rightarrow \quad \#(n \backslash m) \neq \emptyset$
ELEM $\Rightarrow \quad \#((n \backslash m) \times k) \neq \emptyset$
$\langle (n \backslash m) \times k \rangle \hookrightarrow T136 \Rightarrow \quad \mathit{Stat3}: \; (n \backslash m) \times k \neq \emptyset$
$\langle \mathit{Stat1}, \mathit{Stat2}, \mathit{Stat3} \rangle$ ELEM $\Rightarrow \quad m \times k \subseteq n \times k \,\&\, m \times k \neq n \times k$
$\langle n, k \rangle \hookrightarrow T225 \Rightarrow \quad \mathrm{Finite}(n \times k)$
$\langle n \times k, m \times k \rangle \hookrightarrow T167 \Rightarrow \quad \#(m \times k) \in \#(n \times k)$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We now show that arithmetic addition of integers is strictly monotone in its first argument.

-- Strict Monotonicity of Addition

Theorem 279 (247) $M, N, K \in \mathbb{N} \rightarrow (M + K \subseteq N + K \leftrightarrow M \subseteq N)$. PROOF:

Suppose_not$(m, n, k) \Rightarrow \quad m, n, k \in \mathbb{N} \,\&\, \neg(m + k \subseteq n + k \leftrightarrow m \subseteq n)$

-- For suppose that m, n, k constitute a counterexample to our theorem. Since we know that addition is non-strictly monotone, it must be that $m + k \subseteq n + k$, but that m is not a subest of n.

$\langle m, n, k \rangle \hookrightarrow T244 \Rightarrow \quad m + k \subseteq n + k \,\&\, m \nsubseteq n$

-- But since all the quantities involved are integers, and therefore both cardinals and ordinals, n must be smaller than n, and so a contradiction results immediately from Theorem 242 and the commutativity of addition, thereby proving the present theorem.

ALGEBRA $\Rightarrow \quad m + k, n + k \in \mathbb{N}$
$\langle m \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(m)$
$\langle n \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(n)$
$\langle m + k \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(m + k)$

$\langle n+k \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(n+k)$
$\langle m, n \rangle \hookrightarrow T32 \Rightarrow \quad n \in m$
$\langle k, m, n \rangle \hookrightarrow T242 \Rightarrow \quad k+n \in k+m$
$\langle k, n \rangle \hookrightarrow T216 \Rightarrow \quad k+n = n+k$
$\langle k, m \rangle \hookrightarrow T216 \Rightarrow \quad k+m = m+k$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following corollary of Theorem 247 is sometimes more directly useful.

-- Strict Monotonicity of Addition

Theorem 280 (248) $M, N \in \mathbb{N} \,\&\, N \neq \emptyset \rightarrow M \in M + N$. PROOF:

Suppose_not$(m, n) \Rightarrow \quad m, n \in \mathbb{N} \,\&\, n \neq \emptyset \,\&\, m \notin m + n$

-- Since all the quanties involved are integers, and therefore both cardinals and ordinals, The conclusion of our theorem is equivalent to $\neg m + n \subseteq m + \emptyset$.

$\langle n \rangle \hookrightarrow T137 \Rightarrow \quad \emptyset \in \#n$
$\langle m \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(m)$
ALGEBRA $\Rightarrow \quad m + n \in \mathbb{N}$
$\langle m+n \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(m+n)$
$\langle n \rangle \hookrightarrow T180 \Rightarrow \quad \emptyset \in n$
$T182 \Rightarrow \quad \emptyset \in \mathbb{N}$
ELEM $\Rightarrow \quad n \not\subseteq \emptyset$

-- Thus, by Theorem 247, the conclusion ofour theorem is equivalent to $\neg n \subseteq \emptyset$, and so is obvious.

$\langle \emptyset, n, m \rangle \hookrightarrow T247 \Rightarrow \quad \emptyset + m \subseteq n + m$
ALGEBRA $\Rightarrow \quad n + m = m + n \,\&\, \emptyset + m = m$
$\langle n, \emptyset, m \rangle \hookrightarrow T247 \Rightarrow \quad n + m \not\subseteq \emptyset + m$
ALGEBRA $\Rightarrow \quad m + \emptyset = m$
ELEM $\Rightarrow \quad m \subseteq m + n \,\&\, m + n \neq m$
$\langle m, m+n \rangle \hookrightarrow T28 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Strict monotonicity of subtraction

Theorem 281 (249) $N, M \in \mathbb{N} \,\&\, K \in N \,\&\, M \supseteq N \rightarrow M - N \in M - K$. PROOF:

$\mathsf{Suppose\_not}(n,m,k) \Rightarrow \quad n,m \in \mathbb{N} \,\&\, k \in n \,\&\, m \supseteq n \,\&\, m - n \notin m - k$

-- For let n, m, k be a counterexample to our assertion. It then follows by definition that $\#(m\backslash n) \notin \#(m\backslash k)$, so that since all the integers involved must be ordinals $k \in n$ implies that k is a proper subset of n by Theorem 31.

$\mathsf{Use\_def}(-) \Rightarrow \quad \#(m\backslash n) \notin \#(m\backslash k)$
$\langle \mathbb{N} \rangle \hookrightarrow T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, n \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(n)$
$\langle n, k \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(k)$
$\langle m \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(m)$
$\langle n, k \rangle \hookrightarrow T31 \Rightarrow \quad k \subseteq n$
$\mathsf{ELEM} \Rightarrow \quad k \neq n$

-- Hence $m\backslash n$ is a proper subset of $m\backslash k$, and since m and thus $m\backslash k$

$\langle m, m\backslash k \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{Finite}(m\backslash k)$
$\mathsf{ELEM} \Rightarrow \quad m\backslash n \subseteq m\backslash k \,\&\, m\backslash n \neq m\backslash k$
$\langle m\backslash k, m\backslash n \rangle \hookrightarrow T167 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The next theorem tells us that two successive subtractions are equivalent to subtraction of a sum, at least in the positive integer case which it covers.

Theorem 282 (250) $M,N,K \in \mathbb{N} \,\&\, N \supseteq M \,\&\, N - M \supseteq K \rightarrow N \supseteq M + K \,\&\, N - (M+K) = N - M - K$. PROOF:

$\mathsf{Suppose\_not}(m,n,k) \Rightarrow \quad m,n,k \in \mathbb{N} \,\&\, n \supseteq m \,\&\, n - m \supseteq k \,\&\, n \not\supseteq m + k \vee n - (m+k) \neq n - m - k$

-- For consider a potential counterexample n, m, k to our assertion. Plainly n MINUS m and ((n MINUS m) MINUS k) are integers, and so all the quantities we consider are cardinals.

$\langle n, m \rangle \hookrightarrow T239 \Rightarrow \quad n - m \in \mathbb{N}$
$\langle n - m, k \rangle \hookrightarrow T239 \Rightarrow \quad n - m - k \in \mathbb{N}$
$\langle n \rangle \hookrightarrow T180 \Rightarrow \quad n = \#n \,\&\, \mathsf{Finite}(n)$
$\langle m \rangle \hookrightarrow T180 \Rightarrow \quad m = \#m \,\&\, \mathsf{Finite}(m)$
$\langle k \rangle \hookrightarrow T180 \Rightarrow \quad k = \#k \,\&\, \mathsf{Finite}(k)$
$\langle n - m \rangle \hookrightarrow T180 \Rightarrow \quad n - m = \#(n - m)$

-- Theorem 232 used twice, plus a bit of algebra, tells us that n = ((n MINUS m) MINUS k) PLUS (k PLUS m), and so, by the monotonicity of addition, n must include m PLUS k.

$\langle m, n \rangle \hookrightarrow T232 \Rightarrow \quad \#n = \#m + (n - m)$
EQUAL $\Rightarrow \quad n = m + (n - m)$
ALGEBRA $\Rightarrow \quad n = n - m + m$
$\langle k, n - m \rangle \hookrightarrow T232 \Rightarrow \quad \#(n - m) = \#k + (n - m - k)$
EQUAL $\Rightarrow \quad n - m = k + (n - m - k)$
ALGEBRA $\Rightarrow \quad n - m = n - m - k + k$
EQUAL $\Rightarrow \quad n = n - m - k + k + m$
ALGEBRA $\Rightarrow \quad n = n - m - k + (k + m)$
ELEM $\Rightarrow \quad n - m - k \supseteq \emptyset$
$T182 \Rightarrow \quad \emptyset \in \mathbb{N}$
$\langle k, m \rangle \hookrightarrow T239 \Rightarrow \quad k + m \in \mathbb{N}$
$\langle n, m \rangle \hookrightarrow T239 \Rightarrow \quad n - m \in \mathbb{N}$
$\langle n - m, k \rangle \hookrightarrow T239 \Rightarrow \quad n - m - k \in \mathbb{N}$
$\langle \emptyset, n - m - k, k + m \rangle \hookrightarrow T247 \Rightarrow \quad n \supseteq \emptyset + (k + m)$
ALGEBRA $\Rightarrow \quad \emptyset + (k + m) = m + k$
ELEM $\Rightarrow \quad n \supseteq m + k$
$\langle m, k \rangle \hookrightarrow T239 \Rightarrow \quad m + k \in \mathbb{N}$
$\langle m + k \rangle \hookrightarrow T180 \Rightarrow \quad m + k = \#(m + k)$

-- Therefore, using Theorem 232 once more, we see that n = (n MINUS (m PLUS k)) PLUS (k PLUS m) also. Theorem 243 now lets us cancel k PLUS m from these two expressions for n, thereby obtaining te formula asserted by the present theorem.

$\langle m + k, n \rangle \hookrightarrow T232 \Rightarrow \quad n = \#(m + k) + \big(n - (m + k)\big)$
EQUAL $\Rightarrow \quad n = m + k + \big(n - (m + k)\big)$
ALGEBRA $\Rightarrow \quad n = n - (m + k) + (k + m)$
$\langle n, m + k \rangle \hookrightarrow T239 \Rightarrow \quad n - (m + k) \in \mathbb{N}$
$\langle n - m, k \rangle \hookrightarrow T239 \Rightarrow \quad n - m - k \in \mathbb{N}$
$\langle n - (m + k), n - m - k, k + m \rangle \hookrightarrow T243 \Rightarrow$ QED

Theorem 283 (251) $M, N \in \mathbb{N} \to M + N - N = M$. PROOF:

Suppose_not$(m, n) \Rightarrow \quad m, n \in \mathbb{N} \;\&\; m + n - n \neq m$

-- For suppose that m, n is a counterexample to our assertion. $m + n$ and $m + n - n$, and $m + n$ are easily seen to be integers. Since addition is monotone and $\emptyset + n = n$, we have $n \subseteq m + n$.

ALGEBRA $\Rightarrow \quad m + n \in \mathbb{N}$
$\langle m + n, n \rangle \hookrightarrow T239 \Rightarrow \quad m + n - n \in \mathbb{N}$

$\langle \emptyset, \mathsf{m}, \mathsf{n} \rangle \hookrightarrow T244 \Rightarrow \quad \emptyset + \mathsf{n} \subseteq \mathsf{m} + \mathsf{n}$
ALGEBRA $\Rightarrow \quad \emptyset + \mathsf{n} = \mathsf{n}$
ELEM $\Rightarrow \quad \mathsf{n} \subseteq \mathsf{m} + \mathsf{n}$

-- The subtraction lemma (Theorem 232) now tells us that $\#(\mathsf{m}+\mathsf{n}) = \#\mathsf{n} + (\mathsf{m}+\mathsf{n}-\mathsf{n})$, and since n and $\mathsf{m}+\mathsf{n}$ are cardinals, we get $\mathsf{m}+\mathsf{n} = \mathsf{n} + (\mathsf{m}+\mathsf{n}-\mathsf{n}) = \mathsf{m}+\mathsf{n}-\mathsf{n}+\mathsf{n}$, from which $\mathsf{m}+\mathsf{n}-\mathsf{n} = \mathsf{m}$ follows by cancellation, proving our theorem.

$\langle \mathsf{n}, \mathsf{m}+\mathsf{n} \rangle \hookrightarrow T232 \Rightarrow \quad \#(\mathsf{m}+\mathsf{n}) = \#\mathsf{n} + (\mathsf{m}+\mathsf{n}-\mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{n} = \#\mathsf{n}$
$\langle \mathsf{m}+\mathsf{n} \rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{m}+\mathsf{n} = \#(\mathsf{m}+\mathsf{n})$
EQUAL $\Rightarrow \quad \#(\mathsf{m}+\mathsf{n}) = \mathsf{n} + (\mathsf{m}+\mathsf{n}-\mathsf{n})$
EQUAL $\Rightarrow \quad \mathsf{m}+\mathsf{n} = \mathsf{n} + (\mathsf{m}+\mathsf{n}-\mathsf{n})$
ALGEBRA $\Rightarrow \quad \mathsf{m}+\mathsf{n} = \mathsf{m}+\mathsf{n}-\mathsf{n}+\mathsf{n}$
$\langle \mathsf{m}, \mathsf{m}+\mathsf{n}-\mathsf{n}, \mathsf{n} \rangle \hookrightarrow T243 \Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow$ QED

-- Our next, entirely elementary, lemma simply tells us that the cardinality of any singleton is 1.

Theorem 284 (252) $\#\{\mathsf{S}\} = \{\emptyset\}$. PROOF:

Suppose_not(s) $\Rightarrow \quad \#\{\mathsf{s}\} \neq \{\emptyset\}$

-- For {0}, as the sucessor of 0, is clearly a caridnal, and so our assertion is equivalent to # {s} = # {0}.

$T182 \Rightarrow \quad \mathsf{Card}(1)$
Use_def(next) $\Rightarrow \quad \mathsf{next}(\emptyset) = \{\emptyset\}$
Use_def(1) $\Rightarrow \quad 1 = \mathsf{next}(\emptyset)$
Use_def(next) $\Rightarrow \quad 1 = \emptyset \cup \{\emptyset\}$
ELEM $\Rightarrow \quad 1 = \{\emptyset\}$
EQUAL $\Rightarrow \quad \mathsf{Card}(\{\emptyset\})$
$\langle \{\emptyset\} \rangle \hookrightarrow T138 \Rightarrow \quad \#\{\emptyset\} = \{\emptyset\}$

-- But the map {[s, 0]} sevidently sends {s} to {0} and is 1-1, our assertion is immediate from Theorem 131.

Use_def(**domain**) $\Rightarrow \quad Stat1: \ \mathbf{domain}(\{[\mathsf{s}, \emptyset]\}) = \{\mathsf{x}^{[1]} : \mathsf{x} \in \{[\mathsf{s}, \emptyset]\}\}$
SIMPLF $\Rightarrow \quad \{\mathsf{x}^{[1]} : \mathsf{x} \in \{[\mathsf{s}, \emptyset]\}\} = \left\{[\mathsf{s}, \emptyset]^{[1]}\right\}$

ELEM $\Rightarrow$ $\left\{[\mathsf{s},\emptyset]^{[1]}\right\} = \{\mathsf{s}\}$

$\langle$ *Stat1*, $*\rangle$ ELEM $\Rightarrow$ $\mathbf{domain}(\{[\mathsf{s},\emptyset]\}) = \{\mathsf{s}\}$

Use_def (range) $\Rightarrow$ *Stat2*: $\mathbf{range}(\{[\mathsf{s},\emptyset]\}) = \left\{\mathsf{x}^{[2]} : \mathsf{x} \in \{[\mathsf{s},\emptyset]\}\right\}$

SIMPLF $\Rightarrow$ $\left\{\mathsf{x}^{[2]} : \mathsf{x} \in \{[\mathsf{s},\emptyset]\}\right\} = \left\{[\mathsf{s},\emptyset]^{[2]}\right\}$

ELEM $\Rightarrow$ $\left\{[\mathsf{s},\emptyset]^{[2]}\right\} = \{\emptyset\}$

$\langle$ *Stat2*, $*\rangle$ ELEM $\Rightarrow$ $\mathbf{range}(\{[\mathsf{s},\emptyset]\}) = \{\emptyset\}$

ELEM $\Rightarrow$ $1\text{–}1(\{[\mathsf{s},\emptyset]\})$

$\langle \{[\mathsf{s},\emptyset]\} \rangle \hookrightarrow$ *T131* $\Rightarrow$ $\#\mathbf{domain}(\{[\mathsf{s},\emptyset]\}) = \#\mathbf{range}(\{[\mathsf{s},\emptyset]\})$

EQUAL $\Rightarrow$ $\#\{\mathsf{s}\} = \#\{\emptyset\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem combines the preceding results to show that if m and n are integers, n being nonzero, then m can be written as a sum $\mathsf{n} = \mathsf{q} * \mathsf{n} + \mathsf{r}$, where q is the integer quotient of m by n, and r is a remainder less than n.

-- Integer Division with Remainder

Theorem 285 (253) $\mathsf{M},\mathsf{N} \in \mathbb{N}\ \&\ \mathsf{N} \neq \emptyset \rightarrow \mathsf{M}\ \boldsymbol{div}\ \mathsf{N} \in \mathbb{N}\ \&\ \mathsf{M} \supseteq \mathsf{M}\ \boldsymbol{div}\ \mathsf{N} * \mathsf{N}\ \&\ \mathsf{M}\ \boldsymbol{mod}\ \mathsf{N} \in \mathsf{N}$. PROOF:

Suppose_not(m,n) $\Rightarrow$ *Stat1*: $\mathsf{m},\mathsf{n} \in \mathbb{N}\ \&\ \mathsf{n} \neq \emptyset\ \&\ \neg(\mathsf{m}\ \mathbf{div}\ \mathsf{n} \in \mathbb{N}\ \&\ \mathsf{m} \supseteq \mathsf{m}\ \mathbf{div}\ \mathsf{n} * \mathsf{n}\ \&\ \mathsf{m}\ \mathbf{mod}\ \mathsf{n} \in \mathsf{n})$

-- For suppose that our theorem has a counterexample m, n. Consider the set s of all integers of the form $\mathsf{m} - \mathsf{p} * \mathsf{n}$, where p ranges over all integers such that $\mathsf{p} * \mathsf{n}$ is no larger than n. This set cannot be empty, since by setting $\mathsf{p} = \emptyset$ we see that m must belong to it.

$\langle \mathsf{n} \rangle \hookrightarrow$ *T180* $\Rightarrow$ $\mathsf{n} = \#\mathsf{n}\ \&\ \mathcal{O}(\mathsf{n})$

$\langle \mathsf{m} \rangle \hookrightarrow$ *T180* $\Rightarrow$ *Stat2*: $\mathsf{m} = \#\mathsf{m}$

Suppose $\Rightarrow$ *Stat3*: $\{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\} = \emptyset$

$\langle \emptyset \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\neg(\mathsf{m} = \mathsf{m} - \emptyset * \mathsf{n}\ \&\ \emptyset \in \mathbb{N}\ \&\ \mathsf{m} \supseteq \emptyset * \mathsf{n})$

T182 $\Rightarrow$ $\emptyset \in \mathbb{N}$

$\langle \mathsf{n} \rangle \hookrightarrow$ *T210* $\Rightarrow$ $\emptyset * \mathsf{n} = \emptyset$

ELEM $\Rightarrow$ $\mathsf{m} \neq \mathsf{m} - \emptyset * \mathsf{n}$

EQUAL $\Rightarrow$ $\mathsf{m} \neq \mathsf{m} - \emptyset$

$\langle \mathsf{m} \rangle \hookrightarrow$ *T230* $\Rightarrow$ $\mathsf{m} - \emptyset = \mathsf{m}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\} \neq \emptyset$

-- Let $\mathsf{r} = \mathsf{q} * \mathsf{n}$ be the smallest integer in the set s, so that by the axiom of choice r belongs to and is disjoint from s. q and r are plainly integers.

Loc_def $\Rightarrow$ $\mathsf{r} = \mathbf{arb}(\{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\})$
$\langle \{\mathsf{m} - \mathsf{q} * \mathsf{n} : \mathsf{q} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{q} * \mathsf{n}\} \rangle \hookrightarrow$ *T0* $\Rightarrow$ *Stat4* :
$\mathsf{r} \in \{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\} \;\&\; \mathsf{r} \cap \{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\} = \emptyset$
$\langle \mathsf{q} \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ *Stat5* : $\mathsf{r} = \mathsf{m} - \mathsf{q} * \mathsf{n} \;\&\; \mathsf{q} \in \mathbb{N} \;\&\; \mathsf{m} \supseteq \mathsf{q} * \mathsf{n}$
$\langle \mathsf{q} \rangle \hookrightarrow$ *T180* $\Rightarrow$ *Stat6* : $\mathsf{q} = \#\mathsf{q} \;\&\; \mathcal{O}(\mathsf{q})$
ALGEBRA $\Rightarrow$ $\mathsf{q} * \mathsf{n} \in \mathbb{N}$
$\langle \mathsf{m}, \mathsf{q} * \mathsf{n} \rangle \hookrightarrow$ *T239* $\Rightarrow$ $\mathsf{r} \in \mathbb{N}$

-- If r is not smaller than n, it follows by Theorem 250 that m is at least as large as $\mathsf{q} * \mathsf{n} + \mathsf{n}$, and that $\mathsf{m} - (\mathsf{q} * \mathsf{n} + \mathsf{n}) = \mathsf{m} - \mathsf{q} * \mathsf{n} - \mathsf{n}$. It follows, using a little algebra, that m is at least as large as $(\mathsf{q} + 1) * \mathsf{n}$, so $\mathsf{m} - (\mathsf{q} + 1) * \mathsf{n}$ must belong to the set s considered above. Since $\mathsf{r} = \mathbf{arb}(\mathsf{s})$, it follows that $\mathsf{m} - (\mathsf{q} + 1) * \mathsf{n} = \mathsf{r} - \mathsf{n}$ cannot be a member of r, contradicting Theorem 240. Hence we must have $\mathsf{r} \in \mathsf{n}$.

$\langle \mathsf{r} \rangle \hookrightarrow$ *T180* $\Rightarrow$ $\mathsf{r} = \#\mathsf{r} \;\&\; \mathcal{O}(\mathsf{r})$
Suppose $\Rightarrow$ $\mathsf{r} \supseteq \mathsf{n}$
EQUAL $\Rightarrow$ $\mathsf{m} - \mathsf{q} * \mathsf{n} \supseteq \mathsf{n}$
$\langle \mathsf{q} * \mathsf{n}, \mathsf{m}, \mathsf{n} \rangle \hookrightarrow$ *T250* $\Rightarrow$ $\mathsf{m} \supseteq \mathsf{q} * \mathsf{n} + \mathsf{n} \;\&\; \mathsf{m} - (\mathsf{q} * \mathsf{n} + \mathsf{n}) = \mathsf{m} - \mathsf{q} * \mathsf{n} - \mathsf{n}$
ALGEBRA $\Rightarrow$ $\mathsf{q} * \mathsf{n} + \mathsf{n} = (\mathsf{q} + 1) * \mathsf{n}$
EQUAL $\Rightarrow$ $\mathsf{m} \supseteq (\mathsf{q} + 1) * \mathsf{n}$
ALGEBRA $\Rightarrow$ $\mathsf{q} + 1 \in \mathbb{N}$
Suppose $\Rightarrow$ *Stat7* : $\mathsf{m} - (\mathsf{q} + 1) * \mathsf{n} \notin \{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\}$
$\langle \mathsf{q} + 1 \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $\neg(\mathsf{m} - (\mathsf{q} + 1) * \mathsf{n} = \mathsf{m} - (\mathsf{q} + 1) * \mathsf{n} \;\&\; \mathsf{q} + 1 \in \mathbb{N} \;\&\; \mathsf{m} \supseteq (\mathsf{q} + 1) * \mathsf{n})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{m} - (\mathsf{q} + 1) * \mathsf{n} \in \{\mathsf{m} - \mathsf{p} * \mathsf{n} : \mathsf{p} \in \mathbb{N} \,|\, \mathsf{m} \supseteq \mathsf{p} * \mathsf{n}\}$
ELEM $\Rightarrow$ $\mathsf{m} - (\mathsf{q} + 1) * \mathsf{n} \notin \mathsf{r}$
EQUAL $\Rightarrow$ $\mathsf{m} - \mathsf{q} * \mathsf{n} - \mathsf{n} \notin \mathsf{r}$
EQUAL $\Rightarrow$ *Stat8* : $\mathsf{r} - \mathsf{n} \notin \mathsf{r}$
$\langle \mathsf{n}, \mathsf{r} \rangle \hookrightarrow$ *T232* $\Rightarrow$ $\#\mathsf{r} = \#\mathsf{n} + (\mathsf{r} - \mathsf{n})$
EQUAL $\Rightarrow$ $\mathsf{r} = \mathsf{n} + (\mathsf{r} - \mathsf{n})$
ALGEBRA $\Rightarrow$ $\mathsf{r} = \mathsf{r} - \mathsf{n} + \mathsf{n}$
$\langle \mathsf{r}, \mathsf{n} \rangle \hookrightarrow$ *T239* $\Rightarrow$ $\mathsf{r} - \mathsf{n} \in \mathbb{N}$
$\langle \mathsf{r} - \mathsf{n}, \mathsf{n} \rangle \hookrightarrow$ *T240* $\Rightarrow$ *Stat9* : $\mathsf{r} - \mathsf{n} \in \mathsf{r}$
$\langle$ *Stat8*, *Stat9* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{r} \not\supseteq \mathsf{n}$
$\langle \mathsf{n}, \mathsf{r} \rangle \hookrightarrow$ *T32* $\Rightarrow$ *Stat10* : $\mathsf{r} \in \mathsf{n}$

-- Our next aim is to show that $\mathsf{q} = \mathsf{m}\ \mathbf{div}\ \mathsf{n} = \bigcup\{\mathsf{k} \in \mathbb{N} \,|\, \mathsf{k} * \mathsf{n} \subseteq \mathsf{m}\}$. To this end we first note that $\mathsf{q} \in \{\mathsf{k} \in \mathbb{N} \,|\, \mathsf{k} * \mathsf{n} \subseteq \mathsf{m}\}$, and therefore every member of q belongs to $\bigcup\{\mathsf{k} \in \mathbb{N} \,|\, \mathsf{k} * \mathsf{n} \subseteq \mathsf{m}\}$, so that q is included in this set.

Suppose $\Rightarrow$ *Stat11* : $\mathsf{q} \notin \{\mathsf{k} \in \mathbb{N} \,|\, \mathsf{k} * \mathsf{n} \subseteq \mathsf{m}\}$
$\langle\,\rangle \hookrightarrow$ *Stat11* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{q} \in \{\mathsf{k} \in \mathbb{N} \,|\, \mathsf{k} * \mathsf{n} \subseteq \mathsf{m}\}$

$\text{Suppose} \Rightarrow \quad Stat12: \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\} \not\supseteq q$
$\langle c \rangle \hookrightarrow Stat12 \Rightarrow \quad c \in q \,\&\, c \notin \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\}$
$\text{Use\_def}(\bigcup) \Rightarrow \quad Stat13: c \notin \{v: u \in \{k \in \mathbb{N} \mid k * n \subseteq m\}, v \in u\}$
$\text{SIMPLF} \Rightarrow \quad \{v: u \in \{k \in \mathbb{N} \mid k * n \subseteq m\}, v \in u\} = \{v: k \in \mathbb{N}, v \in k \mid k * n \subseteq m\}$
$\langle Stat13 \rangle \text{ ELEM} \Rightarrow \quad Stat14: c \notin \{v: k \in \mathbb{N}, v \in k \mid k * n \subseteq m\}$
$\langle q, c \rangle \hookrightarrow Stat14 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\} \supseteq q$

-- Next suppose that $\bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\}$ is not included in q, so that

$\text{Suppose} \Rightarrow \quad Stat15: (\neg \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\}) \subseteq q$
$\langle d \rangle \hookrightarrow Stat15 \Rightarrow \quad d \in \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\} \,\&\, d \notin q$
$\text{Use\_def}(\bigcup) \Rightarrow \quad d \in \{v: u \in \{k \in \mathbb{N} \mid k * n \subseteq m\}, v \in u\}$
$\text{SIMPLF} \Rightarrow \quad Stat16: d \in \{v: k \in \mathbb{N}, v \in k \mid k * n \subseteq m\}$
$\langle k_1, v_1 \rangle \hookrightarrow Stat16 \Rightarrow \quad Stat17: v_1 \notin q \,\&\, k_1 \in \mathbb{N} \,\&\, v_1 \in k_1 \,\&\, k_1 * n \subseteq m$
$\langle k_1 \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(k_1)$
$\langle k_1, v_1 \rangle \hookrightarrow T11 \Rightarrow \quad Stat18: \mathcal{O}(v_1)$
$\langle k_1, v_1 \rangle \hookrightarrow T12 \Rightarrow \quad Stat19: k_1 \supseteq v_1$
$\langle q, v_1 \rangle \hookrightarrow T28(\langle Stat17, Stat18, Stat6, Stat19 \rangle) \Rightarrow \quad q \in v_1 \vee q = v_1$
$\langle Stat17 \rangle \text{ ELEM} \Rightarrow \quad q \in k_1$
$\langle q, k_1, n \rangle \hookrightarrow T246 \Rightarrow \quad q * n \in k_1 * n$
$\text{ALGEBRA} \Rightarrow \quad k_1 * n \in \mathbb{N}$
$\langle k_1 * n, m, q * n \rangle \hookrightarrow T249 \Rightarrow \quad Stat20: m - k_1 * n \in m - q * n$
$\langle Stat5, Stat20 \rangle \text{ ELEM} \Rightarrow \quad Stat21: m - k_1 * n \in r$
$\text{Suppose} \Rightarrow \quad Stat22: m - k_1 * n \notin \{m - p * n: p \in \mathbb{N} \mid m \supseteq p * n\}$
$\langle k_1 \rangle \hookrightarrow Stat22 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat23: m - k_1 * n \in \{m - p * n: p \in \mathbb{N} \mid m \supseteq p * n\}$

-- This contradiction proves that $q = \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\}$ as asserted, and then by definition $q = m \textbf{ div } n$ and $m \textbf{ mod } n = r$

$\langle Stat4, Stat23, Stat21 \rangle \text{ ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad q = \bigcup\{k \in \mathbb{N} \mid k * n \subseteq m\}$
$\text{Use\_def}(\textbf{ div }) \Rightarrow \quad Stat24: q = m \textbf{ div } n$
$\text{EQUAL } \langle Stat5 \rangle \Rightarrow \quad Stat25: m \textbf{ div } n \in \mathbb{N} \,\&\, m \supseteq m \textbf{ div } n * n$

-- Stat1: ((m MOD n) in n)

$\text{EQUAL } \langle Stat5, Stat24 \rangle \Rightarrow \quad r = m - m \textbf{ div } n * n$
$\text{Use\_def}(\textbf{ mod }) \Rightarrow \quad Stat26: m \textbf{ mod } n = r$
$\text{EQUAL } \langle Stat10, Stat26 \rangle \Rightarrow \quad Stat27: m \textbf{ mod } n \in n$
$\langle Stat1, Stat25, Stat27 \rangle \text{ ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{QED}$

-- Our next theorem states and generalizes the fact that if the product of two cardinal numbers is zero, one of them must be zero.

Theorem 286 (254) $\mathsf{N} * \mathsf{M} = \emptyset \leftrightarrow \mathsf{N} = \emptyset \vee \mathsf{M} = \emptyset$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}, \mathsf{m}) \Rightarrow \quad (\mathsf{n} * \mathsf{m} = \emptyset \;\&\; \mathsf{n} \neq \emptyset \;\&\; \mathsf{m} \neq \emptyset) \vee (\mathsf{n} = \emptyset \vee \mathsf{m} = \emptyset \;\&\; \#\mathsf{n} * \#\mathsf{m} \neq \emptyset)$
$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T199 \Rightarrow \quad \mathsf{n} * \mathsf{m} = \#\mathsf{n} * \#\mathsf{m}$
$\mathsf{EQUAL} \Rightarrow \quad (\#\mathsf{n} * \#\mathsf{m} = \emptyset \;\&\; \mathsf{n} \neq \emptyset \;\&\; \mathsf{m} \neq \emptyset) \vee (\mathsf{n} = \emptyset \vee \mathsf{m} = \emptyset \;\&\; \#\mathsf{n} * \#\mathsf{m} \neq \emptyset)$

-- For let n and m form a counterexample to our assertion, and first suppose that $\mathsf{is}(\#\mathsf{n} * \#\mathsf{m}, \emptyset)$, but neither n nor m is $\emptyset$, so that these sets contain singletons $\{\mathsf{c}\}$ and $\{\mathsf{d}\}$ respectively. Then by Theorems 83 and 180, $\{\emptyset\}$ is a subset of both $\#\mathsf{n}$ and $\#\mathsf{m}$.

$\mathsf{Suppose} \Rightarrow \quad \#\mathsf{n} * \#\mathsf{m} = \emptyset \;\&\; \mathit{Stat1}:\ \mathsf{n} \neq \emptyset \;\&\; \mathit{Stat2}:\ \mathsf{m} \neq \emptyset$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \{\mathsf{c}\} \subseteq \mathsf{n}$
$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \{\mathsf{d}\} \subseteq \mathsf{m}$
$\langle \{\mathsf{c}\}, \mathsf{n} \rangle \hookrightarrow T144 \Rightarrow \quad \#\{\mathsf{c}\} \subseteq \#\mathsf{n}$
$\langle \{\mathsf{d}\}, \mathsf{m} \rangle \hookrightarrow T144 \Rightarrow \quad \#\{\mathsf{d}\} \subseteq \#\mathsf{m}$
$\langle \mathsf{d} \rangle \hookrightarrow T252 \Rightarrow \quad \mathit{Stat3}:\ \{\emptyset\} \subseteq \#\mathsf{m}$
$\langle \mathsf{c} \rangle \hookrightarrow T252 \Rightarrow \quad \mathit{Stat4}:\ \{\emptyset\} \subseteq \#\mathsf{n}$

-- It follows easily that $[\emptyset, \emptyset]$ is a member of $\#\mathsf{n} \times \#\mathsf{m}$, which must therefore have cardinality at least 1. This contradiction shows that if our theorem is false, one of m and n must be $\emptyset$, while $\#\mathsf{n} * \#\mathsf{m}$ is nonzero.

$\mathsf{Suppose} \Rightarrow \quad [\emptyset, \emptyset] \notin \#\mathsf{n} \times \#\mathsf{m}$
$\mathsf{Use\_def}(\times) \Rightarrow \quad \mathit{Stat5}:\ [\emptyset, \emptyset] \notin \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \#\mathsf{n}, \mathsf{y} \in \#\mathsf{m}\}$
$\langle \emptyset, \emptyset \rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad \mathit{Stat6}:\ \neg([\emptyset, \emptyset] = [\emptyset, \emptyset] \;\&\; \emptyset \in \#\mathsf{n} \;\&\; \emptyset \in \#\mathsf{m})$
$\langle \mathit{Stat3}, \mathit{Stat4}, \mathit{Stat6} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad [\emptyset, \emptyset] \in \#\mathsf{n} \times \#\mathsf{m}$
$\mathsf{ELEM} \Rightarrow \quad \{[\emptyset, \emptyset]\} \subseteq \#\mathsf{n} \times \#\mathsf{m}$
$\mathsf{Set\_monot} \Rightarrow \quad \#\{[\emptyset, \emptyset]\} \subseteq \#(\#\mathsf{n} \times \#\mathsf{m})$
$\langle [\emptyset, \emptyset] \rangle \hookrightarrow T252 \Rightarrow \quad \{\emptyset\} \subseteq \#(\#\mathsf{n} \times \#\mathsf{m})$
$\mathsf{Use\_def}(*) \Rightarrow \quad \{\emptyset\} \subseteq \#\mathsf{n} * \#\mathsf{m}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{n} = \emptyset \vee \mathsf{m} = \emptyset \;\&\; \#\mathsf{n} * \#\mathsf{m} \neq \emptyset$

-- But now our conclusion follows immediately from the fact that $\emptyset * \mathsf{m} = \#(\emptyset \times \#\mathsf{m})$ and $\mathsf{m} * \emptyset$ are both $\emptyset$.

$T161 \Rightarrow \quad \mathsf{Card}(\emptyset)$
$\langle \emptyset \rangle \hookrightarrow T138 \Rightarrow \quad \#\emptyset = \emptyset$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{n} = \emptyset$

$\mathsf{EQUAL} \Rightarrow \quad \#n = \#\emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \emptyset * \#m \neq \emptyset$
$\langle \#m \rangle \hookrightarrow T210 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad m = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \#m = \#\emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \#n * \#\emptyset \neq \emptyset$
$\langle n, \emptyset \rangle \hookrightarrow T199 \Rightarrow \quad n * \emptyset \neq \emptyset$
$\langle n \rangle \hookrightarrow T209 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- The following very easy result tells us that arithmetic subtraction is monotone increasing in its first parameter.

Theorem 287 (255) $N \supseteq M \rightarrow N - K \supseteq M - K$. PROOF:

Suppose_not$(n, m, k) \Rightarrow \quad n \supseteq m \,\&\, n - k \not\supseteq m - k$

-- For otherwise, by definition of subtraction and the monotonicity of cardinality we would have $\neg n \backslash k \supseteq m \backslash k$, which is impossible.

Use_def$(-) \Rightarrow \quad \#(n \backslash k) \not\supseteq \#(m \backslash k)$
$\langle m \backslash k, n \backslash k \rangle \hookrightarrow T144 \Rightarrow \quad n \backslash k \not\supseteq m \backslash k$
ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- Next we show that if m is a subset of m and n is finite, the cardinality of the difference set $n \backslash m$ is the cardinality of the difference of the cardinalities $\#n$ and $\#m$.

Theorem 288 (256) $\mathrm{Finite}(N) \,\&\, N \supseteq M \rightarrow \#(N \backslash M) = \#(\#N \backslash \#M)$. PROOF:

Suppose_not$(n, m) \Rightarrow \quad \mathrm{Finite}(n) \,\&\, n \supseteq m \,\&\, \#(n \backslash m) \neq \#(\#n \backslash \#m)$

-- For let n, m be a counterexample. Since $n \supseteq m$, we must have $\#n \supseteq \#m$. Our aim is to prove that $\#(n \backslash m) + \#m \neq \#(\#n \backslash \#m) + \#m$ and then use cancellation to obtain the stated conclusion of our theorem. We begin by using Theorem 232 twice to derive $\#m + \#(\#n \backslash \#m) = \#m + (n - m)$.

$\langle m, n \rangle \hookrightarrow T144 \Rightarrow \quad \#n \supseteq \#m$
Use_def$(-) \Rightarrow \quad \#(n \backslash m) = n - m$
$\langle m, n \rangle \hookrightarrow T232 \Rightarrow \quad \#n = \#m + (n - m)$
$\mathsf{EQUAL} \Rightarrow \quad \#n = \#m + \#(n \backslash m)$
Use_def$(-) \Rightarrow \quad \#(\#n \backslash \#m) = \#n - \#m$
$\langle \#m, \#n \rangle \hookrightarrow T232 \Rightarrow \quad \#\#n = \#\#m + (\#n - \#m)$

$\mathsf{EQUAL} \Rightarrow \quad \#\#n = \#\#m + \#(\#n \backslash \#m)$
$\langle n \rangle \hookrightarrow T140 \Rightarrow \quad \#\#n = \#n$
$\langle m \rangle \hookrightarrow T140 \Rightarrow \quad \#\#m = \#m$

-- But since all the cardinalities involved in the equation mentioned above are integers, the rules of algebra apply and make it clear that the rule of cancellation applies also. Thus our conclusion follows.

$\mathsf{EQUAL} \Rightarrow \quad \#n = \#m + \#(\#n \backslash \#m)$
$\langle n \rangle \hookrightarrow T166 \Rightarrow \quad \mathsf{Finite}(\#n)$
$\langle n, n \backslash m \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{Finite}(n \backslash m)$
$\langle n, m \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{Finite}(m)$
$\langle \#n, \#n \backslash \#m \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{Finite}(\#n \backslash \#m)$
$\langle n \backslash m \rangle \hookrightarrow T166 \Rightarrow \quad \mathsf{Finite}\big(\#(n \backslash m)\big)$
$\langle \#n \backslash \#m \rangle \hookrightarrow T166 \Rightarrow \quad \mathsf{Finite}\big(\#(\#n \backslash \#m)\big)$
$\langle n \backslash m \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}\big(\#(n \backslash m)\big)$
$\langle m \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#m)$
$\langle \#n \backslash \#m \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}\big(\#(\#n \backslash \#m)\big)$
$\langle \#m \rangle \hookrightarrow T179 \Rightarrow \quad \#m \in \mathbb{N}$
$\langle \#(n \backslash m) \rangle \hookrightarrow T179 \Rightarrow \quad \#(n \backslash m) \in \mathbb{N}$
$\langle \#(\#n \backslash \#m) \rangle \hookrightarrow T179 \Rightarrow \quad \#(\#n \backslash \#m) \in \mathbb{N}$
$\mathsf{ALGEBRA} \Rightarrow \quad \#m + \#(n \backslash m) = \#(n \backslash m) + \#m \;\&\; \#m + \#(\#n \backslash \#m) = \#(\#n \backslash \#m) + \#m$
$\mathsf{ELEM} \Rightarrow \quad Stat1: \; \#(n \backslash m) + \#m = \#(\#n \backslash \#m) + \#m$
$\langle Stat1, * \rangle \; \mathsf{ELEM} \Rightarrow \quad n \supseteq n \backslash m \;\&\; \#n \supseteq \#n \backslash \#m$
$\langle \#(n \backslash m), \#(\#n \backslash \#m), \#m \rangle \hookrightarrow T243 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following theorem tells us that for integers an arithmetic subtraction undoes the effect of the corresponding arithmetic addition.

Theorem 289 (257) $\mathsf{N}, \mathsf{M} \in \mathbb{N} \rightarrow \mathsf{N} + \mathsf{M} - \mathsf{M} = \mathsf{N}$. PROOF:

$\mathsf{Suppose\_not}(n, m) \Rightarrow \quad Stat0: \; n, m \in \mathbb{N} \;\&\; n + m - m \neq n$

-- For let two integers n and m be a counterexample to our assertion. It is clear that all the quantities appearing in our theorem are finite, and are cardinals, ordinals, and integers.

$\langle n \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(n) \;\&\; \mathsf{Card}(n)$
$\langle m \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(m) \;\&\; \mathsf{Card}(m)$
$\langle n, m \rangle \hookrightarrow T205 \Rightarrow \quad \mathsf{Finite}(n \cup m)$

$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{n} + \mathsf{m} \in \mathbb{N}$
$\langle \mathsf{n} + \mathsf{m} \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(\mathsf{n} + \mathsf{m})\ \&\ \mathsf{Card}(\mathsf{n} + \mathsf{m})$

-- By definition of addition and subtraction, the negative of our assertion translates into the first set-theoretic inequality seen below, which can be rewritten in the successive forms seen below.

$\mathsf{Use\_def}(+) \Rightarrow \quad \mathsf{Finite}\big(\#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\})\big)\ \&\ \#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}) - \mathsf{m} \neq \mathsf{n}$
$\mathsf{Use\_def}(-) \Rightarrow \quad \#\big(\#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}) \backslash \mathsf{m}\big) \neq \mathsf{n}$
$\langle \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\} \rangle \hookrightarrow T166 \Rightarrow$
$\quad \mathsf{Finite}(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\})$
$\langle \mathsf{m} \rangle \hookrightarrow T138 \Rightarrow \quad \mathsf{m} = \#\mathsf{m}$
$\langle \mathsf{n} \rangle \hookrightarrow T138 \Rightarrow \quad Stat1 : \ \mathsf{n} = \#\mathsf{n}$
$\mathsf{EQUAL} \Rightarrow \quad \#\big(\#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}) \backslash \#\mathsf{m}\big) \neq \mathsf{n}$
$\langle \mathsf{m}, 1 \rangle \hookrightarrow T193 \Rightarrow \quad \#\mathsf{m} = \#(\mathsf{m} \times \{1\})$
$\mathsf{Use\_def}(\times) \Rightarrow \quad \#\mathsf{m} = \#\{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathsf{m}, \mathsf{y} \in \{1\}\}$
$\mathsf{SIMPLF} \Rightarrow \quad \#\{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathsf{m}, \mathsf{y} \in \{1\}\} = \#\{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}$
$\mathsf{EQUAL} \Rightarrow \quad \#\big(\#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}) \backslash \#\{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}\big) \neq \mathsf{n}$
$\langle \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}, \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\} \rangle \hookrightarrow T256([Stat0, \cap]) \Rightarrow$
$\quad \#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\} \setminus \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\}) \neq \mathsf{n}$

-- But since the first two sets appearing in this last inequality are disjoint, our inequality reduces to $\#\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \neq \mathsf{n}$, which is obviously impossible. Hence our theorem is proved.

$T183 \Rightarrow \quad 1 \neq \emptyset$
$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T186 \Rightarrow \quad \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\} \setminus \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{m}\} =$
$\quad \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\}$
$\mathsf{EQUAL} \Rightarrow \quad Stat2 : \ \#\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \neq \mathsf{n}$
$\langle \mathsf{n}, \emptyset \rangle \hookrightarrow T193 \Rightarrow \quad \#\mathsf{n} = \#(\mathsf{n} \times \{\emptyset\})$
$\mathsf{Use\_def}(\times) \Rightarrow \quad Stat3 : \ \#\mathsf{n} = \#\{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathsf{n}, \mathsf{y} \in \{\emptyset\}\}$
$\mathsf{SIMPLF} \Rightarrow \quad Stat4 : \ \#\{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathsf{n}, \mathsf{y} \in \{\emptyset\}\} = \#\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\}$
$\langle Stat2, Stat1, Stat4, Stat3, * \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following theorem states the fact that integer addition is strictly monotone in its first argument.

Theorem 290 (258) $\mathsf{N}, \mathsf{M}, \mathsf{K} \in \mathbb{N} \to (\mathsf{N} \supseteq \mathsf{M} \leftrightarrow \mathsf{N} + \mathsf{K} \supseteq \mathsf{M} + \mathsf{K})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}, \mathsf{m}, \mathsf{k}) \Rightarrow \quad \mathsf{n}, \mathsf{m}, \mathsf{k} \in \mathbb{N}\ \&\ \neg(\mathsf{n} \supseteq \mathsf{m} \leftrightarrow \mathsf{n} + \mathsf{k} \supseteq \mathsf{m} + \mathsf{k})$

-- Suppose that n, m, k are a counterexample to our assertion. If $\mathsf{n} \supseteq \mathsf{m}$ our conclusion follows immediately from the definition of addition and the principle of set monotonicity.

Suppose $\Rightarrow$ $\mathsf{n} \supseteq \mathsf{m}$ & $\mathsf{n} + \mathsf{k} \not\supseteq \mathsf{m} + \mathsf{k}$

Use_def$(+)$ $\Rightarrow$ $\#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{k}\}) \not\supseteq \#(\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{k}\})$

$\langle \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{k}\}, \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{k}\} \rangle \hookrightarrow$ *T144* $\Rightarrow$

$\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{k}\} \not\supseteq \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{m}\} \cup \{[\mathsf{x}, 1] : \mathsf{x} \in \mathsf{k}\}$

Set_monot $\Rightarrow$ $\{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{n}\} \supseteq \{[\mathsf{x}, \emptyset] : \mathsf{x} \in \mathsf{m}\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{n} \not\supseteq \mathsf{m}$ & $\mathsf{n} + \mathsf{k} \supseteq \mathsf{m} + \mathsf{k}$

-- Thus we must have $\mathsf{n} + \mathsf{k} \supseteq \mathsf{m} + \mathsf{k}$ but $\neg \mathsf{n} \supseteq \mathsf{m}$, which since addition is (nonstrictly) monotone is impossible by Theorem 251.

$\langle \mathsf{n} + \mathsf{k}, \mathsf{m} + \mathsf{k}, \mathsf{k} \rangle \hookrightarrow$ *T255* $\Rightarrow$ $\mathsf{n} + \mathsf{k} - \mathsf{k} \supseteq \mathsf{m} + \mathsf{k} - \mathsf{k}$

$\langle \mathsf{n}, \mathsf{k} \rangle \hookrightarrow$ *T251* $\Rightarrow$ $\mathsf{n} \supseteq \mathsf{m} + \mathsf{k} - \mathsf{k}$

$\langle \mathsf{m}, \mathsf{k} \rangle \hookrightarrow$ *T251* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next, very easy result states the inverse relationship between arithmetic addition and subtraction in a very general but slightly unusual way.

Theorem 291 (259) $\mathsf{N} \supseteq \mathsf{M} \rightarrow \#\mathsf{N} = \#\mathsf{M} + \#(\mathsf{N} \backslash \mathsf{M})$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathsf{n} \supseteq \mathsf{m}$ & $\#\mathsf{n} \neq \#\mathsf{m} + \#(\mathsf{n} \backslash \mathsf{m})$

-- For if not we must have # n = # m PLUS # (n-m), and so using the definition of subtraction we find a contradiction With Theorem 232.

$\langle \mathsf{m}, \mathsf{n} \rangle \hookrightarrow$ *T232* $\Rightarrow$ $\#\mathsf{n} = \#\mathsf{m} + (\mathsf{n} - \mathsf{m})$

Use_def$(-)$ $\Rightarrow$ $\mathsf{n} - \mathsf{m} = \#(\mathsf{n} \backslash \mathsf{m})$

EQUAL $\Rightarrow$ $\#(\mathsf{n} - \mathsf{m}) = \#\#(\mathsf{n} \backslash \mathsf{m})$

$\langle \mathsf{n} \backslash \mathsf{m} \rangle \hookrightarrow$ *T140* $\Rightarrow$ $\#(\mathsf{n} - \mathsf{m}) = \#(\mathsf{n} \backslash \mathsf{m})$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next result states an elementary arithmetic relationship between integer subtraction and addition, but only in the case in which all the quantities involved are non-negative.

Theorem 292 (260) $\mathsf{N}, \mathsf{M}, \mathsf{K} \in \mathbb{N}$ & $\mathsf{N} \supseteq \mathsf{M} \rightarrow \mathsf{N} + \mathsf{K} - (\mathsf{M} + \mathsf{K}) = \mathsf{N} - \mathsf{M}$. PROOF:

Suppose_not$(\mathsf{n}, \mathsf{m}, \mathsf{k})$ $\Rightarrow$ $\mathsf{n}, \mathsf{m}, \mathsf{k} \in \mathbb{N}$ & $\mathsf{n} \supseteq \mathsf{m}$ & $\mathsf{n} + \mathsf{k} - (\mathsf{m} + \mathsf{k}) \neq \mathsf{n} - \mathsf{m}$

-- For suppose that n, m, k are a couterexample to our assertion. Since all the quantities involved are integers, we can reassociate to write $\mathsf{n}-\mathsf{m}+(\mathsf{m}+\mathsf{k})$ as $\mathsf{n}-\mathsf{m}+\mathsf{m}+\mathsf{k}$, and then use Theorem 233 to reduce this last quantity to $\mathsf{n}+\mathsf{k}$.

$\langle \mathsf{n},\mathsf{k}\rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{n}+\mathsf{k}\in\mathbb{N}$
$\langle \mathsf{m},\mathsf{k}\rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{m}+\mathsf{k}\in\mathbb{N}$
$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{n}-\mathsf{m}\in\mathbb{N}$
$\langle \mathsf{n}+\mathsf{k},\mathsf{m}+\mathsf{k}\rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{n}+\mathsf{k}-(\mathsf{m}+\mathsf{k})\in\mathbb{N}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{n}-\mathsf{m}+(\mathsf{m}+\mathsf{k})=(\mathsf{n}-\mathsf{m}+\mathsf{m})+\mathsf{k}$
$\langle \mathsf{n}\rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{n}=\#\mathsf{n}$
$\langle \mathsf{n}+\mathsf{k}\rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{n}+\mathsf{k}=\#(\mathsf{n}+\mathsf{k})$
$\langle \mathsf{m},\mathsf{n}\rangle \hookrightarrow T233 \Rightarrow \quad \mathsf{n}-\mathsf{m}+\mathsf{m}=\mathsf{n}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n}-\mathsf{m}+(\mathsf{m}+\mathsf{k})=\mathsf{n}+\mathsf{k}$
$\langle \mathsf{m},\mathsf{n},\mathsf{k}\rangle \hookrightarrow T244 \Rightarrow \quad \mathsf{n}+\mathsf{k}\supseteq\mathsf{m}+\mathsf{k}$

-- Now, subtracting $\mathsf{m}+\mathsf{k}$ from $\mathsf{n}+\mathsf{k}$ and then immediately adding it back, we find that

$$\mathsf{n}-\mathsf{m}+(\mathsf{m}+\mathsf{k})-(\mathsf{m}+\mathsf{k})+(\mathsf{m}+\mathsf{k})=\mathsf{n}+\mathsf{k}$$

so that our conclusion follows immediately from Theorem 243.

$\langle \mathsf{m}+\mathsf{k},\mathsf{n}+\mathsf{k}\rangle \hookrightarrow T233 \Rightarrow \quad \mathsf{n}+\mathsf{k}-(\mathsf{m}+\mathsf{k})+(\mathsf{m}+\mathsf{k})=\mathsf{n}+\mathsf{k}$
$\langle \mathsf{n}+\mathsf{k}-(\mathsf{m}+\mathsf{k}),\mathsf{n}-\mathsf{m},\mathsf{m}+\mathsf{k}\rangle \hookrightarrow T243 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we restate theorem 232 into form useful independent of which of the two sets n and m has the larger cardinality.

Theorem 293 (261) $\mathsf{N},\mathsf{M}\in\mathbb{N}\rightarrow\mathsf{N}=\mathsf{M}+(\mathsf{N}-\mathsf{M})\vee\mathsf{N}=\mathsf{M}-(\mathsf{M}-\mathsf{N})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m}) \Rightarrow \quad \mathsf{n},\mathsf{m}\in\mathbb{N}\ \&\ \neg\big(\mathsf{n}=\mathsf{m}+(\mathsf{n}-\mathsf{m})\vee\mathsf{n}=\mathsf{m}-(\mathsf{m}-\mathsf{n})\big)$

-- Since one of the two ordinals n and m must include the other, our conclusion follows immediately by two applications of Theorem 232 and one of Theorem 251.

$\langle \mathsf{n}\rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{n}=\#\mathsf{n}\ \&\ \mathcal{O}(\mathsf{n})$
$\langle \mathsf{m}\rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{m}=\#\mathsf{m}\ \&\ \mathcal{O}(\mathsf{m})$
$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T26 \Rightarrow \quad \mathsf{n}\subseteq\mathsf{m}\vee\mathsf{m}\subseteq\mathsf{n}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{m}\subseteq\mathsf{n}$
$\langle \mathsf{m},\mathsf{n}\rangle \hookrightarrow T232 \Rightarrow \quad \mathsf{n}=\#\mathsf{m}+(\mathsf{n}-\mathsf{m})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{n}\subseteq\mathsf{m}$
$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T232 \Rightarrow \quad \mathsf{m}=\#\mathsf{n}+(\mathsf{m}-\mathsf{n})$

EQUAL $\Rightarrow$ $\mathsf{m} = \mathsf{n} + (\mathsf{m} - \mathsf{n})$
ALGEBRA $\Rightarrow$ $\mathsf{m} - \mathsf{n} \in \mathbb{N}$
$\langle \mathsf{n}, \mathsf{m} - \mathsf{n} \rangle \hookrightarrow T251 \Rightarrow$ $\mathsf{n} = \mathsf{n} + (\mathsf{m} - \mathsf{n}) - (\mathsf{m} - \mathsf{n})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that the arithmetic increment of any set n by 1, is the cardinality of the immediate successor of n.

Theorem 294 (264) $\mathsf{N} + 1 = \#\mathsf{next}(\mathsf{N})$. PROOF:

-- For, if not, we could take an n whose unitary increment $\mathsf{n} + 1$ differs from the cardinality $\#\mathsf{next}(\mathsf{n})$ of its immediate successor.

Suppose_not$(\mathsf{n}) \Rightarrow$ *Stat1* : $\mathsf{n} + 1 \neq \#\mathsf{next}(\mathsf{n})$

-- But on the other hand, it follows from the theorems $\mathsf{N} + \mathsf{M} = \mathsf{N} + \#\mathsf{M}$ and $\#\{\mathsf{M}\} = \{\emptyset\}$, exploiting the definitions of 1 and next and elementary reasoning, that $\mathsf{n} + \{\mathsf{n}\} = \mathsf{n} + \#\{\mathsf{n}\} = \mathsf{n} + 1$.

$\langle \mathsf{n}, \{\mathsf{n}\} \rangle \hookrightarrow T195 \Rightarrow$ *Stat2* : $\mathsf{n} + \{\mathsf{n}\} = \mathsf{n} + \#\{\mathsf{n}\}$
$\langle \mathsf{n} \rangle \hookrightarrow T252 \Rightarrow$ *Stat3* : $\#\{\mathsf{n}\} = \{\emptyset\}$
Use_def$(1) \Rightarrow$ $1 = \mathsf{next}(\emptyset)$
Use_def$(\mathsf{next}) \Rightarrow$ *Stat4* : $1 = \emptyset \cup \{\emptyset\}$
$\langle$*Stat3*, *Stat4*$\rangle$ ELEM $\Rightarrow$ *Stat5* : $\mathsf{n} \cap \{\mathsf{n}\} = \emptyset \;\&\; \#\{\mathsf{n}\} = 1$
Use_def$(\mathsf{next}) \Rightarrow$ *Stat6* : $\#\mathsf{next}(\mathsf{n}) = \#(\mathsf{n} \cup \{\mathsf{n}\})$

-- Since n and $\{\mathsf{n}\}$ are disjoint, we also have $\mathsf{n} + \{\mathsf{n}\} = \#\mathsf{next}(\mathsf{n})$.

$\langle \mathsf{n}, \{\mathsf{n}\} \rangle \hookrightarrow T189 \Rightarrow$ *Stat7* : $\mathsf{n} + \{\mathsf{n}\} = \#(\mathsf{n} \cup \{\mathsf{n}\})$

-- This leads to a contradiction which proves our assertion.

EQUAL $\langle$*Stat6*, *Stat7*, *Stat2*, *Stat5*$\rangle \Rightarrow$ *Stat8* : $\#\mathsf{next}(\mathsf{n}) = \mathsf{n} + 1$
$\langle$*Stat8*, *Stat1*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- An immediate corollary of Theorem abc is that the increment by 1 of any unsigned integer N, and the immediate successor of N, are the same:

Theorem 295 (265) $\mathsf{N} \in \mathbb{N} \rightarrow \mathsf{N} + 1 = \mathsf{next}(\mathsf{N})$. PROOF:

-- For, if not, we could take an unsigned integer n whose unitary increment $\mathsf{n}+1$ differs from its immediate successor $\mathsf{next}(\mathsf{n})$.

$\mathsf{Suppose\_not}(\mathsf{n}) \Rightarrow \quad Stat1: \ \mathsf{n} \in \mathbb{N} \ \& \ \mathsf{n}+1 \neq \mathsf{next}(\mathsf{n})$

-- However, the unsigned integers are precisely the finite cardinals; so n must be a finite cardinal.

$\langle \mathsf{n} \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(\mathsf{n}) \ \& \ \mathsf{Card}(\mathsf{n})$

-- Therefore $\mathsf{next}(\mathsf{n})$ is a finite cardinal, and accordingly it equals its own cardinality. By Theorem abc, this leads us to a contradiction which proves our theorem.

$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad \mathcal{O}(\mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}\big(\mathsf{next}(\mathsf{n})\big)$
$\langle \mathsf{n} \rangle \hookrightarrow T173 \Rightarrow \quad \mathsf{Finite}\big(\mathsf{next}(\mathsf{n})\big)$
$\langle \mathsf{next}(\mathsf{n}) \rangle \hookrightarrow T178 \Rightarrow \quad \mathsf{Card}\big(\mathsf{next}(\mathsf{n})\big)$
$\langle \mathsf{next}(\mathsf{n}) \rangle \hookrightarrow T138 \Rightarrow \quad Stat2: \ \mathsf{next}(\mathsf{n}) = \#\mathsf{next}(\mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow T264 \Rightarrow \quad Stat3: \ \mathsf{n}+1 = \#\mathsf{next}(\mathsf{n})$
$\langle Stat1, Stat2, Stat3 \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we prove that the union-set of any finite set of unsigned integers is an unsigned integer (actually, a member of M unless M is 0), ...

Theorem 296 (266) $\mathsf{M} \subseteq \mathbb{N} \ \& \ \mathsf{Finite}(\mathsf{M}) \rightarrow (\mathsf{M} \neq \emptyset \longrightarrow \bigcup \mathsf{M} \in \mathsf{M}) \ \& \ \bigcup \mathsf{M} \in \mathbb{N}$. PROOF:

-- Assume that n is a counterexample to our assertion. Then if the first clause of our assertion is violated, it is violated by some inclusion-minimal $\mathsf{m} \subseteq \mathsf{n}$.

$\mathsf{Suppose\_not}(\mathsf{n}) \Rightarrow \quad Stat1: \ \mathsf{n} \subseteq \mathbb{N} \ \& \ \mathsf{Finite}(\mathsf{n}) \ \& \ (\mathsf{n} \neq \emptyset \ \& \ \bigcup \mathsf{n} \notin \mathsf{n}) \vee \bigcup \mathsf{n} \notin \mathbb{N}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{n} \neq \emptyset \ \& \ \bigcup \mathsf{n} \notin \mathsf{n}$
$\mathsf{APPLY} \ \langle \mathsf{m}_\Theta : \ \mathsf{m} \rangle \ \mathsf{finite\_induction}\big(\mathsf{n} \mapsto \mathsf{n}, \mathsf{P}(\mathsf{y}) \mapsto (\mathsf{y} \neq \emptyset \ \& \ \bigcup \mathsf{y} \notin \mathsf{y})\big) \Rightarrow$
$\quad Stat2: \ \mathsf{m} \subseteq \mathsf{n} \ \& \ \mathsf{m} \neq \emptyset \ \& \ \bigcup \mathsf{m} \notin \mathsf{m} \ \& \ \langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \longrightarrow \neg(\mathsf{k} \neq \emptyset \ \& \ \bigcup \mathsf{k} \notin \mathsf{k}) \rangle$

-- Since m is nonempty, $\mathbf{arb}(\mathsf{m})$ is a member of m, and also $\bigcup \mathsf{m} = \mathbf{arb}(\mathsf{m}) \cup \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$ by Theorem 185. Since $\bigcup \mathsf{m} \notin \mathsf{m}$, it follows that m cannot be a singleton, and so it must have some second integer member c.

$\langle Stat2 \rangle \ \mathsf{ELEM} \Rightarrow \quad Stat3: \ \langle \forall \mathsf{k} \subseteq \mathsf{m} \mid \mathsf{k} \neq \mathsf{m} \longrightarrow \neg(\mathsf{k} \neq \emptyset \ \& \ \bigcup \mathsf{k} \notin \mathsf{k}) \rangle \ \& \ \mathsf{m} = \{\mathbf{arb}(\mathsf{m})\} \cup (\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$
$\langle \mathsf{m} \rangle \hookrightarrow T185 \Rightarrow \quad Stat4: \ \bigcup \emptyset = \emptyset \ \& \ \bigcup \mathsf{m} = \mathbf{arb}(\mathsf{m}) \cup \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$
$\mathsf{Suppose} \Rightarrow \quad Stat5: \ \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\} = \emptyset$
$\mathsf{EQUAL} \ \langle Stat5, Stat4 \rangle \Rightarrow \quad Stat6: \ \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}) = \emptyset$

$\langle Stat6, Stat4, Stat3, Stat2\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat7:\ \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\} \neq \emptyset$
$\langle \mathsf{c}\rangle \hookrightarrow Stat7 \Rightarrow$ $Stat8:\ \mathsf{c} \in \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}$
$\langle Stat2, Stat8, Stat1\rangle$ ELEM $\Rightarrow$ $Stat9:\ \mathsf{c}, \mathbf{arb}(\mathsf{m}) \in \mathbb{N} \,\&\, \mathsf{c} \neq \mathbf{arb}(\mathsf{m}) \,\&\, \mathsf{c} \notin \mathbf{arb}(\mathsf{m})$
$T179 \Rightarrow$ $Stat10:\ \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, \mathsf{c}\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(\mathsf{c})$
$\langle \mathbb{N}, \mathbf{arb}(\mathsf{m})\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(\mathbf{arb}(\mathsf{m}))$

-- Since c is not smaller than $\mathbf{arb}(\mathsf{m})$, $\mathbf{arb}(\mathsf{m})$ must be a member of c, and hence a member of a member of $\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}$, i. e. a member of $\bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$.

$\langle \mathsf{c}, \mathbf{arb}(\mathsf{m})\rangle \hookrightarrow T28(\langle Stat9\rangle) \Rightarrow$ $Stat11:\ \mathbf{arb}(\mathsf{m}) \in \mathsf{c}$
Suppose $\Rightarrow$ $\mathbf{arb}(\mathsf{m}) \notin \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$
Use_def $(\bigcup) \Rightarrow$ $Stat12:\ \mathbf{arb}(\mathsf{m}) \notin \{\mathsf{x} :\ \mathsf{y} \in \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}, \mathsf{x} \in \mathsf{y}\}$
$\langle \mathsf{c}, \mathbf{arb}(\mathsf{m})\rangle \hookrightarrow Stat12 \Rightarrow$ $Stat13:\ \neg(\mathsf{c} \in \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{c})\} \,\&\, \mathbf{arb}(\mathsf{m}) \in \mathsf{c})$
$\langle Stat11, Stat8, Stat13\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat14:\ \mathbf{arb}(\mathsf{m}) \in \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$

-- Since $\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}$ is a non-empty subset of m, It follows by Stat15 2 that $\bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}) \in \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}$, and so is an ordinal, whose member $\mathbf{arb}(\mathsf{m})$ must therefore be a subset of it. Thus, by statement 4, we have $\bigcup \mathsf{m} = \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$.

$\langle \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}\rangle \hookrightarrow Stat3(\langle Stat8\rangle) \Rightarrow$ $Stat16:\ \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}) \in \mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}$
$\langle \mathbb{N}, \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})\rangle \hookrightarrow T11(\langle Stat2, Stat1, Stat16, Stat10\rangle) \Rightarrow$ $Stat17:\ \mathcal{O}(\bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}))$
$\langle \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\}), \mathbf{arb}(\mathsf{m})\rangle \hookrightarrow T12([Stat17, Stat14]) \Rightarrow$ $Stat18:\ \mathbf{arb}(\mathsf{m}) \subseteq \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$
$\langle Stat4, Stat18\rangle$ ELEM $\Rightarrow$ $Stat19:\ \bigcup \mathsf{m} = \bigcup(\mathsf{m} \setminus \{\mathbf{arb}(\mathsf{m})\})$

-- But this contradicts Stat15 2. Therefore our intial suppositon must be false, i. e. either n is 0 or $\bigcup \mathsf{n} \in \mathsf{n}$. If $\bigcup \mathsf{n} \in \mathsf{n}$ the assertion of our theorem is clearly satisfied, so only the case $n = 0$ need be considered. But in this case our assertion is obvious. Hence our theorem is valid in all cases.

$\langle Stat19, Stat16, Stat2\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat20:\ \mathsf{n} = \emptyset \vee \bigcup \mathsf{n} \in \mathsf{n}$
Suppose $\Rightarrow$ $\mathsf{n} = \emptyset \,\&\, \bigcup \mathsf{n} \notin \mathbb{N}$
$T185 \Rightarrow$ $\bigcup \emptyset = \emptyset$
EQUAL $\Rightarrow$ $\bigcup \mathsf{n} = \emptyset$
$T183 \Rightarrow$ $Stat21:\ \emptyset \in \mathbb{N}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat22:\ \mathsf{n} = \emptyset \rightarrow \bigcup \mathsf{n} \in \mathbb{N}$
$\langle Stat22, Stat20, Stat1\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It follows easily from the preceding result the union-set of any unsigned integer m is an unsigned integer (in fact, it is always the predecessor of m):

Theorem 297 (267) $\mathsf{M} \in \mathbb{N} \rightarrow (\bigcup \mathsf{M} \in \mathbb{N} \;\&\; \bigcup \mathsf{M} \subseteq \mathsf{M}) \;\&\; (\mathsf{M} \neq \emptyset \rightarrow \bigcup \mathsf{M} \in \mathsf{M})$. Proof:

-- Assume that m is a counterexample to our statement, and recall that the unsigned integers are simply the finite cardinals. It follows that m is a finite ordinal included in $\mathbb{N}$, and then (by excluding the trivial case $m = 0$) we get a contradiction with prior lemmas.

Suppose_not(m) $\Rightarrow$ $Stat1: \; \mathsf{m} \in \mathbb{N} \;\&\; \neg(\bigcup \mathsf{m} \subseteq \mathsf{m} \;\&\; \bigcup \mathsf{m} \in \mathbb{N} \;\&\; (\mathsf{m} \neq \emptyset \rightarrow \bigcup \mathsf{m} \in \mathsf{m}))$
$\langle \mathsf{m} \rangle \hookrightarrow T179 \Rightarrow$ $\mathcal{O}(\mathbb{N}) \;\&\; \mathsf{Card}(\mathsf{m}) \;\&\; \mathsf{Finite}(\mathsf{m})$
Use_def(Card) $\Rightarrow$ $\mathcal{O}(\mathsf{m})$
$\langle \mathbb{N}, \mathsf{m} \rangle \hookrightarrow T12 \Rightarrow$ $Stat2: \; \mathsf{m} \subseteq \mathbb{N}$
$\langle \mathsf{m} \rangle \hookrightarrow T266 \Rightarrow$ $Stat3: \; \bigcup \mathsf{m} \in \mathbb{N} \;\&\; (\mathsf{m} \neq \emptyset \rightarrow \bigcup \mathsf{m} \in \mathsf{m})$
$\langle Stat3, Stat1 \rangle$ ELEM $\Rightarrow$ $Stat4: \; \bigcup \mathsf{m} \not\subseteq \mathsf{m}$
Suppose $\Rightarrow$ $\mathsf{m} = \emptyset$
$T185 \Rightarrow$ $Stat5: \; \bigcup \emptyset = \emptyset$
EQUAL $\Rightarrow$ $Stat6: \; \bigcup \mathsf{m} = \emptyset$
$\langle Stat4, Stat6 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat7: \; \mathsf{m} \neq \emptyset$
$\langle Stat7, Stat3, Stat2 \rangle$ ELEM $\Rightarrow$ $\bigcup \mathsf{m} \in \mathsf{m} \;\&\; \bigcup \mathsf{m} \in \mathbb{N}$
$\langle \mathbb{N}, \bigcup \mathsf{m} \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(\bigcup \mathsf{m})$
$\langle \mathsf{m}, \bigcup \mathsf{m} \rangle \hookrightarrow T12 \Rightarrow$ $Stat8: \; \bigcup \mathsf{m} \subseteq \mathsf{m}$
$\langle Stat8, Stat4 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ Qed

-- Next we prove that for any non-zero unsigned integer m, $\bigcup \mathsf{m}$ is the immediate predecessor of m:

Theorem 298 (268) $\mathsf{M} \in \mathbb{N} \;\&\; \mathsf{M} \neq \emptyset \rightarrow \mathsf{M} = \bigcup \mathsf{M} + 1$. Proof:

Suppose_not(m) $\Rightarrow$ $Stat1: \; \mathsf{m} \in \mathbb{N} \;\&\; \mathsf{m} \neq \emptyset \;\&\; \mathsf{m} \neq \bigcup \mathsf{m} + 1$

-- For let m be a counterexample to our assertion. $\bigcup \mathsf{m}$ is an integer less than m by Theorem 267, and $\bigcup \mathsf{m} + 1$ is the sucessor $\mathsf{next}(\bigcup \mathsf{m})$ of $\bigcup \mathsf{m}$. Thus $\bigcup \mathsf{m} \cup \{\bigcup \mathsf{m}\}$ must be a subset of m, so if our theorem is false it cannot be a superset of m.

$T179 \Rightarrow$ $\mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, \mathsf{m} \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(\mathsf{m})$
$\langle \mathsf{m} \rangle \hookrightarrow T267 \Rightarrow$ $Stat2: \; \bigcup \mathsf{m} \in \mathbb{N} \;\&\; \bigcup \mathsf{m} \in \mathsf{m}$
$\langle \mathsf{m}, \bigcup \mathsf{m} \rangle \hookrightarrow T12 \Rightarrow$ $Stat3: \; \bigcup \mathsf{m} \subseteq \mathsf{m}$
$\langle \bigcup \mathsf{m} \rangle \hookrightarrow T265 \Rightarrow$ $\bigcup \mathsf{m} + 1 = \mathsf{next}(\bigcup \mathsf{m})$
Use_def(next) $\Rightarrow$ $Stat4: \; \bigcup \mathsf{m} + 1 = \bigcup \mathsf{m} \cup \{\bigcup \mathsf{m}\}$
$\langle Stat1, Stat4, Stat2, Stat3 \rangle$ ELEM $\Rightarrow$ $Stat5: \; \mathsf{m} \not\subseteq \bigcup \mathsf{m} \cup \{\bigcup \mathsf{m}\}$

-- But in this case there must exist and integer c less than m but not in $\bigcup\mathsf{m} \cup \{\bigcup\mathsf{m}\}$. Theorem 235 tells us that such a c must be a subset of $\bigcup\mathsf{m}$, and hence either a member of $\bigcup\mathsf{m}$. Hovewver both these cases are clearly impossible, so our theorem is proved.

$\langle\mathsf{c}\rangle\hookrightarrow Stat5 \Rightarrow \quad Stat6:\ \mathsf{c}\in\mathsf{m}\ \&\ \mathsf{c}\notin\bigcup\mathsf{m}\cup\{\bigcup\mathsf{m}\}$
$\langle\mathsf{m},\mathsf{c}\rangle\hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{c})$
$\langle\mathsf{m},\bigcup\mathsf{m}\rangle\hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\bigcup\mathsf{m})$
$\langle\mathsf{m}\rangle\hookrightarrow T235 \Rightarrow \quad Stat7:\ \langle\forall\mathsf{x}\in\mathsf{m}\,|\,\mathsf{x}\subseteq\bigcup\mathsf{m}\rangle$
$\langle\mathsf{c}\rangle\hookrightarrow Stat7(\langle Stat6\rangle) \Rightarrow \quad \mathsf{c}\subseteq\bigcup\mathsf{m}$
$\langle\bigcup\mathsf{m},\mathsf{c}\rangle\hookrightarrow T31(\langle Stat6\rangle) \Rightarrow \quad Stat8:\ \mathsf{c}\in\bigcup\mathsf{m}\vee\mathsf{c}=\bigcup\mathsf{m}$
$\langle Stat8, Stat6\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that the sum of any two integers n and c is the set-theoretic union of n with the set of sums of n with the members of c.

Theorem 299 (269) $\mathsf{X},\mathsf{Y}\in\mathbb{N}\rightarrow\mathsf{X}+\mathsf{Y}=\mathsf{X}\cup\{\mathsf{X}+\mathsf{u}:\ \mathsf{u}\in\mathsf{Y}\}$. PROOF:

Suppose_not$(\mathsf{n},\mathsf{c}) \Rightarrow \quad \mathsf{n},\mathsf{c}\in\mathbb{N}\ \&\ Stat1:\ \mathsf{n}+\mathsf{c}\neq\mathsf{n}\cup\{\mathsf{n}+\mathsf{u}:\ \mathsf{u}\in\mathsf{c}\}$

-- For let n, c be a counterexample to our theorem. It is easily seen that c cannot be $\emptyset$. There must be an element d which is in one but not both of the sets appearing in the inequality seen above.

$\langle\mathsf{d}\rangle\hookrightarrow Stat1 \Rightarrow \quad \neg(\mathsf{d}\in\mathsf{n}+\mathsf{c}\leftrightarrow\mathsf{d}\in\mathsf{n}\cup\{\mathsf{n}+\mathsf{u}:\ \mathsf{u}\in\mathsf{c}\})$
Suppose $\Rightarrow \quad \mathsf{c}=\emptyset$
Suppose $\Rightarrow \quad Stat2:\ \{\mathsf{n}+\mathsf{u}:\ \mathsf{u}\in\emptyset\}\neq\emptyset$
$\langle\mathsf{a}\rangle\hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow \quad \{\mathsf{n}+\mathsf{u}:\ \mathsf{u}\in\emptyset\}=\emptyset$
EQUAL $\Rightarrow \quad \mathsf{n}+\emptyset\neq\mathsf{n}$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow \quad \mathsf{c}\neq\emptyset$

$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle\mathbb{N},\mathsf{c}\rangle\hookrightarrow T12 \Rightarrow \quad \mathsf{c}\subseteq\mathbb{N}$
ALGEBRA $\Rightarrow \quad \mathsf{n}+\mathsf{c}\in\mathbb{N}$
$\langle\mathbb{N},\mathsf{n}+\mathsf{c}\rangle\hookrightarrow T12 \Rightarrow \quad \mathsf{n}+\mathsf{c}\subseteq\mathbb{N}$

-- All of our quantities are integers, and clearly $\mathsf{n}+\mathsf{c}$ is greater than n. First suppose that d is in the second of these sets but not in the first, so that $\mathsf{d}\notin\mathsf{n}+\mathsf{c}$. Then either $\mathsf{d}\in\mathsf{n}$, or there exists an e in c such that $\mathsf{d}=\mathsf{n}+\mathsf{e}$. The first of these cases is impossible, since it would imply $\mathsf{d}\in\mathsf{n}+\mathsf{c}$. But in the second case it is clear that $\mathsf{d}=\mathsf{n}+\mathsf{e}$ is less than, hence a member of, $\mathsf{n}+\mathsf{c}$, a contradiction ruling out this case. Hence we can be sure that d is in the first of our sets but not in the second.

$\langle \mathsf{n} \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(\mathsf{n})$
$\langle \mathsf{c} \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(\mathsf{c})$
$\langle \mathsf{n}+\mathsf{c} \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(\mathsf{n}+\mathsf{c})$
$\langle \mathsf{n},\mathsf{c} \rangle \hookrightarrow T240 \Rightarrow \quad \mathsf{n} \in \mathsf{n}+\mathsf{c}$
$\langle \mathsf{n}+\mathsf{c},\mathsf{n} \rangle \hookrightarrow T31 \Rightarrow \quad \mathsf{n} \subseteq \mathsf{n}+\mathsf{c}$
Suppose $\Rightarrow \quad Stat3: \; \mathsf{d} \in \{\mathsf{n}+\mathsf{u} : \mathsf{u} \in \mathsf{c}\}$
$\langle \mathsf{e} \rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{e} \in \mathsf{c} \;\&\; \mathsf{e} \in \mathbb{N} \;\&\; \mathsf{d} = \mathsf{n}+\mathsf{e}$
$\langle \mathsf{n},\mathsf{c},\mathsf{e} \rangle \hookrightarrow T242 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathsf{d} \in \mathsf{n}+\mathsf{c} \;\&\; \mathsf{d} \in \mathbb{N} \;\&\; Stat4: \; \mathsf{d} \notin \{\mathsf{n}+\mathsf{u} : \mathsf{u} \in \mathsf{c}\}$

-- Since we can now be sure that $\mathsf{d} \notin \mathsf{n}$, we have $\mathsf{d} = \mathsf{d}-\mathsf{n}+\mathsf{n}$. $\mathsf{d}-\mathsf{n}$ is less than c since d is less than $\mathsf{n}+\mathsf{c}$. Hence

$\langle \mathsf{d} \rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{d} = \#\mathsf{d} \;\&\; \mathcal{O}(\mathsf{d})$
$\langle \mathsf{n},\mathsf{d} \rangle \hookrightarrow T32 \Rightarrow \quad \mathsf{d} \supseteq \mathsf{n}$
$\langle \mathsf{n},\mathsf{d} \rangle \hookrightarrow T233 \Rightarrow \quad \mathsf{d} = \mathsf{d}-\mathsf{n}+\mathsf{n}$
$\langle \mathsf{d},\mathsf{n} \rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{d}-\mathsf{n} \in \mathbb{N}$
$\langle \mathsf{d}-\mathsf{n} \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(\mathsf{d}-\mathsf{n})$

-- If we suppose that $\mathsf{d}-\mathsf{n} \notin \mathsf{c}$, then ...

Suppose $\Rightarrow \quad \mathsf{d}-\mathsf{n} \notin \mathsf{c}$
$\langle \mathsf{c},\mathsf{d}-\mathsf{n} \rangle \hookrightarrow T32 \Rightarrow \quad \mathsf{d}-\mathsf{n} \supseteq \mathsf{c}$
$\langle \mathsf{c},\mathsf{d}-\mathsf{n},\mathsf{n} \rangle \hookrightarrow T244 \Rightarrow \quad \mathsf{d}-\mathsf{n}+\mathsf{n} \supseteq \mathsf{c}+\mathsf{n}$
EQUAL $\Rightarrow \quad \mathsf{d} \supseteq \mathsf{c}+\mathsf{n}$

-- ... ALGEBRA gives us $\mathsf{d} \supseteq \mathsf{n}+\mathsf{c}$, leading to a contradiction in this case, because we already know that $\mathsf{d} \in \mathsf{n}+\mathsf{c}$.

ALGEBRA $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathsf{d}-\mathsf{n} \in \mathsf{c}$

-- However, $\mathsf{d}-\mathsf{n} \in \mathsf{c}$ cannot hold either. From this contradiction we get the desired conclusion.

$\langle \mathsf{d}-\mathsf{n} \rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{n}+(\mathsf{d}-\mathsf{n}) \neq \mathsf{d}$
ALGEBRA $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad$ QED

-- Our next result expresses the sum of any two integers x, y as the union of x with the set of all the members of y with x. This is very similar to the preceding result, but reverses the sum which appears inside the setformer in the conclusion of the theorem.

Theorem 300 (270) $\mathsf{X},\mathsf{Y} \in \mathbb{N} \rightarrow \mathsf{X}+\mathsf{Y} = \mathsf{X} \cup \{\mathsf{u}+\mathsf{X} : \mathsf{u} \in \mathsf{Y}\}$. PROOF:

$\mathsf{Suppose\_not}(x,y) \Rightarrow \quad x,y \in \mathbb{N} \,\&\, x+y \neq x \cup \{u+x :\ u \in y\}$

-- Suppose that x, y constitute a counterexample to our theorem, and apply the preceding Theorem to x, y. Since it is easily seen that $\{x+u :\ u \in y\} = \{u+x :\ u \in y\}$, our assertion follows immediately.

$\langle x,y\rangle \hookrightarrow T269 \Rightarrow \quad x+y = x \cup \{x+u :\ u \in y\}$
$\mathsf{Suppose} \Rightarrow \quad Stat1 :\ \{x+u :\ u \in y\} \neq \{u+x :\ u \in y\}$
$\langle c\rangle \hookrightarrow Stat1 \Rightarrow \quad c \in y \,\&\, x+c \neq c+x$
$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N},y\rangle \hookrightarrow T12 \Rightarrow \quad c \in \mathbb{N}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \{x+u :\ u \in y\} = \{u+x :\ u \in y\}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- Next we prove a property of the integer subtraction operator very close to that given by Theorem 233, namely $n = m + (n - m)$ if m is no larger than n.

Theorem 301 (271) $X \in \mathbb{N} \,\&\, X \in Y \vee X = Y \rightarrow Y = X + (Y - X)$. PROOF:

$\mathsf{Suppose\_not}(m,n) \Rightarrow \quad Stat1 :\ n \in \mathbb{N} \,\&\, m \in n \vee m = n \,\&\, n \neq m + (n - m)$

-- For all our quantities are integers, so the conclusion of Theorem 233 is equivalent to that of the present theorem.

$\langle n\rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(n)$
$\langle n,m\rangle \hookrightarrow T12 \Rightarrow \quad m \subseteq n$
$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N},n\rangle \hookrightarrow T12 \Rightarrow \quad n \subseteq \mathbb{N}$
$\mathsf{ELEM} \Rightarrow \quad m \in \mathbb{N}$
$\langle n\rangle \hookrightarrow T180 \Rightarrow \quad n = \#n$
$\langle m,n\rangle \hookrightarrow T233 \Rightarrow \quad \#n = n - m + m$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- Next we show that for unsigned integers $m + n$ is never smaller than n.

Theorem 302 (272) $M,N \in \mathbb{N} \rightarrow M + N \notin N$. PROOF:

$\mathsf{Suppose\_not}(m,n) \Rightarrow \quad Stat1 :\ m,n \in \mathbb{N} \,\&\, m + n \in n$

-- Since by the monotonicity of addition $\mathsf{n} \subseteq \mathsf{m} + \mathsf{n}$.

$T183 \Rightarrow \quad \emptyset \in \mathbb{N}$
ELEM $\Rightarrow \quad \emptyset \subseteq \mathsf{m}$
$\langle \emptyset, \mathsf{m}, \mathsf{n} \rangle \hookrightarrow T244 \Rightarrow \quad \emptyset + \mathsf{n} \subseteq \mathsf{m} + \mathsf{n}$
ALGEBRA $\Rightarrow \quad \mathsf{n} \subseteq \mathsf{m} + \mathsf{n}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following in-recursive formula gives an alternative characterization of signed integer subtraction:

Theorem 303 (10021) $\mathsf{N}, \mathsf{M} \in \mathbb{N} \to \mathsf{N} - \mathsf{M} = \{\mathsf{k} - \mathsf{M} : \mathsf{k} \in \mathsf{N} \mid \mathsf{M} \in \mathsf{k} \vee \mathsf{M} = \mathsf{k}\}$. PROOF:

Suppose_not$(\mathsf{n}, \mathsf{m}) \Rightarrow \quad \mathsf{n}, \mathsf{m} \in \mathbb{N} \,\&\, Stat0 : \mathsf{n} - \mathsf{m} \neq \{\mathsf{k} - \mathsf{m} : \mathsf{k} \in \mathsf{n} \mid \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$

-- For let n, m be a counterexample to our theorem, so that $\mathsf{n}, \mathsf{m}, \mathsf{n} - \mathsf{m}$ are unsigned integers, hence ordinals, and there exists an a belonging to one and only one of $\mathsf{n} - \mathsf{m}$ and $\{\mathsf{k} - \mathsf{m} : \mathsf{k} \in \mathsf{n} \mid \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$.

ALGEBRA $\Rightarrow \quad \mathsf{n} - \mathsf{m} \in \mathbb{N}$
$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, \mathsf{n} - \mathsf{m} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{n} - \mathsf{m})$
$\langle \mathbb{N}, \mathsf{n} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{n})$
$\langle \mathbb{N}, \mathsf{m} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{m})$
$\langle \mathsf{a} \rangle \hookrightarrow Stat0 \Rightarrow \quad \mathsf{a} \in \mathsf{n} - \mathsf{m} \leftrightarrow \mathsf{a} \notin \{\mathsf{k} - \mathsf{m} : \mathsf{k} \in \mathsf{n} \mid \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$
Suppose $\Rightarrow \quad Stat2 : \mathsf{a} \in \{\mathsf{k} - \mathsf{m} : \mathsf{k} \in \mathsf{n} \mid \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$

-- If a belongs to $\{\mathsf{k} - \mathsf{m} : \mathsf{k} \in \mathsf{n} \mid \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$, then a equals $\mathsf{k} - \mathsf{m}$ for some $\mathsf{k} \in \mathsf{n}$ such that m does not exceed k. By monotonicity of integer subtraction relative to its first parameter, $\mathsf{k} - \mathsf{m}$ is less than or equal to $\mathsf{n} - \mathsf{m}$, and hence equal to it. It readily follows that $\mathsf{m} \in \mathsf{n}$ and $\mathsf{k} = \mathsf{n}$, conflicting with $\mathsf{k} \in \mathsf{n}$.

$\langle \mathsf{k} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{k} - \mathsf{m} \notin \mathsf{n} - \mathsf{m} \,\&\, \mathsf{k} \in \mathsf{n} \,\&\, \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}$
$\langle \mathbb{N}, \mathsf{n} \rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{k} \in \mathbb{N}$
$\langle \mathsf{n}, \mathsf{k} \rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{n} \supseteq \mathsf{k}$
$\langle \mathsf{n}, \mathsf{k}, \mathsf{m} \rangle \hookrightarrow T255 \Rightarrow \quad \mathsf{n} - \mathsf{m} \supseteq \mathsf{k} - \mathsf{m}$
ALGEBRA $\Rightarrow \quad \mathsf{k} - \mathsf{m} \in \mathbb{N}$
$\langle \mathbb{N}, \mathsf{k} - \mathsf{m} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{k} - \mathsf{m})$
$\langle \mathsf{n} - \mathsf{m}, \mathsf{k} - \mathsf{m} \rangle \hookrightarrow T31 \Rightarrow \quad \mathsf{k} - \mathsf{m} = \mathsf{n} - \mathsf{m}$
$\langle \mathsf{m}, \mathsf{k} \rangle \hookrightarrow T271 \Rightarrow \quad \mathsf{k} = \mathsf{m} + (\mathsf{k} - \mathsf{m})$
$\langle \mathsf{m}, \mathsf{n} \rangle \hookrightarrow T271 \Rightarrow \quad \mathsf{n} = \mathsf{m} + (\mathsf{n} - \mathsf{m})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat4}: \ \mathsf{a} \notin \{\mathsf{k}-\mathsf{m}: \ \mathsf{k} \in \mathsf{n} \,|\, \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$

-- We are therefore led to consider the opposite case, that a does not belongs to $\{\mathsf{k}-\mathsf{m}: \ \mathsf{k} \in \mathsf{n} \,|\, \mathsf{m} \in \mathsf{k} \vee \mathsf{m} = \mathsf{k}\}$, so that $\mathsf{a} = \mathsf{m}+\mathsf{a}-\mathsf{m}$, and the number $\mathsf{m}+\mathsf{a}$ either does not belong to n or is exceeded by m.

$\langle \mathbb{N}, \mathsf{n}-\mathsf{m}\rangle \hookrightarrow \mathit{T12} \Rightarrow \quad \mathsf{a} \in \mathbb{N}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{a}+\mathsf{m} = \mathsf{m}+\mathsf{a} \ \& \ \mathsf{m}+\mathsf{a} \in \mathbb{N}$
$\langle \mathsf{a}, \mathsf{m}\rangle \hookrightarrow \mathit{T251} \Rightarrow \quad \mathsf{a} = \mathsf{a}+\mathsf{m}-\mathsf{m}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{a} = \mathsf{m}+\mathsf{a}-\mathsf{m}$
$\langle \mathsf{m}+\mathsf{a}\rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad \mathsf{m}+\mathsf{a} \notin \mathsf{n} \vee \neg(\mathsf{m} \in \mathsf{m}+\mathsf{a} \vee \mathsf{m} = \mathsf{m}+\mathsf{a})$

-- We must discard the possibility that m exceeds $\mathsf{m}+\mathsf{a}$.

$\mathsf{Suppose} \Rightarrow \quad \neg(\mathsf{m} \in \mathsf{m}+\mathsf{a} \vee \mathsf{m} = \mathsf{m}+\mathsf{a})$
$\langle \mathbb{N}, \mathsf{m}+\mathsf{a}\rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(\mathsf{m}+\mathsf{a})$
$\langle \mathsf{m}, \mathsf{m}+\mathsf{a}\rangle \hookrightarrow \mathit{T28} \Rightarrow \quad \mathsf{m}+\mathsf{a} \in \mathsf{m}$
$\langle \mathsf{a}, \mathsf{m}\rangle \hookrightarrow \mathit{T272} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{m}+\mathsf{a} \notin \mathsf{n}$

-- We must also discard the possibility $\mathsf{m}+\mathsf{a} \notin \mathsf{n}$, because it enters into direct contradiction with the derivable statements $\mathsf{m}+\mathsf{a} \in \mathsf{m}+(\mathsf{n}-\mathsf{m})$ and $\mathsf{m}+(\mathsf{n}-\mathsf{m}) = \mathsf{n}$. This enables us the draw the desired conclusion.

$\langle \mathsf{m}, \mathsf{n}-\mathsf{m}, \mathsf{a}\rangle \hookrightarrow \mathit{T242} \Rightarrow \quad \mathsf{m}+\mathsf{a} \in \mathsf{m}+(\mathsf{n}-\mathsf{m})$
$\langle \mathsf{n}\rangle \hookrightarrow \mathit{T229} \Rightarrow \quad \mathsf{n}-\mathsf{n} = \emptyset$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{n} = \mathsf{m}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n}-\mathsf{m} = \emptyset$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{n} \neq \mathsf{m}$
$\langle \mathsf{n}, \mathsf{m}\rangle \hookrightarrow \mathit{T28} \Rightarrow \quad \mathsf{n} \in \mathsf{m} \vee \mathsf{m} \in \mathsf{n}$
$\langle \mathsf{m}, \mathsf{n}\rangle \hookrightarrow \mathit{T10020} \Rightarrow \quad \mathsf{m} \in \mathsf{n}$
$\langle \mathsf{m}, \mathsf{n}\rangle \hookrightarrow \mathit{T271} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- Our next goal is to show that every subset of the set $\mathbb{N}$ of unsigned integers which is closed with respect to predecessor formation either belongs to $\mathbb{N}$ or coincides with $\mathbb{N}$. As a preliminary lemma, we show that for any set x closed with respect to predecessor formation, if an element j of $\mathbb{N}$ does not belong to x, then no integer h greater than j belongs to x.

Theorem 304 (10056) $\mathsf{J} \in \mathbb{N} \backslash \mathsf{X} \ \& \ \langle \forall \mathsf{i} \,|\, \mathsf{next}(\mathsf{i}) \in \mathsf{X} \rightarrow \mathsf{i} \in \mathsf{X}\rangle \rightarrow \{\mathsf{h} \in \mathbb{N} \,|\, \mathsf{h} \notin \mathsf{J} \ \& \ \mathsf{h} \in \mathsf{X}\} = \emptyset$. PROOF:

Suppose_not$(\mathsf{x},\mathsf{j}) \Rightarrow$ $\mathsf{x} \subseteq \mathbb{N}$ & $\mathsf{j} \in \mathbb{N} \backslash \mathsf{x}$ & $\{\mathsf{h} \in \mathbb{N} \mid \mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}\} \neq \emptyset$ & *Stat1*: $\langle \forall \mathsf{i} \mid \mathsf{next}(\mathsf{i}) \in \mathsf{x} \rightarrow \mathsf{i} \in \mathsf{x} \rangle$

-- For, suppose that x,j contradict the statement just made and let h be the first integer which is not smaller than j and belongs to $\mathbb{N}$.

Loc_def $\Rightarrow$ $\mathsf{h} = \mathbf{arb}(\{\mathsf{h} \in \mathbb{N} \mid \mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}\})$
ELEM $\Rightarrow$ $\mathsf{h} \cap \{\mathsf{h} \in \mathbb{N} \mid \mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}\} = \emptyset$ & *Stat2*: $\mathsf{h} \in \{\mathsf{h} \in \mathbb{N} \mid \mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}\}$
$\langle\,\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{h} \in \mathbb{N}$ & $\mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}$
T179 $\Rightarrow$ $\mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, \mathsf{j} \rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathsf{j})$
$\langle \mathbb{N}, \mathsf{h} \rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathsf{h})$

-- By T28, either j,h coincide or one of j,h belongs to the other. We must exclude the case $\mathsf{j} = \mathsf{h}$, because h belongs to j and j does not; moreover we know that $\mathsf{h} \notin \mathsf{j}$. Hence $\mathsf{j} \in \mathsf{h}$ and h has the form $\mathsf{next}(\mathsf{k})$ for some unsigned integer k.

$\langle \mathsf{j}, \mathsf{h} \rangle \hookrightarrow$ *T28* $\Rightarrow$ $\mathsf{j} \in \mathsf{h}$
T182 $\Rightarrow$ $\mathcal{O}(\emptyset)$ & $1 \in \mathbb{N}$ & $\mathsf{Card}(1)$
$\langle \mathsf{h}, 1 \rangle \hookrightarrow$ *T239* $\Rightarrow$ $\mathsf{h} - 1 \in \mathbb{N}$
Use_def(Card) $\Rightarrow$ $\mathcal{O}(1)$
Suppose $\Rightarrow$ $\mathsf{h} \neq \mathsf{next}(\mathsf{h} - 1)$
$\langle \mathsf{h}, \emptyset \rangle \hookrightarrow$ *T28* $\Rightarrow$ $\emptyset \in \mathsf{h}$
Use_def(1) $\Rightarrow$ $1 = \mathsf{next}(\emptyset)$
Use_def(next) $\Rightarrow$ $1 \subseteq \mathsf{h}$
$\langle 1, \mathsf{h} \rangle \hookrightarrow$ *T233* $\Rightarrow$ $\#\mathsf{h} = \mathsf{h} - 1 + 1$
$\langle \mathsf{h} \rangle \hookrightarrow$ *T180* $\Rightarrow$ $\mathsf{h} = \mathsf{h} - 1 + 1$
$\langle \mathsf{h} - 1 \rangle \hookrightarrow$ *T265* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{h} = \mathsf{next}(\mathsf{h} - 1)$

-- It turns out that the immediate predecessor of h belongs to h as well as to the set $\{\mathsf{h} \in \mathbb{N} \mid \mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}\}$. However, this contradicts the minimality criterion by which h was selected by means of the **arb** operator. Indeed, we derive $\mathsf{next}(\mathsf{h} - 1) \subseteq \mathsf{j}$, which is incompatible with the earlier proved facts $\mathsf{j} \in \mathsf{h}$, $\mathsf{h} = \mathsf{next}(\mathsf{h} - 1)$.

$\langle \mathsf{h} - 1 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathsf{h} - 1 \in \mathsf{x}$
Use_def(next) $\Rightarrow$ *Stat3*: $\mathsf{h} - 1 \notin \{\mathsf{h} \in \mathbb{N} \mid \mathsf{h} \notin \mathsf{j}$ & $\mathsf{h} \in \mathsf{x}\}$
$\langle\,\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\mathsf{h} - 1 \in \mathsf{j}$
$\langle \mathsf{j}, \mathsf{h} - 1 \rangle \hookrightarrow$ *T12* $\Rightarrow$ $(\mathsf{h} - 1) \cap \{\mathsf{h} - 1\} \subseteq \mathsf{j}$

-- This contradiction leads to the desired conclusion.

Use_def(next) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now we can easily draw the corollary we were aiming at.

Theorem 305 (10057) $X \subseteq \mathbb{N} \,\&\, \langle \forall i \mid next(i) \in X \rightarrow i \in X \rangle \leftrightarrow X \in next(\mathbb{N})$. PROOF:

Suppose_not(x) $\Rightarrow$ $x \subseteq \mathbb{N} \,\&\, \langle \forall i \mid next(i) \in x \rightarrow i \in x \rangle \leftrightarrow x \notin next(\mathbb{N})$

-- Implication in one direction is trivial; for the converse ...

T179 $\Rightarrow$ $\mathcal{O}(\mathbb{N})$
Suppose $\Rightarrow$ $\neg(x \subseteq \mathbb{N} \,\&\, \langle \forall i \mid next(i) \in x \rightarrow i \in x \rangle) \,\&\, x \in next(\mathbb{N})$
Use_def(next) $\Rightarrow$ $x \in \mathbb{N} \vee x = \mathbb{N}$
$\langle \mathbb{N}, x \rangle \hookrightarrow T12 \Rightarrow$ $x \subseteq \mathbb{N}$
ELEM $\Rightarrow$ *Stat1* : $\neg\langle \forall i \mid next(i) \in x \rightarrow i \in x \rangle$
Suppose $\Rightarrow$ $\neg\mathcal{O}(x)$
Suppose $\Rightarrow$ $x = \mathbb{N}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $x \in \mathbb{N}$
$\langle \mathbb{N}, x \rangle \hookrightarrow T11 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathcal{O}(x)$
$\langle i \rangle \hookrightarrow Stat1 \Rightarrow$ $next(i) \in x \,\&\, i \notin x$
$\langle x, next(i) \rangle \hookrightarrow T12 \Rightarrow$ $next(i) \subseteq x$
Use_def(next) $\Rightarrow$ false; Discharge $\Rightarrow$ $x \subseteq \mathbb{N} \,\&\, x \notin next(\mathbb{N}) \,\&\, \langle \forall i \mid next(i) \in x \rightarrow i \in x \rangle$

-- ...reasoning by contradiction, suppose that x is included in the set $\mathbb{N}$ of all unsigned integers and does not belong to $next(\mathbb{N})$, i. e., x differs from $\mathbb{N}$ and does not belong to $\mathbb{N}$. Consider a minimally chosen element j of $\mathbb{N}\backslash j$. Clearly, j must differ from x. However, the preceding lemma makes it impossible to find an element of x which does not belong to j. On the other hand, insofar an an element of the ordinal $\mathbb{N}$, j must be a subset of $\mathbb{N}$; hence, any element h of j not belonging to x must be an element of $\mathbb{N}$ too, violating the supposed minimality of j. Thus we are led to the contradiction $j \neq x$, $j = x$, which proves our statement.

Loc_def $\Rightarrow$ $j = \mathbf{arb}(\mathbb{N}\backslash x)$
Use_def(next) $\Rightarrow$ $j \in \mathbb{N}\backslash x \,\&\, j \cap (\mathbb{N}\backslash x) = \emptyset \,\&\,$ *Stat2* : $x \neq j$
$\langle h \rangle \hookrightarrow Stat2 \Rightarrow$ $h \in x \leftrightarrow h \notin j$
Suppose $\Rightarrow$ $h \notin j \,\&\, h \in x$
Suppose $\Rightarrow$ *Stat3* : $h \notin \{h \in \mathbb{N} \mid h \notin j \,\&\, h \in x\}$
$\langle\, \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $h \in \{h \in \mathbb{N} \mid h \notin j \,\&\, h \in x\}$
$\langle j, x \rangle \hookrightarrow T10056 \Rightarrow$ false; Discharge $\Rightarrow$ $h \in j \,\&\, h \notin x$
$\langle \mathbb{N}, j \rangle \hookrightarrow T12 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 306 (10058) $\{X, Y\} \subseteq \mathsf{next}(\mathbb{N}) \,\&\, \#X \in \mathsf{next}(Y) \rightarrow (X = Y \,\&\, Y = \mathbb{N}) \vee X \in \mathsf{next}(Y) \cap \mathbb{N}$. PROOF:

Suppose_not$(x, y) \Rightarrow \quad \{x, y\} \subseteq \mathsf{next}(\mathbb{N}) \,\&\, \#x \in \mathsf{next}(y) \,\&\, x \neq y \vee y \neq \mathbb{N} \,\&\, x \notin \mathsf{next}(y) \cap \mathbb{N}$
T179 $\Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N} \rangle \hookrightarrow$ *T29* $\Rightarrow \quad \mathcal{O}(\mathsf{next}(\mathbb{N}))$
$\langle \mathsf{next}(\mathbb{N}), x \rangle \hookrightarrow$ *T11* $\Rightarrow \quad \mathcal{O}(x)$
$\langle x \rangle \hookrightarrow$ *T10057* $\Rightarrow \quad x \subseteq \mathbb{N}$
$\langle \mathsf{next}(\mathbb{N}), y \rangle \hookrightarrow$ *T11* $\Rightarrow \quad \mathcal{O}(y)$
$\langle \mathbb{N}, x \rangle \hookrightarrow$ *T24* $\Rightarrow \quad x = \mathbb{N} \vee x = \mathbf{arb}(\mathbb{N} \backslash x)$
Suppose $\Rightarrow \quad x \neq \#x$
$\langle x \rangle \hookrightarrow$ *T138* $\Rightarrow \quad \neg \mathsf{Card}(x)$
Suppose $\Rightarrow \quad x = \mathbb{N}$
T181 $\Rightarrow \quad \mathsf{Card}(\mathbb{N})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow \quad x \in \mathbb{N}$
$\langle x \rangle \hookrightarrow$ *T179* $\Rightarrow$ false; Discharge $\Rightarrow \quad x = \#x$
EQUAL $\Rightarrow \quad x \in \mathsf{next}(y)$
Use_def(next) $\Rightarrow \quad x \in y \vee x = y \,\&\, x \notin y \,\&\, x \neq y$
$\langle y \rangle \hookrightarrow$ *T10057* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

8 Some elementary results concerning finite sequences

-- In this section we develop various elementary properties of finite sequences (of arbitrary elements), constructs useful in a variety of analytic and combinatorial situations. and their conjuctions.

-- Finite sequences

DEF 24*a*. $\quad \mathsf{Fin\_seqs}(X) \quad =_{\mathrm{Def}} \quad \{f \subseteq \mathbb{N} \times X \mid \mathsf{Svm}(f) \,\&\, \mathbf{domain}(f) \in \mathbb{N}\}$

-- Sequence concatenation

DEF 24*b*. $\quad \mathsf{concat}(X, Y) \quad =_{\mathrm{Def}} \quad X \cup \left\{ \left[x^{[1]} + \#X, x^{[2]} \right] : x \in Y \right\}$

-- Subsequences

DEF 24*c*. $\quad \mathsf{Subseqs}(X) \quad =_{\mathrm{Def}} \quad \left\{ X \bullet h : h \subseteq \mathbb{N} \times \mathbb{N} \mid \mathsf{Svm}(h) \,\&\, \left\langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h \upharpoonright i \in h \upharpoonright \mathsf{next}(i) \right\rangle \right\}$

-- Shift operation for sequences

DEF 24*d*. $\quad \mathsf{Shift}(X) \quad =_{\mathrm{Def}} \quad \{[i, X + i] : i \in \mathbb{N}\}$

-- Shifted sequence

DEF 24. $\quad \mathsf{Shifted\_seq}(X, Y) \quad =_{\mathrm{Def}} \quad X \bullet \mathsf{Shift}(Y)$

-- We begin the present collection of lemmas by noting the entirely elementary fact that every element of a finite sequence of elements of a set s is a pair whose first component is an integer and whose second component belongs to s.

Theorem 307 (273) $\mathsf{F} \in \mathsf{Fin\_seqs}(\mathsf{S}) \,\&\, \mathsf{P} \in \mathsf{F} \rightarrow \mathsf{P}^{[1]} \in \mathbb{N} \,\&\, \mathsf{P}^{[2]} \in \mathsf{S}$. Proof:

Suppose_not(f, s, p) $\Rightarrow$ *Stat1*: $\mathsf{f} \in \mathsf{Fin\_seqs}(\mathsf{s}) \,\&\, \mathsf{p} \in \mathsf{f} \,\&\, \neg(\mathsf{p}^{[1]} \in \mathbb{N} \,\&\, \mathsf{p}^{[2]} \in \mathsf{s})$
Use_def(Fin_seqs) $\Rightarrow$ *Stat2*: $\mathsf{f} \in \{\mathsf{f} \subseteq \mathbb{N} \times \mathsf{s} \mid \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{domain}(\mathsf{f}) \in \mathbb{N}\}$
$\langle\,\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3*: $\mathsf{f} \subseteq \mathbb{N} \times \mathsf{s}$
$\langle \mathsf{p}, \mathbb{N}, \mathsf{s}\rangle \hookrightarrow$ *T115*([*Stat1*, *Stat3*]) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we note that the domain of a finite sequence f is just #f. The proof of the present lemma is direct and elementary.

Theorem 308 (274) $\mathsf{F} \in \mathsf{Fin\_seqs}(\mathsf{S}) \rightarrow \mathsf{Finite}(\mathsf{F}) \,\&\, \#\mathsf{F} \in \mathbb{N} \,\&\, \mathbf{domain}(\mathsf{F}) = \#\mathsf{F} \,\&\, \mathsf{Svm}(\mathsf{F}) \,\&\, \mathbf{range}(\mathsf{F}) \subseteq \mathsf{S}$. Proof:

Suppose_not(f, s, y, m) $\Rightarrow$ *Stat0*: $\mathsf{f} \in \mathsf{Fin\_seqs}(\mathsf{s}) \,\&\, \neg\mathsf{Finite}(\mathsf{f}) \vee \#\mathsf{f} \notin \mathbb{N} \vee \mathbf{domain}(\mathsf{f}) \neq \#\mathsf{f} \vee \neg\mathsf{Svm}(\mathsf{f}) \vee \mathbf{range}(\mathsf{F}) \not\subseteq \mathsf{S}$
Use_def(Fin_seqs) $\Rightarrow$ *Stat2*: $\mathsf{f} \in \{\mathsf{f} \subseteq \mathbb{N} \times \mathsf{S} \mid \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{domain}(\mathsf{f}) \in \mathbb{N}\}$
$\langle\,\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{f} \subseteq \mathbb{N} \times \mathsf{s} \,\&\, \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{domain}(\mathsf{f}) \in \mathbb{N}$
$\langle \mathsf{f}\rangle \hookrightarrow$ *T148* $\Rightarrow$ *Stat3*: $\#\mathbf{domain}(\mathsf{f}) = \#\mathsf{f} \,\&\, \mathsf{Svm}(\mathsf{f})$
$\langle \mathbf{domain}(\mathsf{f})\rangle \hookrightarrow$ *T179* $\Rightarrow$ *Stat4*: $\mathsf{Card}(\mathbf{domain}(\mathsf{f})) \,\&\, \mathsf{Finite}(\mathbf{domain}(\mathsf{f}))$
$\langle \mathbf{domain}(\mathsf{f})\rangle \hookrightarrow$ *T138* $\Rightarrow$ *Stat5*: $\mathbf{domain}(\mathsf{f}) = \#\mathbf{domain}(\mathsf{f})$
EQUAL $\langle$*Stat4*, *Stat3*, *Stat5*$\rangle$ $\Rightarrow$ *Stat6*: $\mathsf{Finite}(\#\mathsf{f})$
$\langle \mathsf{f}\rangle \hookrightarrow$ *T130* $\Rightarrow$ $\mathsf{Card}(\#\mathsf{f})$
$\langle \#\mathsf{f}\rangle \hookrightarrow$ *T179* $\Rightarrow$ *Stat7*: $\#\mathsf{f} \in \mathbb{N}$
$\langle \mathsf{f}, \mathbb{N}, \mathsf{s}\rangle \hookrightarrow$ *T116*([*Stat0*, $\cap$]) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Observe that 'Shift' is a parametric function.

Theorem 309 (274*a*) $\mathsf{M} \in \mathbb{N} \rightarrow \mathsf{Svm}\big(\mathsf{Shift}(\mathsf{M})\big)$. Proof:

Suppose_not(m) $\Rightarrow$ $\mathsf{m} \in \mathbb{N} \,\&\, \neg\mathsf{Svm}\big(\mathsf{Shift}(\mathsf{m})\big)$
Use_def(Shift) $\Rightarrow$ $\neg\mathsf{Svm}(\{[\mathsf{i}, \mathsf{m}+\mathsf{i}] : \mathsf{i} \in \mathbb{N}\})$
Use_def(Svm) $\Rightarrow$ *Stat4a*: $\neg\langle\forall \mathsf{p} \in \{[\mathsf{i}, \mathsf{m}+\mathsf{i}] : \mathsf{i} \in \mathbb{N}\}, \mathsf{q} \in \{[\mathsf{i}, \mathsf{m}+\mathsf{i}] : \mathsf{i} \in \mathbb{N}\} \mid \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \rightarrow \mathsf{p} = \mathsf{q}\rangle$
$\langle \mathsf{p}, \mathsf{q}\rangle \hookrightarrow$ *Stat4a* $\Rightarrow$ *Stat5a*: $\mathsf{p}, \mathsf{q} \in \{[\mathsf{i}, \mathsf{m}+\mathsf{i}] : \mathsf{i} \in \mathbb{N}\} \,\&\, \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \,\&\, \mathsf{p} \neq \mathsf{q}$
$\langle \mathsf{i}', \mathsf{iq}\rangle \hookrightarrow$ *Stat5a*([*Stat5a*, $\cap$]) $\Rightarrow$ $\mathsf{i}' \in \mathbb{N} \,\&\, \mathsf{p} = [\mathsf{i}', \mathsf{m}+\mathsf{i}'] \,\&\, \mathsf{iq} \in \mathbb{N} \,\&\, \mathsf{q} = [\mathsf{iq}, \mathsf{m}+\mathsf{iq}]$
$\langle$*Stat5a*, $*\rangle$ ELEM $\Rightarrow$ *Stat6a*: $\mathsf{p} = [\mathsf{i}', \mathsf{m}+\mathsf{i}'] \,\&\, \mathsf{q} = [\mathsf{iq}, \mathsf{m}+\mathsf{iq}] \,\&\, \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \,\&\, \mathsf{p} \neq \mathsf{q}$
$\langle$*Stat6a*$\rangle$ ELEM $\Rightarrow$ $\mathsf{i}' = \mathsf{iq} \,\&\, \mathsf{m}+\mathsf{i}' \neq \mathsf{m}+\mathsf{iq}$
$\langle \mathsf{m}, \mathsf{i}', \mathsf{iq}\rangle \hookrightarrow$ *T243* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 310 (274*b*) $\mathsf{Is\_map}(F) \,\&\, \mathbf{domain}(F) \subseteq \mathbb{N} \rightarrow \mathsf{Shifted\_seq}(F, \emptyset) = F$. PROOF:

Suppose_not(f) ⇒ *Stat0* : $\mathsf{Is\_map}(f) \,\&\, \mathbf{domain}(f) \subseteq \mathbb{N} \,\&\, \mathsf{Shifted\_seq}(f, \emptyset) \neq f$
Use_def(Shifted_seq) ⇒ $f \bullet \mathsf{Shift}(\emptyset) \neq f$
Use_def(Shift) ⇒ $f \bullet \{[i, \emptyset + i] : i \in \mathbb{N}\} \neq f$
Use_def(•) ⇒ $\{[p^{[1]}, q^{[2]}] : p \in \{[i, \emptyset + i] : i \in \mathbb{N}\}, q \in f \,|\, p^{[2]} = q^{[1]}\} \neq f$
SIMPLF ⇒ *Stat1* : $\left\{\left[[i, \emptyset + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, \emptyset + i]^{[2]} = q^{[1]}\right\} \neq f$
$\langle c \rangle \hookrightarrow$ *Stat1* ⇒ $c \in \left\{\left[[i, \emptyset + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, \emptyset + i]^{[2]} = q^{[1]}\right\} \leftrightarrow c \notin f$
Suppose ⇒ *Stat2* : $c \in \left\{\left[[i, \emptyset + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, \emptyset + i]^{[2]} = q^{[1]}\right\}$
$\langle i, q \rangle \hookrightarrow$ *Stat2*($\langle$*Stat2*$\rangle$) ⇒ $c = [i, q^{[2]}] \,\&\, i \in \mathbb{N} \,\&\, q \in f \,\&\, \emptyset + i = q^{[1]}$
ALGEBRA ⇒ $\emptyset + i = i$
$\langle f, q \rangle \hookrightarrow$ *T46*($\langle$*Stat0*$\rangle$) ⇒ false; Discharge ⇒ *Stat3* : $c \notin \left\{\left[[i, \emptyset + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, \emptyset + i]^{[2]} = q^{[1]}\right\} \,\&\, c \in f$
$\langle c^{[1]}, c \rangle \hookrightarrow$ *Stat3*($\langle$*Stat3*$\rangle$) ⇒ $c \neq [c^{[1]}, c^{[2]}] \vee c^{[1]} \notin \mathbb{N} \vee \emptyset + c^{[1]} \neq c^{[1]}$
$\langle c, f \rangle \hookrightarrow$ *T55*([*Stat0*, ∩]) ⇒ $c^{[1]} \in \mathbb{N}$
$\langle f, c \rangle \hookrightarrow$ *T46*([*Stat0*, ∩]) ⇒ $\emptyset + c^{[1]} \neq c^{[1]}$
ALGEBRA ⇒ false; Discharge ⇒ QED

-- The following alternative characterization of the left-shift operator can be useful, e. g., in the proof of Theorem 277 below.

Theorem 311 (275*a*) $M \in \mathbb{N} \,\&\, \mathbf{domain}(F) \subseteq \mathbb{N} \rightarrow \mathsf{Shifted\_seq}(F, M) = \{[x^{[1]} - M, x^{[2]}] : x \in F \,|\, M \in x^{[1]} \vee M = x^{[1]}\}$. PROOF:

Suppose_not(m, f) ⇒ *Stat0* : $m \in \mathbb{N} \,\&\, \mathbf{domain}(f) \subseteq \mathbb{N} \,\&\, \mathsf{Shifted\_seq}(f, m) \neq \{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$
Use_def(Shifted_seq) ⇒ $f \bullet \mathsf{Shift}(m) \neq \{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$
Use_def(Shift) ⇒ $f \bullet \{[i, m + i] : i \in \mathbb{N}\} \neq \{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$
Use_def(•) ⇒ $\{[p^{[1]}, q^{[2]}] : p \in \{[i, m + i] : i \in \mathbb{N}\}, q \in f \,|\, p^{[2]} = q^{[1]}\} \neq$
$\{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$
SIMPLF ⇒ *Stat1* :
$\left\{\left[[i, m + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, m + i]^{[2]} = q^{[1]}\right\} \neq$
$\{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$
$\langle c \rangle \hookrightarrow$ *Stat1* ⇒
$c \in \left\{\left[[i, m + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, m + i]^{[2]} = q^{[1]}\right\} \leftrightarrow c \notin \{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$
Suppose ⇒ *Stat3* :
$c \in \left\{\left[[i, m + i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \,|\, [i, m + i]^{[2]} = q^{[1]}\right\} \,\&$
$c \notin \{[x^{[1]} - m, x^{[2]}] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\}$

$\langle i,q,q\rangle \hookrightarrow \mathit{Stat3}(\langle \mathit{Stat3}\rangle) \Rightarrow \quad c = [i, q^{[2]}] \;\&\; i \in \mathbb{N} \;\&$

$m + i = q^{[1]} \;\&\; c \neq [q^{[1]} - m, q^{[2]}] \vee (m \notin q^{[1]} \;\&\; m \neq q^{[1]})$

ALGEBRA $\Rightarrow \quad m + i \in \mathbb{N} \;\&\; m + i - m = i + m - m$

$\langle i,m\rangle \hookrightarrow \mathit{T251}([\mathit{Stat0}, \cap]) \Rightarrow \quad i + m - m = i$

EQUAL $\langle \mathit{Stat3}\rangle \Rightarrow \quad q^{[1]} \in \mathbb{N} \;\&\; m \notin q^{[1]} \;\&\; m \neq q^{[1]}$

Suppose $\Rightarrow \quad i = \emptyset$

ALGEBRA $\Rightarrow \quad m + \emptyset = m$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow \quad i \neq \emptyset$

$\langle m,i\rangle \hookrightarrow \mathit{T240} \Rightarrow$ false; Discharge $\Rightarrow$

$\mathit{Stat4}:\; c \in \{[x^{[1]} - m, x^{[2]}] : x \in f \mid m \in x^{[1]} \vee m = x^{[1]}\} \;\&\; c \notin \left\{\left[[i, m+i]^{[1]}, q^{[2]}\right] : i \in \mathbb{N}, q \in f \mid [i, m+i]^{[2]} = q^{[1]}\right\}$

$\langle x, x^{[1]} - m, x\rangle \hookrightarrow \mathit{Stat4}(\langle \mathit{Stat4}\rangle) \Rightarrow \quad c = [x^{[1]} - m, x^{[2]}] \;\&\; x \in f \;\&$

$m \in x^{[1]} \vee m = x^{[1]} \;\&\; x^{[1]} - m \notin \mathbb{N} \vee m + (x^{[1]} - m) \neq x^{[1]}$

$\langle x,f\rangle \hookrightarrow \mathit{T55} \Rightarrow \quad x^{[1]} \in \mathbb{N}$

ALGEBRA $\Rightarrow \quad x^{[1]} - m \in \mathbb{N}$

$\langle \mathit{Stat4}, *\rangle$ ELEM $\Rightarrow \quad m + (x^{[1]} - m) \neq x^{[1]}$

$\langle m, x^{[1]}\rangle \hookrightarrow \mathit{T271}([\mathit{Stat0}, \cap]) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma will be used just below to show that the concatenation of two finite sequences is also a finite sequence. Among others, it shows that the left-shift operator defined above always produces a single-valued map.

Theorem 312 (275) $F, G \in \mathrm{Fin\_seqs}(S) \;\&\; M \in \mathbb{N} \rightarrow$
$\mathrm{Svm}(\{[x^{[1]} + \#F, x^{[2]}] : x \in G\}) \;\&\; \mathrm{Svm}(\mathrm{Shifted\_seq}(F, M)) \;\&\; \mathbf{domain}(F) \cap \mathbf{domain}(\{[x^{[1]} + \#F, x^{[2]}] : x \in G\}) = \emptyset$. PROOF:

Suppose_not$(f, s, g, m) \Rightarrow \quad f, g \in \mathrm{Fin_seqs}(s) \;\&$

$m \in \mathbb{N} \;\&\; \neg\mathrm{Svm}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) \vee \neg\mathrm{Svm}(\mathrm{Shifted\_seq}(f, m)) \vee \mathbf{domain}(f) \cap \mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) \neq \emptyset$

Use_def(Fin_seqs) $\Rightarrow \quad \mathit{Stat1}:\; f \in \{h : h \subseteq \mathbb{N} \times S \mid \mathrm{Svm}(h) \;\&\; \mathbf{domain}(h) \in \mathbb{N}\} \;\&\; \mathit{Stat2}:\; g \in \{h : h \subseteq \mathbb{N} \times S \mid \mathrm{Svm}(h) \;\&\; \mathbf{domain}(h) \in \mathbb{N}\}$

$\langle h_0\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad h_0 \subseteq \mathbb{N} \times S \;\&\; h_0 = f \;\&\; \mathrm{Svm}(h_0) \;\&\; \mathbf{domain}(h_0) \in \mathbb{N}$

$\langle h_2\rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad h_2 \subseteq \mathbb{N} \times S \;\&\; h_2 = g \;\&\; \mathrm{Svm}(h_2) \;\&\; \mathbf{domain}(h_2) \in \mathbb{N}$

EQUAL $\Rightarrow \quad f \subseteq \mathbb{N} \times S \;\&\; \mathrm{Svm}(f) \;\&\; \mathbf{domain}(f) \in \mathbb{N}$

EQUAL $\Rightarrow \quad g \subseteq \mathbb{N} \times S \;\&\; \mathrm{Svm}(g) \;\&\; \mathbf{domain}(g) \in \mathbb{N}$

$\langle f\rangle \hookrightarrow \mathit{T274} \Rightarrow \quad \mathbf{domain}(f) = \#f \;\&\; \#f \in \mathbb{N}$

Suppose $\Rightarrow \quad \neg\mathrm{Svm}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\})$

APPLY $\langle x_\Theta : c, y_\Theta : d\rangle$ Svm_test$(a(x) \mapsto x^{[1]} + \#f, b(x) \mapsto x^{[2]}, s \mapsto g) \Rightarrow$

$(c, d \in g \;\&\; c^{[1]} + \#f = d^{[1]} + \#f \;\&\; c^{[2]} \neq d^{[2]}) \vee \mathrm{Svm}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\})$

ELEM $\Rightarrow \quad c, d \in \mathbb{N} \times s$

$\langle c, \mathbb{N}, s\rangle \hookrightarrow \mathit{T115} \Rightarrow \quad c^{[1]} \in \mathbb{N}$

$\langle d, \mathbb{N}, s\rangle \hookrightarrow \mathit{T115} \Rightarrow \quad d^{[1]} \in \mathbb{N}$

Use_def(Svm) $\Rightarrow \quad \mathit{Stat5}:\; \langle \forall x \in g, y \in g \mid x^{[1]} = y^{[1]} \rightarrow x = y\rangle$

$\langle c, d\rangle \hookrightarrow Stat5 \Rightarrow \quad Stat6:\ c^{[1]} = d^{[1]} \to c = d$
$\langle c^{[1]}, d^{[1]}, \#f\rangle \hookrightarrow T243 \Rightarrow \quad Stat7:\ c = d$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{Svm}\left(\mathsf{Shifted\_seq}(f, m)\right) \vee \mathbf{domain}(f) \cap \mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) \neq \emptyset$
Suppose $\Rightarrow$ $Stat8:\ \mathbf{domain}(f) \cap \mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) \neq \emptyset$
$\langle c_1\rangle \hookrightarrow Stat8 \Rightarrow \quad c_1 \in \mathbf{domain}(f) \cap \mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\})$
EQUAL $\Rightarrow$ $c_1 \in \#f \cap \mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\})$
Use_def(**domain**) $\Rightarrow$ $c_1 \in \#f \cap \{x^{[1]} : x \in \{[x^{[1]} + \#f, x^{[2]}] : x \in g\}\}$
SIMPLF $\Rightarrow$ $Stat9:\ c_1 \in \#f \cap \left\{[x^{[1]} + \#f, x^{[2]}]^{[1]} : x \in g\right\}$
ELEM $\Rightarrow$ $Stat10:\ c_1 \in \left\{[x^{[1]} + \#f, x^{[2]}]^{[1]} : x \in g\right\}\ \&\ Stat11:\ c_1 \in \#f$
$\langle d_1\rangle \hookrightarrow Stat10 \Rightarrow \quad Stat12:\ d_1 \in g\ \&\ c_1 = \left[d_1^{[1]} + \#f, d_1^{[2]}\right]^{[1]}$
ELEM $\Rightarrow$ $Stat13:\ c_1 = d_1^{[1]} + \#f$
ELEM $\Rightarrow$ $d_1 \in \mathbb{N} \times s$
$\langle d_1, \mathbb{N}, s\rangle \hookrightarrow T115 \Rightarrow \quad d_1^{[1]} \in \mathbb{N}$
$\langle d_1^{[1]}, \#f\rangle \hookrightarrow T272 \Rightarrow \quad Stat14:\ d_1^{[1]} + \#f \notin \#f$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{Svm}\left(\mathsf{Shifted\_seq}(f, m)\right)$
Use_def(Shifted_seq) $\Rightarrow$ $\neg\mathsf{Svm}\left(f \bullet \mathsf{Shift}(m)\right)$
$\langle f, \mathsf{Shift}(m)\rangle \hookrightarrow T103 \Rightarrow \quad \neg\mathsf{Svm}\left(\mathsf{Shift}(m)\right)$
$\langle m\rangle \hookrightarrow T274a \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now we show that the concatenation of two finite sequences must be a finite sequence.

Theorem 313 (276) $F, G \in \mathsf{Fin\_seqs}(S) \to \mathsf{concat}(F, G) \in \mathsf{Fin\_seqs}(S)\ \&\ \#\mathsf{concat}(F, G) = \#F + \#G$. PROOF:

Suppose_not(f, s, g) $\Rightarrow$ $f, g \in \mathsf{Fin\_seqs}(s)\ \&\ \mathsf{concat}(f, g) \notin \mathsf{Fin\_seqs}(s) \vee \#\mathsf{concat}(f, g) \neq \#f + \#g$

-- For suppose the contrary.

$\langle f, s\rangle \hookrightarrow T274 \Rightarrow \quad Stat1:\ \#f \in \mathbb{N}\ \&\ \mathbf{domain}(f) = \#f$
$\langle g, s\rangle \hookrightarrow T274 \Rightarrow \quad Stat2:\ \#g \in \mathbb{N}\ \&\ \mathbf{domain}(g) = \#g$
Suppose $\Rightarrow$ $\mathbf{domain}(\mathsf{concat})(f, g) \neq \#f + \#g$
Use_def(concat) $\Rightarrow$ $Stat3:\ \mathbf{domain}(f \cup \{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) \neq \#f + \#g$
$\langle f, \left\{[x^{[1]} + \#f, x^{[2]}] : x \in g\right\}\rangle \hookrightarrow T70 \Rightarrow \quad Stat4:\ \mathbf{domain}(f \cup \{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) =$
$\mathbf{domain}(f) \cup \mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\})$
Suppose $\Rightarrow$ $\mathbf{domain}(\{[x^{[1]} + \#f, x^{[2]}] : x \in g\}) \neq \{u + \#f : u \in \mathbf{domain}(g)\}$
Use_def(**domain**) $\Rightarrow$ $\{x^{[1]} : x \in \{[x^{[1]} + \#f, x^{[2]}] : x \in g\}\} \neq \{u + \#f : u \in \{x^{[1]} : x \in g\}\}$

$\mathsf{SIMPLF} \Rightarrow \quad \mathit{Stat5}: \left\{\left[x^{[1]} + \#f, x^{[2]}\right]^{[1]} : x \in g\right\} \neq \left\{x^{[1]} + \#f : x \in g\right\}$

$\langle a_1 \rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad \mathit{Stat6}:$

$(a_1 \in \left\{\left[x^{[1]} + \#f, x^{[2]}\right]^{[1]} : x \in g\right\} \;\&\; a_1 \notin \left\{x^{[1]} + \#f : x \in g\right\}) \vee$

$a_1 \notin \left\{\left[x^{[1]} + \#f, x^{[2]}\right]^{[1]} : x \in g\right\} \;\&\; a_1 \in \left\{x^{[1]} + \#f : x \in g\right\}$

$\langle b_1, b_1, b_2, b_2 \rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad \mathit{Stat7}:$

$((b_1 \in g \;\&\; a_1 = \left[{b_1}^{[1]} + \#f, {b_1}^{[2]}\right]^{[1]}) \;\&\; (b_1 \in g \rightarrow a_1 \neq {b_1}^{[1]} + \#f)) \vee$

$(b_2 \in g \;\&\; a_2 = \left[{b_2}^{[1]} + \#f, {b_2}^{[2]}\right]^{[1]}) \;\&\; (b_2 \in g \rightarrow a_2 \neq {b_2}^{[1]} + \#f)$

$\langle \mathit{Stat7} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat8}: \ \mathbf{domain}(\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\}) = \{u + \#f : u \in \mathbf{domain}(g)\}$

$\mathsf{EQUAL}\ \langle \mathit{Stat2}, \mathit{Stat8} \rangle \Rightarrow \quad \mathit{Stat9}: \ \mathbf{domain}(\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\}) = \{u + \#f : u \in \#g\}$

$\langle \mathit{Stat3}, \mathit{Stat4}, \mathit{Stat1}, \mathit{Stat9} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat10}: \ \#f \cup \{u + \#f : u \in \#g\} \neq \#f + \#g$

$\langle \#f, \#g \rangle \hookrightarrow \mathit{T270} \Rightarrow \quad \mathit{Stat11}: \ \#f + \#g = \#f \cup \{u + \#f : u \in \#g\}$

$\langle \mathit{Stat10}, \mathit{Stat11} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat12}: \ \mathbf{domain}(\mathsf{concat})(f, g) = \#f + \#g$

$\langle \#f, \#g \rangle \hookrightarrow \mathit{T239} \Rightarrow \quad \mathit{Stat13}: \ \#f + \#g \in \mathbb{N}$

$\langle \mathit{Stat12}, \mathit{Stat13} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat14}: \ \mathbf{domain}(\mathsf{concat})(f, g) \in \mathbb{N}$

$\langle \mathbf{domain}(\mathsf{concat})(f, g) \rangle \hookrightarrow \mathit{T180} \Rightarrow \quad \mathit{Stat15}: \ \mathbf{domain}(\mathsf{concat})(f, g) = \#\mathbf{domain}(\mathsf{concat})(f, g)$

$\mathsf{Use\_def}(\mathsf{Fin\_seqs}) \Rightarrow \quad \mathit{Stat16}: \ f \in \{f \subseteq \mathbb{N} \times s \mid \mathsf{Svm}(f) \;\&\; \mathbf{domain}(f) \in \mathbb{N}\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat16} \Rightarrow \quad \mathit{Stat17}: \ \mathsf{Svm}(f) \;\&\; \mathbf{domain}(f) \in \mathbb{N} \;\&\; \mathit{Stat18}: \ f \subseteq \mathbb{N} \times s$

$\mathsf{Use\_def}(\times) \Rightarrow \quad f \subseteq \{[x, y] : x \in \mathbb{N}, y \in s\}$

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat19}: \ \neg\langle \forall u \in f, \exists x \in \mathbb{N}, y \in s \mid u = [x, y] \rangle$

$\langle u \rangle \hookrightarrow \mathit{Stat19} \Rightarrow \quad \mathit{Stat20}: \ \neg\langle \exists x \in \mathbb{N}, y \in s \mid u = [x, y] \rangle \;\&\; u \in \{[x, y] : x \in \mathbb{N}, y \in s\}$

$\langle x, y, x, y \rangle \hookrightarrow \mathit{Stat20} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat21}: \ \langle \forall u \in f, \exists x \in \mathbb{N}, y \in s \mid u = [x, y] \rangle$

$\mathsf{Use\_def}(\mathsf{Fin\_seqs}) \Rightarrow \quad \mathit{Stat22}: \ g \in \{g \subseteq \mathbb{N} \times s \mid \mathsf{Svm}(g) \;\&\; \mathbf{domain}(g) \in \mathbb{N}\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat22} \Rightarrow \quad \mathit{Stat23}: \ \mathsf{Svm}(g) \;\&\; \mathbf{domain}(g) \in \mathbb{N} \;\&\; \mathit{Stat24}: \ g \subseteq \mathbb{N} \times s$

$\mathsf{Use\_def}(\times) \Rightarrow \quad g \subseteq \{[x, y] : x \in \mathbb{N}, y \in s\}$

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat25}: \ \neg\langle \forall u \in g, \exists x \in \mathbb{N}, y \in s \mid u = [x, y] \rangle$

$\langle w \rangle \hookrightarrow \mathit{Stat25} \Rightarrow \quad \mathit{Stat26}: \ \neg\langle \exists x \in \mathbb{N}, y \in s \mid w = [x, y] \rangle \;\&\; w \in \{[x, y] : x \in \mathbb{N}, y \in s\}$

$\langle x', y', x', y' \rangle \hookrightarrow \mathit{Stat26} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat27}: \ \langle \forall u \in g, \exists x \in \mathbb{N}, y \in s \mid u = [x, y] \rangle$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{concat}(f, g) \notin \mathsf{Fin\_seqs}(s)$

$\mathsf{Use\_def}(\mathsf{Fin\_seqs}) \Rightarrow \quad \mathit{Stat28}: \ \mathsf{concat}(f, g) \notin \{f \subseteq \mathbb{N} \times s \mid \mathsf{Svm}(f) \;\&\; \mathbf{domain}(f) \in \mathbb{N}\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat28} \Rightarrow \quad \mathit{Stat29}: \ \mathsf{concat}(f, g) \not\subseteq \mathbb{N} \times s \vee \neg\mathsf{Svm}(\mathsf{concat}(f, g)) \vee \mathbf{domain}(\mathsf{concat})(f, g) \notin \mathbb{N}$

$\langle \mathit{Stat14}, \mathit{Stat29} \rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{concat}(f, g) \not\subseteq \mathbb{N} \times s \vee \neg\mathsf{Svm}(\mathsf{concat}(f, g))$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{concat}(f, g) \not\subseteq \mathbb{N} \times s$

$\mathsf{Use\_def}(\mathsf{concat}) \Rightarrow \quad \mathit{Stat30}: \ f \cup \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\} \not\subseteq \mathbb{N} \times s$

$\langle \mathit{Stat17}, \mathit{Stat30} \rangle\ \mathsf{ELEM} \Rightarrow \quad \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\} \not\subseteq \mathbb{N} \times s$

$\mathsf{Use\_def}(\times) \Rightarrow \quad \mathit{Stat31}: \ \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\} \not\subseteq \{[u, v] : u \in \mathbb{N}, v \in s\}$

$\langle c\rangle \hookrightarrow Stat31 \Rightarrow \quad Stat32: \ c \in \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\} \ \& \ c \notin \{[u,v] : u \in \mathbb{N}, v \in s\}$

$\langle a, a^{[1]} + \#f, a^{[2]}\rangle \hookrightarrow Stat32 \Rightarrow \quad Stat33:$

$\quad a \in g \ \& \ c = \left[a^{[1]} + \#f, a^{[2]}\right] \ \& \ a^{[1]} + \#f \notin \mathbb{N} \vee a^{[2]} \notin s$

$\langle a, e', f'\rangle \hookrightarrow Stat27 \Rightarrow \quad Stat34: \ e' \in \mathbb{N} \ \& \ f' \in s \ \& \ a = [e', f']$

$\langle Stat34, Stat33\rangle$ ELEM $\Rightarrow \quad Stat35: \ a^{[1]} = e' \ \& \ a^{[1]} + \#f \notin \mathbb{N}$

EQUAL $\langle Stat35\rangle \Rightarrow \quad Stat36: \ e' + \#f \notin \mathbb{N}$

$\langle e', \#f\rangle \hookrightarrow T239(\langle Stat34, Stat1, Stat36\rangle) \Rightarrow$ false; Discharge $\Rightarrow \quad \neg\mathsf{Svm}(\mathsf{concat}(f, g))$

Use_def(concat) $\Rightarrow \quad Stat37: \ \neg\mathsf{Svm}(f \cup \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\})$

$\langle f, s, g, \#f\rangle \hookrightarrow T275 \Rightarrow \quad Stat38:$

$\quad \mathsf{Svm}(\left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\}) \ \& \ \mathbf{domain}(f) \cap \mathbf{domain}(\left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\}) = \emptyset$

$\langle f, \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\}\rangle \hookrightarrow T79([Stat38, Stat17]) \Rightarrow$

$\quad \mathsf{Svm}(f \cup \left\{\left[x^{[1]} + \#f, x^{[2]}\right] : x \in g\right\})$

$\langle Stat37\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad Stat39: \ \mathsf{concat}(f, g) \in \mathsf{Fin\_seqs}(s) \ \& \ \#\mathsf{concat}(f, g) \neq \#f + \#g$

Use_def(Fin_seqs) $\Rightarrow \quad Stat40: \ \mathsf{concat}(f, g) \in \{f \subseteq \mathbb{N} \times s \mid \mathsf{Svm}(f) \ \& \ \mathbf{domain}(f) \in \mathbb{N}\}$

$\langle\ \rangle \hookrightarrow Stat40 \Rightarrow \quad \mathsf{Svm}(\mathsf{concat}(f, g))$

$\langle \mathsf{concat}(f, g)\rangle \hookrightarrow T148 \Rightarrow \quad Stat41: \ \#\mathbf{domain}(\mathsf{concat})(f, g) = \#\mathsf{concat}(f, g)$

$\langle Stat12, Stat41, Stat15, Stat39\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 314 (277) $F \in \mathsf{Fin\_seqs}(S) \ \& \ M \in \mathbf{domain}(F) \rightarrow \mathsf{Shifted\_seq}(F, M) \in \mathsf{Fin\_seqs}(S) \ \& \ F = \mathsf{concat}(F_{|M}, \mathsf{Shifted\_seq}(F, M))$. PROOF:

Suppose_not(f, s, m) $\Rightarrow \quad f \in \mathsf{Fin\_seqs}(s) \ \& \ m \in \mathbf{domain}(f) \ \& \ \mathsf{Shifted\_seq}(f, m) \notin \mathsf{Fin\_seqs}(s) \vee f \neq \mathsf{concat}(f_{|m}, \mathsf{Shifted\_seq}(f, m))$

-- For, assuming the triple f, s, m to be a counterexample to our theorem, we reach a contradiction leading to the desired conclusion arguing as follows. We begin by observing that m is an unsigned integer insofar as an element of the domain of f; hence, by the earlier Theorem 275, $\mathsf{Shifted\_seq}(f, m)$ is a single-valued-map. Hence, if we assume that $\mathsf{Shifted\_seq}(f, m) \notin \mathsf{Fin\_seqs}(s)$, the reason can either be that $\mathsf{Shifted\_seq}(f, m)$ is not included in $\mathbb{N} \times s$ or that its domain is not an insigned integer.

$\langle f, s\rangle \hookrightarrow T274 \Rightarrow \quad \#f \in \mathbb{N} \ \& \ \mathbf{domain}(f) = \#f \ \& \ \mathsf{Svm}(f) \ \& \ \mathbf{range}(f) \subseteq s$

$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$

$\langle \mathbb{N}, \mathbf{domain}(f)\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathbf{domain}(f))$

$\langle \mathbb{N}, \mathbf{domain}(f)\rangle \hookrightarrow T12 \Rightarrow \quad \mathbf{domain}(f) \subseteq \mathbb{N} \ \& \ m \in \mathbb{N}$

$\langle \mathbf{domain}(f), m\rangle \hookrightarrow T12 \Rightarrow \quad \mathbf{domain}(f) \supseteq m$

$\langle f, s, f, m\rangle \hookrightarrow T275 \Rightarrow \quad \mathsf{Svm}(\mathsf{Shifted\_seq}(f, m))$

Use_def(Svm) $\Rightarrow \quad \mathsf{Is\_map}(f) \ \& \ \mathsf{Is\_map}(\mathsf{Shifted\_seq}(f, m))$

$\langle m, f\rangle \hookrightarrow T275a \Rightarrow \quad Stat2a: \ \mathsf{Shifted\_seq}(f, m) = \left\{\left[x^{[1]} - m, x^{[2]}\right] : x \in F \mid m \in x^{[1]} \vee m = x^{[1]}\right\}$

Suppose $\Rightarrow \quad \mathsf{Shifted\_seq}(f, m) \notin \mathsf{Fin\_seqs}(s)$

$\langle \mathsf{f}, \mathsf{s}, \mathsf{f}, \mathsf{m} \rangle \hookrightarrow T275 \Rightarrow \quad \mathsf{Svm}(\mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}))$
$\mathsf{Use\_def}(\mathsf{Fin\_seqs}) \Rightarrow \quad Stat1: \ \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}) \notin \{\mathsf{f} \subseteq \mathbb{N} \times \mathsf{s} \mid \mathsf{Svm}(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) \in \mathbb{N}\}$
$\langle \ \rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}) \nsubseteq \mathbb{N} \times \mathsf{s} \vee \mathbf{domain}(\mathsf{Shifted\_seq})(\mathsf{f}, \mathsf{m}) \notin \mathbb{N}$

-- However, assuming that $\mathbb{N} \times \mathsf{s}$ does not include $\mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m})$ conflicts with the very definition of $\mathsf{Shifted\_seq}$.

$\mathsf{Suppose} \Rightarrow \quad Stat2: \ \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}) \nsubseteq \mathbb{N} \times \mathsf{s}$
$\langle \mathsf{c} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{c} \notin \mathbb{N} \times \mathsf{s} \ \& \ \mathsf{c} \in \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m})$
$\langle Stat2a \rangle \ \mathsf{ELEM} \Rightarrow \quad Stat3: \ \mathsf{c} \in \{[\mathsf{x}^{[1]} - \mathsf{m}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{f} \mid \mathsf{m} \in \mathsf{x}^{[1]} \vee \mathsf{m} = \mathsf{x}^{[1]}\}$
$\mathsf{Use\_def}(\times) \Rightarrow \quad Stat4: \ \mathsf{c} \notin \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathsf{s}\}$
$\langle \mathsf{p} \rangle \hookrightarrow Stat3 \Rightarrow \quad \mathsf{p} \in \mathsf{f} \ \& \ \mathsf{c} = [\mathsf{p}^{[1]} - \mathsf{m}, \mathsf{p}^{[2]}] \ \& \ \mathsf{m} \in \mathsf{p}^{[1]} \vee \mathsf{m} = \mathsf{p}^{[1]}$
$\langle \mathsf{p}^{[1]} - \mathsf{m}, \mathsf{p}^{[2]} \rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{p}^{[1]} - \mathsf{m} \notin \mathbb{N} \vee \mathsf{p}^{[2]} \notin \mathsf{s}$
$\langle \mathsf{p}, \mathsf{f} \rangle \hookrightarrow T55 \Rightarrow \quad \mathsf{p}^{[1]} \in \mathbb{N}$
$\langle \mathsf{p}^{[1]}, \mathsf{m} \rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{p}^{[2]} \notin \mathsf{s}$
$\langle \mathsf{p}, \mathsf{f} \rangle \hookrightarrow T56 \Rightarrow \quad \mathsf{p}^{[2]} \in \mathbf{range}(\mathsf{f})$
$\langle \mathsf{p}, \mathsf{f} \rangle \hookrightarrow T56 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathbf{domain}(\mathsf{Shifted\_seq})(\mathsf{f}, \mathsf{m}) \notin \mathbb{N}$

-- Likewise, we must discard the possibility that $\mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m})$ is not included in $\mathbb{N} \times \mathsf{s}$.

$\mathsf{Suppose} \Rightarrow \quad \{\mathsf{x}^{[1]} - \mathsf{m} : \mathsf{x} \in \mathsf{f} \mid \mathsf{m} \in \mathsf{x}^{[1]} \vee \mathsf{m} = \mathsf{x}^{[1]}\} \neq \{\mathsf{y} - \mathsf{m} : \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{m} \in \mathsf{y} \vee \mathsf{m} = \mathsf{y}\}$
$\mathsf{Use\_def}(\mathbf{domain}) \Rightarrow \quad \{\mathsf{y} - \mathsf{m} : \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{m} \in \mathsf{y} \vee \mathsf{m} = \mathsf{y}\} = \{\mathsf{y} - \mathsf{m} : \mathsf{y} \in \{\mathsf{p}^{[1]} : \mathsf{p} \in \mathsf{f}\} \mid \mathsf{m} \in \mathsf{y} \vee \mathsf{m} = \mathsf{y}\}$
$\mathsf{SIMPLF} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \{\mathsf{x}^{[1]} - \mathsf{m} : \mathsf{x} \in \mathsf{f} \mid \mathsf{m} \in \mathsf{x}^{[1]} \vee \mathsf{m} = \mathsf{x}^{[1]}\} = \{\mathsf{y} - \mathsf{m} : \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{m} \in \mathsf{y} \vee \mathsf{m} = \mathsf{y}\}$
$\mathsf{ELEM} \Rightarrow \quad Stat4a: \ \mathbf{domain}(\{[\mathsf{x}^{[1]} - \mathsf{M}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{F} \mid \mathsf{M} \in \mathsf{x}^{[1]} \vee \mathsf{M} = \mathsf{x}^{[1]}\}) = \{\mathsf{y} - \mathsf{m} : \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{m} \in \mathsf{y} \vee \mathsf{m} = \mathsf{y}\}$
$\mathsf{EQUAL} \ \langle Stat2a, Stat4a \rangle \Rightarrow \quad \mathbf{domain}(\mathsf{Shifted\_seq})(\mathsf{f}, \mathsf{m}) = \{\mathsf{y} - \mathsf{m} : \mathsf{y} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{m} \in \mathsf{y} \vee \mathsf{m} = \mathsf{y}\}$
$\langle \mathbf{domain}(\mathsf{f}), \mathsf{m} \rangle \hookrightarrow T10021 \Rightarrow \quad \mathbf{domain}(\mathsf{f}) - \mathsf{m} \notin \mathbb{N}$
$\langle \mathbf{domain}(\mathsf{f}), \mathsf{m} \rangle \hookrightarrow T239 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat5: \ \mathsf{f} \neq \mathsf{concat}(\mathsf{f}_{|\mathsf{m}}, \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}))$

-- Having at this point eliminated the possibility that $\mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}) \notin \mathsf{Fin\_seqs}(\mathsf{s})$, we now proceed under the temporary assumption that $\mathsf{f} \neq \mathsf{concat}(\mathsf{f}_{|\mathsf{m}}, \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}))$. This will lead us to a contradiction, too. Indeed, we must assume the existence of a q such that q belongs to either f or $\mathsf{concat}(\mathsf{f}_{|\mathsf{m}}, \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}))$ but does not belong to both of them, and in particular does not belong to $\mathsf{f}_{|\mathsf{m}}$.

$\langle \mathsf{q} \rangle \hookrightarrow Stat5 \Rightarrow \quad Stat10: \ \mathsf{q} \in \mathsf{f} \leftrightarrow \mathsf{q} \notin \mathsf{concat}(\mathsf{f}_{|\mathsf{m}}, \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m}))$
$\mathsf{Use\_def}(\mathsf{concat}) \Rightarrow \quad Stat11: \ \mathsf{concat}(\mathsf{f}_{|\mathsf{m}}, \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m})) = \mathsf{f}_{|\mathsf{m}} \cup \{[\mathsf{x}^{[1]} + \#\mathsf{f}_{|\mathsf{m}}, \mathsf{x}^{[2]}] : \mathsf{x} \in \mathsf{Shifted\_seq}(\mathsf{f}, \mathsf{m})\}$
$\langle \mathsf{f}, \mathsf{m} \rangle \hookrightarrow T43 \Rightarrow \quad Stat12: \ \mathsf{q} \notin \mathsf{f}_{|\mathsf{m}}$
$\langle \mathsf{m} \rangle \hookrightarrow T180 \Rightarrow \quad \mathsf{m} = \#\mathsf{m} \ \& \ \mathcal{O}(\mathsf{m})$
$\langle \mathsf{f}, \mathsf{m} \rangle \hookrightarrow T10000a \Rightarrow \quad \#\mathsf{f}_{|\mathsf{m}} = \mathsf{m}$

$\mathsf{EQUAL} \Rightarrow \quad \mathit{Stat13}: \ \left\{\left[x^{[1]} + \#f_{|m}, x^{[2]}\right] : x \in \mathsf{Shifted\_seq}(f, m)\right\} = \left\{\left[x^{[1]} + m, x^{[2]}\right] : x \in \mathsf{Shifted\_seq}(f, m)\right\}$

-- Assuming that q belongs to f, we reach the following contradiction:

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat14}: \ q \notin \left\{\left[x^{[1]} + m, x^{[2]}\right] : x \in \mathsf{Shifted\_seq}(f, m)\right\}$
$\mathsf{Use\_def}(|) \Rightarrow \quad \mathit{Stat15}: \ q \notin \left\{x : x \in f \,|\, x^{[1]} \in m\right\}$
$\langle q \rangle \hookrightarrow \mathit{Stat15} \Rightarrow \quad q^{[1]} \notin m$
$\langle q, f \rangle \hookrightarrow \mathit{T55} \Rightarrow \quad q^{[1]} \in \mathbb{N}$
$\langle f, q \rangle \hookrightarrow \mathit{T46} \Rightarrow \quad q = \left[q^{[1]}, q^{[2]}\right]$
$\langle q^{[1]} \rangle \hookrightarrow \mathit{T180} \Rightarrow \quad \mathcal{O}(q^{[1]})$
$\langle q^{[1]}, m \rangle \hookrightarrow \mathit{T28} \Rightarrow \quad m \in q^{[1]} \vee m = q^{[1]}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat14a}: \ \left[q^{[1]} - m, q^{[2]}\right] \notin \mathsf{Shifted\_seq}(f, m)$
$\langle \mathit{Stat2a}, \mathit{Stat14a}, * \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat16}: \ \left[q^{[1]} - m, q^{[2]}\right] \notin \left\{\left[x^{[1]} - m, x^{[2]}\right] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\right\}$
$\langle q \rangle \hookrightarrow \mathit{Stat16} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \left[q^{[1]} - m, q^{[2]}\right] \in \mathsf{Shifted\_seq}(f, m)$
$\left\langle \left[q^{[1]} - m, q^{[2]}\right] \right\rangle \hookrightarrow \mathit{Stat14} \Rightarrow \quad q \neq \left[q^{[1]} - m + m, q^{[2]}\right]$
$\langle m, q^{[1]} \rangle \hookrightarrow \mathit{T271} \Rightarrow \quad q^{[1]} = m + (q^{[1]} - m)$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat17}: \ q \in \left\{\left[x^{[1]} + m, x^{[2]}\right] : x \in \mathsf{Shifted\_seq}(f, m)\right\}$

-- On the other hand, if we make the opposite assumption, that f belongs to $\mathsf{concat}\left(f_{|m}, \mathsf{Shifted\_seq}(f, m)\right)$, then we reach the following contradiction:

$\langle r \rangle \hookrightarrow \mathit{Stat17} \Rightarrow \quad \mathit{Stat18}: \ q = \left[r^{[1]} + m, r^{[2]}\right] \ \& \ r \in \mathsf{Shifted\_seq}(f, m)$
$\langle \mathit{Stat2a}, \mathit{Stat18}, * \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat19}: \ r \in \left\{\left[x^{[1]} - m, x^{[2]}\right] : x \in f \,|\, m \in x^{[1]} \vee m = x^{[1]}\right\}$
$\langle a \rangle \hookrightarrow \mathit{Stat19}([]) \Rightarrow \quad \mathit{Stat20}: \ a \in f \ \& \ r = \left[a^{[1]} - m, a^{[2]}\right] \ \& \ m \in a^{[1]} \vee m = a^{[1]}$
$\langle \mathit{Stat20} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat21}: \ r^{[1]} = a^{[1]} - m \ \& \ a^{[2]} = r^{[2]}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat22}: \ q \neq a$
$\langle m, a^{[1]} \rangle \hookrightarrow \mathit{T271} \Rightarrow \quad a^{[1]} = m + (a^{[1]} - m)$
$\mathsf{EQUAL} \ \langle \mathit{Stat20} \rangle \Rightarrow \quad \mathit{Stat23}: \ a^{[1]} = m + r^{[1]}$
$\langle a, f \rangle \hookrightarrow \mathit{T55} \Rightarrow \quad a^{[1]} \in \mathbb{N}$
$\langle f, a \rangle \hookrightarrow \mathit{T46} \Rightarrow \quad \mathit{Stat24}: \ a = \left[a^{[1]}, a^{[2]}\right]$
$\mathsf{ALGEBRA} \Rightarrow \quad a^{[1]} - m \in \mathbb{N}$
$\mathsf{EQUAL} \Rightarrow \quad r^{[1]} \in \mathbb{N}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathit{Stat25}: \ m + r^{[1]} = r^{[1]} + m$
$\langle \mathit{Stat18}, \mathit{Stat23}, \mathit{Stat21}, \mathit{Stat24}, \mathit{Stat25}, \mathit{Stat22} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat26}: \ q \in f$
$\langle \mathit{Stat10}, \mathit{Stat11}, \mathit{Stat12}, \mathit{Stat13}, \mathit{Stat17}, \mathit{Stat26} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

THEORY $\mathsf{subseq}(g, f)$

-- Subsequence of a finite or denumerable sequence

$\mathsf{Svm}(f) \ \& \ \mathbf{domain}(f) \in \mathsf{next}(\mathbb{N})$

$g \in \mathsf{Subseqs}(f)$
END subseq

ENTER_THEORY subseq

-- Subsequence generator

DEF subseq · 0. $h_\Theta =_{Def} \{p \in \mathbf{arb}(\{h \subseteq \mathbb{N}\times\mathbb{N} \mid g = f\bullet h \,\&\, \mathsf{Svm}(h) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h\restriction i \in h\restriction \mathsf{next}(i)\rangle\}) \mid p^{[2]} \in \mathbf{domain}(f)\}$

Theorem 315 (subseq · 1) $g = f\bullet h_\Theta \,\&\, 1\text{–}1(h_\Theta) \,\&\, \mathbf{domain}(h_\Theta) \in \mathsf{next}(\mathbb{N}) \,\&\, \mathbf{range}(h_\Theta) \subseteq \mathbf{domain}(f) \,\&\, \langle \forall i \in \mathbf{domain}(h_\Theta), j \in \mathbf{domain}(h_\Theta) \mid i \in j \rightarrow h_\Theta\restriction i \in h_\Theta\restriction j\rangle$. PROOF:

Suppose_not $\Rightarrow$ *Stat0* : $g \neq f\bullet h_\Theta \vee \neg 1\text{–}1(h_\Theta) \vee \mathbf{domain}(h_\Theta) \notin \mathsf{next}(\mathbb{N}) \vee \mathbf{range}(h_\Theta) \not\subseteq \mathbf{domain}(f) \vee \neg\langle \forall i \in \mathbf{domain}(h_\Theta), j \in \mathbf{domain}(h_\Theta) \mid i \in j \rightarrow h_\Theta\restriction i \in h_\Theta\restriction j\rangle$

-- We start with expanding the definition of 'Subseqs', which requires g to be of the form $f\bullet h$, with h meeting various conditions which, however, do not include the condition that the components of h belong to **domain**(f). Starting with the specific h from which h_Θ has been obtained by simply putting $h_\Theta = \{p \in h \mid p^{[2]} \in \mathbf{domain}(f)\}$, we must check that h_Θ is a (finite or infinite) sequence enjoying the same properties as h, plus the additional one that its components belong to the domain of f.

Assump $\Rightarrow$ $\mathsf{Svm}(f) \,\&\, \mathbf{domain}(f) \in \mathsf{next}(\mathbb{N}) \,\&\, g \in \mathsf{Subseqs}(f)$

$\langle \mathbf{domain}(f)\rangle \hookrightarrow T10057 \Rightarrow$ $\mathbf{domain}(f) \subseteq \mathbb{N}$

Use_def(Subseqs) $\Rightarrow$ *Stat3a* : $g \in \{f\bullet h : h \subseteq \mathbb{N}\times\mathbb{N} \mid \mathsf{Svm}(h) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h\restriction i \in h\restriction \mathsf{next}(i)\rangle\}$

Loc_def $\Rightarrow$ $h_0 = \mathbf{arb}(\{h \subseteq \mathbb{N}\times\mathbb{N} \mid g = f\bullet h \,\&\, \mathsf{Svm}(h) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h\restriction i \in h\restriction \mathsf{next}(i)\rangle\})$

Suppose $\Rightarrow$ *Stat2a* : $\emptyset = \{h \subseteq \mathbb{N}\times\mathbb{N} \mid g = f\bullet h \,\&\, \mathsf{Svm}(h) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h\restriction i \in h\restriction \mathsf{next}(i)\rangle\}$

$\langle h_1\rangle \hookrightarrow Stat3a \Rightarrow$ $h_1 \subseteq \mathbb{N}\times\mathbb{N} \,\&\, g = f\bullet h_1 \,\&\, \mathsf{Svm}(h_1) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h_1) \rightarrow i \in \mathbf{domain}(h_1) \,\&\, h_1\restriction i \in h_1\restriction \mathsf{next}(i)\rangle$

$\langle h_1\rangle \hookrightarrow Stat2a \Rightarrow$ false; Discharge $\Rightarrow$ *Stat1a* : $h_0 \in \{h \subseteq \mathbb{N}\times\mathbb{N} \mid g = f\bullet h \,\&\, \mathsf{Svm}(h) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h\restriction i \in h\restriction \mathsf{next}(i)\rangle\}$

$\langle\,\rangle \hookrightarrow Stat1a \Rightarrow$ $g = f\bullet h_0 \,\&\, h_0 \subseteq \mathbb{N}\times\mathbb{N} \,\&\, \mathsf{Svm}(h_0) \,\&\,$ *Stat4a* : $\langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h_0) \rightarrow i \in \mathbf{domain}(h_0) \,\&\, h_0\restriction i \in h_0\restriction \mathsf{next}(i)\rangle$

$\langle h_0, \mathbb{N}, \mathbb{N}\rangle \hookrightarrow T116 \Rightarrow$ $\mathbf{domain}(h_0) \subseteq \mathbb{N}$

Use_def(h_Θ) $\Rightarrow$

$h_\Theta = \{p \in \mathbf{arb}(\{h \subseteq \mathbb{N}\times\mathbb{N} \mid g = f\bullet h \,\&\, \mathsf{Svm}(h) \,\&\, \langle \forall i \mid \mathsf{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h) \,\&\, h\restriction i \in h\restriction \mathsf{next}(i)\rangle\}) \mid p^{[2]} \in \mathbf{domain}(f)\}$

EQUAL $\Rightarrow$ $h_\Theta = \{p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\}$

Suppose $\Rightarrow$ *Stat5a* : $h_\Theta \not\subseteq h_0$

$\langle c\rangle \hookrightarrow Stat5a \Rightarrow$ $c \notin h_0 \,\&\,$ *Stat6a* : $c \in \{p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\}$

$\langle\,\rangle \hookrightarrow Stat6a(\langle Stat5a\rangle) \Rightarrow$ false; Discharge $\Rightarrow$ $h_\Theta \subseteq h_0$

$\langle h_\Theta, h_0\rangle \hookrightarrow T60 \Rightarrow$ $\mathbf{domain}(h_\Theta) \subseteq \mathbf{domain}(h_0) \,\&\, \mathbf{domain}(h_\Theta) \subseteq \mathbb{N}$

$\langle h_\Theta, h_0\rangle \hookrightarrow T48 \Rightarrow$ *Stat3* : $\mathsf{Svm}(h_\Theta)$

Use_def(Svm) $\Rightarrow$ $\mathsf{Is\_map}(h_\Theta)$

Suppose $\Rightarrow$ *Stat7a* : $\mathbf{range}(h_\Theta) \not\subseteq \mathbf{domain}(f)$

$\langle d \rangle \hookrightarrow Stat7a \Rightarrow \quad d \notin \mathbf{domain}(f) \;\&\; d \in \mathbf{range}(h_\Theta)$

EQUAL $\Rightarrow \quad d \in \mathbf{range}(\{p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\})$

Use_def(**range**) $\Rightarrow \quad d \in \{p^{[2]} : p \in \{p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\}\}$

SIMPLF $\Rightarrow \quad Stat8a: \; d \in \{p^{[2]} : p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\}$

$\langle p' \rangle \hookrightarrow Stat8a(\langle Stat7a \rangle) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \mathbf{range}(h_\Theta) \subseteq \mathbf{domain}(f)$

Suppose $\Rightarrow \quad f \bullet h_0 \neq f \bullet h_\Theta$

Use_def($\bullet$) $\Rightarrow \quad \{[x^{[1]}, y^{[2]}] : x \in h_0, y \in f \mid x^{[2]} = y^{[1]}\} \neq \{[x^{[1]}, y^{[2]}] : x \in h_\Theta, y \in f \mid x^{[2]} = y^{[1]}\}$

Set_monot $\Rightarrow \quad \{[x^{[1]}, y^{[2]}] : x \in h_\Theta, y \in f \mid x^{[2]} = y^{[1]}\} \subseteq \{[x^{[1]}, y^{[2]}] : x \in h_0, y \in f \mid x^{[2]} = y^{[1]}\}$

ELEM $\Rightarrow \quad Stat9a: \; \{[x^{[1]}, y^{[2]}] : x \in h_0, y \in f \mid x^{[2]} = y^{[1]}\} \not\subseteq \{[x^{[1]}, y^{[2]}] : x \in h_\Theta, y \in f \mid x^{[2]} = y^{[1]}\}$

$\langle e \rangle \hookrightarrow Stat9a \Rightarrow \quad Stat10a:$

$e \in \{[x^{[1]}, y^{[2]}] : x \in h_0, y \in f \mid x^{[2]} = y^{[1]}\} \;\&\; e \notin \{[x^{[1]}, y^{[2]}] : x \in h_\Theta, y \in f \mid x^{[2]} = y^{[1]}\}$

$\langle x_0, y_0, x_0, y_0 \rangle \hookrightarrow Stat10a \Rightarrow \quad e = [x_0{}^{[1]}, y_0{}^{[2]}] \;\&\; x_0 \in h_0 \;\&\; y_0 \in f \;\&\; x_0{}^{[2]} = y_0{}^{[1]} \;\&\; Stat11a:$

$x_0 \notin \{p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\}$

$\langle\,\rangle \hookrightarrow Stat11a([Stat10a, \cap]) \Rightarrow \quad x_0{}^{[2]} \notin \mathbf{domain}(f)$

Use_def(**domain**) $\Rightarrow \quad Stat12a: \; x_0{}^{[2]} \notin \{y^{[1]} : y \in f\}$

$\langle y_0 \rangle \hookrightarrow Stat12a([Stat10a, \cap]) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad f \bullet h_\Theta = f \bullet h_0 \;\&\; g = f \bullet h_\Theta$

$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$

$\langle \mathbb{N} \rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}(\mathrm{next}(\mathbb{N}))$

$\langle \mathrm{next}(\mathbb{N}), \mathbf{domain}(f) \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathbf{domain}(f))$

Suppose $\Rightarrow \quad Stat13a: \; \neg\langle \forall i \mid \mathrm{next}(i) \in \mathbf{domain}(h_\Theta) \rightarrow i \in \mathbf{domain}(h_\Theta) \;\&\; h_\Theta \restriction i \in h_\Theta \restriction \mathrm{next}(i) \rangle$

$\langle iq \rangle \hookrightarrow Stat13a(\langle Stat13a \rangle) \Rightarrow \quad \mathrm{next}(iq) \in \mathbf{domain}(h_\Theta) \;\&\; iq \notin \mathbf{domain}(h_\Theta) \vee h_\Theta \restriction iq \notin h_\Theta \restriction \mathrm{next}(iq)$

$\langle iq \rangle \hookrightarrow Stat4a(\langle Stat3a \rangle) \Rightarrow \quad \mathrm{next}(iq), iq \in \mathbf{domain}(h_0) \;\&\; h_0 \restriction iq \in h_0 \restriction \mathrm{next}(iq)$

$\langle h_\Theta, \mathrm{next}(iq) \rangle \hookrightarrow T69([Stat3a, \cap]) \Rightarrow \quad [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)] \in h_0$

$\langle h_0, [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)] \rangle \hookrightarrow T67(\langle Stat3a \rangle) \Rightarrow \quad h_0 \restriction [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)]^{[1]} =$

$[\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)]^{[2]}$

$TELEM \Rightarrow \quad [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)]^{[1]} = \mathrm{next}(iq) \;\&\; [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)]^{[2]} = h_\Theta \restriction \mathrm{next}(iq)$

EQUAL $\langle Stat13a \rangle \Rightarrow \quad h_0 \restriction \mathrm{next}(iq) = h_\Theta \restriction \mathrm{next}(iq)$

$\langle h_\Theta, \mathrm{next}(iq) \rangle \hookrightarrow T69(\langle Stat3 \rangle) \Rightarrow \quad [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)] \in h_\Theta$

$\langle [\mathrm{next}(iq), h_\Theta \restriction \mathrm{next}(iq)], h_\Theta \rangle \hookrightarrow T56(\langle Stat3 \rangle) \Rightarrow \quad h_\Theta \restriction \mathrm{next}(iq) \in \mathbf{domain}(f)$

$\langle \mathbf{domain}(f), h_\Theta \restriction \mathrm{next}(iq) \rangle \hookrightarrow T12([Stat3, \cap]) \Rightarrow \quad h_0 \restriction iq \in \mathbf{domain}(f)$

Use_def(Svm) $\Rightarrow \quad \mathrm{Is\_map}(h_0)$

$\langle h_0, iq \rangle \hookrightarrow T69([Stat3a, \cap]) \Rightarrow \quad [iq, h_0 \restriction iq] \in h_0$

Suppose $\Rightarrow \quad [iq, h_0 \restriction iq] \notin h_\Theta$

EQUAL $\Rightarrow \quad Stat14a: \; [iq, h_0 \restriction iq] \notin \{p \in h_0 \mid p^{[2]} \in \mathbf{domain}(f)\}$

$\langle\,\rangle \hookrightarrow Stat14a([Stat13a, \cap]) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad [iq, h_0 \restriction iq] \in h_\Theta$

$\langle h_\Theta, [iq, h_0 \restriction iq] \rangle \hookrightarrow T67([Stat3a, \cap]) \Rightarrow \quad h_\Theta \restriction [iq, h_0 \restriction iq]^{[1]} =$

$[iq, h_0 \restriction iq]^{[2]}$

$\mathit{TELEM} \Rightarrow \quad [\mathsf{iq}, \mathsf{h}_0 \restriction \mathsf{iq}]^{[1]} = \mathsf{iq} \;\&\; [\mathsf{iq}, \mathsf{h}_0 \restriction \mathsf{iq}]^{[2]} = \mathsf{h}_0 \restriction \mathsf{iq}$
$\mathsf{EQUAL} \ \langle \mathit{Stat13a}, * \rangle \Rightarrow \quad \mathsf{h}_\Theta \restriction \mathsf{iq} = \mathsf{h}_0 \restriction \mathsf{iq}$
$\langle \mathit{Stat13a}, * \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{iq} \notin \mathbf{domain}(\mathsf{h}_\Theta)$
$\langle [\mathsf{iq}, \mathsf{h}_0 \restriction \mathsf{iq}], \mathsf{h}_\Theta \rangle \hookrightarrow \mathit{T55}([\mathit{Stat13a}, \cap]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat4}: \ \langle \forall \mathsf{i} \,|\, \mathsf{next}(\mathsf{i}) \in \mathbf{domain}(\mathsf{h}_\Theta) \rightarrow \mathsf{i} \in \mathbf{domain}(\mathsf{h}_\Theta) \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i} \in \mathsf{h}_\Theta \restriction \mathsf{next}(\mathsf{i}) \rangle$

$\langle \mathsf{h}_\Theta, \mathbb{N}, \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow \mathit{T116} \Rightarrow \quad \mathbf{range}(\mathsf{h}_\Theta) \subseteq \mathbf{domain}(\mathsf{f})$
$\langle \mathsf{h}_\Theta, \mathsf{f} \rangle \hookrightarrow \mathit{T85} \Rightarrow \quad \mathbf{domain}(\mathsf{f} \bullet \mathsf{h}_\Theta) = \mathbf{domain}(\mathsf{h}_\Theta) \;\&\; \mathbf{range}(\mathsf{f} \bullet \mathsf{h}_\Theta) = \mathbf{range}(\mathsf{f}_{|\mathbf{range}(\mathsf{h}_\Theta)})$
$\langle \mathsf{f}, \mathsf{h}_\Theta \rangle \hookrightarrow \mathit{T103} \Rightarrow \quad \mathsf{Svm}(\mathsf{f} \bullet \mathsf{h}_\Theta)$

-- Observe that f and h_Θ have ordinal numbers not exceeding $\mathbb{N}$ as their domains. This is obvious (as has been proved above, en passant) for f, which either equals $\mathbb{N}$ or belongs to it. Concerning h_Θ (and hence g, which has the same domain as h_Θ), what claimed follows from the earlier Theorem 10057, because its domain is closed relative to predecessor formation.

$\mathsf{Suppose} \Rightarrow \quad \neg\big(\mathcal{O}(\mathbf{domain}(\mathsf{h}_\Theta)) \;\&\; \mathbf{domain}(\mathsf{h}_\Theta) \in \mathsf{next}(\mathbb{N})\big)$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat5}: \ \neg\langle \forall \mathsf{i} \,|\, \mathsf{next}(\mathsf{i}) \in \mathbf{domain}(\mathsf{h}_\Theta) \rightarrow \mathsf{i} \in \mathbf{domain}(\mathsf{h}_\Theta) \rangle$
$\langle \mathsf{i}' \rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad \mathsf{next}(\mathsf{i}') \in \mathbf{domain}(\mathsf{h}_\Theta) \;\&\; \mathsf{i}' \notin \mathbf{domain}(\mathsf{h}_\Theta)$
$\langle \mathsf{i}' \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \langle \forall \mathsf{i} \,|\, \mathsf{next}(\mathsf{i}) \in \mathbf{domain}(\mathsf{h}_\Theta) \rightarrow \mathsf{i} \in \mathbf{domain}(\mathsf{h}_\Theta) \rangle$
$\langle \mathbf{domain}(\mathsf{h}_\Theta) \rangle \hookrightarrow \mathit{T10057} \Rightarrow \quad \mathbf{domain}(\mathsf{h}_\Theta) \in \mathsf{next}(\mathbb{N})$
$\langle \mathsf{next}(\mathbb{N}), \mathbf{domain}(\mathsf{h}_\Theta) \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathcal{O}(\mathbf{domain}(\mathsf{h}_\Theta)) \;\&\; \mathbf{domain}(\mathsf{h}_\Theta) \in \mathsf{next}(\mathbb{N})$

-- We will now see that h_Θ is a strictly increasing function, from which its one-one-ness will follow.

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat14}: \ \neg\langle \forall \mathsf{i} \in \mathbf{domain}(\mathsf{h}_\Theta), \mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i} \in \mathsf{j} \rightarrow \mathsf{h}_\Theta \restriction \mathsf{i} \in \mathsf{h}_\Theta \restriction \mathsf{j} \rangle$

-- Indeed, assuming the contrary, we could choose unsigned integers $\mathsf{i}_1, \mathsf{j}_2$ such that i_1 precedes j_2 and $\mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}_2$. Moreover, we can find the smallest j_1 such that i_1 precedes j_1 and $\mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}_1$. Notice that j_1 cannot be the immediate successor of i_1.

$\langle \mathsf{i}_1, \mathsf{j}_2 \rangle \hookrightarrow \mathit{Stat14} \Rightarrow \quad \mathsf{i}_1, \mathsf{j}_2 \in \mathbf{domain}(\mathsf{h}_\Theta) \;\&\; \mathsf{i}_1 \in \mathsf{j}_2 \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}_2$
$\mathsf{Loc\_def} \Rightarrow \quad \mathsf{j}_1 = \mathbf{arb}(\{\mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i}_1 \in \mathsf{j} \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}\})$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat15}: \ \{\mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i}_1 \in \mathsf{j} \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}\} = \emptyset$
$\langle \mathsf{j}_2 \rangle \hookrightarrow \mathit{Stat15} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat16}: \ \mathsf{j}_1 \in \{\mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i}_1 \in \mathsf{j} \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}\} \;\&\; \mathsf{j}_1 \cap \{\mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i}_1 \in \mathsf{j} \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}\} = \emptyset$
$\langle\, \rangle \hookrightarrow \mathit{Stat16} \Rightarrow \quad \mathsf{j}_1 \in \mathbf{domain}(\mathsf{h}_\Theta) \;\&\; \mathsf{i}_1 \in \mathsf{j}_1 \;\&\; \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{j}_1$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{j}_1 = \mathsf{next}(\mathsf{i}_1)$
$\mathsf{EQUAL} \ \langle \mathit{Stat14} \rangle \Rightarrow \quad \mathsf{h}_\Theta \restriction \mathsf{i}_1 \notin \mathsf{h}_\Theta \restriction \mathsf{next}(\mathsf{i}_1) \;\&\; \mathsf{next}(\mathsf{i}_1) \in \mathbf{domain}(\mathsf{h}_\Theta)$
$\langle \mathsf{i}_1 \rangle \hookrightarrow \mathit{Stat4}([\mathit{Stat14}, \cap]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{j}_1 \neq \mathsf{next}(\mathsf{i}_1)$

-- From the fact $\mathsf{h}_\Theta \upharpoonright \mathsf{i}_1 \in \mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1)$ (due to the minimality in the choice of j_1) and the fact $\mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1) \in \mathsf{h}_\Theta \upharpoonright \mathsf{j}_1$ (due to an assumed property of h_Θ, whose domain is closed relative to predecessor extraction), it follows that $\mathsf{h}_\Theta \upharpoonright \mathsf{i}_1 \in \mathsf{h}_\Theta \upharpoonright \mathsf{j}_1$, contrary to a fact just established. Thanks to this contradiction, we can discharge our temporary assumption concluding that h_Θ fails to be one-one.

$T182([Stat14, \cap]) \Rightarrow \quad 1 \in \mathbb{N} \,\&\, \mathcal{O}(\emptyset)$
$\langle \mathsf{j}_1, 1 \rangle \hookrightarrow T239([Stat3a, \cap]) \Rightarrow \quad \mathsf{j}_1 - 1 \in \mathbb{N}$
$\langle \mathbb{N}, \mathsf{j}_1 \rangle \hookrightarrow T11([Stat3a, \cap]) \Rightarrow \quad \mathcal{O}(\mathsf{j}_1)$
Suppose $\Rightarrow \quad \mathsf{j}_1 \neq \mathsf{next}(\mathsf{j}_1 - 1)$
$\langle \mathsf{j}_1, \emptyset \rangle \hookrightarrow T28([Stat16, \cap]) \Rightarrow \quad \emptyset \in \mathsf{j}_1$
Use_def(1) $\Rightarrow \quad 1 = \mathsf{next}(\emptyset)$
Use_def(next) $\Rightarrow \quad 1 \subseteq \mathsf{j}_1$
$\langle 1, \mathsf{j}_1 \rangle \hookrightarrow T233([Stat16, \cap]) \Rightarrow \quad \#\mathsf{j}_1 = \mathsf{j}_1 - 1 + 1$
$\langle \mathsf{j}_1 \rangle \hookrightarrow T180([Stat3a, \cap]) \Rightarrow \quad \mathsf{j}_1 = \mathsf{j}_1 - 1 + 1$
$\langle \mathsf{j}_1 - 1 \rangle \hookrightarrow T265([Stat16, \cap]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{j}_1 = \mathsf{next}(\mathsf{j}_1 - 1)$
Use_def(next) $\Rightarrow \quad \mathsf{j}_1 - 1 \in \mathsf{j}_1$
EQUAL $\langle Stat16, * \rangle \Rightarrow \quad \mathsf{next}(\mathsf{j}_1 - 1) \in \mathbf{domain}(\mathsf{h}_\Theta)$
$\langle \mathsf{j}_1 - 1 \rangle \hookrightarrow Stat4([Stat16, \cap]) \Rightarrow \quad \mathsf{j}_1 - 1 \in \mathbf{domain}(\mathsf{h}_\Theta) \,\&\, \mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1) \in \mathsf{h}_\Theta \upharpoonright \mathsf{next}(\mathsf{j}_1 - 1)$
EQUAL $\langle Stat16 \rangle \Rightarrow \quad \mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1) \in \mathsf{h}_\Theta \upharpoonright \mathsf{j}_1$
Suppose $\Rightarrow \quad Stat17: \ \mathsf{h}_\Theta \upharpoonright \mathsf{i}_1 \notin \mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1)$
EQUAL $\langle Stat16, * \rangle \Rightarrow \quad \mathsf{i}_1 \in \mathsf{next}(\mathsf{j}_1 - 1)$
Use_def(next) $\Rightarrow \quad \mathsf{i}_1 \in \mathsf{j}_1 - 1 \cup \{\mathsf{j}_1 - 1\}$
Suppose $\Rightarrow \quad \mathsf{i}_1 = \mathsf{j}_1 - 1$
EQUAL $\langle Stat16 \rangle \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{i}_1 \in \mathsf{j}_1 - 1$
Suppose $\Rightarrow \quad Stat18: \ \mathsf{j}_1 - 1 \notin \{\mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i}_1 \in \mathsf{j} \,\&\, \mathsf{h}_\Theta \upharpoonright \mathsf{i}_1 \notin \mathsf{h}_\Theta \upharpoonright \mathsf{j}\}$
$\langle\ \rangle \hookrightarrow Stat18([Stat16, \cap]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{j}_1 - 1 \in \{\mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i}_1 \in \mathsf{j} \,\&\, \mathsf{h}_\Theta \upharpoonright \mathsf{i}_1 \notin \mathsf{h}_\Theta \upharpoonright \mathsf{j}\}$
$\langle Stat16, * \rangle$ ELEM $\Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{h}_\Theta \upharpoonright \mathsf{i}_1 \in \mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1)$
$\langle \mathsf{j}_1, \mathsf{h}_\Theta \rangle \hookrightarrow T64([Stat0, \cap]) \Rightarrow \quad \mathsf{h}_\Theta \upharpoonright \mathsf{j}_1 \in \mathbb{N}$
$\langle \mathbb{N}, \mathsf{h}_\Theta \upharpoonright \mathsf{j}_1 \rangle \hookrightarrow T11([Stat3, \cap]) \Rightarrow \quad \mathcal{O}(\mathsf{h}_\Theta \upharpoonright \mathsf{j}_1)$
$\langle \mathsf{h}_\Theta \upharpoonright \mathsf{j}_1, \mathsf{h}_\Theta \upharpoonright (\mathsf{j}_1 - 1) \rangle \hookrightarrow T12([Stat16, \cap]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \neg \mathsf{1\text{–}1}(\mathsf{h}_\Theta) \,\&\, Stat20: \ \langle \forall \mathsf{i} \in \mathbf{domain}(\mathsf{h}_\Theta), \mathsf{j} \in \mathbf{domain}(\mathsf{h}_\Theta) \,|\, \mathsf{i} \in \mathsf{j} \rightarrow \mathsf{h}_\Theta \upharpoonright \mathsf{i} \in \mathsf{h}_\Theta \upharpoonright \mathsf{j} \rangle$

-- To now see that h_Θ is a one-one function, contradicting the conclusion just reached and thus leading to the desired overall conclusion, we can exploit the strict monotonicity of h_Θ. In the first place, if we assume that h_Θ is not one-one, then we must admit the existence of distinct pairs p, q in h_Θ sharing the same second component.

Use_def(1–1) $\Rightarrow \quad Stat6: \ \neg \langle \forall \mathsf{x} \in \mathsf{h}_\Theta, \mathsf{y} \in \mathsf{h}_\Theta \,|\, \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{p}, \mathsf{q} \rangle \hookrightarrow Stat6 \Rightarrow \quad \mathsf{p}, \mathsf{q} \in \mathsf{h}_\Theta \,\&\, \mathsf{p}^{[2]} = \mathsf{q}^{[2]} \,\&\, \mathsf{p} \neq \mathsf{q}$
$\langle \mathsf{h}_\Theta, \mathsf{p} \rangle \hookrightarrow T46 \Rightarrow \quad \mathsf{p} = [\mathsf{p}^{[1]}, \mathsf{p}^{[2]}]$
$\langle \mathsf{h}_\Theta, \mathsf{q} \rangle \hookrightarrow T46 \Rightarrow \quad \mathsf{q} = [\mathsf{q}^{[1]}, \mathsf{q}^{[2]}]$

$\langle p, h_\Theta \rangle \hookrightarrow T55 \Rightarrow \quad p^{[1]} \in \mathbb{N}$
$\langle q, h_\Theta \rangle \hookrightarrow T55 \Rightarrow \quad q^{[1]} \in \mathbb{N}$
$\langle \mathbb{N}, p^{[1]} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(p^{[1]})$
$\langle \mathbb{N}, q^{[1]} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(q^{[1]})$
$\langle p^{[1]}, q^{[1]} \rangle \hookrightarrow T28(\langle Stat6 \rangle) \Rightarrow \quad Stat7: \; p^{[1]} \in q^{[1]} \vee q^{[1]} \in p^{[1]}$
$\langle Stat6, * \rangle$ ELEM $\Rightarrow \quad p, q \in h_\Theta \;\&\; p^{[2]} = q^{[2]}$

-- Let p_0, p_1 be such pairs p, q ordered so that the first component of p_0 precedes the first component of p_1. Then $p_0{}^{[2]}$ must precede $p_1{}^{[2]}$, which leads to a contradiction.

Loc_def $\Rightarrow \quad p_0 = \textbf{if } p^{[1]} \in q^{[1]} \textbf{ then } p \textbf{ else } q \textbf{ fi}$
Loc_def $\Rightarrow \quad p_1 = \textbf{if } p^{[1]} \in q^{[1]} \textbf{ then } q \textbf{ else } p \textbf{ fi}$
$\langle p_0, h_\Theta \rangle \hookrightarrow T55([Stat7, \cap]) \Rightarrow \quad p_0{}^{[1]} \in \mathbf{domain}(h_\Theta)$
$\langle p_1, h_\Theta \rangle \hookrightarrow T55([Stat7, \cap]) \Rightarrow \quad p_1{}^{[1]} \in \mathbf{domain}(h_\Theta)$
Suppose $\Rightarrow \quad \neg(p_0{}^{[1]} \in p_1{}^{[1]} \;\&\; p_0{}^{[2]} \notin p_1{}^{[2]})$
Suppose $\Rightarrow \quad p^{[1]} \in q^{[1]}$
$\langle Stat7, * \rangle$ ELEM $\Rightarrow \quad p_0 = p \;\&\; p_1 = q \;\&\; p^{[2]} \notin q^{[2]}$
EQUAL $\langle Stat7, * \rangle \Rightarrow$ false; Discharge $\Rightarrow \quad q^{[1]} \in p^{[1]}$
$\langle Stat7, * \rangle$ ELEM $\Rightarrow \quad p_0 = q \;\&\; p_1 = p \;\&\; q^{[2]} \notin p^{[2]}$
EQUAL $\langle Stat7, * \rangle \Rightarrow$ false; Discharge $\Rightarrow \quad Stat8: \; p_0{}^{[1]} \in p_1{}^{[1]} \;\&\; p_0{}^{[2]} \notin p_1{}^{[2]}$
$\langle h_\Theta, p_0 \rangle \hookrightarrow T67([Stat3, \cap]) \Rightarrow \quad h_\Theta \restriction p_0{}^{[1]} = p_0{}^{[2]}$
$\langle h_\Theta, p_1 \rangle \hookrightarrow T67([Stat3, \cap]) \Rightarrow \quad h_\Theta \restriction p_1{}^{[1]} = p_1{}^{[2]}$
EQUAL $\langle Stat8 \rangle \Rightarrow \quad h_\Theta \restriction p_0{}^{[1]} \notin h_\Theta \restriction p_1{}^{[1]}$
$\langle p_0{}^{[1]}, p_1{}^{[1]} \rangle \hookrightarrow Stat20 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 316 (subseq · 2) $\{i \in \mathbf{domain}(h_\Theta) \,|\, i \not\subseteq h_\Theta \restriction i\} = \emptyset$. PROOF:

Suppose_not $\Rightarrow \quad Stat0: \; \{i \in \mathbf{domain}(h_\Theta) \,|\, i \not\subseteq h_\Theta \restriction i\} \neq \emptyset$
Loc_def $\Rightarrow \quad i_0 = \mathbf{arb}(\{i \in \mathbf{domain}(h_\Theta) \,|\, i \not\subseteq h_\Theta \restriction i\})$
ELEM $\Rightarrow \quad Stat1: \; i_0 \cap \{i \in \mathbf{domain}(h_\Theta) \,|\, i \not\subseteq h_\Theta \restriction i\} = \emptyset \;\&\; Stat1a: \; i_0 \in \{i \in \mathbf{domain}(h_\Theta) \,|\, i \not\subseteq h_\Theta \restriction i\}$
$\langle \, \rangle \hookrightarrow Stat1a \Rightarrow \quad i_0 \in \mathbf{domain}(h_\Theta) \;\&\; i_0 \not\subseteq h_\Theta \restriction i_0$
Assump $\Rightarrow \quad \mathbf{domain}(f) \in \mathsf{next}(\mathbb{N})$
$\langle \mathbf{domain}(f) \rangle \hookrightarrow T10057 \Rightarrow \quad \mathbf{domain}(f) \subseteq \mathbb{N}$
$Tsubseq \cdot 1 \Rightarrow \quad \mathbf{domain}(h_\Theta) \in \mathsf{next}(\mathbb{N}) \;\&\; \mathbf{range}(h_\Theta) \subseteq \mathbb{N} \;\&\; Stat2: \; \langle \forall i \in \mathbf{domain}(h_\Theta), j \in \mathbf{domain}(h_\Theta) \,|\, i \in j \rightarrow h_\Theta \restriction i \in h_\Theta \restriction j \rangle$
$\langle \mathbf{domain}(h_\Theta) \rangle \hookrightarrow T10057 \Rightarrow \quad \mathbf{domain}(h_\Theta) \subseteq \mathbb{N}$
$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
ELEM $\Rightarrow \quad i_0 \in \mathbb{N} \;\&\; i_0 \neq \emptyset$
$T182([Stat1, \cap]) \Rightarrow \quad 1 \in \mathbb{N} \;\&\; \mathcal{O}(\emptyset)$
$\langle i_0, 1 \rangle \hookrightarrow T239([Stat1, \cap]) \Rightarrow \quad i_0 - 1 \in \mathbb{N}$

$\langle \mathbb{N}, i_0 \rangle \hookrightarrow T11([Stat1, \cap]) \Rightarrow \quad \mathcal{O}(i_0)$
Suppose $\Rightarrow \quad i_0 \neq \mathsf{next}(i_0 - 1)$
$\langle i_0, \emptyset \rangle \hookrightarrow T28([Stat1, \cap]) \Rightarrow \quad \emptyset \in i_0$
Use_def(1) $\Rightarrow \quad 1 = \mathsf{next}(\emptyset)$
Use_def(next) $\Rightarrow \quad 1 \subseteq i_0$
$\langle 1, i_0 \rangle \hookrightarrow T233([Stat1, \cap]) \Rightarrow \quad \#i_0 = i_0 - 1 + 1$
$\langle i_0 \rangle \hookrightarrow T180([Stat1, \cap]) \Rightarrow \quad i_0 = i_0 - 1 + 1$
$\langle i_0 - 1 \rangle \hookrightarrow T265([Stat1, \cap]) \Rightarrow$ false; Discharge $\Rightarrow \quad i_0 = \mathsf{next}(i_0 - 1)$
Use_def(next) $\Rightarrow \quad i_0 = i_0 - 1 \cup \{i_0 - 1\}$
$\langle \mathbb{N} \rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}(\mathsf{next}(\mathbb{N}))$
$\langle \mathsf{next}(\mathbb{N}), \mathbf{domain}(h_\Theta) \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathbf{domain}(h_\Theta))$
$\langle \mathbf{domain}(h_\Theta), i_0 \rangle \hookrightarrow T12 \Rightarrow \quad i_0 - 1 \in \mathbf{domain}(h_\Theta)$
$\langle i_0 - 1, i_0 \rangle \hookrightarrow Stat2 \Rightarrow \quad h_\Theta \restriction (i_0 - 1) \in h_\Theta \restriction i_0$
Suppose $\Rightarrow \quad i_0 - 1 \not\subseteq h_\Theta \restriction (i_0 - 1)$
Suppose $\Rightarrow \quad Stat3: \ i_0 - 1 \notin \{i \in \mathbf{domain}(h_\Theta) \mid i \not\subseteq h_\Theta \restriction i\}$
$\langle \, \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow \quad i_0 - 1 \in \{i \in \mathbf{domain}(h_\Theta) \mid i \not\subseteq h_\Theta \restriction i\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad i_0 - 1 \subseteq h_\Theta \restriction (i_0 - 1)$
$\langle i_0, h_\Theta \rangle \hookrightarrow T64 \Rightarrow \quad h_\Theta \restriction i_0 \in \mathbb{N}$
$\langle \mathbb{N}, h_\Theta \restriction i_0 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(h_\Theta \restriction i_0)$
$\langle h_\Theta \restriction i_0, h_\Theta \restriction (i_0 - 1) \rangle \hookrightarrow T12 \Rightarrow \quad i_0 - 1 \subseteq h_\Theta \restriction i_0 \ \& \ i_0 - 1 \neq h_\Theta \restriction i_0$
$\langle h_\Theta \restriction i_0, i_0 - 1 \rangle \hookrightarrow T12 \Rightarrow \quad i_0 - 1 \subseteq h_\Theta \restriction i_0$
$\langle i_0, i_0 - 1 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(i_0 - 1)$
$\langle h_\Theta \restriction i_0, i_0 - 1 \rangle \hookrightarrow T31 \Rightarrow \quad i_0 - 1 \in h_\Theta \restriction i_0$
Use_def(next) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that any subsequence g of a (finite or infinite) sequence f of elements of a set s is also a sequence of elements of s, having a (finite or infinite) length not exceeding the length of f.

Theorem 317 (subseq $\cdot$ 3) $\mathsf{Svm}(g) \ \& \ g \subseteq \mathbf{domain}(f) \times \mathbf{range}(f) \ \& \ \mathbf{domain}(g) \in \mathsf{next}(\mathbb{N}) \cap \mathsf{next}(\mathbf{domain}(f))$. PROOF:

Suppose_not $\Rightarrow \quad \neg\mathsf{Svm}(g) \lor g \not\subseteq \mathbf{domain}(f) \times \mathbf{range}(f) \lor \mathbf{domain}(g) \notin \mathsf{next}(\mathbb{N}) \cap \mathsf{next}(\mathbf{domain}(f))$
Assump $\Rightarrow \quad \mathsf{Svm}(f) \ \& \ \mathbf{domain}(f) \in \mathsf{next}(\mathbb{N})$
$Tsubseq \cdot 1 \Rightarrow \quad g = f \bullet h_\Theta \ \& \ 1\text{–}1(h_\Theta) \ \& \ \mathbf{domain}(h_\Theta) \in \mathsf{next}(\mathbb{N}) \ \& \ \mathbf{range}(h_\Theta) \subseteq \mathbf{domain}(f)$
$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N} \rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}(\mathsf{next}(\mathbb{N}))$
$\langle \mathsf{next}(\mathbb{N}), \mathbf{domain}(f) \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathbf{domain}(f))$
Use_def(1–1) $\Rightarrow \quad Stat3: \ \mathsf{Svm}(h_\Theta)$
$\langle h_\Theta, f \rangle \hookrightarrow T85 \Rightarrow \quad \mathbf{domain}(f \bullet h_\Theta) = \mathbf{domain}(h_\Theta) \ \& \ \mathbf{range}(f \bullet h_\Theta) = \mathbf{range}(f_{|\mathbf{range}(h_\Theta)})$
$\langle f, h_\Theta \rangle \hookrightarrow T103 \Rightarrow \quad \mathsf{Svm}(f \bullet h_\Theta)$

$\langle \mathsf{f}, \mathbf{range}(\mathsf{h}_\Theta)\rangle \hookrightarrow T72 \Rightarrow \quad \mathbf{range}(\mathsf{f}\bullet\mathsf{h}_\Theta) \subseteq \mathbf{range}(\mathsf{f})$
EQUAL $\Rightarrow \quad \mathsf{Svm}(\mathsf{g})\ \&\ \mathbf{domain}(\mathsf{g}) = \mathbf{domain}(\mathsf{h}_\Theta)\ \&\ \mathbf{range}(\mathsf{g}) \subseteq \mathbf{range}(\mathsf{f})$

-- This is, in detail, how we derive from the one-oneness of h_Θ the fact that $\mathbf{domain}(\mathsf{g})$ cannot be a larger ordinal than $\mathbf{domain}(\mathsf{f})$. Then only the second alternative of the or-statement in the Suppose_not statement, namely $\neg \mathsf{g} \subseteq \mathbf{domain}(\mathsf{f}) \times \mathbf{range}(\mathsf{f})$, survives.

$\langle \mathsf{h}_\Theta\rangle \hookrightarrow T131 \Rightarrow \quad \#\mathbf{range}(\mathsf{h}_\Theta) = \#\mathbf{domain}(\mathsf{h}_\Theta)$
$\langle \mathbf{domain}(\mathsf{f}), \mathbf{range}(\mathsf{h}_\Theta)\rangle \hookrightarrow T143 \Rightarrow \quad \#\mathbf{domain}(\mathsf{h}_\Theta) \subseteq \mathbf{domain}(\mathsf{f})$
$\langle \mathbf{domain}(\mathsf{h}_\Theta)\rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#\mathbf{domain}(\mathsf{h}_\Theta))$
$\langle \mathbf{domain}(\mathsf{f}), \#\mathbf{domain}(\mathsf{h}_\Theta)\rangle \hookrightarrow T31 \Rightarrow \quad \#\mathbf{domain}(\mathsf{h}_\Theta) \in \mathbf{domain}(\mathsf{f}) \vee \#\mathbf{domain}(\mathsf{h}_\Theta) = \mathbf{domain}(\mathsf{f})$
Use_def(next) $\Rightarrow \quad \#\mathbf{domain}(\mathsf{h}_\Theta) \in \mathsf{next}(\mathbf{domain}(\mathsf{f}))$
$\langle \mathbf{domain}(\mathsf{h}_\Theta), \mathbf{domain}(\mathsf{f})\rangle \hookrightarrow T10058 \Rightarrow \quad (\mathbf{domain}(\mathsf{h}_\Theta) = \mathbf{domain}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) = \mathbb{N}) \vee \mathbf{domain}(\mathsf{h}_\Theta) \in \mathsf{next}(\mathbf{domain}(\mathsf{f})) \cap \mathbb{N}$

Suppose $\Rightarrow \quad \mathbf{domain}(\mathsf{g}) \notin \mathsf{next}(\mathbf{domain}(\mathsf{f}))$
$\langle Stat3\rangle$ ELEM $\Rightarrow \quad \mathbf{domain}(\mathsf{h}_\Theta) = \mathbf{domain}(\mathsf{f})$
Use_def(next) $\Rightarrow \quad \mathbf{domain}(\mathsf{f}) \in \mathsf{next}(\mathbf{domain}(\mathsf{f}))$
$\langle Stat3\rangle$ ELEM $\Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow \quad \mathsf{g} \not\subseteq \mathbf{domain}(\mathsf{f}) \times \mathbf{range}(\mathsf{f})$

-- However, $\neg \mathsf{g} \subseteq \mathbf{domain}(\mathsf{f}) \times \mathbf{range}(\mathsf{f})$ amounts to one of the three possibilities $\neg \mathsf{Is\_map}(\mathsf{g})$, $\neg \mathbf{domain}(\mathsf{g}) \subseteq \mathbf{domain}(\mathsf{f})$, or $\neg \mathbf{range}(\mathsf{g}) \subseteq \mathbf{range}(\mathsf{f})$, the third of which has already been eliminated. The first possibility must be discarded too, because g is known to be a single-valued map. The possibility $\neg \mathbf{domain}(\mathsf{g}) \subseteq \mathbf{domain}(\mathsf{f})$ would yield that $\mathbf{domain}(\mathsf{f})$ precedes $\mathbf{domain}(\mathsf{g})$ in the standard order of ordinals, but this conflicts with the fact, just derived above, that $\mathbf{domain}(\mathsf{g}) \in \mathsf{next}(\mathbf{domain}(\mathsf{f}))$.

Use_def(Svm) $\Rightarrow \quad \mathsf{Is\_map}(\mathsf{g})$
$\langle \mathsf{g}, \mathbf{domain}(\mathsf{f}), \mathbf{range}(\mathsf{f})\rangle \hookrightarrow T116 \Rightarrow \quad \mathbf{domain}(\mathsf{g}) \not\subseteq \mathbf{domain}(\mathsf{f})$
$\langle \mathbf{domain}(\mathsf{f})\rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}\big(\mathsf{next}(\mathbf{domain}(\mathsf{f}))\big)$
$\langle \mathsf{next}(\mathbf{domain}(\mathsf{f})), \mathbf{domain}(\mathsf{g})\rangle \hookrightarrow T12 \Rightarrow \quad \mathbf{domain}(\mathsf{g}) \subseteq \mathsf{next}(\mathbf{domain}(\mathsf{f}))$
Use_def(next) $\Rightarrow \quad \mathbf{domain}(\mathsf{f}) \in \mathbf{domain}(\mathsf{g})$
Use_def(next) $\Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow$ QED

Theorem 318 (subseq · 4) $\mathbf{domain}(\mathsf{h}_\Theta) \neq \mathbb{N} \rightarrow \mathsf{Finite}(\mathsf{g})$. PROOF:

Suppose_not $\Rightarrow \quad \mathbf{domain}(\mathsf{h}_\Theta) \neq \mathbb{N}\ \&\ \neg\mathsf{Finite}(\mathsf{g})$
$Tsubseq \cdot 1 \Rightarrow \quad \mathsf{g} = \mathsf{f}\bullet\mathsf{h}_\Theta\ \&\ 1\text{–}1(\mathsf{h}_\Theta)\ \&\ \mathbf{domain}(\mathsf{h}_\Theta) \in \mathsf{next}(\mathbb{N})\ \&\ \mathbf{range}(\mathsf{h}_\Theta) \subseteq \mathbf{domain}(\mathsf{f})$
$\langle \mathbf{domain}(\mathsf{g})\rangle \hookrightarrow T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})\ \&\ \neg\mathsf{Finite}(\mathbb{N})\ \&\ \big(\mathbf{domain}(\mathsf{g}) \in \mathbb{N} \rightarrow \mathsf{Finite}(\mathbf{domain}(\mathsf{g}))\big)$
Assump $\Rightarrow \quad \mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) \in \mathsf{next}(\mathbb{N})$

$\langle \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow T10057 \Rightarrow \quad \mathbf{domain}(\mathsf{f}) \subseteq \mathbb{N}$
Use_def(next) $\Rightarrow \quad \mathbf{domain}(\mathsf{h}_\ominus) \in \mathbb{N}$
$\langle \mathbb{N}, \mathbf{domain}(\mathsf{h}_\ominus) \rangle \hookrightarrow T12 \Rightarrow \quad \mathbf{domain}(\mathsf{h}_\ominus) \subseteq \mathbb{N} \ \& \ \mathbf{range}(\mathsf{h}_\ominus) \subseteq \mathbb{N}$
$\langle \mathsf{h}_\ominus, \mathsf{f} \rangle \hookrightarrow T85 \Rightarrow \quad \mathbf{domain}(\mathsf{f} \bullet \mathsf{h}_\ominus) = \mathbf{domain}(\mathsf{h}_\ominus)$
Use_def(next) $\Rightarrow \quad Stat10: \ \mathbf{domain}(\mathsf{h}_\ominus) \in \mathbb{N}$
$\langle \mathsf{f}, \mathsf{h}_\ominus \rangle \hookrightarrow T50 \Rightarrow \quad \mathsf{Is\_map}(\mathsf{f} \bullet \mathsf{h}_\ominus)$
Use_def(1–1) $\Rightarrow \quad \mathsf{Svm}(\mathsf{h}_\ominus)$
$\langle \mathsf{f}, \mathsf{h}_\ominus \rangle \hookrightarrow T103 \Rightarrow \quad \mathsf{Svm}(\mathsf{f} \bullet \mathsf{h}_\ominus)$
EQUAL $\Rightarrow \quad \mathbf{domain}(\mathsf{g}) \in \mathbb{N} \ \& \ \mathsf{Is\_map}(\mathsf{g}) \ \& \ \mathsf{Svm}(\mathsf{g})$
$\langle \mathsf{g} \rangle \hookrightarrow T165 \Rightarrow \quad \mathsf{Finite}(\mathbf{range}(\mathsf{g}))$
$\langle \mathbf{domain}(\mathsf{g}), \mathbf{range}(\mathsf{g}) \rangle \hookrightarrow T225 \Rightarrow \quad \mathsf{Finite}(\mathbf{domain}(\mathsf{g}) \times \mathbf{range}(\mathsf{g}))$
$\langle \mathsf{g}, \mathbf{domain}(\mathsf{g}), \mathbf{range}(\mathsf{g}) \rangle \hookrightarrow T116(\langle Stat10 \rangle) \Rightarrow \quad \mathsf{g} \subseteq \mathbf{domain}(\mathsf{g}) \times \mathbf{range}(\mathsf{g})$
$\langle \mathbf{domain}(\mathsf{g}) \times \mathbf{range}(\mathsf{g}), \mathsf{g} \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY subseq

THEORY subseq(g, f)

-- Subsequence of a finite or denumerable sequence

$\mathsf{Svm}(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) \in \mathsf{next}(\mathbb{N})$
$\mathsf{g} \in \mathsf{Subseqs}(\mathsf{f})$
$\Rightarrow (\mathsf{h}_\ominus)$
$\mathsf{g} = \mathsf{f} \bullet \mathsf{h}_\ominus \ \& \ 1\text{–}1(\mathsf{h}_\ominus) \ \& \ \mathbf{domain}(\mathsf{h}_\ominus) \in \mathsf{next}(\mathbb{N}) \ \& \ \mathbf{range}(\mathsf{h}_\ominus) \subseteq \mathbf{domain}(\mathsf{f})$
$\langle \forall \mathsf{i} \in \mathbf{domain}(\mathsf{h}_\ominus), \mathsf{j} \in \mathbf{domain}(\mathsf{h}_\ominus) \mid \mathsf{i} \in \mathsf{j} \rightarrow \mathsf{h}_\ominus \upharpoonright \mathsf{i} \in \mathsf{h}_\ominus \upharpoonright \mathsf{j} \rangle$
$\{\mathsf{i} \in \mathbf{domain}(\mathsf{h}_\ominus) \mid \mathsf{i} \not\subseteq \mathsf{h}_\ominus \upharpoonright \mathsf{i}\} = \emptyset$
$\mathsf{Svm}(\mathsf{g}) \ \& \ \mathsf{g} \subseteq \mathbf{domain}(\mathsf{f}) \times \mathbf{range}(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{g}) \in \mathsf{next}(\mathbb{N}) \cap \mathsf{next}(\mathbf{domain}(\mathsf{f}))$
$\mathbf{domain}(\mathsf{h}_\ominus) \neq \mathbb{N} \rightarrow \mathsf{Finite}(\mathsf{g})$
END subseq

-- Our next theorem states that every subsequence of a finite sequence f of elements of a set s is an alike sequence, whose length does not exceed the length of f.

Theorem 319 (10071) $\mathsf{F} \in \mathsf{Fin\_seqs}(\mathsf{S}) \ \& \ \mathsf{G} \in \mathsf{Subseqs}(\mathsf{F}) \rightarrow \mathsf{G} \in \mathsf{Fin\_seqs}(\mathsf{S}) \ \& \ \mathbf{domain}(\mathsf{G}) \in \mathsf{next}(\mathbf{domain}(\mathsf{F}))$. PROOF:

Suppose_not(f, s, g) $\Rightarrow \quad \mathsf{f} \in \mathsf{Fin\_seqs}(\mathsf{s}) \ \& \ \mathsf{g} \in \mathsf{Subseqs}(\mathsf{f}) \ \& \ \mathsf{g} \notin \mathsf{Fin\_seqs}(\mathsf{s}) \vee \mathbf{domain}(\mathsf{g}) \notin \mathsf{next}(\mathbf{domain}(\mathsf{f}))$

-- For, assuming by contradiction that $\mathsf{f}, \mathsf{s}, \mathsf{g}$ are a counterexample, from the THEORY subseq we get that g is a function included in $\mathbf{domain}(\mathsf{f}) \times \mathsf{s}$, having as its domain an ordinal not exceeding either $\mathbb{N}$ or $\mathbf{domain}(\mathsf{f})$. It turns out readily that g is included in $\mathbb{N} \times \mathsf{s}$; hence, in order that g does not belong to $\mathsf{Fin\_seqs}(\mathsf{s})$, its domain must equal $\mathbb{N}$.

$\mathsf{Use\_def}(\mathsf{Fin\_seqs}) \Rightarrow \quad \mathit{Stat1}: \ \mathsf{f} \in \{\mathsf{f} \subseteq \mathbb{N} \times \mathsf{s} \mid \mathsf{Svm}(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) \in \mathbb{N}\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{f} \subseteq \mathbb{N} \times \mathsf{s} \ \& \ \mathsf{Svm}(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) \in \mathbb{N}$

$\mathsf{Use\_def}(\mathsf{next}) \Rightarrow \quad \mathbf{domain}(\mathsf{f}) \in \mathsf{next}(\mathbb{N})$

$\mathsf{APPLY}\ \langle \mathsf{h}_\Theta : \mathsf{h} \rangle\ \mathsf{subseq}(\mathsf{g} \mapsto \mathsf{g}, \mathsf{f} \mapsto \mathsf{f}) \Rightarrow$

$\quad \mathsf{g} \subseteq \mathbf{domain}(\mathsf{f}) \times \mathbf{range}(\mathsf{f}) \ \& \ \mathsf{Svm}(\mathsf{g}) \ \& \ \mathbf{domain}(\mathsf{g}) \in \mathsf{next}(\mathbb{N}) \cap \mathsf{next}(\mathbf{domain}(\mathsf{f}))$

$\langle \mathsf{f}, \mathbb{N}, \mathsf{s} \rangle \hookrightarrow \mathit{T116} \Rightarrow \quad \mathbf{domain}(\mathsf{f}) \subseteq \mathbb{N} \ \& \ \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s}$

$\langle \mathbf{domain}(\mathsf{f}), \mathbb{N}, \mathbf{range}(\mathsf{f}), \mathsf{s} \rangle \hookrightarrow \mathit{T219} \Rightarrow \quad \mathsf{g} \subseteq \mathbb{N} \times \mathsf{s}$

$\langle \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow \mathit{T179} \Rightarrow \quad \mathcal{O}(\mathbb{N}) \ \& \ \mathsf{Card}(\mathbf{domain}(\mathsf{f})) \ \& \ \mathsf{Finite}(\mathbf{domain}(\mathsf{f}))$

$\langle \mathbb{N} \rangle \hookrightarrow \mathit{T29} \Rightarrow \quad \mathcal{O}(\mathsf{next}(\mathbb{N}))$

$\mathsf{Suppose} \Rightarrow \quad \mathbb{N} = \mathbf{domain}(\mathsf{g})$

-- However, if we assume that $\mathbf{domain}(\mathsf{g}) = \mathbb{N}$, then ...

$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad \mathcal{O}(\mathbf{domain}(\mathsf{f}))$

$\langle \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow \mathit{T29} \Rightarrow \quad \mathcal{O}(\mathsf{next}(\mathbf{domain}(\mathsf{f})))$

$\langle \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow \mathit{T173} \Rightarrow \quad \mathsf{Finite}(\mathsf{next}(\mathbf{domain}(\mathsf{f})))$

$\langle \mathsf{next}(\mathbf{domain}(\mathsf{f})) \rangle \hookrightarrow \mathit{T178} \Rightarrow \quad \mathsf{Card}(\mathsf{next}(\mathbf{domain}(\mathsf{f})))$

-- ...we get that $\mathsf{next}(\mathbf{domain}(\mathsf{f})) \in \mathbf{domain}(\mathsf{g})$, conflicting with $\mathbf{domain}(\mathsf{g}) \in \mathsf{next}(\mathbf{domain}(\mathsf{f}))$.

$\langle \mathsf{next}(\mathbf{domain}(\mathsf{f})) \rangle \hookrightarrow \mathit{T179} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathbb{N} \neq \mathbf{domain}(\mathsf{g})$

$\mathsf{Use\_def}(\mathsf{next}) \Rightarrow \quad \mathbf{domain}(\mathsf{g}) \in \mathbb{N}$

$\mathsf{Use\_def}(\mathsf{Fin\_seqs}) \Rightarrow \quad \mathit{Stat3}: \ \mathsf{g} \notin \{\mathsf{f} \subseteq \mathbb{N} \times \mathsf{S} \mid \mathsf{Svm}(\mathsf{f}) \ \& \ \mathbf{domain}(\mathsf{f}) \in \mathbb{N}\}$

-- We have reached a contradiction showing that the statement of this lemma is true.

$\langle\ \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next theorem states that every shifted sequence is a subsequence of the original sequence.

Theorem 320 (10072) $\mathsf{M} \in \mathbb{N} \rightarrow \mathsf{Shifted\_seq}(\mathsf{F}, \mathsf{M}) \in \mathsf{Subseqs}(\mathsf{F})$. PROOF:

$\text{Suppose\_not}(m, f) \Rightarrow$ $\mathit{Stat0}:\ m \in \mathbb{N}\ \&\ \text{Shifted\_seq}(f, m) \notin \text{Subseqs}(f)$
$\text{Use\_def}(\text{Shifted\_seq}) \Rightarrow$ $f \bullet \text{Shift}(m) \notin \text{Subseqs}(f)$

-- For, assuming that the contrary is true for m, f and exploiting the definition of 'Subseqs', we get that $\text{Shift}(m)$ is not a function, or it is not includedin $\mathbb{N} \times \mathbb{N}$, or its domain is not closed relative to predecessor extraction, or it is not increasing.

$\text{Use\_def}(\text{Subseqs}) \Rightarrow$ $\mathit{Stat1}:\ f \bullet \text{Shift}(m) \notin \{f \bullet h :\ h \subseteq \mathbb{N} \times \mathbb{N} \mid \text{Svm}(h)\ \&\ \langle \forall i \mid \text{next}(i) \in \mathbf{domain}(h) \rightarrow i \in \mathbf{domain}(h)\ \&\ h \upharpoonright i \in h \upharpoonright \text{next}(i) \rangle \}$
$\langle \text{Shift}(m) \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\text{Shift}(m) \not\subseteq \mathbb{N} \times \mathbb{N} \vee \neg \text{Svm}(\text{Shift}(m)) \vee \neg \langle \forall i \mid \text{next}(i) \in \mathbf{domain}(\text{Shift})(m) \rightarrow i \in \mathbf{domain}(\text{Shift})(m)\ \&\ \text{Shift}(m) \upharpoonright i \in \text{Shift}(m) \upharpoonright \text{next}(i) \rangle$
$\text{Use\_def}(\text{Shift}) \Rightarrow$ $\text{Shift}(m) = \{[i, m + i] :\ i \in \mathbb{N}\}$

-- The first two possibilities are discarded quite straightforwardly.

$\text{Suppose} \Rightarrow$ $\mathit{Stat1a}:\ \{[i, m + i] :\ i \in \mathbb{N}\} \not\subseteq \mathbb{N} \times \mathbb{N}$
$\langle c \rangle \hookrightarrow \mathit{Stat1a} \Rightarrow$ $\mathit{Stat2a}:\ c \in \{[i, m + i] :\ i \in \mathbb{N}\}\ \&\ c \notin \mathbb{N} \times \mathbb{N}$
$\langle ir \rangle \hookrightarrow \mathit{Stat2a} \Rightarrow$ $ir \in \mathbb{N}\ \&\ [ir, m + ir] \notin \mathbb{N} \times \mathbb{N}$
$\text{ALGEBRA} \Rightarrow$ $m + ir \in \mathbb{N}$
$\text{Use\_def}(\times) \Rightarrow$ $\mathit{Stat3a}:\ [ir, m + ir] \notin \{[i, j] :\ i \in \mathbb{N}, j \in \mathbb{N}\}$
$\langle ir, m + ir \rangle \hookrightarrow \mathit{Stat3a} \Rightarrow$ false; $\text{Discharge} \Rightarrow$ $\text{Shift}(m) \subseteq \mathbb{N} \times \mathbb{N}$
$\langle m \rangle \hookrightarrow \mathit{T274a} \Rightarrow$ $\mathit{Stat7}:\ \text{Svm}(\text{Shift}(m))\ \&\ \mathit{Stat8a}:\ \neg \langle \forall i \mid \text{next}(i) \in \mathbf{domain}(\text{Shift})(m) \rightarrow i \in \mathbf{domain}(\text{Shift})(m)\ \&\ \text{Shift}(m) \upharpoonright i \in \text{Shift}(m) \upharpoonright \text{next}(i) \rangle$

-- The third possibility is also discarded easily, so only the fourth one must be considered.

$\langle i \rangle \hookrightarrow \mathit{Stat8a} \Rightarrow$ $\text{next}(i) \in \mathbf{domain}(\text{Shift})(m)\ \&\ i \notin \mathbf{domain}(\text{Shift})(m) \vee \text{Shift}(m) \upharpoonright i \notin \text{Shift}(m) \upharpoonright \text{next}(i)$
$\text{ELEM} \Rightarrow$ $\mathbf{domain}(\{[i, m + i] :\ i \in \mathbb{N}\}) = \mathbb{N}$
$\text{EQUAL} \Rightarrow$ $\mathbf{domain}(\text{Shift})(m) = \mathbf{domain}(\{[i, m + i] :\ i \in \mathbb{N}\})$
$\text{ELEM} \Rightarrow$ $\text{next}(i) \in \mathbb{N}$
$\mathit{T179} \Rightarrow$ $\mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, \text{next}(i) \rangle \hookrightarrow \mathit{T12} \Rightarrow$ $\text{next}(i) \subseteq \mathbb{N}$
$\text{Use\_def}(\text{next}) \Rightarrow$ $\mathit{Stat8}:\ i \in \mathbb{N}\ \&\ \text{Shift}(m) \upharpoonright i \notin \text{Shift}(m) \upharpoonright \text{next}(i)$

-- By eliminating the fourth possibility, we get a contradiction proving the statement of the present theorem.

$\text{Suppose} \Rightarrow$ $\mathit{Stat9}:\ [i, m + i] \notin \{[i, m + i] :\ i \in \mathbb{N}\}$
$\langle i \rangle \hookrightarrow \mathit{Stat9} \Rightarrow$ false; $\text{Discharge} \Rightarrow$ $\mathit{Stat10}:\ [i, m + i] \in \text{Shift}(m)$
$\langle \text{Shift}(m), [i, m + i] \rangle \hookrightarrow \mathit{T67}(\langle \mathit{Stat7}, \mathit{Stat10}, * \rangle) \Rightarrow$ $\text{Shift}(m) \upharpoonright [i, m + i]^{[1]} =$
$[i, m + i]^{[2]}$
$\text{Suppose} \Rightarrow$ $\mathit{Stat11}:\ [\text{next}(i), m + \text{next}(i)] \notin \{[j, m + j] :\ j \in \mathbb{N}\}$
$\langle \text{next}(i) \rangle \hookrightarrow \mathit{Stat11} \Rightarrow$ false; $\text{Discharge} \Rightarrow$ $\mathit{Stat12}:\ [\text{next}(i), m + \text{next}(i)] \in \text{Shift}(m)$

$\langle \mathsf{Shift}(m), [\mathsf{next}(i), m + \mathsf{next}(i)] \rangle \hookrightarrow T67(\langle Stat7, Stat12, * \rangle) \Rightarrow$ $\mathsf{Shift}(m) \upharpoonright [\mathsf{next}(i), m + \mathsf{next}(i)]^{[1]} =$
$[\mathsf{next}(i), m + \mathsf{next}(i)]^{[2]}$
$TELEM \Rightarrow$ $[i, m + i]^{[1]} = i \;\&\; [i, m + i]^{[2]} = m + i \;\&\; [\mathsf{next}(i), m + \mathsf{next}(i)]^{[1]} = \mathsf{next}(i) \;\&\; [\mathsf{next}(i), m + \mathsf{next}(i)]^{[2]} = m + \mathsf{next}(i)$
$\langle i \rangle \hookrightarrow T265([Stat8, Stat8]) \Rightarrow$ $i + 1 = \mathsf{next}(i)$
ALGEBRA $\Rightarrow$ $Stat13: \; m + i \in \mathbb{N} \;\&\; m + (i + 1) = (m + i) + 1$
$\langle m + i \rangle \hookrightarrow T265(\langle Stat13 \rangle) \Rightarrow$ $m + i + 1 = \mathsf{next}(m + i)$
EQUAL $\langle Stat8, * \rangle \Rightarrow$ $m + i \notin \mathsf{next}(m + i)$
Use_def(next) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- In the special case of a finite sequence f, the preceding theorem specializes into one stating that every sequence obtained by shifting f is a (finite) subsequence of f.

Theorem 321 (10073) $F \in \mathsf{Fin\_seqs}(S) \;\&\; M \in \mathbb{N} \rightarrow \mathsf{Shifted\_seq}(F, M) \in \mathsf{Fin\_seqs}(S) \;\&\; \mathbf{domain}(F \bullet \mathsf{Shift}(M)) \in \mathsf{next}(\mathbf{domain}(F))$. PROOF:

Suppose_not(f, s, m) $\Rightarrow$ $f \in \mathsf{Fin\_seqs}(s) \;\&\; m \in \mathbb{N} \;\&\; \mathsf{Shifted\_seq}(f, m) \notin \mathsf{Fin\_seqs}(s) \vee \mathbf{domain}(\mathsf{Shifted\_seq})(f, m) \notin \mathsf{next}(\mathbf{domain}(f))$
$\langle m, f \rangle \hookrightarrow T10072 \Rightarrow$ $\mathsf{Shifted\_seq}(f, m) \in \mathsf{Subseqs}(f)$
Use_def(Shifted_seq) $\Rightarrow$ $f \bullet \mathsf{Shift}(m) \in \mathsf{Subseqs}(f) \;\&\; f \bullet \mathsf{Shift}(m) \notin \mathsf{Fin\_seqs}(s) \vee \mathbf{domain}(f \bullet \mathsf{Shift}(m)) \notin \mathsf{next}(\mathbf{domain}(f))$
$\langle f, s, f \bullet \mathsf{Shift}(m) \rangle \hookrightarrow T10071 \Rightarrow$ $\mathbf{domain}(f \bullet \mathsf{Shift}(m)) \notin \mathsf{next}(\mathbf{domain}(f))$
Use_def(Fin_seqs) $\Rightarrow$ $Stat1: \; f \in \{f \subseteq \mathbb{N} \times s \mid \mathsf{Svm}(f) \;\&\; \mathbf{domain}(f) \in \mathbb{N}\}$
$\langle \, \rangle \hookrightarrow Stat1 \Rightarrow$ $f \subseteq \mathbb{N} \times s \;\&\; \mathsf{Svm}(f) \;\&\; \mathbf{domain}(f) \in \mathbb{N}$
Use_def(next) $\Rightarrow$ $\mathbf{domain}(f) \in \mathsf{next}(\mathbb{N})$
APPLY $\langle h_\Theta : h \rangle$ subseq $(g \mapsto f \bullet \mathsf{Shift}(m), f \mapsto f) \Rightarrow$ $\mathbf{domain}(f \bullet \mathsf{Shift}(m)) \in \mathsf{next}(\mathbb{N}) \cap \mathsf{next}(\mathbf{domain}(f))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

9 The cardinal product theorem (a digression)

-- Next we start to prepare for proof of the cardinal product theorem (Theorem 195 below), which asserts that the product of two infinite cardinals is always the larger of the two. This proof uses properties of the product ordering of the Cartesian product of two ordinals which the following theory begins to lay out.

THEORY ordval_fcn$(s, f(x))$
-- Elementary properties of ordinal - valued functions
$s \neq \emptyset \;\&\; \langle \forall x \in s \mid \mathcal{O}(f(x)) \rangle$
END ordval_fcn

ENTER_THEORY ordval_fcn

-- Points at which f attains its minimum

DEF 00a. $\mathsf{rng}_\Theta \;=_{\text{Def}}\; \{x :\ x \in s \mid f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})\}$

-- The first result we prove within the present theory is that the ordinal-valued function f assumed by the theory always attains its minimum value $\mathbf{arb}(\{f(u) :\ u \in s\})$.

-- An ordinal - valued function attains its minimum

Theorem 322 (ordval_fcn$_1$) $\mathsf{rng}_\Theta \neq \emptyset \;\&\; \langle \forall x \in \mathsf{rng}_\Theta, y \in s \mid f(x) \subseteq f(y)\rangle \;\&\; \mathsf{rng}_\Theta = \{x :\ x \in s \mid f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})\}$. PROOF:

Suppose_not(s) $\Rightarrow$ $\mathsf{rng}_\Theta = \emptyset \vee \neg\langle \forall x \in \mathsf{rng}_\Theta, y \in s \mid f(x) \subseteq f(y)\rangle \vee \mathsf{rng}_\Theta \neq \{x :\ x \in s \mid f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})\}$

-- Assume the contrary. Since $\{f(u) :\ u \in s\}$ is clearly nonempty, $\mathbf{arb}(\{f(y) :\ y \in s\}) \in \{f(u) :\ u \in s\}$ by the axiom of choice. Hence $\mathbf{arb}(\{f(y) :\ y \in s\})$ can be written as $f(d)$ with $d \in s$, and then it is clear by definition that $d \in \mathsf{rng}_\Theta$ so that $\mathsf{rng}_\Theta \neq \emptyset$, and hence only the second clause of out theorem can be false, i. e there must exist $x \in \mathsf{rng}_\Theta$, $y \in s$ such that $f(y)$ is less than $f(x)$.

Use_def(rng_Θ) $\Rightarrow$ $\mathsf{rng}_\Theta = \{x :\ x \in s \mid f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})\}$
Assump $\Rightarrow$ *Stat1* : $s \neq \emptyset$ & *Stat2* : $\langle \forall x \in s \mid \mathcal{O}(f(x))\rangle$
$\langle c'\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $c' \in s$
Suppose $\Rightarrow$ *Stat3* : $f(c') \notin \{f(u) :\ u \in s\}$
$\langle c'\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ $\{f(u) :\ u \in s\} \neq \emptyset$
$\langle \{f(y) :\ y \in s\}\rangle \hookrightarrow$ *T0* $\Rightarrow$ *Stat4* :
$\mathbf{arb}(\{f(y) :\ y \in s\}) \in \{f(u) :\ u \in s\}$ & $\mathbf{arb}(\{f(y) :\ y \in s\}) \cap \{f(u) :\ u \in s\} = \emptyset$
$\langle d\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $d \in s$ & $\mathbf{arb}(\{f(y) :\ y \in s\}) = f(d)$
Suppose $\Rightarrow$ $d \notin \mathsf{rng}_\Theta$
Use_def(rng_Θ) $\Rightarrow$ *Stat5* : $d \notin \{x :\ x \in s \mid f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})\}$
$\langle d\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6* : $\neg\langle \forall x \in \mathsf{rng}_\Theta, y \in s \mid f(x) \subseteq f(y)\rangle$
$\langle x, y\rangle \hookrightarrow$ *Stat6* $\Rightarrow$ *Stat7* : $x \in \mathsf{rng}_\Theta$ & $y \in s$ & $f(x) \not\subseteq f(y)$

-- rng_Θ is clearly a subset of s, while $f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})$ and $f(y)$ are both ordinals.

Suppose $\Rightarrow$ *Stat8* : $s \not\supseteq \mathsf{rng}_\Theta$
$\langle c\rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $c \notin s$ & $c \in \mathsf{rng}_\Theta$
Use_def(rng_Θ) $\Rightarrow$ *Stat9* : $c \in \{x \in s \mid f(x) = \mathbf{arb}(\{f(u) :\ u \in s\})\}$
$\langle\ \rangle \hookrightarrow$ *Stat9* $\Rightarrow$ false; Discharge $\Rightarrow$ $s \supseteq \mathsf{rng}_\Theta$
$\langle x\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathcal{O}(f(x))$
$\langle y\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathcal{O}(f(y))$

$\langle f(x), f(y)\rangle \hookrightarrow T32 \Rightarrow \quad f(y) \in f(x)$
Use_def$(\mathrm{rng}_\Theta) \Rightarrow \quad Stat10: \; x \in \{v : v \in s \,|\, f(v) = \mathbf{arb}(\{f(u) : u \in s\})\}$
$\langle v\rangle \hookrightarrow Stat10 \Rightarrow \quad x = v \;\&\; f(v) = \mathbf{arb}(\{f(u) : u \in s\})$
EQUAL $\Rightarrow \quad f(x) = \mathbf{arb}(\{f(u) : u \in s\})$

-- But $f(y)$ is clearly in $\{f(u) : u \in s\}$, and since this violates the disjointness clause of the axiom of choice as applied above, have a contradiction which completes our proof.

Suppose $\Rightarrow \quad Stat11: \; f(y) \notin \{f(u) : u \in s\}$
$\langle y\rangle \hookrightarrow Stat11 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad f(y) \in \{f(u) : u \in s\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is also obvious that the set rng_Θ on which $f(x)$ attains its minimum value is a subset of s.

Theorem 323 (ordval_fcn$_2$) $\mathrm{rng}_\Theta \subseteq s$. PROOF:

Suppose_not $\Rightarrow \quad \mathrm{rng}_\Theta \not\subseteq s$
Use_def$(\mathrm{rng}_\Theta) \Rightarrow \quad \mathrm{rng}_\Theta = \{x : x \in s \,|\, f(x) = \mathbf{arb}(\{f(y) : y \in s\})\}$
Set_monot $\Rightarrow \quad \{x : x \in s \,|\, f(x) = \mathbf{arb}(\{f(y) : y \in s\})\} \subseteq \{x : x \in s\}$
ELEM $\Rightarrow \quad \mathrm{rng}_\Theta \subseteq \{x : x \in s\}$
SIMPLF $\Rightarrow \quad \{x : x \in s\} = s$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Finally it is clear that any $y \in s$ at which the minimum of f is attained belongs to the set rng_Θ.

Theorem 324 (ordval_fcn$_3$) $\langle \forall x \in \mathrm{rng}_\Theta, y \in s \,|\, f(x) = f(y) \rightarrow y \in \mathrm{rng}_\Theta\rangle$. PROOF:

Suppose_not$(s) \Rightarrow \quad Stat1: \; \neg\langle \forall x \in \mathrm{rng}_\Theta, y \in s \,|\, f(x) = f(y) \rightarrow y \in \mathrm{rng}_\Theta\rangle$
$\langle x, y\rangle \hookrightarrow Stat1 \Rightarrow \quad x \in \mathrm{rng}_\Theta \;\&\; y \in s \;\&\; f(x) = f(y) \;\&\; y \notin \mathrm{rng}_\Theta$
Use_def$(\mathrm{rng}_\Theta) \Rightarrow \quad Stat2: \; x \in \{x \in s \,|\, f(x) = \mathbf{arb}(\{f(u) : u \in s\})\}$
$\langle\,\rangle \hookrightarrow Stat2 \Rightarrow \quad f(x) = \mathbf{arb}(\{f(u) : u \in s\})$
ELEM $\Rightarrow \quad f(y) = \mathbf{arb}(\{f(u) : u \in s\})$
Suppose $\Rightarrow \quad Stat3: \; y \notin \{x : x \in s \,|\, f(x) = \mathbf{arb}(\{f(u) : u \in s\})\}$
$\langle y\rangle \hookrightarrow Stat3 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad y \in \{x : x \in s \,|\, f(x) = \mathbf{arb}(\{f(u) : u \in s\})\}$
Use_def$(\mathrm{rng}_\Theta) \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY ordval_fcn

THEORY ordval_fcn(s, f)

-- Elementary functions of ordinal - valued functions

$s \neq \emptyset \;\&\; \langle \forall x \in s \mid \mathcal{O}(f(x)) \rangle$

$\Rightarrow (rng_\Theta)$

$rng_\Theta = \{x : x \in s \mid f(x) = \mathbf{arb}(\{f(y) : y \in s\})\} \;\&\; rng \neq \emptyset \;\&\; \langle \forall x \in rng_\Theta, y \in s \mid f(x) \subseteq f(y) \rangle$

$rng \subseteq s$

END ordval_fcn

-- —- —- Our next theory concerns binary relations R which are well-founded on a given domain s. This means that R is irreflexive on s, and that each non-null subset t of this domain has a 'minimal' element, i. e. an element not greater than any other element of t in the ordering defined by R. We will show that any such relation can be extended into one which is isomorphic to the membership relator on an ordinal in 1-1 ordered correspondence with the set s. —- —-

THEORY well_founded_set(s, x ◁ y)

$\langle \forall t \mid t \subseteq s \;\&\; t \neq \emptyset \longrightarrow \langle \exists w \in t, \forall v \in t \mid \neg v \lhd w \rangle \rangle$

END well_founded_set

ENTER_THEORY well_founded_set

-- —- Our first theorem shows that the relation is antisymmetric, and hence irreflexive. —-

Theorem 325 (well_founded_set · 0) $X, Y \in s \rightarrow \neg(X \lhd Y \;\&\; Y \lhd X) \;\&\; \neg X \lhd X$. PROOF:

-- We proceed by contradiction. Suppose that our theorem is false, and let x, y be a counterexample.

Suppose_not(x, s, y) ⇒ $x, y \in s \;\&\; (x \lhd y \;\&\; y \lhd x) \vee x \lhd x$

ELEM ⇒ $\{x, y\} \subseteq s \;\&\; \{x, y\} \neq \emptyset \;\&\; \{x\} \subseteq s \;\&\; \{x\} \neq \emptyset$

-- Suppose first that the pair x,y violates antisymmetry, so that x precedes y and y precedes x. Any minimal element u of the doubleton $\{\mathsf{x},\mathsf{y}\}$ must be either x or y; but if it is x this conflicts with the fact that y precedes x; and symmetrically if the minimal element is y.

Suppose $\Rightarrow$ $\mathsf{x} \lhd \mathsf{y} \;\&\; \mathsf{y} \lhd \mathsf{x}$
Assump $\Rightarrow$ *Stat1* : $\langle \forall \mathsf{t} \mid \mathsf{t} \subseteq \mathsf{s} \;\&\; \mathsf{t} \neq \emptyset \rightarrow \langle \exists \mathsf{w} \in \mathsf{t}, \forall \mathsf{v} \in \mathsf{t} \mid \neg \mathsf{v} \lhd \mathsf{w} \rangle \rangle$
$\langle \{\mathsf{x},\mathsf{y}\} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $\langle \exists \mathsf{w} \in \{\mathsf{x},\mathsf{y}\}, \forall \mathsf{v} \in \{\mathsf{x},\mathsf{y}\} \mid \neg \mathsf{v} \lhd \mathsf{w} \rangle$
$\langle \mathsf{u} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{u} \in \{\mathsf{x},\mathsf{y}\} \;\&\;$ *Stat3* : $\langle \forall \mathsf{v} \in \{\mathsf{x},\mathsf{y}\} \mid \neg \mathsf{v} \lhd \mathsf{u} \rangle$
$\langle \mathsf{x} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\mathsf{x} \in \{\mathsf{x},\mathsf{y}\} \rightarrow \neg \mathsf{x} \lhd \mathsf{u}$
$\langle \mathsf{y} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\mathsf{y} \in \{\mathsf{x},\mathsf{y}\} \rightarrow \neg \mathsf{y} \lhd \mathsf{u}$
ELEM $\Rightarrow$ $(\mathsf{u} = \mathsf{y} \;\&\; \neg \mathsf{x} \lhd \mathsf{u}) \vee (\mathsf{u} = \mathsf{x} \;\&\; \neg \mathsf{y} \lhd \mathsf{u})$
Suppose $\Rightarrow$ $\mathsf{u} = \mathsf{y} \;\&\; \neg \mathsf{x} \lhd \mathsf{u}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{u} = \mathsf{x} \;\&\; \neg \mathsf{y} \lhd \mathsf{u}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{x} \lhd \mathsf{x}$

-- Similarly the minimal element of the singleton $\{\mathsf{x}\}$ must be x, so x cannot precede x, completing our proof.

$\langle \{\mathsf{x}\} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat4* : $\langle \exists \mathsf{w} \in \{\mathsf{x}\}, \forall \mathsf{v} \in \{\mathsf{x}\} \mid \neg \mathsf{v} \lhd \mathsf{w} \rangle$
$\langle \mathsf{u}_2 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ *Stat5* : $\mathsf{u}_2 \in \{\mathsf{x}\} \;\&\; \langle \forall \mathsf{v} \in \{\mathsf{x}\} \mid \neg \mathsf{v} \lhd \mathsf{u}_2 \rangle$
$\langle \mathsf{u}_2 \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\mathsf{u}_2 \in \{\mathsf{x}\} \;\&\; (\mathsf{u}_2 \in \{\mathsf{x}\} \rightarrow \neg \mathsf{u}_2 \lhd \mathsf{u}_2)$
ELEM $\Rightarrow$ $\mathsf{u}_2 = \mathsf{x} \;\&\; \neg \mathsf{u}_2 \lhd \mathsf{u}_2$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —- By assumption, we can define a selector Minrel_Θ which picks a minimal element from each non-null subset of the domain s. —-

DEF 10000. $\mathsf{Minrel}_\Theta(\mathsf{X}) =_{\mathrm{Def}}$ **if** $\mathsf{X} \subseteq \mathsf{s} \;\&\; \mathsf{X} \neq \emptyset$ **then arb**$(\{\mathsf{m} : \mathsf{m} \in \mathsf{X} \mid \langle \forall \mathsf{y} \in \mathsf{X} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle \})$ **else** s **fi**

-- The following statement merely captures the definition of Minrel_thryvar as a theorem for use outside the present theory.

Theorem 326 (well_founded_set · 00) $\mathsf{Minrel}_\Theta(T) =$ **if** $T \subseteq \mathsf{s} \;\&\; T \neq \emptyset$ **then arb**$(\{\mathsf{m} : \mathsf{m} \in T \mid \langle \forall \mathsf{y} \in T \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle \})$ **else** s **fi**. PROOF:

Suppose_not(t,s) $\Rightarrow$ $\mathsf{Minrel}_\Theta(T) \neq$ **if** $\mathsf{t} \subseteq \mathsf{s} \;\&\; \mathsf{t} \neq \emptyset$ **then arb**$(\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle \})$ **else** s **fi**
Use_def(Minrel_Θ) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 327 (well_founded_set · $0a$) $T \subseteq \mathsf{s} \,\&\, T \neq \emptyset \rightarrow \mathsf{Minrel}_\Theta(T) \in T \,\&\, \langle \forall \mathsf{y} \in T \mid \neg \mathsf{y} \lhd \mathsf{Minrel}_\Theta(T) \rangle$. PROOF:

-- Proceeding by contradiction, assume that there is a non-null subset t of the domain s of our relation for which the operator Minrel_Θ just introduced fails to select a minimal element from t.

Suppose_not$(\mathsf{t}, \mathsf{s}) \Rightarrow$ $\mathsf{t} \subseteq \mathsf{s} \,\&\, \mathsf{t} \neq \emptyset \,\&\, \neg \big(\mathsf{Minrel}_\Theta(\mathsf{t}) \in \mathsf{t} \,\&\, \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{Minrel}_\Theta(\mathsf{t}) \rangle \big)$
Assump $\Rightarrow$ *Stat1* : $\langle \forall \mathsf{t} \mid \mathsf{t} \subseteq \mathsf{s} \,\&\, \mathsf{t} \neq \emptyset \rightarrow \langle \exists \mathsf{w} \in \mathsf{t}, \forall \mathsf{v} \in \mathsf{t} \mid \neg \mathsf{v} \lhd \mathsf{w} \rangle \rangle$
$\langle \mathsf{t} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $\langle \exists \mathsf{x} \in \mathsf{t}, \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{x} \rangle$
$\langle \mathsf{m} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{m} \in \mathsf{t} \,\&\, \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle$

-- This conflicts with the fact that by assumption the set of minimal elements of t cannot be empty, and so the present lemma follows.

Suppose $\Rightarrow$ *Stat3* : $\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle\} = \emptyset$
$\langle \mathsf{m} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{arb}(\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle\}) \in \{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle\}$
Use_def$(\mathsf{Minrel}_\Theta) \Rightarrow$ $\mathsf{Minrel}_\Theta(\mathsf{t}) = \mathbf{arb}(\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle\})$
EQUAL $\Rightarrow$ *Stat4* : $\mathsf{Minrel}_\Theta(\mathsf{t}) \in \{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m} \rangle\}$
$\langle \mathsf{m}' \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\mathsf{m}' = \mathsf{Minrel}_\Theta(\mathsf{t}) \,\&\, \mathsf{Minrel}_\Theta(\mathsf{t}) \in \mathsf{t} \,\&\, \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}' \rangle$
EQUAL $\Rightarrow$ $\langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{Minrel}_\Theta(\mathsf{t}) \rangle$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —
— —- Next we prove the elementary fact that $\mathsf{Minrel}_\Theta(\mathsf{t})$ cannot be less than $\mathsf{Minrel}_\Theta(\mathsf{r})$ if t and s are both non-null subsets of s and t is a subset of r: — — — — — — — —
— —-

-- Monotonicity of Minrel_thryvar

Theorem 328 (well_founded_set · $0b$) $\mathsf{R} \subseteq \mathsf{s} \,\&\, T \subseteq \mathsf{R} \,\&\, T \neq \emptyset \rightarrow \neg \mathsf{Minrel}_\Theta(T) \lhd \mathsf{Minrel}_\Theta(\mathsf{R})$. PROOF:

-- Assuming the contrary, it follows that a minimal element of the subset t of the set r would precede some minimal element of r:

Suppose_not$(\mathsf{r}, \mathsf{s}, \mathsf{t}) \Rightarrow$ $\mathsf{r} \subseteq \mathsf{s} \,\&\, \mathsf{t} \subseteq \mathsf{r} \,\&\, \mathsf{t} \neq \emptyset \,\&\, \mathsf{Minrel}_\Theta(\mathsf{t}) \lhd \mathsf{Minrel}_\Theta(\mathsf{r})$
ELEM $\Rightarrow$ $\mathsf{r} \neq \emptyset \,\&\, \mathsf{t} \subseteq \mathsf{s}$

-- now use the preceding theorem twice, to derive a contradiction from the fact that $\mathsf{Minrel}_\Theta(\mathsf{t})$ comes before $\mathsf{Minrel}_\Theta(\mathsf{r})$

$\langle \mathsf{t} \rangle \hookrightarrow$ *Twell_founded_set* $\cdot 0a \Rightarrow$ $\mathsf{Minrel}_\Theta(\mathsf{t}) \in \mathsf{t}$
$\langle \mathsf{r} \rangle \hookrightarrow$ *Twell_founded_set* $\cdot 0a \Rightarrow$ $\mathsf{Minrel}_\Theta(\mathsf{r}) \in \mathsf{r} \,\&\,$ *Stat1* : $\langle \forall \mathsf{y} \in \mathsf{r} \mid \neg \mathsf{y} \lhd \mathsf{Minrel}_\Theta(\mathsf{r}) \rangle$

ELEM $\Rightarrow$ $\mathsf{Minrel}_\Theta(\mathsf{t}) \in \mathsf{r}$
$\langle \mathsf{Minrel}_\Theta(\mathsf{t})\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\neg\mathsf{Minrel}_\Theta(\mathsf{t}) \lhd \mathsf{Minrel}_\Theta(\mathsf{r})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- — The fact that for any non-null subset t of s, $\mathsf{Minrel}_\Theta(\mathsf{t})$ belongs to t is an even more elementary consequence of the one assumption of the present theory. —

Theorem 329 (well_founded_set $\cdot$ $0c$) $\mathsf{S} \supseteq T \;\&\; T \neq \emptyset \rightarrow \mathsf{Minrel}_\Theta(T) \in T$. PROOF:

Suppose_not$(\mathsf{s},\mathsf{t}) \Rightarrow$ $\mathsf{s} \supseteq \mathsf{t} \;\&\; \mathsf{t} \neq \emptyset \;\&\; \mathsf{Minrel}_\Theta(\mathsf{t}) \notin \mathsf{t}$
Use_def$(\mathsf{Minrel}_\Theta) \Rightarrow$ $\mathsf{Minrel}_\Theta(\mathsf{t}) = \mathbf{arb}\big(\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\}\big)$
Assump $\Rightarrow$ $\mathit{Stat1}: \langle \forall \mathsf{t} \mid \mathsf{t} \subseteq \mathsf{s} \;\&\; \mathsf{t} \neq \emptyset \rightarrow \langle \exists \mathsf{w} \in \mathsf{t}, \forall \mathsf{v} \in \mathsf{t} \mid \neg \mathsf{v} \lhd \mathsf{w}\rangle\rangle$
$\langle \mathsf{t}\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}: \langle \exists \mathsf{x} \in \mathsf{t}, \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{x}\rangle$
$\langle \mathsf{x}\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathit{Stat3}: \mathsf{x} \in \mathsf{t} \;\&\; \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{x}\rangle$
Suppose $\Rightarrow$ $\mathit{Stat4}: \{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\} = \emptyset$
$\langle \mathsf{x}\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\} \neq \emptyset$
$\langle \{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\}\rangle \hookrightarrow \mathit{T0} \Rightarrow$ $\mathbf{arb}\big(\{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\}\big) \in \{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\}$
EQUAL $\Rightarrow$ $\mathit{Stat5}: \mathsf{Minrel}_\Theta(\mathsf{t}) \in \{\mathsf{m} : \mathsf{m} \in \mathsf{t} \mid \langle \forall \mathsf{y} \in \mathsf{t} \mid \neg \mathsf{y} \lhd \mathsf{m}\rangle\}$
$\langle \mathsf{m}\rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ $\mathsf{Minrel}_\Theta(\mathsf{t}) = \mathsf{m} \;\&\; \mathsf{m} \in \mathsf{t}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- — We now use the Minrel_Θ selector to define a function from ordinals into, and eventually onto, our ordered set s, and prove a first elementary property of this function. —

DEF $00b$. $\mathsf{orden}_\Theta(\mathsf{X}) =_{\mathrm{Def}} \mathsf{Minrel}_\Theta\big(\mathsf{s} \setminus \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{X}\}\big)$

-- The following statement captures the definition of orden_thryvar as a theorem for use outside the present theory.

Theorem 330 (well_founded_set $\cdot$ $1a$) $\mathsf{orden}_\Theta(\mathsf{X}) = \mathsf{Minrel}_\Theta\big(\mathsf{s} \setminus \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{X}\}\big)$. PROOF:

Suppose_not$(\mathsf{x},\mathsf{s}) \Rightarrow$ $\mathsf{orden}_\Theta(\mathsf{x}) \neq \mathsf{Minrel}_\Theta\big(\mathsf{s} \setminus \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}\big)$
Use_def$(\mathsf{orden}_\Theta) \Rightarrow$ $\mathsf{orden}_\Theta(\mathsf{x}) = \mathsf{Minrel}_\Theta\big(\mathsf{s} \setminus \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}\big)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 331 (well_founded_set · 1) $s \not\subseteq \{\mathsf{orden}_\Theta(y) : y \in X\} \rightarrow \mathsf{orden}_\Theta(X) \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in X\} \;\&\; \langle \forall y \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in X\} \mid \neg y \lhd \mathsf{orden}_\Theta(X)\rangle$. PROOF:

Suppose_not(s, x) $\Rightarrow$ $s \not\subseteq \{\mathsf{orden}_\Theta(y) : y \in x\} \;\&\; \neg\big(\mathsf{orden}_\Theta(x) \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \;\&\; \langle \forall y \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \mid \neg y \lhd \mathsf{orden}_\Theta(x)\rangle\big)$

-- The proof, which is by contradiction, just uses the definition of 'orden_thryvar' and Lemma well_founded_set. 0a.

ELEM $\Rightarrow$ $s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \subseteq s \;\&\; s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \neq \emptyset$

-- For since $t = s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\}$ is a non-null subset of s, $\mathsf{Minrel}_\Theta(t) = \mathsf{orden}_\Theta(x)$ is a minimal element of t, which is what we assert.

$\langle s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\}\rangle \hookrightarrow$ *Twell_founded_set · 0a* $\Rightarrow$ $\mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\}) \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \;\&\;$
$\langle \forall y \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \mid \neg y \lhd \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\})\rangle$

Use_def(orden_Θ) $\Rightarrow$ $\mathsf{orden}_\Theta(x) = \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in X\})$

EQUAL $\Rightarrow$ $\mathsf{orden}_\Theta(x) \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \;\&\; \langle \forall y \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \mid \neg y \lhd \mathsf{orden}_\Theta(x)\rangle$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —
— — —- Next we show that $\mathsf{orden}_\Theta(x) = s$ only if $s \subseteq \{\mathsf{orden}_\Theta(y) : y \in X\}$ — — — —
— —-

Theorem 332 (well_founded_set · 2) $s \subseteq \{\mathsf{orden}_\Theta(y) : y \in X\} \leftrightarrow \mathsf{orden}_\Theta(X) = s$. PROOF:

-- For assume the contrary, and first consider the case in which s incin {orden_thryvar (y): y in x}

Suppose_not(s, x) $\Rightarrow$ $\neg\big(s \subseteq \{\mathsf{orden}_\Theta(y) : y \in x\} \leftrightarrow \mathsf{orden}_\Theta(x) = s\big)$

Use_def(orden_Θ) $\Rightarrow$ $\mathsf{orden}_\Theta(x) = \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\})$

Suppose $\Rightarrow$ $s \subseteq \{\mathsf{orden}_\Theta(y) : y \in x\} \;\&\; \mathsf{orden}_\Theta(x) \neq s$

-- In this case, the definitions of the operators involved lead to an immediate contradiction, ruling it out, and so leave only the case $\mathsf{orden}_\Theta(x) = s$ to be considered.

ELEM $\Rightarrow$ $\mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\}) \neq s$

Use_def(Minrel_Θ) $\Rightarrow$ $s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \neq \emptyset$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $s \setminus \{\mathsf{orden}_\Theta(y) : y \in x\} \neq \emptyset \;\&\; \mathsf{orden}_\Theta(x) = s$

-- but then $\mathsf{s} = \mathsf{Minrel}_\Theta\big(\mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}\big)$ by definition of orden_Θ, and so must belong to $\mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}$

ELEM $\Rightarrow$ $\mathsf{s} = \mathsf{Minrel}_\Theta\big(\mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}\big)$
ELEM $\Rightarrow$ $\mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\} \subseteq \mathsf{s}\ \&\ \mathsf{s} \neq \emptyset$
$\big\langle \mathsf{s}, \mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}\big\rangle \hookrightarrow$ *Twell_founded_set* $\cdot$ 0c $\Rightarrow$ $\mathsf{Minrel}_\Theta\big(\mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}\big) \in \mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —- Our next result shows that the image, via the enumerator, of any x is either s or an element of s: —-

Theorem 333 (well_founded_set $\cdot$ 3) $\mathsf{orden}_\Theta(\mathsf{X}) \neq \mathsf{s} \to \mathsf{orden}_\Theta(\mathsf{X}) \in \mathsf{s}$. PROOF:

-- Suppose x is a counterexample. Then Theorem well_founded_set. 2 tells us that some element of s is not the image, via the enumerator orden_Θ, of any element of x.

Suppose_not(x, s) $\Rightarrow$ $\mathsf{orden}_\Theta(\mathsf{x}) \neq \mathsf{s}\ \&\ \mathsf{orden}_\Theta(\mathsf{x}) \notin \mathsf{s}$
$\langle \mathsf{s}, \mathsf{x}\rangle \hookrightarrow$ *Twell_founded_set* $\cdot$ 2 $\Rightarrow$ $\mathsf{s} \not\subseteq \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}$

-- Thus, by Theorem well_founded_set. 1, $\mathsf{orden}_\Theta(\mathsf{x})$ belongs to s, a contradiction which proves our statement.

$\langle \mathsf{s}, \mathsf{x}\rangle \hookrightarrow$ *Twell_founded_set* $\cdot$ 1 $\Rightarrow$ $\mathsf{orden}_\Theta(\mathsf{x}) \in \mathsf{s}\backslash\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{x}\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —- Now we show that orden_thryvar, restricted to ordinals, is a monotone mapping into the ordered set s: —-

-- Ordinal enumeration is monotone on ordinals

Theorem 334 (well_founded_set $\cdot$ 5) $\mathcal{O}(\mathsf{U})\ \&\ \mathcal{O}(\mathsf{V})\ \&\ \mathsf{orden}_\Theta(\mathsf{U}) \neq \mathsf{s}\ \&\ \mathsf{orden}_\Theta(\mathsf{U}) \lhd \mathsf{orden}_\Theta(\mathsf{V}) \to \mathsf{U} \in \mathsf{V}$. PROOF:

-- For if the ordinals o_1, o_2 are a counterexample to the asserted statement ...

Suppose_not($\mathsf{o}_1, \mathsf{o}_2, \mathsf{s}$) $\Rightarrow$ $\mathcal{O}(\mathsf{o}_1)\ \&\ \mathcal{O}(\mathsf{o}_2)\ \&\ \mathsf{orden}_\Theta(\mathsf{o}_1) \neq \mathsf{s}\ \&\ \mathsf{orden}_\Theta(\mathsf{o}_1) \lhd \mathsf{orden}_\Theta(\mathsf{o}_2)\ \&\ \mathsf{o}_1 \notin \mathsf{o}_2$

-- ...then it follows from Theorem well_founded_set. 2 that the images, via the enumerator, of the elements of o_1 fail to exhaust the elements of s:

$\langle s, o_1 \rangle \hookrightarrow$ *Twell_founded_set* $\cdot 2 \Rightarrow \quad s \not\subseteq \{\mathsf{orden}_\Theta(y) : y \in o_1\}$

-- A contradiction will now be derived from the assumption that o_1 does not precede o2. Since ordinals o_1, o_2 can always be compared, this assumption implies that o_2 either precedes o_1 or is equal to it.

$\langle o_1, o_2 \rangle \hookrightarrow$ *T28* $\Rightarrow \quad o_2 \in o_1 \vee o_1 = o_2$

-- But since, for ordinals, membership implies inclusion, Theorem well_ordered_set_0b shows that the o_1-th element of s cannot precede the o_2-th element of s. This contradiction proves our theorem.

$\langle o_1, o_2 \rangle \hookrightarrow$ *T31* $\Rightarrow \quad o_2 \in o_1 \rightarrow o_2 \subseteq o_1$
ELEM $\Rightarrow \quad o_2 \subseteq o_1$
Set_monot $\Rightarrow \quad s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_1\} \subseteq s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_2\}$
ELEM $\Rightarrow \quad s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_1\} \neq \emptyset \;\&\; s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_2\} \subseteq s$
$\langle s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_2\}, s, s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_1\} \rangle \hookrightarrow$ *Twell_founded_set* $\cdot$ *0b* $\Rightarrow$
$\quad \neg \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_1\}) \lhd \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_2\})$
Use_def$(\mathsf{orden}_\Theta) \Rightarrow \quad \mathsf{orden}_\Theta(o_2) = \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_2\})$
Use_def$(\mathsf{orden}_\Theta) \Rightarrow \quad \mathsf{orden}_\Theta(o_1) = \mathsf{Minrel}_\Theta(s \setminus \{\mathsf{orden}_\Theta(y) : y \in o_1\})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —- Next we show that the enumerator, restricted to any ordinal v, enumerates a superset of the 'segment' of s consisting of all elements of s which precede the v-th: —-

Theorem 335 (well_founded_set $\cdot$ 6) $\{u : u \in s \mid u \lhd \mathsf{orden}_\Theta(V)\} \subseteq \{\mathsf{orden}_\Theta(x) : x \in V\}$. PROOF:

-- Proceed by contradiction, and let v be a counterexample, so that there is some element b of s such that $b \lhd \mathsf{orden}_\Theta(v)$ which differs from the image $\mathsf{orden}_\Theta(x)$ of any $x \in v$.

Suppose_not$(s, v) \Rightarrow \quad$ *Stat1* : $\{u : u \in s \mid u \lhd \mathsf{orden}_\Theta(v)\} \not\subseteq \{\mathsf{orden}_\Theta(u) : u \in v\}$
$\langle b \rangle \hookrightarrow$ *Stat1* $\Rightarrow \quad$ *Stat2* : $b \in \{u : u \in s \mid u \lhd \mathsf{orden}_\Theta(v)\} \;\&\;$ *Stat3* : $b \notin \{\mathsf{orden}_\Theta(u) : u \in v\}$
$\langle b' \rangle \hookrightarrow$ *Stat2* $\Rightarrow \quad b = b' \;\&\; b \in s \;\&\; b' \lhd \mathsf{orden}_\Theta(v)$
EQUAL $\Rightarrow \quad b \lhd \mathsf{orden}_\Theta(v)$

-- By Theorem well_founded_set. 1, no element of $s \setminus \{\mathsf{orden}_\Theta(y) : y \in v\}$ can precede orden_thryvar (v). In particular, b cannot precede $\mathsf{orden}_\Theta(v)$. This contradiction proves our theorem.

$\mathsf{ELEM} \Rightarrow \quad b \in s \backslash \{orden_\Theta(u) : u \in v\} \;\&\; s \not\subseteq \{orden_\Theta(u) : u \in v\}$
$\langle s, v\rangle \hookrightarrow Twell\_founded\_set \cdot 1 \Rightarrow \quad Stat4 : \langle \forall y \in s \backslash \{orden_\Theta(y) : y \in v\} \mid \neg y \lhd orden_\Theta(v)\rangle$
$\langle b\rangle \hookrightarrow Stat4 \Rightarrow \quad \neg b \lhd orden_\Theta(v)$
$\mathsf{ELEM} \Rightarrow \quad false; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- —- Our next theorem asserts that distinct ordinals whose images under the enumerator differ from (and hence belong to) s, have different images. —-

-- Well - ordering is initially 1 - 1

Theorem 336 (well_founded_set · 7) $\mathcal{O}(U) \;\&\; \mathcal{O}(V) \;\&\; orden_\Theta(U) \neq s \;\&\; orden_\Theta(V) \neq s \;\&\; U \neq V \rightarrow orden_\Theta(U) \neq orden_\Theta(V)$. PROOF:

$\mathsf{Suppose\_not}(o_1, o_2, s) \Rightarrow \quad \mathcal{O}(o_1) \;\&\; \mathcal{O}(o_2) \;\&\; orden_\Theta(o_1) \neq s \;\&\; orden_\Theta(o_2) \neq s \;\&\; o_1 \neq o_2 \;\&\; orden_\Theta(o_1) = orden_\Theta(o_2)$

-- Proceed by contradiction, and let the ordinals o_1, o_2 be a counterexample. It follows by theorems well_founded_set. 1 and well_founded_set. 2 that the image of o_1 (resp. o_2) lies outside $\{orden_\Theta(y) : y \in o_1\}$ (resp. $\{orden_\Theta(y) : y \in o_2\}$).

$\langle s, o_1\rangle \hookrightarrow Twell\_founded\_set \cdot 2 \Rightarrow \quad s \not\subseteq \{orden_\Theta(y) : y \in o_1\}$
$\langle s, o_2\rangle \hookrightarrow Twell\_founded\_set \cdot 2 \Rightarrow \quad s \not\subseteq \{orden_\Theta(y) : y \in o_2\}$
$\langle s, o_1\rangle \hookrightarrow Twell\_founded\_set \cdot 1 \Rightarrow \quad orden_\Theta(o_1) \in s \backslash \{orden_\Theta(y) : y \in o_1\}$
$\langle s, o_2\rangle \hookrightarrow Twell\_founded\_set \cdot 1 \Rightarrow \quad orden_\Theta(o_2) \in s \backslash \{orden_\Theta(y) : y \in o_2\}$

-- Given two distinct ordinals, one always precedes the other: assume first that o_1 belongs to o_2. Then obviously the image of o_1 is image of an element of o_2, a contradiction.

$\langle o_1, o_2\rangle \hookrightarrow T28 \Rightarrow \quad o_1 \in o_2 \vee o_2 \in o_1$
$\mathsf{Suppose} \Rightarrow \quad o_1 \in o_2$
$\mathsf{Suppose} \Rightarrow \quad Stat1 : orden_\Theta(o_1) \notin \{orden_\Theta(y) : y \in o_2\}$
$\langle o_1\rangle \hookrightarrow Stat1 \Rightarrow \quad false; \qquad \mathsf{Discharge} \Rightarrow \quad orden_\Theta(o_1) \in \{orden_\Theta(y) : y \in o_2\}$
$\mathsf{ELEM} \Rightarrow \quad false; \qquad \mathsf{Discharge} \Rightarrow \quad o_2 \in o_1$

-- A similar contradiction results if o_2 belongs to o_1. This shows that our original assumption is false, and so completes our proof.

$\mathsf{Suppose} \Rightarrow \quad Stat2 : orden_\Theta(o_2) \notin \{orden_\Theta(y) : y \in o_1\}$
$\langle o_2\rangle \hookrightarrow Stat2 \Rightarrow \quad false; \qquad \mathsf{Discharge} \Rightarrow \quad orden_\Theta(o_2) \in \{orden_\Theta(y) : y \in o_1\}$
$\mathsf{ELEM} \Rightarrow \quad false; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- —
— — —- The following theorem asserts that for every s there is an ordinal o such that the restriction to o of the enumerator is a 1-1 map from s onto s. —-

Theorem 337 (well_founded_set · 8) $\langle \exists o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o) \,\&\, s = \{\mathsf{orden}_\Theta(x) : x \in o\} \,\&\, \langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s \rangle \,\&\, 1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\}) \rangle$. Proof:

-- Proceeding by contradiction, assume the theorem to be false.

Suppose_not(s) $\Rightarrow$ $\mathit{Stat1}: \neg\langle \exists o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o) \,\&\, s = \{\mathsf{orden}_\Theta(x) : x \in o\} \,\&\, \langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s \rangle \,\&\, 1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\}) \rangle$

-- We first show that there is at least one ordinal in next($\#\mathcal{P}s$) whose image under the enumerator is all of s. For assume the contrary. The cardinality of $\mathcal{P}s$ is an ordinal which must be different from #s. Hence by Theorem well_founded_set. 2 the set $\{\mathsf{orden}_\Theta(y) : y \in \#\mathcal{P}s\}$ of images of elements of $\#\mathcal{P}s$ cannot include s.

$\langle \mathcal{P}s \rangle \hookrightarrow T130 \Rightarrow$ $\mathsf{Card}(\#\mathcal{P}s) \,\&\, \mathcal{O}(\#\mathcal{P}s)$
$\langle \#\mathcal{P}s \rangle \hookrightarrow T29 \Rightarrow$ $\mathcal{O}(\mathsf{next}(\#\mathcal{P}s))$
Suppose $\Rightarrow$ $\mathit{Stat2}: \neg\langle \exists o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o) \,\&\, \mathsf{orden}_\Theta(o) = s \rangle$
Use_def(next) $\Rightarrow$ $\mathsf{next}(\#\mathcal{P}s) = \#\mathcal{P}s \cup \{\#\mathcal{P}s\}$
ELEM $\Rightarrow$ $\#\mathcal{P}s \in \mathsf{next}(\#\mathcal{P}s)$
$\langle \#\mathcal{P}s \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathsf{orden}_\Theta(\#\mathcal{P}s) \neq s$
$\langle s, \#\mathcal{P}s \rangle \hookrightarrow \mathit{Twell\_founded\_set} \cdot 2 \Rightarrow$ $s \not\subseteq \{\mathsf{orden}_\Theta(y) : y \in \#\mathcal{P}s\}$

-- But $\{\mathsf{orden}_\Theta(y) : y \in \#\mathcal{P}s\}$ must be included in s, for otherwise some element $\mathsf{orden}_\Theta(b)$, with $b \in \#\mathcal{P}s$ would lie outside s, whereas by definition $\mathsf{orden}_\Theta(b)$ must belong to s.

Suppose $\Rightarrow$ $\mathit{Stat3}: \{\mathsf{orden}_\Theta(o) : o \in \#\mathcal{P}s\} \not\subseteq s$
$\langle b_1 \rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $b_1 \notin s \,\&\, \mathit{Stat4}: b_1 \in \{\mathsf{orden}_\Theta(o) : o \in \#\mathcal{P}s\}$
$\langle b \rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $b \in \#\mathcal{P}s \,\&\, \mathsf{orden}_\Theta(b) \notin s$
$\langle \mathsf{next}(\#\mathcal{P}s), b \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(b)$
$\langle \mathsf{next}(\#\mathcal{P}s), b \rangle \hookrightarrow T12 \Rightarrow$ $b \subseteq \#\mathcal{P}s$
Set_monot $\Rightarrow$ $s \setminus \{\mathsf{orden}_\Theta(y) : y \in \#\mathcal{P}s\} \subseteq s \setminus \{\mathsf{orden}_\Theta(y) : y \in b\}$
ELEM $\Rightarrow$ $s \not\subseteq \{\mathsf{orden}_\Theta(y) : y \in b\}$
$\langle s, b \rangle \hookrightarrow \mathit{Twell\_founded\_set} \cdot 1 \Rightarrow$ $\mathsf{orden}_\Theta(b) \in s \setminus \{\mathsf{orden}_\Theta(y) : y \in b\}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{orden}_\Theta(o) : o \in \#\mathcal{P}s\} \subseteq s$

-- We can use our general 'fcn_symbol' theory to form a single-valued map f which pairs each element of $\#\mathcal{P}s$ to its image under the enumerator. The following properties of this map result automatically:

$\mathsf{Loc\_def} \Rightarrow \quad \mathsf{f} = \{[\mathsf{x}, \mathsf{orden}_\Theta(\mathsf{x})] : \mathsf{x} \in \#\mathcal{P}\mathsf{s}\}$

$\mathsf{APPLY}\ \langle \mathsf{x}_\Theta : \mathsf{d}_1, \mathsf{y}_\Theta : \mathsf{d}_2 \rangle\ \mathsf{fcn\_symbol}(\mathsf{f}(\mathsf{x}) \mapsto \mathsf{orden}_\Theta(\mathsf{x}), \mathsf{g} \mapsto \mathsf{f}, \mathsf{s} \mapsto \#\mathcal{P}\mathsf{s}) \Rightarrow$

$\mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) = \#\mathcal{P}\mathsf{s}\ \&\ \mathbf{range}(\mathsf{f}) = \{\mathsf{orden}_\Theta(\mathsf{x}) : \mathsf{x} \in \#\mathcal{P}\mathsf{s}\}\ \&\ \big(\mathsf{d}_1, \mathsf{d}_2 \in \#\mathcal{P}\mathsf{s}\ \&\ \mathsf{orden}_\Theta(\mathsf{d}_1) = \mathsf{orden}_\Theta(\mathsf{d}_2)\ \&\ \mathsf{d}_1 \neq \mathsf{d}_2\big) \vee 1\text{--}1(\mathsf{f})$

-- But then the distinct elements d_1, d_2 cannot have the same image under the map f. Indeed, since d_1 and d_2 belong to the ordinal $\#\mathcal{P}\mathsf{s}$, they are ordinals included in $\#\mathcal{P}\mathsf{s}$; which by set monotonicity tells us that neither $\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{d}_1\}$ nor $\{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{d}_2\}$ can include s.

$\mathsf{Suppose} \Rightarrow \quad \mathsf{d}_1, \mathsf{d}_2 \in \#\mathcal{P}\mathsf{s}\ \&\ \mathsf{orden}_\Theta(\mathsf{d}_1) = \mathsf{orden}_\Theta(\mathsf{d}_2)\ \&\ \mathsf{d}_1 \neq \mathsf{d}_2$

$\langle \#\mathcal{P}\mathsf{s}, \mathsf{d}_1 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{d}_1)$

$\langle \#\mathcal{P}\mathsf{s}, \mathsf{d}_2 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\mathsf{d}_2)$

$\langle \#\mathcal{P}\mathsf{s}, \mathsf{d}_1 \rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{d}_1 \subseteq \#\mathcal{P}\mathsf{s}$

$\langle \#\mathcal{P}\mathsf{s}, \mathsf{d}_2 \rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{d}_2 \subseteq \#\mathcal{P}\mathsf{s}$

$\mathsf{Set\_monot} \Rightarrow \quad \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{d}_1\} \subseteq \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \#\mathcal{P}\mathsf{s}\}$

$\mathsf{Set\_monot} \Rightarrow \quad \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{d}_2\} \subseteq \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \#\mathcal{P}\mathsf{s}\}$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{s} \not\subseteq \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{d}_1\}\ \&\ \mathsf{s} \not\subseteq \{\mathsf{orden}_\Theta(\mathsf{y}) : \mathsf{y} \in \mathsf{d}_2\}$

-- But now Theorems well_founded_set. 2 and well_founded_set. 7 tell us that $\mathsf{orden}_\Theta(\mathsf{d}_1)$ and $\mathsf{orden}_\Theta(\mathsf{d}_1)$ must be different, a contradiction which shows f is one-one.

$\langle \mathsf{s}, \mathsf{d}_1 \rangle \hookrightarrow Twell\_founded\_set \cdot 2 \Rightarrow \quad \mathsf{orden}_\Theta(\mathsf{d}_1) \neq \mathsf{s}$

$\langle \mathsf{s}, \mathsf{d}_2 \rangle \hookrightarrow Twell\_founded\_set \cdot 2 \Rightarrow \quad \mathsf{orden}_\Theta(\mathsf{d}_2) \neq \mathsf{s}$

$\langle \mathsf{d}_1, \mathsf{d}_2 \rangle \hookrightarrow Twell\_founded\_set \cdot 7 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad 1\text{--}1(\mathsf{f})$

-- Since the domain and range of the 1-1 mapping f have the same cardinality but are $\mathsf{next}(\#\mathcal{P}\mathsf{s})$ and a subset of s respectively, it follows that next (# pow (s)) is less than or equal to the cardinality of s, violating Cantor's theorem. This refutes our assumption that there is no ordinal in $\mathsf{next}(\#\mathcal{P}\mathsf{s})$ whose image under the enumerator is s.

$\langle \mathsf{f} \rangle \hookrightarrow T131 \Rightarrow \quad \#\mathbf{range}(\mathsf{f}) = \#\mathbf{domain}(\mathsf{f})$

$\mathsf{EQUAL} \Rightarrow \quad \#\mathbf{range}(\mathsf{f}) = \#\#\mathcal{P}\mathsf{s}$

$\mathsf{EQUAL} \Rightarrow \quad \#\{\mathsf{orden}_\Theta(\mathsf{x}) : \mathsf{x} \in \#\mathcal{P}\mathsf{s}\} = \#\#\mathcal{P}\mathsf{s}$

$\langle \mathcal{P}\mathsf{s} \rangle \hookrightarrow T140 \Rightarrow \quad \#\{\mathsf{orden}_\Theta(\mathsf{x}) : \mathsf{x} \in \#\mathcal{P}\mathsf{s}\} = \#\mathcal{P}\mathsf{s}$

$\langle \{\mathsf{orden}_\Theta(\mathsf{x}) : \mathsf{x} \in \#\mathcal{P}\mathsf{s}\}, \mathsf{s} \rangle \hookrightarrow T144 \Rightarrow \quad \#\mathcal{P}\mathsf{s} \subseteq \#\mathsf{s}$

$\langle \mathsf{s} \rangle \hookrightarrow T228 \Rightarrow \quad \#\mathsf{s} \in \#\mathcal{P}\mathsf{s}$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat5 : \langle \exists \mathsf{o} \in \mathsf{next}(\#\mathcal{P}\mathsf{s}) \mid \mathcal{O}(\mathsf{o})\ \&\ \mathsf{orden}_\Theta(\mathsf{o}) = \mathsf{s} \rangle$

-- Having now established that there is an ordinal in $\mathsf{next}(\#\mathcal{P}\mathsf{s})$ whose image under the enumerator is s, we give the smallest such ordinal the name o.

$\langle o_1 \rangle \hookrightarrow Stat5 \Rightarrow \quad o_1 \in \mathsf{next}(\#\mathcal{P}s) \;\&\; \mathcal{O}(o_1) \;\&\; \mathsf{orden}_\Theta(o_1) = s$

APPLY $\langle mt_\Theta : o \rangle$ transfinite_induction$\Big(n \mapsto o_1, p(x) \mapsto \big(x \in \mathsf{next}(\#\mathcal{P}s) \;\&\; \mathcal{O}(x) \;\&\; \mathsf{orden}_\Theta(x) = s\big)\Big) \Rightarrow$

$$Stat6: \; \langle \forall x \,|\, o \in \mathsf{next}(\#\mathcal{P}s) \;\&\; \mathcal{O}(o) \;\&\; \mathsf{orden}_\Theta(o) = s \;\&\; \Big(x \in o \rightarrow \neg\big(x \in \mathsf{next}(\#\mathcal{P}s) \;\&\; \mathcal{O}(x) \;\&\; \mathsf{orden}_\Theta(x) = s\big)\Big)\rangle$$

-- It follows from our initial hypothesis that either there is some element of $o \in \mathsf{next}(\#\mathcal{P}s)$ whose image is s, or the set of images of the elements of o is different from s, or the restriction of the enumerator to o fails to be 1-1. We consider these three possibilities in turn.

$\langle a_0 \rangle \hookrightarrow Stat6 \Rightarrow \quad o \in \mathsf{next}(\#\mathcal{P}s) \;\&\; \mathcal{O}(o) \;\&\; \mathsf{orden}_\Theta(o) = s$

$\langle o \rangle \hookrightarrow Stat1 \Rightarrow \quad s \neq \{\mathsf{orden}_\Theta(x) : x \in o\} \vee \neg\langle \forall x \in o \,|\, \mathsf{orden}_\Theta(x) \neq s \rangle \vee \neg 1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})$

-- Case 1. Suppose that there is some element x of o whose image under the enumerator is s. Since x must be an ordinal, this conflicts with the assumed minimality of o, and so disposes of Case 1.

Suppose $\Rightarrow \quad Stat7: \; \neg\langle \forall x \in o \,|\, \mathsf{orden}_\Theta(x) \neq s \rangle$

$\langle \#\mathcal{P}s \rangle \hookrightarrow T29 \Rightarrow \quad \mathcal{O}(\mathsf{next}(\#\mathcal{P}s))$

$\langle x \rangle \hookrightarrow Stat7 \Rightarrow \quad x \in o \;\&\; \mathsf{orden}_\Theta(x) = s$

$\langle \mathsf{next}(\#\mathcal{P}s), o \rangle \hookrightarrow T12 \Rightarrow \quad o \subseteq \mathsf{next}(\#\mathcal{P}s)$

ELEM $\Rightarrow \quad x \in \mathsf{next}(\#\mathcal{P}s)$

$\langle o, x \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(x)$

$\langle x \rangle \hookrightarrow Stat6 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad Stat8: \; \langle \forall x \in o \,|\, \mathsf{orden}_\Theta(x) \neq s \rangle$

-- Theorem well_founded_set. 3 now tells us $\{\mathsf{orden}_\Theta(y) : y \in o\}$ must be a subset of s;

Suppose $\Rightarrow \quad Stat9: \; \{\mathsf{orden}_\Theta(y) : y \in o\} \not\subseteq s$

$\langle c \rangle \hookrightarrow Stat9 \Rightarrow \quad Stat10: \; c \in \{\mathsf{orden}_\Theta(y) : y \in o\} \;\&\; c \notin s$

$\langle y \rangle \hookrightarrow Stat10 \Rightarrow \quad y \in o \;\&\; \mathsf{orden}_\Theta(y) \notin s$

$\langle y \rangle \hookrightarrow Stat8 \Rightarrow \quad \mathsf{orden}_\Theta(y) \neq s$

$\langle y \rangle \hookrightarrow Twell\_founded\_set \cdot 3 \Rightarrow \quad \mathsf{orden}_\Theta(y) \in s$

ELEM $\Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \{\mathsf{orden}_\Theta(y) : y \in o\} \subseteq s$

-- Case 2. Now suppose that $\{\mathsf{orden}_\Theta(y) : y \in o\}$ is a proper subset of s. Then Theorem well_founded_set. 2 tells us that $\mathsf{orden}_\Theta(o)$ is a member of s and so is different from s, ruling out this case.

Suppose $\Rightarrow \quad s \neq \{\mathsf{orden}_\Theta(y) : y \in o\}$

ELEM $\Rightarrow \quad s \not\subseteq \{\mathsf{orden}_\Theta(y) : y \in o\}$

$\langle s, o \rangle \hookrightarrow Twell\_founded\_set \cdot 2 \Rightarrow \quad \mathsf{orden}_\Theta(o) \neq s$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg 1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})$

-- Case 3. Finally, suppose that the restriction h of the enumerator to o is not 1-1, and let u, v be different elements of o which have the same image under the map h. Then u and v belong to o and so must both be ordinals. Moreover, since they belong to o and $\mathsf{orden}_\Theta(x) \neq s$ for every element x of o, $\mathsf{orden}_\Theta(u)$ and $\mathsf{orden}_\Theta(v)$ both differ from s. Hence by well_founded_set. 7 $\mathsf{orden}_\Theta(u) \neq \mathsf{orden}_\Theta(v)$. This contradiction shows that h is one-one, eliminating the last of our three cases and so proving our theorem.

Loc_def $\Rightarrow$ $h = \{[x, \mathsf{orden}_\Theta(x)] : x \in o\}$
APPLY $\langle x_\Theta : u, y_\Theta : v\rangle$ fcn_symbol $(f(x) \mapsto \mathsf{orden}_\Theta(x), g \mapsto h, s \mapsto o)$ $\Rightarrow$
$\mathsf{Svm}(h)$ & $\mathbf{domain}(h) = o$ & $\mathbf{range}(h) = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\big(u, v \in o$ & $\mathsf{orden}_\Theta(u) = \mathsf{orden}_\Theta(v)$ & $u \neq v\big) \vee 1\text{–}1(h)$
Suppose $\Rightarrow$ *Stat11* : $u, v \in o$ & $\mathsf{orden}_\Theta(u) = \mathsf{orden}_\Theta(v)$ & $u \neq v$
$\langle o, u\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(u)$
$\langle o, v\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(v)$
$\langle u\rangle \hookrightarrow Stat8 \Rightarrow$ $\mathsf{orden}_\Theta(u) \neq s$
$\langle v\rangle \hookrightarrow Stat8 \Rightarrow$ $\mathsf{orden}_\Theta(v) \neq s$
$\langle u, v\rangle \hookrightarrow Twell\_founded\_set \cdot 7 \Rightarrow$ $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We can now make the following definition, convenient for subsequent application.

DEF well_founded_set · b. $\mathsf{ord}_\Theta =_{\mathrm{Def}} \mathbf{arb}\big(\{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}\big)$

-- This last definition lets us reformulate the preceding theorem as follows.

Theorem 338 (well_founded_set · 9) $\mathsf{ord}_\Theta \in \mathsf{next}(\#\mathcal{P}s)$ & $\mathcal{O}(\mathsf{ord}_\Theta)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in \mathsf{ord}_\Theta\}$ & $\langle \forall x \in \mathsf{ord}_\Theta \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in \mathsf{ord}_\Theta\})$. PROOF:

Suppose_not $\Rightarrow$ $\neg\big(\mathsf{ord}_\Theta \in \mathsf{next}(\#\mathcal{P}s)$ & $\mathcal{O}(\mathsf{ord}_\Theta)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in \mathsf{ord}_\Theta\}$ & $\langle \forall x \in \mathsf{ord}_\Theta \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in \mathsf{ord}_\Theta\})\big)$
Twell_founded_set $\cdot 8 \Rightarrow$ *Stat1* : $\langle \exists o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\rangle$
$\langle o\rangle \hookrightarrow Stat1 \Rightarrow$ $o \in \mathsf{next}(\#\mathcal{P}s)$ & $\mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})$
Suppose $\Rightarrow$ *Stat2* : $o \notin \{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}$
$\langle o\rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ $o \in \{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}$
ELEM $\Rightarrow$ $\emptyset \neq \{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}$
$\big\langle \{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}\big\rangle \hookrightarrow T0 \Rightarrow$
$\mathbf{arb}\big(\{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}\big) \in$
$\{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}$
Use_def $(\mathsf{ord}_\Theta) \Rightarrow$ *Stat3* : $\mathsf{ord}_\Theta \in \{o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)$ & $s = \{\mathsf{orden}_\Theta(x) : x \in o\}$ & $\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\rangle$ & $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in o\})\}$

$\langle\,\rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The results just proved can be summarized as follows:

ENTER_THEORY Set_theory

DISPLAY well_founded_set

THEORY well_founded_set$(s, y \lhd x)$
$\Rightarrow (\mathrm{Minrel}_\Theta, \mathrm{orden}_\Theta, \mathrm{ord}_\Theta)$
$\langle \forall t \subseteq s \mid t \neq \emptyset \rightarrow \langle \exists x \in t, \forall y \in t \mid \neg y \lhd x \rangle \rangle$
$\langle \forall x \in s, y \in s \mid (x \lhd y \rightarrow \neg y \lhd x) \;\&\; \neg x \lhd x \rangle$
$\langle \forall x \mid s \not\subseteq \{\mathrm{orden}_\Theta(y) : y \in X\} \rightarrow \mathrm{orden}_\Theta(X) \in s \backslash \{\mathrm{orden}_\Theta(y) : y \in X\} \;\&\; \langle \forall y \in s \backslash \{\mathrm{orden}_\Theta(y) : y \in X\} \mid \neg y \lhd \mathrm{orden}_\Theta(X) \rangle \rangle$
$\langle \forall x \mid s \subseteq \{\mathrm{orden}_\Theta(y) : y \in X\} \leftrightarrow \mathrm{orden}_\Theta(X) = s \rangle$
$\langle \forall x \mid \mathrm{orden}_\Theta(X) \neq s \rightarrow \mathrm{orden}_\Theta(X) \in s \rangle$
$\langle \forall u, v \mid \mathcal{O}(U) \;\&\; \mathcal{O}(V) \;\&\; \mathrm{orden}_\Theta(U) \neq s \;\&\; \mathrm{orden}_\Theta(U) \lhd \mathrm{orden}_\Theta(V) \rightarrow U \in V \rangle$
$\langle \forall v \mid \{u : u \in s \mid u \lhd \mathrm{orden}_\Theta(V)\} \subseteq \{\mathrm{orden}_\Theta(x) : x \in V\} \rangle$
$\langle \forall u, v \mid \mathcal{O}(U) \;\&\; \mathcal{O}(V) \;\&\; \mathrm{orden}_\Theta(U) \neq s \;\&\; \mathrm{orden}_\Theta(V) \neq s \;\&\; U \neq V \rightarrow \mathrm{orden}_\Theta(U) \neq \mathrm{orden}_\Theta(V) \rangle$
$\mathrm{ord}_\Theta \in \mathrm{next}(\#\mathcal{P}s) \;\&\; \mathcal{O}(o) \;\&\; s = \{\mathrm{orden}_\Theta(x) : x \in \mathrm{ord}_\Theta\}$
$\;\&\; (\langle \forall x \in \mathrm{ord}_\Theta \mid \mathrm{orden}_\Theta(x) \neq s \rangle) \;\&\; 1\text{–}1(\{[x, \mathrm{orden}_\Theta(x)] : x \in \mathrm{ord}_\Theta\})$
$\langle \forall t \mid \mathrm{Minrel}_\Theta(t) = \textbf{if } t \subseteq s \;\&\; t \neq \emptyset \textbf{ then arb}(\{m : m \in t \mid \langle \forall y \in t \mid \neg y \lhd m \rangle\}) \textbf{ else } s \textbf{ fi} \rangle$
END well_founded_set

-- —
— —- —
— — —- Our next theory, which extends the previous, concerns binary relations which strictly well-order a given domain s. This means that the relation is transitive linear (sometimes called 'trichotomic') and irreflexive on s, and that each non-null subset t of this domain has an element which precedes every other element of t in the ordering. We show that any such relation is order-isomorphic to the membership relator on an ordinal in 1-1 ordered correspondence with the set.

THEORY well_ordered_set$(s, x \lhd y)$
$\langle \forall x \in s \mid \neg x \lhd x \rangle$
$\langle \forall x \in s, y \in s, zz \in s \mid x \lhd y \;\&\; y \lhd zz \rightarrow x \lhd zz \rangle$
$\langle \forall t \subseteq s \mid t \neq \emptyset \rightarrow \langle \exists x \in t, \forall y \in t \mid x \lhd y \vee x = y \rangle \rangle$
END well_ordered_set

ENTER_THEORY well_ordered_set

-- —- Trichotomy, which does not appear as an explicit theory assumption, will now be derived from the assumption concerning minima; moreover, we will show that each non-null set t of the ordered domain has a minimal element, that is an element which does not follow any other element of t in the ordering. —-

Theorem 339 (well_ordered_set · 0) $X, Y \in s \rightarrow X \lhd Y \vee Y \lhd X \vee X = Y$. PROOF:

-- We proceed by contradiction. Suppose that our theorem is false, and let x, y, and t be a counterexample.

Suppose_not$(x, s, y) \Rightarrow \quad x, y \in s \ \& \ \neg x \lhd y \ \& \ \neg y \lhd x \ \& \ x \neq y$

-- If the asserted trichotomy does not hold, there must exist elements x, y of s which cannot be compared. But these would form a doubleton without minimum, contradicting one of our assumptions.

Suppose $\Rightarrow \quad x, y \in s \ \& \ \neg x \lhd y \ \& \ \neg y \lhd x \ \& \ x \neq y$
Assump $\Rightarrow \quad$ *Stat1* : $\langle \forall t \subseteq s \mid t \neq \emptyset \rightarrow \langle \exists u \in t, \forall v \in t \mid u \lhd v \vee u = v \rangle \rangle$
ELEM $\Rightarrow \quad \{x, y\} \subseteq s \ \& \ \{x, y\} \neq \emptyset$
$\langle \{x, y\} \rangle \hookrightarrow$ *Stat1* $\Rightarrow \quad$ *Stat2* : $\langle \exists x_1 \in \{x, y\}, \forall y_1 \in \{x, y\} \mid x_1 \lhd y_1 \vee x_1 = y_1 \rangle$
$\langle u \rangle \hookrightarrow$ *Stat2* $\Rightarrow \quad u \in \{x, y\} \ \&$ *Stat3* : $\langle \forall v \in \{x, y\} \mid u \lhd v \vee u = v \rangle$
ELEM $\Rightarrow \quad u = x \vee u = y \ \& \ x, y \in \{x, y\}$
Suppose $\Rightarrow \quad u = x$
$\langle y \rangle \hookrightarrow$ *Stat3* $\Rightarrow \quad u \lhd y \vee u = y$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad u = y$
$\langle x \rangle \hookrightarrow$ *Stat3* $\Rightarrow \quad u \lhd x \vee u = x$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad$ QED

Theorem 340 (well_ordered_set · 0a) $T \subseteq s \ \& \ T \neq \emptyset \rightarrow \langle \exists x \in T, \forall y \in T \mid \neg y \lhd x \rangle$. PROOF:

Suppose_not$(t, s) \Rightarrow \quad t \subseteq s \ \& \ t \neq \emptyset \ \&$ *Stat4* : $\neg \langle \exists x \in t, \forall y \in t \mid \neg y \lhd x \rangle$

-- Assuming the contrary, there must exist a non-null subset t of s each of whose elements has at least one predecessor. On the other hand, we know that t has an element v which precedes every other element of t: this element cannot have any predecessors, else the irreflexivity of the ordering relation would be violated.

Assump $\Rightarrow \quad$ *Stat5* : $\langle \forall t \subseteq s \mid t \neq \emptyset \rightarrow \langle \exists x \in t, \forall y \in t \mid x \lhd y \vee x = y \rangle \rangle$
$\langle t \rangle \hookrightarrow$ *Stat5* $\Rightarrow \quad$ *Stat6* : $\langle \exists x \in t, \forall y \in t \mid x \lhd y \vee x = y \rangle$

$\langle v\rangle\hookrightarrow Stat6 \Rightarrow \quad v \in t \,\&\, Stat7:\ \langle\forall y \in t \mid v \lhd y \vee v = y\rangle$
$\langle v\rangle\hookrightarrow Stat4 \Rightarrow \quad Stat8:\ \neg\langle\forall y \in t \mid \neg y \lhd v\rangle$
$\langle w\rangle\hookrightarrow Stat8 \Rightarrow \quad w \in t \,\&\, w \lhd v$
$\langle w\rangle\hookrightarrow Stat7 \Rightarrow \quad v \lhd w \vee v = w$
ELEM $\Rightarrow \quad v, w \in s$
Assump $\Rightarrow \quad Stat9:\ \langle\forall x \in s \mid \neg x \lhd x\rangle$
$\langle v\rangle\hookrightarrow Stat9 \Rightarrow \quad \neg v \lhd v$
Suppose $\Rightarrow \quad v = w$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad v \lhd w$
Assump $\Rightarrow \quad Stat10:\ \langle\forall x \in s, y \in s, zz \in s \mid x \lhd y \,\&\, y \lhd zz \rightarrow x \lhd zz\rangle$
$\langle v, w, v\rangle\hookrightarrow Stat10 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- We can now import all theorems of the theory well_founded_set into the present theory.

APPLY $\langle \mathrm{Minrel}_\Theta : \mathrm{Minrel}_\Theta, \mathrm{orden}_\Theta : \mathrm{orden}_\Theta\rangle$ well_founded_set$(s \mapsto s, x \lhd y \mapsto x \lhd y) \Rightarrow$

Theorem 341 (well_ordered_set · 100)
$\langle\forall x, y \mid x, y \in s \rightarrow \neg(x \lhd y \,\&\, y \lhd x) \,\&\, \neg x \lhd x\rangle \,\&\, \langle\forall x \mid s \not\subseteq \{\mathrm{orden}_\Theta(y) :\ y \in x\} \rightarrow \mathrm{orden}_\Theta(x) \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in x\} \,\&\, \langle\forall y \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in x\} \mid \neg y \lhd \mathrm{orden}_\Theta(x)\rangle\rangle \,\&\, \langle\forall x \mid$
$\langle\forall u, v \mid \mathcal{O}(u) \,\&\, \mathcal{O}(v) \,\&\, \mathrm{orden}_\Theta(u) \neq s \,\&\, \mathrm{orden}_\Theta(v) \neq s \,\&\, u \neq v \rightarrow \mathrm{orden}_\Theta(u) \neq \mathrm{orden}_\Theta(v)\rangle \,\&\, \langle\exists o \in \mathrm{next}(\#\mathcal{P}s) \mid \mathcal{O}(o) \,\&\, s = \{\mathrm{orden}_\Theta(x) :\ x \in o\} \,\&\, \langle\forall x \in o \mid \mathrm{orden}_\Theta(x) \neq s\rangle \,\&$

Theorem 342 (well_ordered_set · 10) $X, Y \in s \rightarrow (X \lhd Y \rightarrow \neg Y \lhd X) \,\&\, \neg X \lhd X$. PROOF:

Suppose_not$(x, s, y) \Rightarrow \quad x, y \in s \,\&\, \neg((x \lhd y \rightarrow \neg y \lhd x) \,\&\, \neg x \lhd x)$
Twell_ordered_set · 100 $\Rightarrow \quad Stat1:\ \langle\forall x, y \mid x, y \in s \rightarrow \neg(x \lhd y \,\&\, y \lhd x) \,\&\, \neg x \lhd x\rangle$
$\langle x, y\rangle\hookrightarrow Stat1 \Rightarrow \quad x, y \in s \rightarrow (x \lhd y \rightarrow \neg y \lhd x) \,\&\, \neg x \lhd x$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

Theorem 343 (well_ordered_set · 1) $s \not\subseteq \{\mathrm{orden}_\Theta(y) :\ y \in X\} \rightarrow \mathrm{orden}_\Theta(X) \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in X\} \,\&\, \langle\forall y \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in X\} \mid \neg y \lhd \mathrm{orden}_\Theta(X)\rangle$. PROOF:

Suppose_not$(s, x) \Rightarrow \quad \neg s \not\subseteq \{\mathrm{orden}_\Theta(y) :\ y \in X\} \rightarrow \mathrm{orden}_\Theta(x) \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in x\} \,\&\, \langle\forall y \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in x\} \mid \neg y \lhd \mathrm{orden}_\Theta(x)\rangle$
Twell_ordered_set · 100 $\Rightarrow \quad Stat1:\ \langle\forall x \mid s \not\subseteq \{\mathrm{orden}_\Theta(y) :\ y \in X\} \rightarrow \mathrm{orden}_\Theta(X) \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in X\} \,\&\, \langle\forall y \in s \setminus \{\mathrm{orden}_\Theta(y) :\ y \in X\} \mid \neg y \lhd \mathrm{orden}_\Theta(X)\rangle\rangle$
$\langle x\rangle\hookrightarrow Stat1 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

Theorem 344 (well_ordered_set · 2) $s \subseteq \{\mathrm{orden}_\Theta(y) :\ y \in X\} \leftrightarrow \mathrm{orden}_\Theta(X) = s$. PROOF:

$\mathsf{Suppose\_not}(s,x) \Rightarrow \quad \neg\big(s \subseteq \{\mathsf{orden}_\Theta(y) : y \in x\} \leftrightarrow \mathsf{orden}_\Theta(x) = s\big)$

$\mathit{Twell\_ordered\_set} \cdot 100 \Rightarrow \quad \mathit{Stat1} : \big\langle \forall x \mid s \subseteq \{\mathsf{orden}_\Theta(y) : y \in X\} \leftrightarrow \mathsf{orden}_\Theta(X) = s\big\rangle$

$\langle x \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 345 (well_ordered_set · 3) $\mathsf{orden}_\Theta(X) \neq s \rightarrow \mathsf{orden}_\Theta(X) \in s$. PROOF:

$\mathsf{Suppose\_not}(x,s) \Rightarrow \quad \neg\big(\mathsf{orden}_\Theta(x) \neq s \longrightarrow \mathsf{orden}_\Theta(x) \in s\big)$

$\mathit{Twell\_ordered\_set} \cdot 100 \Rightarrow \quad \mathit{Stat1} : \big\langle \forall x \mid \mathsf{orden}_\Theta(X) \neq s \longrightarrow \mathsf{orden}_\Theta(X) \in s\big\rangle$

$\langle x \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 346 (well_ordered_set · 5a) $\mathcal{O}(U)\ \&\ \mathcal{O}(V)\ \&\ \mathsf{orden}_\Theta(U) \neq s\ \&\ \mathsf{orden}_\Theta(U) \lhd \mathsf{orden}_\Theta(V) \rightarrow U \in V$. PROOF:

$\mathsf{Suppose\_not}(u,v,s) \Rightarrow \quad \neg\big(\mathcal{O}(u)\ \&\ \mathcal{O}(v)\ \&\ \mathsf{orden}_\Theta(u) \neq s\ \&\ \mathsf{orden}_\Theta(u) \lhd \mathsf{orden}_\Theta(v) \longrightarrow u \in v\big)$

$\mathit{Twell\_ordered\_set} \cdot 100 \Rightarrow \quad \mathit{Stat1} : \big\langle \forall u,v \mid \mathcal{O}(u)\ \&\ \mathcal{O}(v)\ \&\ \mathsf{orden}_\Theta(u) \neq s\ \&\ \mathsf{orden}_\Theta(u) \lhd \mathsf{orden}_\Theta(v) \longrightarrow u \in v\big\rangle$

$\langle u,v \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 347 (well_ordered_set · 6a) $\{u : u \in s \mid u \lhd \mathsf{orden}_\Theta(V)\} \subseteq \{\mathsf{orden}_\Theta(x) : x \in V\}$. PROOF:

$\mathsf{Suppose\_not}(s,v) \Rightarrow \quad \{u : u \in s \mid u \lhd \mathsf{orden}_\Theta(v)\} \not\subseteq \{\mathsf{orden}_\Theta(x) : x \in v\}$

$\mathit{Twell\_ordered\_set} \cdot 100 \Rightarrow \quad \mathit{Stat1} : \big\langle \forall v \mid \{u : u \in s \mid u \lhd \mathsf{orden}_\Theta(V)\} \subseteq \{\mathsf{orden}_\Theta(x) : x \in V\}\big\rangle$

$\langle v \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 348 (well_ordered_set · 7) $\mathcal{O}(U)\ \&\ \mathcal{O}(V)\ \&\ \mathsf{orden}_\Theta(U) \neq s\ \&\ \mathsf{orden}_\Theta(V) \neq s\ \&\ U \neq V \rightarrow \mathsf{orden}_\Theta(U) \neq \mathsf{orden}_\Theta(V)$. PROOF:

$\mathsf{Suppose\_not}(u,v,s) \Rightarrow \quad \neg\big(\mathcal{O}(u)\ \&\ \mathcal{O}(v)\ \&\ \mathsf{orden}_\Theta(u) \neq s\ \&\ \mathsf{orden}_\Theta(v) \neq s\ \&\ u \neq v \longrightarrow \mathsf{orden}_\Theta(u) \neq \mathsf{orden}_\Theta(v)\big)$

$\mathit{Twell\_ordered\_set} \cdot 100 \Rightarrow \quad \mathit{Stat1} : \big\langle \forall u,v \mid \mathcal{O}(U)\ \&\ \mathcal{O}(V)\ \&\ \mathsf{orden}_\Theta(U) \neq s\ \&\ \mathsf{orden}_\Theta(V) \neq s\ \&\ U \neq V \longrightarrow \mathsf{orden}_\Theta(U) \neq \mathsf{orden}_\Theta(V)\big\rangle$

$\langle u,v \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 349 (well_ordered_set · 8) $\big\langle \exists o \mid \mathcal{O}(o)\ \&\ s = \{\mathsf{orden}_\Theta(x) : x \in o\}\ \&\ \big\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\big\rangle\ \&\ 1\text{–}1(\{[x,\mathsf{orden}_\Theta(x)] : x \in o\})\big\rangle$. PROOF:

$\mathsf{Suppose\_not}(s) \Rightarrow \quad \mathit{Stat2} : \neg\big\langle \exists o \mid \mathcal{O}(o)\ \&\ s = \{\mathsf{orden}_\Theta(x) : x \in o\}\ \&\ \big\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\big\rangle\ \&\ 1\text{–}1(\{[x,\mathsf{orden}_\Theta(x)] : x \in o\})\big\rangle$

$\mathit{Twell\_ordered\_set} \cdot 100 \Rightarrow \quad \mathit{Stat1} : \big\langle \exists o \in \mathsf{next}(\#\mathcal{P}s) \mid \mathcal{O}(o)\ \&\ s = \{\mathsf{orden}_\Theta(x) : x \in o\}\ \&\ \big\langle \forall x \in o \mid \mathsf{orden}_\Theta(x) \neq s\big\rangle\ \&\ 1\text{–}1(\{[x,\mathsf{orden}_\Theta(x)] : x \in o\})\big\rangle$

$\langle o_1 \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathcal{O}(o_1)\ \&\ s = \{\mathsf{orden}_\Theta(x) : x \in o_1\}\ \&\ \big\langle \forall x \in o_1 \mid \mathsf{orden}_\Theta(x) \neq s\big\rangle\ \&\ 1\text{–}1(\{[x,\mathsf{orden}_\Theta(x)] : x \in o_1\})$

$\langle o_1 \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 350 (well_ordered_set · 8a) $\text{orden}_\Theta(X) = \text{Minrel}_\Theta(s \setminus \{\text{orden}_\Theta(y) : y \in X\})$. PROOF:

Suppose_not$(x, s) \Rightarrow$ $\text{orden}_\Theta(x) \neq \text{Minrel}_\Theta(s \setminus \{\text{orden}_\Theta(y) : y \in x\})$
$Twell\_ordered\_set \cdot 100 \Rightarrow$ $Stat1 : \langle \forall x \mid \text{orden}_\Theta(x) = \text{Minrel}_\Theta(s \setminus \{\text{orden}_\Theta(y) : y \in x\}) \rangle$
$\langle x \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 351 (well_ordered_set · 8b) $\text{Minrel}_\Theta(T) = \textbf{if } T \subseteq s \ \& \ T \neq \emptyset \textbf{ then arb}(\{m : m \in T \mid \langle \forall y \in T \mid \neg y \lhd m \rangle\}) \textbf{ else } s \textbf{ fi}$. PROOF:

Suppose_not$(t, s) \Rightarrow$ $\text{Minrel}_\Theta(t) \neq \textbf{if } t \subseteq s \ \& \ t \neq \emptyset \textbf{ then arb}(\{m : m \in t \mid \langle \forall y \in t \mid \neg y \lhd m \rangle\}) \textbf{ else } s \textbf{ fi}$
$Twell\_ordered\_set \cdot 100 \Rightarrow$ $Stat1 : \langle \forall t \mid \text{Minrel}_\Theta(t) = \textbf{if } t \subseteq s \ \& \ t \neq \emptyset \textbf{ then arb}(\{m : m \in t \mid \langle \forall y \in t \mid \neg y \lhd m \rangle\}) \textbf{ else } s \textbf{ fi} \rangle$
$\langle t \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- This mechanical transition being accomplished, our next theorem tells us that the enumerator, restricted to ordinals, conforms to the linear ordering of s:

-- Well - ordering is isomorphic to ordinal enumeration

Theorem 352 (well_ordered_set · 5) $\mathcal{O}(U) \ \& \ \mathcal{O}(V) \ \& \ \text{orden}_\Theta(U) \neq s \ \& \ \text{orden}_\Theta(V) \neq s \rightarrow (\text{orden}_\Theta(U) \lhd \text{orden}_\Theta(V) \leftrightarrow U \in V)$. PROOF:

-- For if the ordinals o_1, o_2 are a counterexample to the asserted statement ...

Suppose_not$(o_1, o_2, s) \Rightarrow$ $\mathcal{O}(o_1) \ \& \ \mathcal{O}(o_2) \ \& \ \text{orden}_\Theta(o_1) \neq s \ \& \ \text{orden}_\Theta(o_2) \neq s \ \& \ \neg(\text{orden}_\Theta(o_1) \lhd \text{orden}_\Theta(o_2) \leftrightarrow o_1 \in o_2)$

-- ...then it follows from well_ordered_set. 5a that the negated equivalence is satisfied with its left-hand side false and its right-hand side true

Suppose $\Rightarrow$ $\text{orden}_\Theta(o_1) \lhd \text{orden}_\Theta(o_2) \ \& \ o_1 \notin o_2$
$\langle o_1, o_2 \rangle \hookrightarrow Twell\_ordered\_set \cdot 5a \Rightarrow$ false; Discharge $\Rightarrow$ $\neg\text{orden}_\Theta(o_1) \lhd \text{orden}_\Theta(o_2) \ \& \ o_1 \in o_2$

-- However, since the images of o1 an o2 via the enumerator orden_thryvar both belong to s, trichotomy implies that orden_thryvar (o1) and orden_thryvar (o2) can be compared.

$\langle o_1 \rangle \hookrightarrow Twell\_ordered\_set \cdot 3 \Rightarrow$ $\text{orden}_\Theta(o_1) \in s$
$\langle o_2 \rangle \hookrightarrow Twell\_ordered\_set \cdot 3 \Rightarrow$ $\text{orden}_\Theta(o_2) \in s$
$\langle \text{orden}_\Theta(o_1), s, \text{orden}_\Theta(o_2) \rangle \hookrightarrow Twell\_ordered\_set \cdot 0 \Rightarrow$ $\text{orden}_\Theta(o_2) \lhd \text{orden}_\Theta(o_1) \lor \text{orden}_\Theta(o_1) = \text{orden}_\Theta(o_2)$

-- Neither orden_thryvar (o1) = orden_thryvar (o2) nor the only residual alternative can hold, though;

$\langle o_1, o_2 \rangle \hookrightarrow$ *Twell_ordered_set* · 7 ⇒ $\text{orden}_\Theta(o_2) \lhd \text{orden}_\Theta(o_1)$

-- and hence we are led to the desired contradiction.

$\langle o_2, o_1 \rangle \hookrightarrow$ *Twell_ordered_set* · 5a ⇒ false; Discharge ⇒ QED

-- — Next we show that the enumerator orden_Θ, restricted to an ordinal v for which $\text{orden}_\Theta(v)$ belongs to s, enumerates the segment of s consisting of all elements which precede the v-th: —-

Theorem 353 (well_ordered_set · 6) $\mathcal{O}(V) \,\&\, \text{orden}_\Theta(V) \neq s \rightarrow \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(V)\} = \{\text{orden}_\Theta(x) : x \in V\}$. PROOF:

Suppose_not(o, s) ⇒ $\mathcal{O}(o) \,\&\, \text{orden}_\Theta(o) \neq s \,\&\, \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(o)\} \neq \{\text{orden}_\Theta(u) : u \in o\}$

-- Proceed by contradiction, and use transfinite induction to find the least ordinal o for which our assertion is false.

$\langle s, o \rangle \hookrightarrow$ *Twell_ordered_set* · 6a ⇒ $\{u : u \in s \,|\, u \lhd \text{orden}_\Theta(o)\} \not\supseteq \{\text{orden}_\Theta(u) : u \in o\}$

APPLY $\langle \text{mt}_\Theta : o_1 \rangle$ transfinite_induction$\Big(n \mapsto o, P(x) \mapsto \big(\mathcal{O}(x) \,\&\, \text{orden}_\Theta(x) \neq s \,\&\, \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(x)\} \not\supseteq \{\text{orden}_\Theta(u) : u \in x\}\big)\Big)$ ⇒

Stat0 : $\langle \forall x \,|\, \mathcal{O}(o_1) \,\&\, \text{orden}_\Theta(o_1) \neq s \,\&\, \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(o_1)\} \not\supseteq \{\text{orden}_\Theta(u) : u \in o_1\} \,\&\, \Big(x \in o_1 \longrightarrow \neg\big(\mathcal{O}(x) \,\&\, \text{orden}_\Theta(x) \neq s \,\&\, \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(x)\} \not\supseteq \{\text{ord}$

$\langle \emptyset \rangle \hookrightarrow$ *Stat0* ⇒ $\mathcal{O}(o_1) \,\&\, \text{orden}_\Theta(o_1) \neq s \,\&\, \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(o_1)\} \not\supseteq \{\text{orden}_\Theta(u) : u \in o_1\}$

-- Then since $\text{orden}_\Theta(o_1)$ belongs to s, the range of orden_Θ on o_1 is a proper subset of s; and then by definition $\text{orden}_\Theta(o_1)$ must be the least element of s not in this range.

$\langle s, o_1 \rangle \hookrightarrow$ *Twell_ordered_set* · 2 ⇒ $s \not\subseteq \{\text{orden}_\Theta(y) : y \in o_1\}$
$\langle s, o_1 \rangle \hookrightarrow$ *Twell_ordered_set* · 1 ⇒ $\text{orden}_\Theta(o_1) \in s \setminus \{\text{orden}_\Theta(y) : y \in o_1\} \,\&\,$ *Stat1* : $\langle \forall y \in s \setminus \{\text{orden}_\Theta(y) : y \in o_1\} \,|\, \neg y \lhd \text{orden}_\Theta(o_1) \rangle$

-- First suppose that there is an $o_2 \in o_1$ such that $\text{orden}_\Theta(o_2)$ does not precede $\text{orden}_\Theta(o_1)$. Then plainly o_2 is an ordinal and a proper subset of o_1.

Suppose ⇒ *Stat2* : $\{u : u \in s \,|\, u \lhd \text{orden}_\Theta(o_1)\} \not\supseteq \{\text{orden}_\Theta(o_2) : o_2 \in o_1\}$
$\langle c \rangle \hookrightarrow$ *Stat2* ⇒ *Stat3* : $c \in \{\text{orden}_\Theta(o_2) : o_2 \in o_1\} \,\&\,$ *Stat4* : $c \notin \{u : u \in s \,|\, u \lhd \text{orden}_\Theta(o_1)\}$
$\langle o_2 \rangle \hookrightarrow$ *Stat3* ⇒ $c = \text{orden}_\Theta(o_2) \,\&\, o_2 \in o_1$
$\langle o_1, o_2 \rangle \hookrightarrow$ *T11* ⇒ $\mathcal{O}(o_2)$
$\langle o_1, o_2 \rangle \hookrightarrow$ *T12* ⇒ $o_2 \subseteq o_1$

-- Then it follows, since the range of orden_Θ on o_1 is a proper subset of s, that if we remove all images of elements of o_2 from s, a nonempty set r will remain. Then the image of u will be the least elements of r By to its minimality, the image of o_1 must precede the image of u, which leads to a contradiction.

Set_monot $\Rightarrow$ $\{\text{orden}_\Theta(y) : y \in o_1\} \supseteq \{\text{orden}_\Theta(y) : y \in o_2\}$
ELEM $\Rightarrow$ $s \not\subseteq \{\text{orden}_\Theta(y) : y \in o_1\}\ \&\ s \setminus \{\text{orden}_\Theta(y) : y \in o_2\} \neq \emptyset$
$\langle o_2 \rangle \hookrightarrow$ *Twell_ordered_set* $\cdot$ *8a* $\Rightarrow$ $\text{orden}_\Theta(o_2) = \text{Minrel}_\Theta\big(s \setminus \{\text{orden}_\Theta(y) : y \in o_2\}\big)$
$\langle s, o_2 \rangle \hookrightarrow$ *Twell_ordered_set* $\cdot$ *1* $\Rightarrow$ $\text{orden}_\Theta(o_2) \in s$
$\langle \text{orden}_\Theta(o_2) \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\neg \text{orden}_\Theta(o_2) \lhd \text{orden}_\Theta(o_1)$

-- On the other hand, if all images of elements of o_1 precede the image of o_1, our initial assumption implies that some predecessor b of the image of o_1 in s is not the image of an element of o_1.

$\langle o_2, o_1 \rangle \hookrightarrow$ *Twell_ordered_set* $\cdot$ *5* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —
— —- Our next theorem combines the preceding results to prove that when the image of an ordinal under the enumerator differs from (and hence belongs to) s, the restriction g of 'orden_thryvar' to v is a 1-1 map whose range consists of all predecessors in s of the image of v. —
— — — — — — —-

Theorem 354 (well_ordered_set $\cdot$ 9) $\mathcal{O}(V)\ \&\ \text{orden}_\Theta(V) \neq s \rightarrow$
$1\text{–}1(\{[x, \text{orden}_\Theta(x)] : x \in V\})\ \&\ \mathbf{domain}(\{[x, \text{orden}_\Theta(x)] : x \in V\}) = V\ \&\ \mathbf{range}(\{[x, \text{orden}_\Theta(x)] : x \in V\}) = \{u : u \in s \mid u \lhd \text{orden}_\Theta(V)\}\ \&\ \{u : u \in s \mid u \lhd \text{orden}_\Theta(V)\} = \{\text{or}$

-- For consider a counterexample v. Using fcn_symbol, we see that $g = \{[x, \text{orden}_\Theta(x)] : x \in v\}$ is single-valued and has domain v. Moreover, if g is not 1-1, there are distinct elements x, y of v such that $\text{orden}_\Theta(x) = \text{orden}_\Theta(y)$.

Suppose_not(v, s) $\Rightarrow$
$\mathcal{O}(v)\ \&\ \text{orden}_\Theta(v) \neq s\ \&$
$\neg\big(1\text{–}1(\{[x, \text{orden}_\Theta(x)] : x \in v\})\ \&\ \mathbf{domain}(\{[x, \text{orden}_\Theta(x)] : x \in v\}) = v\ \&\ \mathbf{range}(\{[x, \text{orden}_\Theta(x)] : x \in v\}) = \{u : u \in s \mid u \lhd \text{orden}_\Theta(v)\}\ \&\ \{u : u \in s \mid u \lhd \text{orden}_\Theta(v)$
Loc_def $\Rightarrow$ $g = \{[x, \text{orden}_\Theta(x)] : x \in v\}$
APPLY $\langle x_\Theta : x, y_\Theta : y \rangle$ fcn_symbol$(f(x) \mapsto \text{orden}_\Theta(x), g \mapsto g, s \mapsto v)$ $\Rightarrow$
$\text{Svm}(g)\ \&\ \mathbf{range}(g) = \{\text{orden}_\Theta(x) : x \in v\}\ \&\ \mathbf{domain}(g) = v\ \&\ \big(x, y \in v\ \&\ \text{orden}_\Theta(x) = \text{orden}_\Theta(y)\ \&\ x \neq y\big) \vee 1\text{–}1(g)$
EQUAL $\Rightarrow$ $\mathbf{range}(\{[x, \text{orden}_\Theta(x)] : x \in v\}) = \{\text{orden}_\Theta(x) : x \in v\}\ \&\ \mathbf{domain}(\{[x, \text{orden}_\Theta(x)] : x \in v\}) = v$

-- But the final conjunct in this last formula leads to the following contradiction: since x and y belong to the ordinal v they are ordinals included in v. But since $\text{orden}_\Theta(v) \neq s$, Theorem well_ordered_set. 2 tells us that s is not included in $\{\text{orden}_\Theta(u) : u \in v\}$.

Suppose $\Rightarrow$ $x, y \in v \;\&\; \mathsf{orden}_\Theta(x) = \mathsf{orden}_\Theta(y) \;\&\; x \neq y$
$\langle v, x\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(x)$
$\langle v, y\rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(y)$
$\langle v, x\rangle \hookrightarrow T12 \Rightarrow$ $x \subseteq v$
$\langle v, y\rangle \hookrightarrow T12 \Rightarrow$ $y \subseteq v$
$\langle s, v\rangle \hookrightarrow Twell\_ordered\_set \cdot 2 \Rightarrow$ $s \not\subseteq \{\mathsf{orden}_\Theta(u) : u \in v\}$

-- It follows that the images under g of x and y both belong to s, and then by well_ordered_set. 7 they are the same. This contradiction shows that g is one-one.

Set_monot $\Rightarrow$ $\{\mathsf{orden}_\Theta(u) : u \in v\} \supseteq \{\mathsf{orden}_\Theta(u) : u \in x\}$
Set_monot $\Rightarrow$ $\{\mathsf{orden}_\Theta(u) : u \in v\} \supseteq \{\mathsf{orden}_\Theta(u) : u \in y\}$
ELEM $\Rightarrow$ $s \not\subseteq \{\mathsf{orden}_\Theta(u) : u \in x\} \;\&\; s \not\subseteq \{\mathsf{orden}_\Theta(u) : u \in y\}$
$\langle s, x\rangle \hookrightarrow Twell\_ordered\_set \cdot 2 \Rightarrow$ $\mathsf{orden}_\Theta(x) \neq s$
$\langle s, y\rangle \hookrightarrow Twell\_ordered\_set \cdot 2 \Rightarrow$ $\mathsf{orden}_\Theta(y) \neq s$
$\langle x, y\rangle \hookrightarrow Twell\_ordered\_set \cdot 7 \Rightarrow$ false; Discharge $\Rightarrow$ 1–1(g)
EQUAL $\Rightarrow$ $1\text{–}1(\{[x, \mathsf{orden}_\Theta(x)] : x \in v\})$

-- We still must prove that g has the stated range, but this is immediate from Theorem well_ordered_set. 6, completing the proof of the present theorem.

$\langle v\rangle \hookrightarrow Twell\_ordered\_set \cdot 6 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- —
— — —- —
— — — — —- The results just proved can be summarized as follows:

ENTER_THEORY Set_theory

DISPLAY well_ordered_set

THEORY well_ordered_set$(s, y \lhd x)$
$\langle \forall x \in s \mid \neg x \lhd x\rangle \;\&\; \langle \forall x \in s, y \in s, z \in s \mid x \lhd y \;\&\; y \lhd z \rightarrow x \lhd z\rangle \;\&\; \langle \forall t \subseteq s \mid t \neq \emptyset \rightarrow \langle \exists x \in t, y \in t \mid x \lhd y \vee x = y\rangle\rangle$
$\Rightarrow$ (orden)
$\langle \forall x \in s, y \in s \mid x \lhd y \vee y \lhd x \vee x = y\rangle$
$s \subseteq \{\mathsf{orden}(y) : y \in X\} \leftrightarrow \mathsf{orden}(X) = s$
$\mathsf{orden}(X) \neq s \rightarrow \mathsf{orden}(X) \in s$
-- Well - ordering is isomorphic to ordinal enumeration
$\mathcal{O}(U) \;\&\; \mathcal{O}(V) \;\&\; \mathsf{orden}(U) \neq s \;\&\; \mathsf{orden}(V) \neq s \rightarrow (\mathsf{orden}(U) \lhd \mathsf{orden}(V) \leftrightarrow U \in V)$
$\mathcal{O}(V) \;\&\; \mathsf{orden}(V) \neq s \rightarrow \{u : u \in s \mid u \lhd \mathsf{orden}(V)\} = \{\mathsf{orden}(x) : x \in V\}$
$\mathcal{O}(U) \;\&\; \mathcal{O}(V) \;\&\; \mathsf{orden}(U) \neq s \;\&\; \mathsf{orden}(V) \neq s \;\&\; U \neq V \rightarrow \mathsf{orden}(U) \neq \mathsf{orden}(V)$
$\langle \exists o \mid \mathcal{O}(o) \;\&\; s = \{\mathsf{orden}(x) : x \in o\} \;\&\; \langle \forall x \in o \mid \mathsf{orden}(x) \neq s\rangle \;\&\; 1\text{–}1(\{[x, \mathsf{orden}(x)] : x \in o\})\rangle$

$\mathcal{O}(\mathsf{V})$ & $\mathsf{orden}(\mathsf{V}) \neq \mathsf{s} \rightarrow$
$1\text{–}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{V}\})$ & $\mathbf{domain}(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{V}\}) = \mathsf{V}$ & $\mathbf{range}(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{V}\}) = \{\mathsf{u} : \mathsf{u} \in \mathsf{s} \mid \mathsf{u} \lhd \mathsf{orden}(\mathsf{V})\}$ & $\{\mathsf{u} : \mathsf{u} \in \mathsf{s} \mid \mathsf{u} \lhd \mathsf{orden}(\mathsf{V})\} = \{\mathsf{orden}$

END well_ordered_set

-- Next, in more direct preparation for the proof of the cardinal product theorem at which we aim, we prove various properties of a modified lexicographic ordering of the Cartesian product set s of two ordinals. Our key goal is to show that s is well-ordered by this ordering.

THEORY product_order$(\mathsf{o}_1, \mathsf{o}_2)$
$\mathcal{O}(\mathsf{o}_1)$ & $\mathcal{O}(\mathsf{o}_2)$
END product_order

ENTER_THEORY product_order

-- The following definition introduces the ordering that we will use: one pair $[\mathsf{x}, \mathsf{y}]$ of ordinals is less than another pair $[\mathsf{u}, \mathsf{v}]$ iff either the maximum of x and y is less than the maximum of u, v, or if these maxima are equal and x is less than u, or if the maxima are equal, $\mathsf{x} = \mathsf{u}$, and y is less than v.

DEF product_order_c. $\mathsf{X} <_\Theta \mathsf{Y} \quad \leftrightarrow_{\text{Def}} \quad \mathsf{X}^{[1]} \cup \mathsf{X}^{[2]} \in \mathsf{Y}^{[1]} \cup \mathsf{Y}^{[2]} \lor (\mathsf{X}^{[1]} \cup \mathsf{X}^{[2]} = \mathsf{Y}^{[1]} \cup \mathsf{Y}^{[2]}$ & $\mathsf{X}^{[1]} \in \mathsf{Y}^{[1]}) \lor (\mathsf{X}^{[1]} \cup \mathsf{X}^{[2]} = \mathsf{Y}^{[1]} \cup \mathsf{Y}^{[2]}$ & $\mathsf{X}^{[1]} = \mathsf{Y}^{[1]}$ & $\mathsf{X}^{[2]} \in \mathsf{Y}^{[2]})$

-- We first note, in the three following Lemmas, that for all pairs $[\mathsf{x}, \mathsf{y}]$ in the Cartesian product of our two ordinals, x, y, the minimum and the maximum of x and y are ordinals. Our first results are trivial consequences of the fact that any member of an ordinal is also an ordinal.

Theorem 355 (product_order$_1$) $\mathsf{X} \in \mathsf{o}_1 \times \mathsf{o}_2 \rightarrow \mathcal{O}(\mathsf{X}^{[1]})$. PROOF:

Suppose_not$(\mathsf{x}, \mathsf{o}_1, \mathsf{o}_2) \Rightarrow \quad \mathsf{x} \in \mathsf{o}_1 \times \mathsf{o}_2$ & $\neg\mathcal{O}(\mathsf{x}^{[1]})$
Use_def$(\times) \Rightarrow \quad$ *Stat1* : $\mathsf{x} \in \{[\mathsf{u}, \mathsf{v}] : \mathsf{u} \in \mathsf{o}_1, \mathsf{v} \in \mathsf{o}_2\}$
$\langle \mathsf{a}, \mathsf{b} \rangle \hookrightarrow$ *Stat1* $\Rightarrow \quad \mathsf{x} = [\mathsf{a}, \mathsf{b}]$ & $\mathsf{a} \in \mathsf{o}_1$ & $\mathsf{b} \in \mathsf{o}_2$
Assump $\Rightarrow \quad \mathcal{O}(\mathsf{o}_1)$ & $\mathcal{O}(\mathsf{o}_2)$
$\langle \mathsf{o}_1, \mathsf{a} \rangle \hookrightarrow$ *T11* $\Rightarrow \quad \mathcal{O}(\mathsf{a})$
ELEM $\Rightarrow \quad \mathsf{x}^{[1]} = \mathsf{a}$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

Theorem 356 (product_order$_2$) $\mathsf{X} \in \mathsf{o}_1 \times \mathsf{o}_2 \rightarrow \mathcal{O}(\mathsf{X}^{[2]})$. PROOF:

$\mathsf{Suppose\_not}(x, o_1, o_2) \Rightarrow \quad x \in o_1 \times o_2 \,\&\, \neg\mathcal{O}(x^{[2]})$
$\mathsf{Use\_def}(\times) \Rightarrow \quad \mathit{Stat1}:\ x \in \{[u, v] : u \in o_1, v \in o_2\}$
$\langle a, b\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad x = [a, b] \,\&\, a \in o_1 \,\&\, b \in o_2$
$\mathsf{Assump} \Rightarrow \quad \mathcal{O}(o_1) \,\&\, \mathcal{O}(o_2)$
$\langle o_2, b\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(b)$
$\mathsf{ELEM} \Rightarrow \quad x^{[2]} = b$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Now we note a trivial consequence of the fact that the maximum of two ordinals is an ordinal.

Theorem 357 (product_order$_3$) $X \in o_1 \times o_2 \rightarrow \mathcal{O}(X^{[1]} \cup X^{[2]})$. PROOF:

$\mathsf{Suppose\_not}(x, o_1, o_2) \Rightarrow \quad x \in o_1 \times o_2 \,\&\, \neg\mathcal{O}(x^{[1]} \cup x^{[2]})$
$\mathsf{Use\_def}(\times) \Rightarrow \quad \mathit{Stat1}:\ x \in \{[u, v] : u \in o_1, v \in o_2\}$
$\langle a, b\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad x = [a, b] \,\&\, a \in o_1 \,\&\, b \in o_2$
$\mathsf{Assump} \Rightarrow \quad \mathcal{O}(o_1) \,\&\, \mathcal{O}(o_2)$
$\langle o_1, a\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(a)$
$\langle o_2, b\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(b)$
$\langle a, b\rangle \hookrightarrow T27 \Rightarrow \quad \mathcal{O}(a \cup b)$
$\mathsf{ELEM} \Rightarrow \quad x^{[1]} \cup x^{[2]} = a \cup b$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we show that the binary relationship $<_\Theta$ defined above has the properties required of a linear ordering. The following theorem asserts that any two distinct elements of our Cartesian product set are related by this ordering, and product_order_5 tells us that this ordering is transitive.

Theorem 358 (product_order$_4$) $X, Y \in o_1 \times o_2 \rightarrow X <_\Theta Y \vee Y <_\Theta X \vee X = Y \,\&\, \neg X <_\Theta X$. PROOF:

$\mathsf{Suppose\_not}(x, o_1, o_2, y) \Rightarrow \quad \mathit{Stat1}:\ x, y \in o_1 \times o_2 \,\&\, \neg(x <_\Theta y \vee y <_\Theta x \vee x = y) \vee x <_\Theta x$

-- For let x, o_1, o_2, y be a counterexample to our assertion, and usee the definitions of the ordering operations involved to translate the negative of our assertion into the following statements:

$\mathsf{Use\_def}(<_\Theta) \Rightarrow \quad \mathit{Stat2}:$
$x <_\Theta x \leftrightarrow x^{[1]} \cup x^{[2]} \in x^{[1]} \cup x^{[2]} \vee (x^{[1]} \cup x^{[2]} = y^{[1]} \cup x^{[2]} \,\&\, x^{[1]} \in x^{[1]}) \vee (x^{[1]} \cup x^{[2]} = x^{[1]} \cup x^{[2]} \,\&\, x^{[1]} = x^{[1]} \,\&\, x^{[2]} \in x^{[2]})$
$\mathsf{Use\_def}(<_\Theta) \Rightarrow \quad \mathit{Stat3}:$
$x <_\Theta y \leftrightarrow x^{[1]} \cup x^{[2]} \in y^{[1]} \cup y^{[2]} \vee (x^{[1]} \cup x^{[2]} = y^{[1]} \cup y^{[2]} \,\&\, x^{[1]} \in y^{[1]}) \vee (x^{[1]} \cup x^{[2]} = y^{[1]} \cup y^{[2]} \,\&\, x^{[1]} = y^{[1]} \,\&\, x^{[2]} \in y^{[2]})$

Use_def($<_\Theta$) $\Rightarrow$ *Stat4* :

$y <_\Theta x \leftrightarrow y^{[1]} \cup y^{[2]} \in x^{[1]} \cup x^{[2]} \vee (y^{[1]} \cup y^{[2]} = x^{[1]} \cup x^{[2]} \,\&\, y^{[1]} \in x^{[1]}) \vee (y^{[1]} \cup y^{[2]} = x^{[1]} \cup x^{[2]} \,\&\, y^{[1]} = x^{[1]} \,\&\, y^{[2]} \in x^{[2]})$

-- Use the fact that x and y are members of the Cartesian product set o1 PROD o2 to translate the first of these statements into assertions about the components a, b, c, d of x and y, as follows:

Use_def($\times$) $\Rightarrow$ *Stat5* : $x, y \in \{[x_1, y_1] : x_1 \in o_1, y_1 \in o_2\}$

$\langle a, b, c, d \rangle \hookrightarrow$ *Stat5*([]) $\Rightarrow$ *Stat6* :

$x = [a, b] \,\&\, a \in o_1 \,\&\, b \in o_2 \,\&\, y = [c, d] \,\&\, c \in o_1 \,\&\, d \in o_2$

$\langle$*Stat6*$\rangle$ ELEM $\Rightarrow$ *Stat7* : $x^{[1]} = a \,\&\, x^{[2]} = b$

$\langle$*Stat6, Stat6*$\rangle$ ELEM $\Rightarrow$ *Stat8* : $y^{[1]} = c \,\&\, y^{[2]} = d$

$\langle$*Stat7, Stat2*, $*\rangle$ ELEM $\Rightarrow$ *Stat9* :

$x <_\Theta x \leftrightarrow a \cup b \in a \cup b \vee (a \cup b = a \cup b \,\&\, a \in a) \vee (a \cup b = a \cup b \,\&\, a = a \,\&\, b \in b)$

-- Since the right-hand side of the resulting assertion is obviously impossible, it follows that $x <_\Theta x$ must be false, so that we need only consider the first clause of our assertion.

$\langle$*Stat9*, $*\rangle$ ELEM $\Rightarrow$ *Stat10* : $\neg x <_\Theta x$

$\langle$*Stat10, Stat1*, $*\rangle$ ELEM $\Rightarrow$ *Stat11* : $\neg(x <_\Theta y \vee y <_\Theta x \vee x = y)$

-- This translates easily into the statement about a, b, c, and d seen below.

$\langle$*Stat7, Stat8, Stat3*, $*\rangle$ ELEM $\Rightarrow$ *Stat12* :

$x <_\Theta y \leftrightarrow a \cup b \in c \cup d \vee (a \cup b = c \cup d \,\&\, a \in c) \vee (a \cup b = c \cup d \,\&\, a = c \,\&\, b \in d)$

$\langle$*Stat7, Stat8, Stat4*, $*\rangle$ ELEM $\Rightarrow$ *Stat13* :

$y <_\Theta x \leftrightarrow c \cup d \in a \cup b \vee (c \cup d = a \cup b \,\&\, c \in a) \vee (c \cup d = a \cup b \,\&\, c = a \,\&\, d \in b)$

$\langle$*Stat11, Stat12, Stat13, Stat6*, $*\rangle$ ELEM $\Rightarrow$ *Stat14* :

$\neg\Big((a \cup b \in c \cup d \vee (a \cup b = c \cup d \,\&\, a \in c) \vee (a \cup b = c \cup d \,\&\, a = c \,\&\, b \in d)) \vee (c \cup d \in a \cup b \vee (c \cup d = a \cup b \,\&\, c \in a) \vee (c \cup d = a \cup b \,\&\, c = a \,\&\, d \in b)) \vee [a, b] =$

-- But now, since a, b, c, and d are all ordinals (so that $a \cup b$ and $c \cup d$ are ordinals also), it follows that $a \cup b = c \cup d$, so that statement 50 reduces to the simpler form seen as statement 53 below.

Assump $\Rightarrow$ $\mathcal{O}(o_1) \,\&\, \mathcal{O}(o_2)$

$\langle o_1, a \rangle \hookrightarrow$ *T11* $\Rightarrow$ *Stat15* : $\mathcal{O}(a)$

$\langle o_2, b \rangle \hookrightarrow$ *T11* $\Rightarrow$ *Stat16* : $\mathcal{O}(b)$

$\langle a, b \rangle \hookrightarrow$ *T27* $\Rightarrow$ *Stat17* : $\mathcal{O}(a \cup b)$

$\langle o_1, c \rangle \hookrightarrow$ *T11* $\Rightarrow$ *Stat18* : $\mathcal{O}(c)$

$\langle o_2, d \rangle \hookrightarrow$ *T11* $\Rightarrow$ *Stat19* : $\mathcal{O}(d)$

$\langle c, d \rangle \hookrightarrow$ *T27* $\Rightarrow$ *Stat20* : $\mathcal{O}(c \cup d)$

$\langle \mathit{Stat14}, \mathit{Stat14}, * \rangle$ ELEM $\Rightarrow$ $\mathit{Stat21}:\ \neg(a \cup b \in c \cup d \vee c \cup d \in a \cup b)$
$\langle a \cup b, c \cup d \rangle \hookrightarrow T28(\langle \mathit{Stat21}, \mathit{Stat17}, \mathit{Stat20} \rangle) \Rightarrow$ $\mathit{Stat22}:\ a \cup b = c \cup d$
$\langle \mathit{Stat14}, \mathit{Stat22}, * \rangle$ ELEM $\Rightarrow$ $\mathit{Stat23}:\ \neg\big(a \in c \vee (a = c \,\&\, b \in d) \vee c \in a \vee (c = a \,\&\, d \in b) \vee [a, b] = [c, d]\big)$

-- Repeating this same argument for the pair a, c of ordinals, we find that $a = c$, so that statement 53 reduces to statement 55 as seen below.

$\langle a, c \rangle \hookrightarrow T28(\langle \mathit{Stat23}, \mathit{Stat15}, \mathit{Stat18}, * \rangle) \Rightarrow$ $\mathit{Stat24}:\ a = c$
$\langle \mathit{Stat24}, \mathit{Stat23}, * \rangle$ ELEM $\Rightarrow$ $\mathit{Stat25}:\ \neg(b \in d \vee d \in b \vee [a, b] = [c, d])$

-- Repeating this same argument once more for b, d, we find that $b = d$, which implies $[a, b] = [c, d]$, and so proves our theorem.

$\langle b, d \rangle \hookrightarrow T28(\langle \mathit{Stat25}, \mathit{Stat16}, \mathit{Stat19}, * \rangle) \Rightarrow$ $\mathit{Stat26}:\ b = d$
$\langle \mathit{Stat24}, \mathit{Stat25}, \mathit{Stat26} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the binary relationship Ord1p2_thryvar has the transitivity property required of an ordering.

Theorem 359 (product_order$_5$) $X, Y, ZZ \in o_1 \times o_2 \,\&\, X <_\Theta Y \,\&\, Y <_\Theta ZZ \rightarrow X <_\Theta ZZ$. PROOF:

Suppose_not$(x, o_1, o_2, y, zz) \Rightarrow$ $\mathit{Stat1}:\ x, y, zz \in o_1 \times o_2 \,\&\, x <_\Theta y \,\&\, y <_\Theta zz \,\&\, \neg x <_\Theta zz$

-- For let $x = [a, b]$, $y = [c, d]$, $zz = [e, f]$ be a counterexample. Since all the quantities a, b, c, d, e, f, and also $a \cup b$, $c \cup d$, $e \cup f$ are ordinals, they are all comparable both by membership and inclusion. Since the ordering of pairs like [a. b] by Ord1p2_thryvar is first of all by membership, and hence inclusion, of a + b, We must have $a \cup b \subseteq c \cup d$, $c \cup d \subseteq e \cup f$, but since $x <_\Theta z$ is false, $e \cup f$ cannot be a proper superset of $a \cup b$; hence all the sets $a \cup b$, $c \cup d$, $e \cup f$ must be equal.

$\langle x, o_1, o_2, zz \rangle \hookrightarrow \mathit{Tproduct\_order\_4} \Rightarrow$ $zz <_\Theta x \vee zz = x$
Use_def$(<_\Theta) \Rightarrow$ $x^{[1]} \cup x^{[2]} \in y^{[1]} \cup y^{[2]} \vee$
$(x^{[1]} \cup x^{[2]} = y^{[1]} \cup y^{[2]} \,\&\, x^{[1]} \in y^{[1]}) \vee (x^{[1]} \cup x^{[2]} = y^{[1]} \cup y^{[2]} \,\&\, x^{[1]} = y^{[1]} \,\&\, x^{[2]} \in y^{[2]})$
Use_def$(<_\Theta) \Rightarrow$ $y^{[1]} \cup y^{[2]} \in zz^{[1]} \cup zz^{[2]} \vee$
$(y^{[1]} \cup y^{[2]} = zz^{[1]} \cup zz^{[2]} \,\&\, y^{[1]} \in zz^{[1]}) \vee (y^{[1]} \cup y^{[2]} = zz^{[1]} \cup zz^{[2]} \,\&\, y^{[1]} = zz^{[1]} \,\&\, y^{[2]} \in zz^{[2]})$
Use_def$(<_\Theta) \Rightarrow$ $\big(zz^{[1]} \cup zz^{[2]} \in x^{[1]} \cup x^{[2]} \vee (zz^{[1]} \cup zz^{[2]} = x^{[1]} \cup x^{[2]} \,\&\, zz^{[1]} \in x^{[1]}) \vee (zz^{[1]} \cup zz^{[2]} = x^{[1]} \cup x^{[2]} \,\&\, zz^{[1]} = x^{[1]} \,\&\, zz^{[2]} \in x^{[2]})\big) \vee$
$zz = x$
Use_def$(\times) \Rightarrow$ $\mathit{Stat2}:\ x, y \in \{[x_1, y_1] : x_1 \in o_1, y_1 \in o_2\} \,\&\, zz \in \{[x, y] : x \in o_1, y \in o_2\}$
$\langle a, b, c, d, e, f \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathit{Stat3}:$
$x = [a, b] \,\&\, a \in o_1 \,\&\, b \in o_2 \,\&\, y = [c, d] \,\&\, c \in o_1 \,\&\, d \in o_2 \,\&\, zz = [e, f] \,\&\, e \in o_1 \,\&\, f \in o_2$
ELEM $\Rightarrow$ $[a, b]^{[1]} = a \,\&\, [a, b]^{[2]} = b \,\&\, [c, d]^{[1]} = c \,\&\, [c, d]^{[2]} = d \,\&\, [e, f]^{[1]} = e \,\&\, [e, f]^{[2]} = f$

$\text{EQUAL} \Rightarrow \quad x^{[1]} = a \,\&\, x^{[2]} = b \,\&\, y^{[1]} = c \,\&\, y^{[2]} = d \,\&\, zz^{[1]} = e \,\&\, zz^{[2]} = f$
$\text{EQUAL} \Rightarrow \quad \mathit{Stat4}: \; a \cup b \in c \cup d \vee (a \cup b = c \cup d \,\&\, a \in c) \vee (a \cup b = c \cup d \,\&\, a = c \,\&\, b \in d)$
$\text{EQUAL} \Rightarrow \quad \mathit{Stat5}: \; c \cup d \in e \cup f \vee (c \cup d = e \cup f \,\&\, c \in e) \vee (c \cup d = e \cup f \,\&\, c = e \,\&\, d \in f)$
$\text{EQUAL} \Rightarrow \quad \mathit{Stat6}: \; \big(e \cup f \in a \cup b \vee (e \cup f = a \cup b \,\&\, e \in a) \vee (e \cup f = a \cup b \,\&\, e = a \,\&\, f \in b)\big) \vee \; [e, f] \; = \; [a, b]$
$\langle \mathit{Stat6} \rangle \; \text{ELEM} \Rightarrow \quad \mathit{Stat7}: \; e \cup f \in a \cup b \vee e \cup f = a \cup b$
$\text{Assump} \Rightarrow \quad \mathcal{O}(o_1) \,\&\, \mathcal{O}(o_2)$
$\langle o_1, a \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(a)$
$\langle o_2, b \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(b)$
$\langle o_1, c \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(c)$
$\langle o_2, d \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(d)$
$\langle o_1, e \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(e)$
$\langle o_2, f \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(f)$
$\text{ELEM} \Rightarrow \quad \mathit{Stat8}: \; \mathcal{O}(a) \,\&\, \mathcal{O}(b) \,\&\, \mathcal{O}(c) \,\&\, \mathcal{O}(d) \,\&\, \mathcal{O}(e) \,\&\, \mathcal{O}(f)$
$\langle a, b \rangle \hookrightarrow \mathit{T26} \Rightarrow \quad a \subseteq b \vee b \subseteq a$
$\langle c, d \rangle \hookrightarrow \mathit{T26} \Rightarrow \quad c \subseteq d \vee d \subseteq c$
$\langle e, f \rangle \hookrightarrow \mathit{T26} \Rightarrow \quad e \subseteq f \vee f \subseteq e$
$\langle a, e \rangle \hookrightarrow \mathit{T28} \Rightarrow \quad a \in e \vee e \in a \vee a = e$
$\langle b, f \rangle \hookrightarrow \mathit{T28} \Rightarrow \quad b \in f \vee f \in b \vee b = f$
$\langle a, b \rangle \hookrightarrow \mathit{T27} \Rightarrow \quad \mathcal{O}(a \cup b)$
$\langle c, d \rangle \hookrightarrow \mathit{T27} \Rightarrow \quad \mathcal{O}(c \cup d)$
$\langle e, f \rangle \hookrightarrow \mathit{T27} \Rightarrow \quad \mathcal{O}(e \cup f)$
$\text{ELEM} \Rightarrow \quad \mathit{Stat9}: \; \mathcal{O}(a \cup b) \,\&\, \mathcal{O}(c \cup d) \,\&\, \mathcal{O}(e \cup f)$
$\langle c \cup d, a \cup b \rangle \hookrightarrow \mathit{T31} \Rightarrow \quad \mathit{Stat10}: \; a \cup b \in c \cup d \rightarrow a \cup b \subseteq c \cup d$
$\langle e \cup f, c \cup d \rangle \hookrightarrow \mathit{T31} \Rightarrow \quad \mathit{Stat11}: \; c \cup d \in e \cup f \rightarrow c \cup d \subseteq e \cup f$
$\langle a \cup b, e \cup f \rangle \hookrightarrow \mathit{T31} \Rightarrow \quad \mathit{Stat12}: \; e \cup f \in a \cup b \rightarrow e \cup f \subseteq a \cup b$

-- The known product-order relationships between $[a, b]$, $[c, d]$, and $[e, f]$ translate into the following membership and equality statements.

$\langle \mathit{Stat6} \rangle \; \text{ELEM} \Rightarrow \quad \mathit{Stat13}: \; e \cup f \in a \cup b \vee (e \cup f = a \cup b \,\&\, e \in a) \vee (e \cup f = a \cup b \,\&\, e = a \,\&\, f \in b) \vee (e = a \,\&\, f = b)$

-- The known ordering of these same pairs implies the following inclusions, from which the equality of all three union sets follows. This in turn implies the membership relations, and hence the inclusions, seen below.

$\langle \mathit{Stat4}, \mathit{Stat10}, * \rangle \; \text{ELEM} \Rightarrow \quad \mathit{Stat14}: \; a \cup b \subseteq c \cup d$
$\langle \mathit{Stat5}, \mathit{Stat11}, * \rangle \; \text{ELEM} \Rightarrow \quad \mathit{Stat15}: \; c \cup d \subseteq e \cup f$
$\langle \mathit{Stat7}, \mathit{Stat12}, * \rangle \; \text{ELEM} \Rightarrow \quad \mathit{Stat16}: \; e \cup f \subseteq a \cup b$
$\langle \mathit{Stat16}, \mathit{Stat14}, \mathit{Stat15}, \mathit{Stat4}, \mathit{Stat5}, * \rangle \; \text{ELEM} \Rightarrow \quad \mathit{Stat17}:$
$a \cup b = c \cup d \,\&\, c \cup d = e \cup f \,\&\, a \in c \vee (a = c \,\&\, b \in d) \,\&\, c \in e \vee (c = e \,\&\, d \in f)$
$\langle c, a \rangle \hookrightarrow \mathit{T31}(\langle \mathit{Stat8}, \mathit{Stat17}, * \rangle) \Rightarrow \quad \mathit{Stat18}: \; c \supseteq a$

$\langle e, c \rangle \hookrightarrow T31(\langle Stat8, Stat17, * \rangle) \Rightarrow \quad Stat19: \ e \supseteq c$
$\langle a, e \rangle \hookrightarrow T31(\langle Stat13, Stat8, Stat17, * \rangle) \Rightarrow \quad Stat20: \ a \supseteq e$

-- But then clearly the sets a, c, and e are all equal, so we must have $b \in d \ \& \ d \in f \ \& \ b \notin f$, which contradicts the fact that, for ordinals, the membership relationship is transitive.

$\langle Stat13, Stat17, Stat18, Stat19, Stat20, * \rangle$ ELEM $\Rightarrow \quad Stat21: \ a = c \ \& \ c = e \ \& \ b \in d \ \& \ d \in f \ \& \ b \notin f$
$\langle f, d \rangle \hookrightarrow T31 \Rightarrow \quad f \supseteq d$
$\langle Stat21 \rangle$ ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that $<_\Theta$ has the well-ordering property: any subset of our Cartesian product set contains a element minimal in the ordering Ord1p2_thryvar.

Theorem 360 (product_order$_6$) $T \subseteq o_1 \times o_2 \ \& \ T \neq \emptyset \rightarrow \langle \exists x \in T, \forall y \in t \mid x <_\Theta y \vee x = y \rangle$. PROOF:

Suppose_not$(t, o_1, o_2) \Rightarrow \quad Stat1: \ t \subseteq o_1 \times o_2 \ \& \ t \neq \emptyset \ \& \ Stat2: \ \neg\langle \exists x \in t, \forall y \in t \mid x <_\Theta y \vee x = y \rangle$

-- For in the contrary case some Cartesian product $o_1 \times o_2$ of two ordinals has a subset t having no minimal element. Plainly, $x^{[1]} \cup x^{[2]}$ is an ordinal for every x in t. Thus the set rel_1 of elements of t on which $x^{[1]} \cup x^{[2]}$ takes on its minimum value is nonempty.

Assump $\Rightarrow \quad \mathcal{O}(o_1) \ \& \ \mathcal{O}(o_2)$
Suppose $\Rightarrow \quad Stat3: \ \neg\langle \forall x \in t \mid \mathcal{O}(x^{[1]} \cup x^{[2]}) \rangle$
$\langle a \rangle \hookrightarrow Stat3 \Rightarrow \quad a \in t \ \& \ \neg\mathcal{O}(a^{[1]} \cup a^{[2]})$
ELEM $\Rightarrow \quad a \in o_1 \times o_2$
Use_def$(\times) \Rightarrow \quad Stat4: \ a \in \{[x, y] : \ x \in o_1, y \in o_2\}$
$\langle b, c \rangle \hookrightarrow Stat4 \Rightarrow \quad a = [b, c] \ \& \ b \in o_1 \ \& \ c \in o_2$
ELEM $\Rightarrow \quad a^{[1]} \cup a^{[2]} = b \cup c$
EQUAL $\Rightarrow \quad \neg\mathcal{O}(b \cup c)$
$\langle o_1, b \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(b)$
$\langle o_2, c \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(c)$
$\langle b, c \rangle \hookrightarrow T27 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \langle \forall x \in t \mid \mathcal{O}(x^{[1]} \cup x^{[2]}) \rangle$
APPLY $\langle rng_\Theta : \ rel_1 \rangle$ ordval_fcn$(s \mapsto t, f(x) \mapsto x^{[1]} \cup x^{[2]}) \Rightarrow$
$Stat5: \ rel_1 \neq \emptyset \ \& \ rel_1 \subseteq t \ \& \ rel_1 = \{x : \ x \in t \mid x^{[1]} \cup x^{[2]} = \mathbf{arb}(\{y^{[1]} \cup y^{[2]} : \ y \in t\})\} \ \& \ Stat6: \ \langle \forall x \in rel_1, y \in t \mid x^{[1]} \cup x^{[2]} \subseteq y^{[1]} \cup y^{[2]} \rangle$

-- Since the function $x \mapsto x^{[1]}$ is also ordinal-valued, rel_1 admits a nonempty subset rel_2 on which this function takes on its minimum value.

Suppose $\Rightarrow \quad Stat7: \ \neg\langle \forall x \in rel_1 \mid \mathcal{O}(x^{[1]}) \rangle$
$\langle d \rangle \hookrightarrow Stat7 \Rightarrow \quad d \in rel_1 \ \& \ \neg\mathcal{O}(d^{[1]})$
ELEM $\Rightarrow \quad d \in o_1 \times o_2$

Use_def($\times$) $\Rightarrow$ $\quad Stat8:\ \mathsf{d} \in \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathsf{o}_1, \mathsf{y} \in \mathsf{o}_2\}$
$\langle \mathsf{b}_2, \mathsf{c}_2 \rangle \hookrightarrow Stat8 \Rightarrow$ $\quad \mathsf{d} = [\mathsf{b}_2, \mathsf{c}_2]\ \&\ \mathsf{b}_2 \in \mathsf{o}_1$
ELEM $\Rightarrow$ $\quad \mathsf{d}^{[1]} = \mathsf{b}_2$
EQUAL $\Rightarrow$ $\quad \neg\mathcal{O}(\mathsf{b}_2)$
Assump $\Rightarrow$ $\quad \mathcal{O}(\mathsf{o}_1)\ \&\ \mathcal{O}(\mathsf{o}_2)$
$\langle \mathsf{o}_1, \mathsf{b}_2 \rangle \hookrightarrow T11 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \langle \forall \mathsf{x} \in \mathsf{rel}_1 \mid \mathcal{O}(\mathsf{x}^{[1]}) \rangle$
APPLY $\langle \mathsf{rng}_\Theta : \mathsf{rel}_2 \rangle$ ordval_fcn $(\mathsf{s} \mapsto \mathsf{rel}_1, \mathsf{f}(\mathsf{x}) \mapsto \mathsf{x}^{[1]}) \Rightarrow$
$\quad Stat9:\ \mathsf{rel}_2 \neq \emptyset\ \&\ \mathsf{rel}_2 \subseteq \mathsf{rel}_1\ \&\ \mathsf{rel}_2 = \left\{\mathsf{x} : \mathsf{x} \in \mathsf{rel}_1 \mid \mathsf{x}^{[1]} = \mathbf{arb}\left(\left\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{rel}_1\right\}\right)\right\}\ \&\ Stat10:\ \langle \forall \mathsf{x} \in \mathsf{rel}_2, \mathsf{y} \in \mathsf{rel}_1 \mid \mathsf{x}^{[1]} \subseteq \mathsf{y}^{[1]} \rangle$

-- Similarly, since the function $\mathsf{x} \mapsto \mathsf{x}^{[2]}$ is also ordinal-valued, rel_2 admits a nonempty subset rel_3 on which this function takes on its minimum value.

Suppose $\Rightarrow$ $\quad Stat11:\ \neg\langle \forall \mathsf{x} \in \mathsf{rel}_2 \mid \mathcal{O}(\mathsf{x}^{[2]}) \rangle$
$\langle \mathsf{e} \rangle \hookrightarrow Stat11 \Rightarrow$ $\quad \mathsf{e} \in \mathsf{rel}_2\ \&\ \neg\mathcal{O}(\mathsf{e}^{[2]})$
ELEM $\Rightarrow$ $\quad \mathsf{e} \in \mathsf{o}_1 \times \mathsf{o}_2$
Use_def($\times$) $\Rightarrow$ $\quad Stat12:\ \mathsf{e} \in \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathsf{o}_1, \mathsf{y} \in \mathsf{o}_2\}$
$\langle \mathsf{b}_3, \mathsf{c}_3 \rangle \hookrightarrow Stat12 \Rightarrow$ $\quad \mathsf{e} = [\mathsf{b}_3, \mathsf{c}_3]\ \&\ \mathsf{c}_3 \in \mathsf{o}_2$
ELEM $\Rightarrow$ $\quad \mathsf{e}^{[2]} = \mathsf{c}_3$
EQUAL $\Rightarrow$ $\quad \neg\mathcal{O}(\mathsf{c}_3)$
$\langle \mathsf{o}_2, \mathsf{c}_3 \rangle \hookrightarrow T11 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \langle \forall \mathsf{x} \in \mathsf{rel}_2 \mid \mathcal{O}(\mathsf{x}^{[2]}) \rangle$
APPLY $\langle \mathsf{rng}_\Theta : \mathsf{rel}_3 \rangle$ ordval_fcn $(\mathsf{s} \mapsto \mathsf{rel}_2, \mathsf{f}(\mathsf{x}) \mapsto \mathsf{x}^{[2]}) \Rightarrow$
$\quad Stat13:\ \mathsf{rel}_3 \neq \emptyset\ \&\ \mathsf{rel}_3 \subseteq \mathsf{rel}_2\ \&\ Stat14:\ \langle \forall \mathsf{x} \in \mathsf{rel}_3, \mathsf{y} \in \mathsf{rel}_2 \mid \mathsf{x}^{[2]} \subseteq \mathsf{y}^{[2]} \rangle$

-- But it is easily seen that any element of the set rel_3 is minimal over t in the product order $<_\Theta$.

$\langle \mathsf{x} \rangle \hookrightarrow Stat13 \Rightarrow$ $\quad Stat15:\ \mathsf{x} \in \mathsf{rel}_3\ \&\ \mathsf{x} \in \mathsf{rel}_2\ \&\ \mathsf{x} \in \mathsf{rel}_1$
$\langle Stat15, Stat5 \rangle$ ELEM $\Rightarrow$ $\quad Stat16:\ \mathsf{x} \in \left\{\mathsf{u} : \mathsf{u} \in \mathsf{t} \mid \mathsf{u}^{[1]} \cup \mathsf{u}^{[2]} = \mathbf{arb}\left(\left\{\mathsf{v}^{[1]} \cup \mathsf{v}^{[2]} : \mathsf{v} \in \mathsf{t}\right\}\right)\right\}$
$\langle Stat15, Stat9 \rangle$ ELEM $\Rightarrow$ $\quad Stat17:\ \mathsf{x} \in \left\{\mathsf{v} : \mathsf{v} \in \mathsf{rel}_1 \mid \mathsf{v}^{[1]} = \mathbf{arb}\left(\left\{\mathsf{u}^{[1]} : \mathsf{u} \in \mathsf{rel}_1\right\}\right)\right\}$
$\langle \mathsf{x}_2 \rangle \hookrightarrow Stat16 \Rightarrow$ $\quad Stat18:\ \mathsf{x} = \mathsf{x}_2\ \&\ \mathsf{x}_2{}^{[1]} \cup \mathsf{x}_2{}^{[2]} = \mathbf{arb}\left(\left\{\mathsf{y}^{[1]} \cup \mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{t}\right\}\right)$
EQUAL $\Rightarrow$ $\quad Stat19:\ \mathsf{x}^{[1]} \cup \mathsf{x}^{[2]} = \mathbf{arb}\left(\left\{\mathsf{y}^{[1]} \cup \mathsf{y}^{[2]} : \mathsf{y} \in \mathsf{t}\right\}\right)$
$\langle \mathsf{x}_3 \rangle \hookrightarrow Stat17 \Rightarrow$ $\quad Stat20:\ \mathsf{x} = \mathsf{x}_3\ \&\ \mathsf{x}_3{}^{[1]} = \mathbf{arb}\left(\left\{\mathsf{y}^{[1]} : \mathsf{y} \in \mathsf{rel}_1\right\}\right)$
EQUAL $\Rightarrow$ $\quad Stat21:\ \mathsf{x}^{[1]} = \mathbf{arb}\left(\left\{\mathsf{y}^{[1]} : \mathsf{y} \in \mathsf{rel}_1\right\}\right)$
Suppose $\Rightarrow$ $\quad Stat22:\ \neg\langle \forall \mathsf{y} \in \mathsf{t} \mid \mathsf{x} <_\Theta \mathsf{y} \vee \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{y} \rangle \hookrightarrow Stat22 \Rightarrow$ $\quad Stat23:\ \mathsf{y} \in \mathsf{t}\ \&\ \neg(\mathsf{x} <_\Theta \mathsf{y} \vee \mathsf{x} = \mathsf{y})$
Use_def($<_\Theta$) $\Rightarrow$ $\quad Stat24:\ \neg$
$\quad \mathsf{x}^{[1]} \cup \mathsf{x}^{[2]} \in \mathsf{y}^{[1]} \cup \mathsf{y}^{[2]} \vee (\mathsf{x}^{[1]} \cup \mathsf{x}^{[2]} = \mathsf{y}^{[1]} \cup \mathsf{y}^{[2]}\ \&\ \mathsf{x}^{[1]} \in \mathsf{y}^{[1]}) \vee (\mathsf{x}^{[1]} \cup \mathsf{x}^{[2]} = \mathsf{y}^{[1]} \cup \mathsf{y}^{[2]}\ \&\ \mathsf{x}^{[1]} = \mathsf{y}^{[1]}\ \&\ \mathsf{x}^{[2]} \in \mathsf{y}^{[2]}) \vee \mathsf{x} = \mathsf{y}$
$\langle Stat1, Stat23, Stat15, Stat5 \rangle$ ELEM $\Rightarrow$ $\quad \mathsf{x}, \mathsf{y} \in \mathsf{o}_1 \times \mathsf{o}_2$
Use_def($\times$) $\Rightarrow$ $\quad Stat25:\ \mathsf{x}, \mathsf{y} \in \{[\mathsf{u},\mathsf{v}] : \mathsf{u} \in \mathsf{o}_1, \mathsf{v} \in \mathsf{o}_2\}$
$\langle \mathsf{u}_1, \mathsf{v}_1, \mathsf{u}_2, \mathsf{v}_2 \rangle \hookrightarrow Stat25 \Rightarrow$ $\quad Stat26:\ \mathsf{x} = [\mathsf{u}_1, \mathsf{v}_1]\ \&\ \mathsf{u}_1 \in \mathsf{o}_1\ \&\ \mathsf{v}_1 \in \mathsf{o}_2\ \&\ \mathsf{y} = [\mathsf{u}_2, \mathsf{v}_2]\ \&\ \mathsf{u}_2 \in \mathsf{o}_1\ \&\ \mathsf{v}_2 \in \mathsf{o}_2$

$\langle Stat26\rangle$ ELEM $\Rightarrow$ $Stat27:\ x^{[1]} = u_1$
$\langle Stat26, Stat26\rangle$ ELEM $\Rightarrow$ $Stat28:\ x^{[2]} = v_1$
$\langle Stat26, Stat26\rangle$ ELEM $\Rightarrow$ $Stat29:\ y^{[1]} = u_2$
$\langle Stat26, Stat26\rangle$ ELEM $\Rightarrow$ $Stat30:\ y^{[2]} = v_2$
$\langle Stat26, *\rangle$ ELEM $\Rightarrow$ $Stat31:\ x = [u_1, v_1]\ \&\ y = [u_2, v_2]$
EQUAL $\Rightarrow$ $Stat32:\ \neg(u_1 \cup v_1 \in u_2 \cup v_2 \vee (u_1 \cup v_1 = u_2 \cup v_2\ \&\ u_1 \in u_2) \vee (u_1 \cup v_1 = u_2 \cup v_2\ \&\ u_1 = u_2\ \&\ v_1 \in v_2) \vee [u_1, v_1] = [u_2, v_2])$
$\langle x, y\rangle \hookrightarrow Stat6 \Rightarrow$ $x^{[1]} \cup x^{[2]} \subseteq y^{[1]} \cup y^{[2]}$
EQUAL $\Rightarrow$ $Stat33:\ u_1 \cup v_1 \subseteq u_2 \cup v_2$
$\langle Stat32\rangle$ ELEM $\Rightarrow$ $Stat34:\ u_1 \cup v_1 \notin u_2 \cup v_2$

-- By statement 776 all of u1, v1, u2, v2 are ordinals, and therefore u1 + v1 and u2 + v2 are also ordinals.

$\langle o_1, u_1\rangle \hookrightarrow T11 \Rightarrow$ $Stat35:\ \mathcal{O}(u_1)$
$\langle o_2, v_1\rangle \hookrightarrow T11 \Rightarrow$ $Stat36:\ \mathcal{O}(v_1)$
$\langle o_1, u_2\rangle \hookrightarrow T11 \Rightarrow$ $Stat37:\ \mathcal{O}(u_2)$
$\langle o_2, v_2\rangle \hookrightarrow T11 \Rightarrow$ $Stat38:\ \mathcal{O}(v_2)$
$\langle u_1, v_1\rangle \hookrightarrow T27 \Rightarrow$ $Stat39:\ \mathcal{O}(u_1 \cup v_1)$
$\langle u_2, v_2\rangle \hookrightarrow T27 \Rightarrow$ $Stat40:\ \mathcal{O}(u_2 \cup v_2)$

-- It therefore follows by statements 777 and 785 and by Theorem 32 that $u_1 \cup v_1 = u_2 \cup v_2$, so that y, like x, must be a member of the subset rel_1 of t.

$\langle u_2 \cup v_2, u_1 \cup v_1\rangle \hookrightarrow T32(\langle Stat39, Stat40, Stat34\rangle) \Rightarrow$ $Stat41:\ u_1 \cup v_1 \supseteq u_2 \cup v_2$
$\langle Stat41, Stat33\rangle$ ELEM $\Rightarrow$ $Stat42:\ u_2 \cup v_2 = u_1 \cup v_1$
EQUAL $\langle Stat42, Stat27, Stat28, Stat29, Stat30\rangle \Rightarrow$ $Stat43:\ y^{[1]} \cup y^{[2]} = x^{[1]} \cup x^{[2]}$
Suppose $\Rightarrow$ $Stat44:\ y \notin rel_1$
$\langle Stat5, Stat44\rangle$ ELEM $\Rightarrow$ $Stat45:\ y \notin \{x : x \in t \mid x^{[1]} \cup x^{[2]} = \mathbf{arb}(\{u^{[1]} \cup u^{[2]} : u \in t\})\}$
$\langle y\rangle \hookrightarrow Stat45 \Rightarrow$ $Stat46:\ y^{[1]} \cup y^{[2]} \neq \mathbf{arb}(\{x^{[1]} \cup x^{[2]} : x \in t\})$
$\langle Stat43, Stat46, Stat19\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $y \in rel_1$
$\langle Stat32, Stat42\rangle$ ELEM $\Rightarrow$ $Stat47:\ \neg(u_1 \in u_2 \vee (u_1 = u_2\ \&\ v_1 \in v_2) \vee [u_1, v_1] = [u_2, v_2])$
$\langle u_2, u_1\rangle \hookrightarrow T32(\langle Stat47, Stat35, Stat37\rangle) \Rightarrow$ $Stat48:\ u_1 \supseteq u_2$
$\langle x, y\rangle \hookrightarrow Stat10 \Rightarrow$ $x^{[1]} \subseteq y^{[1]}$
EQUAL $\Rightarrow$ $u_1 \subseteq u_2$
$\langle Stat48\rangle$ ELEM $\Rightarrow$ $Stat49:\ u_1 = u_2$
EQUAL $\langle Stat49, Stat27, Stat28, Stat29, Stat30\rangle \Rightarrow$ $Stat50:\ y^{[1]} = x^{[1]}$
Suppose $\Rightarrow$ $Stat51:\ y \notin rel_2$
EQUAL $\langle Stat51, Stat9\rangle \Rightarrow$ $Stat52:\ y \notin \{x : x \in rel_1 \mid x^{[1]} = \mathbf{arb}(\{u^{[1]} : u \in rel_1\})\}$
$\langle y\rangle \hookrightarrow Stat52 \Rightarrow$ $Stat53:\ y^{[1]} \neq \mathbf{arb}(\{x^{[1]} : x \in rel_1\})$
$\langle Stat21, Stat50, Stat53\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $y \in rel_2$

$\langle x, y \rangle \hookrightarrow Stat14 \Rightarrow \quad Stat54: \; x^{[2]} \subseteq y^{[2]}$
EQUAL $\langle Stat54, Stat27, Stat28, Stat29, Stat30 \rangle \Rightarrow \quad Stat55: \; v_1 \subseteq v_2$
$\langle Stat49, Stat47 \rangle$ ELEM $\Rightarrow \quad Stat56: \; v_1 \notin v_2$
$\langle v_2, v_1 \rangle \hookrightarrow T32(\langle Stat56, Stat36, Stat38 \rangle) \Rightarrow \quad Stat57: \; v_1 \supseteq v_2$
$\langle Stat55, Stat57 \rangle$ ELEM $\Rightarrow \quad Stat58: \; v_1 = v_2$
$\langle Stat47, Stat58, Stat49 \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The results just established can be summarized as follows.

DISPLAY product_order

THEORY product_order(o_1, o_2)
-- The product - ordering of two ordinals is a well - ordering
$\mathcal{O}(o_1) \;\&\; \mathcal{O}(o_2)$
$\Rightarrow (\; <_\Theta \;)$
$\langle \forall x \in o_1 \times o_2 \mid \mathcal{O}(x^{[1]}) \rangle$
$\langle \forall x \in o_1 \times o_2 \mid \mathcal{O}(x^{[2]}) \rangle$
$\langle \forall x \in o_1 \times o_2 \mid \mathcal{O}(x^{[1]} \cup x^{[2]}) \rangle$
$X <_\Theta Y \leftrightarrow X^{[1]} \cup X^{[2]} \in Y^{[1]} \cup Y^{[2]} \vee (X^{[1]} \cup X^{[2]} = Y^{[1]} \cup Y^{[2]} \;\&\; X^{[1]} \in Y^{[1]}) \vee (X^{[1]} \cup X^{[2]} = Y^{[1]} \cup Y^{[2]} \;\&\; X^{[1]} = Y^{[1]} \;\&\; X^{[2]} \in Y^{[2]})$
$\langle \forall x \in o_1 \times o_2, y \in o_1 \times o_2 \mid x <_\Theta y \vee y <_\Theta x \vee x = y \;\&\; \neg x <_\Theta x \rangle$
$\langle \forall x \in o_1 \times o_2, y \in o_1 \times o_2, z \in o_1 \times o_2 \mid x <_\Theta y \;\&\; y <_\Theta z \longrightarrow x <_\Theta z \rangle$
$T \subseteq o_1 \times o_2 \;\&\; T \neq \emptyset \longrightarrow \langle \exists x \in T, y \in t \mid x <_\Theta y \vee x = y \rangle$
END product_order

-- Next we show that the addition of any single element to an infinite set does not change its cardinality.

-- One - more Lemma

Theorem 361 (278) $\neg \mathrm{Finite}(S) \rightarrow \#S = \#(S \cup \{C\})$. PROOF:

Suppose_not$(s, c) \Rightarrow \quad \neg \mathrm{Finite}(s) \;\&\; \#s \neq \#(s \cup \{c\})$

-- For suppose that s, c is a counterexample to our assertion. Since s is infinite, it is the single-valued image of a proper subset of itself, whose domain therefore omits some element b of s. The mapping $\{[b,c]\}$ defined only on b which maps b to c is plainly single-valued, and since $b \notin s$, $f \cup \{[b,c]\}$ is a single-valued mapping of a subset of s onto $\#(s \cup \{c\})$. Thus $\#\mathbf{domain}(f \cup \{[b,c]\})$ is not greater than $\#s$, while $\#\mathbf{range}(f \cup \{[b,c]\})$ is $\#(s \cup \{c\})$. Hence by theorem 145 $\#(s \cup \{c\})$ is no more than $\#s$, proving our assertion.

$\langle s\rangle \hookrightarrow T168 \Rightarrow \quad Stat1: \langle \exists f \mid \mathsf{Svm}(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) \subseteq s \;\&\; s \neq \mathbf{domain}(f)\rangle$
$\langle f\rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{Svm}(f) \;\&\; \mathbf{range}(f) = s \;\&\; \mathbf{domain}(f) \subseteq s \;\&\; Stat2: s \not\subseteq \mathbf{domain}(f)$
$\langle b\rangle \hookrightarrow Stat2 \Rightarrow \quad \{b\} \subseteq s \;\&\; \{b\} \cap \mathbf{domain}(f) = \emptyset$
ELEM $\Rightarrow \quad \mathsf{Svm}(\{[b,c]\}) \;\&\; \mathbf{range}(\{[b,c]\}) = \{c\} \;\&\; \mathbf{domain}(\{[b,c]\}) = \{b\}$
$\langle f, \{[b,c]\}\rangle \hookrightarrow T79 \Rightarrow \quad \mathsf{Svm}(f \cup \{[b,c]\})$
$\langle f, \{[b,c]\}\rangle \hookrightarrow T70 \Rightarrow \quad \mathbf{domain}(f \cup \{[b,c]\}) \subseteq s$
$\langle f, \{[b,c]\}\rangle \hookrightarrow T71 \Rightarrow \quad \mathbf{range}(f \cup \{[b,c]\}) = s \cup \{c\}$
Set_monot $\Rightarrow \quad Stat3: \#\mathbf{domain}(f \cup \{[b,c]\}) \subseteq \#s$
$\langle f \cup \{[b,c]\}\rangle \hookrightarrow T145 \Rightarrow \quad \#\mathbf{range}(f \cup \{[b,c]\}) \subseteq \#\mathbf{domain}(f \cup \{[b,c]\})$
EQUAL $\Rightarrow \quad Stat4: \#(s \cup \{c\}) \subseteq \#\mathbf{domain}(f \cup \{[b,c]\})$
$\langle Stat3, Stat4\rangle$ ELEM $\Rightarrow \quad Stat5: \#(s \cup \{c\}) \subseteq \#s$
$\langle Stat5, *\rangle$ ELEM $\Rightarrow \quad s \subseteq s \cup \{c\}$
$\langle s, s \cup \{c\}\rangle \hookrightarrow T144 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Using theorem 278, it is easy to prove inductively that the addition of finitely many elements to an infinite set s never changes the cardinality of s.

-- Few - more Lemma

Theorem 362 (279) $\neg\mathsf{Finite}(S) \;\&\; \mathsf{Finite}(T) \rightarrow \#S = \#(S \cup T)$. PROOF:

Suppose_not(s, t) $\Rightarrow \quad \neg\mathsf{Finite}(s) \;\&\; \mathsf{Finite}(t) \;\&\; \#s \neq \#(s \cup t)$

-- For if s, t are a counterexample to our assertion, it follows by the principle of finite induction proved earlier that there is a smallest finite x for which there exists an infinite v for which $\#v \neq \#(v \cup x)$

Suppose $\Rightarrow \quad Stat1: \neg\langle \exists u \mid \neg\mathsf{Finite}(u) \;\&\; \#u \neq \#(u \cup t)\rangle$
$\langle s\rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \langle \exists u \mid \neg\mathsf{Finite}(u) \;\&\; \#u \neq \#(u \cup t)\rangle$
APPLY $\langle m_\Theta : x\rangle$ finite_induction $(n \mapsto t, P(x) \mapsto \langle \exists u \mid \neg\mathsf{Finite}(u) \;\&\; \#u \neq \#(u \cup x)\rangle) \Rightarrow$
$x \subseteq t \;\&\; Stat2: \langle \exists u \mid \neg\mathsf{Finite}(u) \;\&\; \#u \neq \#(u \cup x)\rangle \;\&\; Stat3: \langle \forall y \subseteq x \mid y \neq x \rightarrow \neg\langle \exists u \mid \neg\mathsf{Finite}(u) \;\&\; \#u \neq \#(u \cup y)\rangle\rangle$
$\langle v\rangle \hookrightarrow Stat2 \Rightarrow \quad \neg\mathsf{Finite}(v) \;\&\; \#v \neq \#(v \cup x)$

-- Since x plainly cannot be empty, it must have some element c. Then $\#(\mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\})) = \#\mathsf{v}$ by the minimality of x, and so $\#(\mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\})) \cup \{\mathsf{c}\} = \#\mathsf{v}$ by the preceding theorem, completing our proof.

Suppose $\Rightarrow$ $\mathsf{x} = \emptyset$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat4* : $\mathsf{x} \neq \emptyset$
$\langle \mathsf{c} \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\mathsf{c} \in \mathsf{x}$
$\langle \mathsf{x} \setminus \{\mathsf{c}\} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat5* : $\neg \langle \exists \mathsf{u} \mid \neg \mathsf{Finite}(\mathsf{u}) \,\&\, \#\mathsf{u} \neq \#(\mathsf{u} \cup (\mathsf{x} \setminus \{\mathsf{c}\})) \rangle$
$\langle \mathsf{v} \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\#\mathsf{v} = \#(\mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\}))$
$\langle \mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\}), \mathsf{v} \rangle \hookrightarrow$ *T162* $\Rightarrow$ $\neg \mathsf{Finite}(\mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\}))$
$\langle \mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\}), \mathsf{c} \rangle \hookrightarrow$ *T278* $\Rightarrow$ $\#(\mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\}) \cup \{\mathsf{c}\}) = \#\mathsf{v}$
ELEM $\Rightarrow$ $\mathsf{v} \cup (\mathsf{x} \setminus \{\mathsf{c}\}) \cup \{\mathsf{c}\} = \mathsf{v} \cup \mathsf{x}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 363 (280) $\neg \mathsf{Finite}(\mathsf{S}) \,\&\, \mathsf{X} \in \#\mathsf{S} \rightarrow \mathsf{X} \cup \{\mathsf{X}\} \in \#\mathsf{S}$. PROOF:

Suppose_not(s, x) $\Rightarrow$ $\neg \mathsf{Finite}(\mathsf{s}) \,\&\, \mathsf{x} \in \#\mathsf{s} \,\&\, \mathsf{x} \cup \{\mathsf{x}\} \notin \#\mathsf{s}$

-- For let s, x be a counterexample to our assertion, so $\mathsf{x} \cup \{\mathsf{x}\} \notin \#\mathsf{s}$. Plainly both $\#\mathsf{s}$ and its member x are ordinals, while $\#\mathsf{s}$ must be infinite. If x is finite, so is its ordinal successor $\mathsf{x} \cup \{\mathsf{x}\}$, and therefore $\mathsf{x} \cup \{\mathsf{x}\}$ must be a member of the infinite ordinal $\#\mathsf{s}$ by Theorem 170, ruling out this possibility.

$\langle \mathsf{s} \rangle \hookrightarrow$ *T130* $\Rightarrow$ $\mathsf{Card}(\#\mathsf{s}) \,\&\, \mathcal{O}(\#\mathsf{s})$
$\langle \#\mathsf{s}, \mathsf{x} \rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathsf{x})$
$\langle \mathsf{x} \rangle \hookrightarrow$ *T29* $\Rightarrow$ $\mathcal{O}(\mathsf{next}(\mathsf{x}))$
Use_def(next) $\Rightarrow$ $\mathcal{O}(\mathsf{x} \cup \{\mathsf{x}\})$
Suppose $\Rightarrow$ $\mathsf{Finite}(\mathsf{x})$
$\langle \mathsf{x} \rangle \hookrightarrow$ *T172* $\Rightarrow$ $\mathsf{Finite}(\mathsf{x} \cup \{\mathsf{x}\})$

-- Hence x is infinite, implying $\#(\mathsf{x} \cup \{\mathsf{x}\}) = \#\mathsf{x}$ by the One-more lemma. Since by assumption x is less than, i. e. a proper subset of, $\#\mathsf{s}$, and since $\#\mathsf{x}$ is no more than x by Theorem 143, it follows that $\#\mathsf{x} = \#(\mathsf{x} \cup \{\mathsf{x}\})$ is also a proper subset of $\#\mathsf{s}$. But if $\mathsf{x} \cup \{\mathsf{x}\} \notin \#\mathsf{s}$, then $\#\mathsf{s} \subseteq \mathsf{x} \cup \{\mathsf{x}\}$ by Theorem 32, in which case it follows that $\#\#\mathsf{s} \subseteq \#(\mathsf{x} \cup \{\mathsf{x}\})$, so $\#\mathsf{s} \subseteq \#(\mathsf{x} \cup \{\mathsf{x}\})$ giving the contradiction $\#\mathsf{s} = \#(\mathsf{x} \cup \{\mathsf{x}\})$, and so proving the present corollary.

$\langle \#\mathsf{s}, \mathsf{x} \cup \{\mathsf{x}\} \rangle \hookrightarrow$ *T170* $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \mathsf{Finite}(\mathsf{x})$
$\langle \#\mathsf{s}, \mathsf{x} \rangle \hookrightarrow$ *T31* $\Rightarrow$ *Stat1* : $\mathsf{x} \subseteq \#\mathsf{s} \,\&\, \mathsf{x} \neq \#\mathsf{s}$
$\langle \mathsf{x}, \mathsf{x} \rangle \hookrightarrow$ *T143* $\Rightarrow$ $\#\mathsf{x} \subseteq \mathsf{x}$

$\langle x, x\rangle \hookrightarrow T278 \Rightarrow \quad \#(x \cup \{x\}) = \#x$
$\langle Stat1, *\rangle$ ELEM $\Rightarrow \quad \#(x \cup \{x\}) \subseteq \#s \;\&\; \#(x \cup \{x\}) \neq \#s$
$\langle \#s, x \cup \{x\}\rangle \hookrightarrow T32 \Rightarrow \quad \#s \subseteq x \cup \{x\}$
$\langle \#s, x \cup \{x\}\rangle \hookrightarrow T144 \Rightarrow \quad \#\#s \subseteq \#(x \cup \{x\})$
$\langle s\rangle \hookrightarrow T140 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next main aim is to prove the cardinal Division-by-2 Lemma which appears as Theorem 285 below. Since this result lies a bit deeper than most of the theorems proved up to the present point, we give an informal outline of its proof before entering upon its formal details. The theorem asserts that for every infinite set s, $\#s = \#(s \times \{\emptyset, 1\})$. If this is false, then $\#s \neq \#(s \times \{\emptyset, 1\}) = s * \{\emptyset, 1\} = \#s * \{\emptyset, 1\} = \#(\#s \times \{\emptyset, 1\})$, so our assertion is also false for the infinite cardinal $\#s$, and hence for some infinite ordinal. It follows by the axiom of choice that our assertion is also false for some smallest infinite ordinal s_1. s_1 must be a cardinal, since otherwise $\#s_1 \in s_1$, but also $s_1 \in \#(s_1 \times \{\emptyset, 1\})$ $= \#(\#s_1 \times \{\emptyset, 1\})$, so we would have $\#s_1 \in \#(\#s_1 \times \{\emptyset, 1\})$ also, contradicting the minimality of s_1. Order $s_1 \times \{\emptyset, 1\}$ by the product ordering described above. The theory we have just established tells us that this is a well-ordering, so that by our previous theory of well-orderings there exists a 1-1 map f from some ordinal o onto $s_1 \times \{\emptyset, 1\}$ which is monotone increasing if o is given its standard ordering and $s_1 \times \{\emptyset, 1\}$ is given its product ordering. If o_1 is a finite member of the ordinal o, then obviously $\#o_1$ is a member of the infinite cardinal s_1. If o_1 is an infinite member of the ordinal o, then it is a proper subset of o; so $\mathbf{range}(f_{|o_1})$ is a proper subset t of $s_1 \times \{\emptyset, 1\}$. If $[n, \emptyset] \in \mathbf{range}(f_{|o_1})$ and $m \in n$, then $[m, k]$ is less than $[n, \emptyset]$ for each $k = \emptyset, 1$. Hence there must exist some $[n, \emptyset] \in s_1 \times \{\emptyset, 1\}$ such that $[n, \emptyset] \notin \mathbf{range}(f_{|o_1})$, since otherwise $\mathbf{range}(f_{|o_1})$ would be all of $s_1 \times \{\emptyset, 1\}$ rather than a proper subset. It follows that for every $[m, k] \in t$, m is a member of n. That is, $\mathbf{range}(f_{|o_1})$ is a subset of $n \times \{\emptyset, 1\}$, and so it follows from the minimality of s_1 that $\#t \subseteq \#(n \times \{\emptyset, 1\}) = n$ in s_1. Hence $\#o_1$ is a member of s_1 for each o_1 in o, proving that $\#o \subseteq s_1$ in this case also. Since o is in $1\backslash 1$ correspondence with $s_1 \times \{\emptyset, 1\}$ it follows that $\#(s_1 \times \{\emptyset, 1\}) \subseteq s_1$, contrary to our assumption. This contradiction proves our desired theorem. To keep its details under control, we precede the formal proof of Theorem 285 with several lemmas, as suggested by the preceding discussion. Our first lemma asserts that if s and t are ordinals, s being infinite, and if no member of t has a cardinality larger than $\#s$, then t has a cardinality no larger than $\#s$.

Theorem 364 (281) $\mathcal{O}(S) \;\&\; \mathcal{O}(T) \;\&\; \neg\mathrm{Finite}(S) \;\&\; \langle \forall u \in T \mid \#u \in \#S\rangle \rightarrow \#T \subseteq \#S$. PROOF:

Suppose_not$(s, t) \Rightarrow \quad \mathcal{O}(s) \;\&\; \mathcal{O}(t) \;\&\; \neg\mathrm{Finite}(s) \;\&\; Stat1: \langle \forall x \in t \mid \#x \in \#s\rangle \;\&\; Stat2: \#t \not\subseteq \#s$
$\langle s\rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#s)$
$\langle t\rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#t)$

$\langle \#t, \#s \rangle \hookrightarrow T32 \Rightarrow \quad \#s \in \#t$
$\langle t, t \rangle \hookrightarrow T143 \Rightarrow \quad \#t \subseteq t$
$\langle \#s \rangle \hookrightarrow Stat1 \Rightarrow \quad \#\#s \in \#s$
$\langle s \rangle \hookrightarrow T140 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- We will also find the following elementary property of Cartesian product useful.

Theorem 365 (282) $T \neq \emptyset \rightarrow \#S \subseteq \#(S \times T)$. PROOF:

Suppose_not$(t, s) \Rightarrow \quad Stat1: \; t \neq \emptyset \; \& \; \#s \not\subseteq \#(s \times t)$
$\langle c \rangle \hookrightarrow Stat1 \Rightarrow \quad c \in t$
$\langle s, s, \{c\}, t \rangle \hookrightarrow T219 \Rightarrow \quad s \times \{c\} \subseteq s \times t$
$\langle s \times \{c\}, s \times t \rangle \hookrightarrow T144 \Rightarrow \quad \#(s \times \{c\}) \subseteq \#(s \times t)$
$\langle s, c \rangle \hookrightarrow T193 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Our next theorem, which simply combines results available through the product_order and well_ordered_set theories developed above, tells us that the Cartesian product $o_1 \times o_2$ of any two ordinals is order-isomorphic to a third ordinal o via a 1-1 map that is strictly monotone relative to the product order of $o_1 \times o_2$.

Theorem 366 (283) $\mathcal{O}(O_1) \; \& \; \mathcal{O}(O_2) \rightarrow$
$\langle \exists f \mid 1\text{–}1(f) \; \& \; \mathcal{O}(\mathbf{domain}(f)) \; \& \; \mathbf{range}(f) = O_1 \times O_2 \; \& \; \langle \forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow f\restriction x^{[1]} \cup f\restriction x^{[2]} \in f\restriction y^{[1]} \cup f\restriction y^{[2]} \vee (f\restriction x^{[1]} \cup f\restriction x^{[2]} = f\restriction y^{[1]} \cup f\restriction y^{[2]} \; \& \; f\restriction x^{[1]} \in f\restriction y^{[$

-- For suppose the contrary,

Suppose_not$(o_1, o_2) \Rightarrow \quad \mathcal{O}(o_1) \; \& \; \mathcal{O}(o_2) \; \& \; Stat1:$
$\neg\langle \exists f \mid 1\text{–}1(f) \; \& \; \mathcal{O}(\mathbf{domain}(f)) \; \& \; \mathbf{range}(f) = o_1 \times o_2 \; \& \; \langle \forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow f\restriction x^{[1]} \cup f\restriction x^{[2]} \in f\restriction y^{[1]} \cup f\restriction y^{[2]} \vee (f\restriction x^{[1]} \cup f\restriction x^{[2]} = f\restriction y^{[1]} \cup f\restriction y^{[2]} \; \& \; f\restriction x$

-- And consider the product ordering of $o_1 \times o_2$, which is a well-ordering.

APPLY $\langle <_\Theta : \text{prod\_order} \rangle$ product_order$(o_1 \mapsto o_1, o_2 \mapsto o_2) \Rightarrow$
$Stat2: \; \langle \forall x, y \mid \text{prod\_order}(x, y) \leftrightarrow x^{[1]} \cup x^{[2]} \in y^{[1]} \cup y^{[2]} \vee (x^{[1]} \cup x^{[2]} = y^{[1]} \cup y^{[2]} \; \& \; x^{[1]} \in y^{[1]}) \vee (x^{[1]} \cup x^{[2]} = y^{[1]} \cup y^{[2]} \; \& \; x^{[1]} = y^{[1]} \; \& \; x^{[2]} \in y^{[2]}) \rangle \; \& \; Stat80: \; \langle \forall x, y \mid x, y \in$

-- Adjust unrestricted quantifiers to the syntax used in the hypotheses of the THEORY well_ordered_set.

Suppose $\Rightarrow \quad Stat90: \; \neg\langle \forall x \in o_1 \times o_2 \mid \neg\text{prod\_order}(x, x) \rangle$
$\langle x_0 \rangle \hookrightarrow Stat90 \Rightarrow \quad x_0 \in o_1 \times o_2 \; \& \; \text{prod\_order}(x_0, x_0)$
$\langle x_0, x_0 \rangle \hookrightarrow Stat80 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \langle \forall x \in o_1 \times o_2 \mid \neg\text{prod\_order}(x, x) \rangle$

Suppose $\Rightarrow$ $\mathit{Stat91}:\ \neg\langle\forall x \in o_1 \times o_2, y \in o_1 \times o_2, z \in o_1 \times o_2 \mid \mathsf{prod\_order}(x,y)\ \&\ \mathsf{prod\_order}(y,z) \rightarrow \mathsf{prod\_order}(x,z)\rangle$

$\langle x_1, y_1, z_1\rangle \hookrightarrow \mathit{Stat91} \Rightarrow$ $x_1, y_1, z_1 \in o_1 \times o_2\ \&\ \mathsf{prod\_order}(x_1,y_1)\ \&\ \mathsf{prod\_order}(y_1,z_1)\ \&\ \neg\mathsf{prod\_order}(x_1,z_1)$

$\langle x_1, y_1, z_1\rangle \hookrightarrow \mathit{Stat81} \Rightarrow$ false; Discharge $\Rightarrow$ $\langle\forall x \in o_1 \times o_2, y \in o_1 \times o_2, z \in o_1 \times o_2 \mid \mathsf{prod\_order}(x,y)\ \&\ \mathsf{prod\_order}(y,z) \rightarrow \mathsf{prod\_order}(x,z)\rangle$

Suppose $\Rightarrow$ $\mathit{Stat92}:\ \neg\langle\forall t \subseteq o_1 \times o_2 \mid t \neq \emptyset \rightarrow \langle\exists x \in t, \forall y \in t \mid \mathsf{prod\_order}(x,y) \vee x = y\rangle\rangle$

$\langle t_0\rangle \hookrightarrow \mathit{Stat92} \Rightarrow$ $t_0 \subseteq o_1 \times o_2\ \&\ t_0 \neq \emptyset\ \&\ \neg\langle\exists x \in t_0, \forall y \in t_0 \mid \mathsf{prod\_order}(x,y) \vee x = y\rangle$

$\langle t_0\rangle \hookrightarrow \mathit{Stat82} \Rightarrow$ false; Discharge $\Rightarrow$ $\langle\forall t \subseteq o_1 \times o_2 \mid t \neq \emptyset \rightarrow \langle\exists x \in t, \forall y \in t \mid \mathsf{prod\_order}(x,y) \vee x = y\rangle\rangle$

-- It follows by the theory of well-ordered sets developed above that there exists an ordinal o and a function $\mathsf{orden}(x)$ such that $f = \{[x, \mathsf{orden}(x)] : x \in o\}$ is a 1-1 mapping of o onto $o_1 \times o_2$, which puts the ordering of o by membership into isomorphism with the product-ordering of $o_1 \times o_2$.

APPLY $\langle \mathsf{orden}_\Theta : \mathsf{orden}\rangle$ $\mathsf{well\_ordered\_set}(s \mapsto o_1 \times o_2, x \lhd y \mapsto \mathsf{prod\_order}(x,y)) \Rightarrow$

$\mathit{Stat3}:\ \langle\forall u, v \mid \mathcal{O}(u)\ \&\ \mathcal{O}(v)\ \&\ \mathsf{orden}(u) \neq o_1 \times o_2\ \&\ \mathsf{orden}(v) \neq o_1 \times o_2 \rightarrow \big(\mathsf{prod\_order}(\mathsf{orden}(u), \mathsf{orden}(v)) \leftrightarrow u \in v\big)\rangle\ \&\ \mathit{Stat4}:\ \langle\exists o \mid \mathcal{O}(o)\ \&\ o_1 \times o_2 = \{\mathsf{orden}(x) : x \in o$

$\langle o\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $\mathcal{O}(o)\ \&\ o_1 \times o_2 = \{\mathsf{orden}(x) : x \in o\}\ \&\ \mathit{Stat5}:\ \langle\forall x \in o \mid \mathsf{orden}(x) \neq o_1 \times o_2\rangle\ \&\ 1\text{–}1(\{[x, \mathsf{orden}(x)] : x \in o\})$

Loc_def $\Rightarrow$ $f = \{[x, \mathsf{orden}(x)] : x \in o\}$

APPLY $\langle\ \rangle$ $\mathsf{fcn\_symbol}(f(x) \mapsto \mathsf{orden}(x), g \mapsto f, s \mapsto o) \Rightarrow$

$\mathbf{domain}(f) = o\ \&\ \mathbf{range}(f) = \{\mathsf{orden}(x) : x \in o\}\ \&\ \mathit{Stat6}:\ \langle\forall x \mid x \in o \rightarrow f{\upharpoonright}x = \mathsf{orden}(x)\rangle$

EQUAL $\Rightarrow$ $\mathbf{domain}(f) = o\ \&\ \mathcal{O}(\mathbf{domain}(f))\ \&\ \mathbf{range}(f) = o_1 \times o_2\ \&\ 1\text{–}1(f)$

Suppose $\Rightarrow$ $\mathit{Stat7}:\ \neg\langle\forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow \mathsf{prod\_order}(\mathsf{orden}(x), \mathsf{orden}(y))\rangle$

$\langle x, y\rangle \hookrightarrow \mathit{Stat7} \Rightarrow$ $x, y \in o\ \&\ \neg\big(x \in y \leftrightarrow \mathsf{prod\_order}(\mathsf{orden}(x), \mathsf{orden}(y))\big)$

$\langle x\rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ $\mathsf{orden}(x) \neq o_1 \times o_2$

$\langle y\rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ $\mathsf{orden}(y) \neq o_1 \times o_2$

$\langle o, x\rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(x)$

$\langle o, y\rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(y)$

$\langle x, y\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat8}:\ \langle\forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow \mathsf{prod\_order}(\mathsf{orden}(x), \mathsf{orden}(y))\rangle$

-- But it is now obvious that (x in y) has the explicit form asserted in our theorem, completing the present proof.

$\langle f\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat9}:$

$\neg\langle\forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow f{\upharpoonright}x^{[1]} \cup f{\upharpoonright}x^{[2]} \in f{\upharpoonright}y^{[1]} \cup f{\upharpoonright}y^{[2]} \vee (f{\upharpoonright}x^{[1]} \cup f{\upharpoonright}x^{[2]} = f{\upharpoonright}y^{[1]} \cup f{\upharpoonright}y^{[2]}\ \&\ f{\upharpoonright}x^{[1]} \in f{\upharpoonright}y^{[1]}) \vee (f{\upharpoonright}x^{[1]} \cup f{\upharpoonright}x^{[2]} = f{\upharpoonright}y^{[1]} \cup f{\upharpoonright}y^{[2]}\ \&\ f{\upharpoonright}x^{[1]} =$

$\langle x_2, y_2\rangle \hookrightarrow \mathit{Stat9} \Rightarrow$

$x_2, y_2 \in \mathbf{domain}(f)\ \&$

$\neg\big(x_2 \in y_2 \leftrightarrow f{\upharpoonright}x_2^{[1]} \cup f{\upharpoonright}x_2^{[2]} \in f{\upharpoonright}y_2^{[1]} \cup f{\upharpoonright}y_2^{[2]} \vee (f{\upharpoonright}x_2^{[1]} \cup f{\upharpoonright}x_2^{[2]} = f{\upharpoonright}y_2^{[1]} \cup f{\upharpoonright}y_2^{[2]}\ \&\ f{\upharpoonright}x_2^{[1]} \in f{\upharpoonright}y_2^{[1]}) \vee (f{\upharpoonright}x_2^{[1]} \cup f{\upharpoonright}x_2^{[2]} = f{\upharpoonright}y_2^{[1]} \cup f{\upharpoonright}y_2^{[2]}\ \&\ f{\upharpoonright}x_2^{[1]} = f{\upharpoonright}y_2^{[1]}\ \&\ f{\upharpoonright}x_2^{[2]} \in f$

$\langle x_2, y_2\rangle \hookrightarrow \mathit{Stat8} \Rightarrow$ $x_2 \in y_2 \leftrightarrow \mathsf{prod\_order}(\mathsf{orden}(x_2), \mathsf{orden}(y_2))$

$\langle \mathsf{orden}(x_2), \mathsf{orden}(y_2)\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$

$\mathsf{prod\_order}(\mathsf{orden}(x_2), \mathsf{orden}(y_2)) \leftrightarrow \mathsf{orden}^{[1]}(x_2) \cup \mathsf{orden}^{[2]}(x_2) \in \mathsf{orden}^{[1]}(y_2) \cup \mathsf{orden}^{[2]}(y_2) \vee (\mathsf{orden}^{[1]}(x_2) \cup \mathsf{orden}^{[2]}(x_2) = \mathsf{orden}^{[1]}(y_2) \cup \mathsf{orden}^{[2]}(y_2) \;\&\; \mathsf{orden}^{[1]}(x_2) \in \mathsf{or}$

$\langle x_2 \rangle \hookrightarrow Stat6 \Rightarrow \quad f \restriction x_2 = \mathsf{orden}(x_2)$

$\langle y_2 \rangle \hookrightarrow Stat6 \Rightarrow \quad f \restriction y_2 = \mathsf{orden}(y_2)$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following extension of Theorem 283 gives us the subsequently needed property of the map f whose existence is asserted by Theorem 283: that the range of the restriction of f to any element of its domain is contained in $y \times y$ for some y less than the maximum of o_1 and o_2.

Theorem 367 (284) $\mathsf{Card}(O_1) \;\&\; \mathsf{Card}(O_2) \;\&\; \neg\mathsf{Finite}(O_1 \cup O_2) \to \langle \exists f \mid 1\text{–}1(f) \;\&\; \mathcal{O}(\mathbf{domain}(f)) \;\&\; \mathbf{range}(f) = O_1 \times O_2 \;\&\; \langle \forall x \in \mathbf{domain}(f), \exists y \in O_1 \cup O_2 \mid \mathbf{range}(f_{|x}) \subseteq y \times y \rangle \rangle$

Suppose_not$(o_1, o_2) \Rightarrow \quad \mathsf{Card}(o_1) \;\&\; \mathsf{Card}(o_2) \;\&\; \neg\mathsf{Finite}(o_1 \cup o_2) \;\&\; Stat1: \neg$
$\langle \exists f \mid 1\text{–}1(f) \;\&\; \mathcal{O}(\mathbf{domain}(f)) \;\&\; \mathbf{range}(f) = o_1 \times o_2 \;\&\; \langle \forall x \in \mathbf{domain}(f), \exists y \in o_1 \cup o_2 \mid \mathbf{range}(f_{|x}) \subseteq y \times y \rangle \rangle$

-- For suppose not. Take f to be the mapping whose existence is given by the preceding theorem, and let w be an element of its domain which contradicts our present assertion.

Use_def(Card) $\Rightarrow \quad \mathcal{O}(o_1) \;\&\; \mathcal{O}(o_2)$

$\langle o_1, o_2 \rangle \hookrightarrow T283 \Rightarrow \quad Stat2:$
$\langle \exists f \mid 1\text{–}1(f) \;\&\; \mathcal{O}(\mathbf{domain}(f)) \;\&\; \mathbf{range}(f) = o_1 \times o_2 \;\&\; \langle \forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow f\restriction x^{[1]} \cup f\restriction x^{[2]} \in f\restriction y^{[1]} \cup f\restriction y^{[2]} \vee (f\restriction x^{[1]} \cup f\restriction x^{[2]} = f\restriction y^{[1]} \cup f\restriction y^{[2]} \;\&\; f\restriction x^{[1]}$

$\langle f \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat3:$
$1\text{–}1(f) \;\&\; \mathcal{O}(\mathbf{domain}(f)) \;\&\; \mathbf{range}(f) = o_1 \times o_2 \;\&\; Stat4:$
$\langle \forall x \in \mathbf{domain}(f), y \in \mathbf{domain}(f) \mid x \in y \leftrightarrow f\restriction x^{[1]} \cup f\restriction x^{[2]} \in f\restriction y^{[1]} \cup f\restriction y^{[2]} \vee (f\restriction x^{[1]} \cup f\restriction x^{[2]} = f\restriction y^{[1]} \cup f\restriction y^{[2]} \;\&\; f\restriction x^{[1]} \in f\restriction y^{[1]}) \vee (f\restriction x^{[1]} \cup f\restriction x^{[2]} = f\restriction y^{[1]} \cup f\restriction y^{[2]} \;\&\; f\restriction x^{[1]} = f\restriction$

$\langle f \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat5: \neg\langle \forall x \in \mathbf{domain}(f), \exists y \in o_1 \cup o_2 \mid \mathbf{range}(f_{|x}) \subseteq y \times y \rangle$

$\langle w \rangle \hookrightarrow Stat5 \Rightarrow \quad w \in \mathbf{domain}(f) \;\&\; Stat6: \neg\langle \exists y \in o_1 \cup o_2 \mid \mathbf{range}(f_{|w}) \subseteq y \times y \rangle$

-- Since w is a member of $\mathbf{domain}(f)$ it is a proper subset of $\mathbf{domain}(f)$. Hence the image of w by f is a proper subset of $o_1 \times o_2$. i. e. there exists some element $b = [c, d]$ in $o_1 \times o_2$ which does not belong to $\mathbf{range}(f_{|w})$. Write $[c, d]$ as $[c, d] = e^{[2]}$ where $e \in f$.

Use_def(range) $\Rightarrow \quad \mathbf{range}(f_{|w}) = \{x^{[2]} : x \in f_{|w}\}$

Use_def(|) $\Rightarrow \quad \mathbf{range}(f_{|w}) = \{x^{[2]} : x \in \{p \in f \mid p^{[1]} \in w\}\}$

SIMPLF $\Rightarrow \quad \mathbf{range}(f_{|w}) = \{p^{[2]} : p \in f \mid p^{[1]} \in w\}$

$\langle \mathbf{domain}(f), w \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(w)$

$\langle \mathbf{domain}(f), w \rangle \hookrightarrow T31 \Rightarrow \quad w \subseteq \mathbf{domain}(f) \;\&\; Stat7: w \neq \mathbf{domain}(f)$

$\langle f, w \rangle \hookrightarrow T88 \Rightarrow \quad \mathbf{range}(f_{|w}) \subseteq \mathbf{range}(f) \;\&\; Stat8: \mathbf{range}(f_{|w}) \neq \mathbf{range}(f)$

$\langle b \rangle \hookrightarrow Stat8 \Rightarrow \quad b \in \mathbf{range}(f) \;\&\; b \notin \mathbf{range}(f_{|w})$

Use_def(range) $\Rightarrow \quad Stat9: b \in \{x^{[2]} : x \in f\}$

$\langle e \rangle \hookrightarrow Stat9 \Rightarrow \quad Stat10:\ e \in f \ \& \ b = e^{[2]}$
ELEM $\Rightarrow \quad b \in o_1 \times o_2$
Use_def $(\times) \Rightarrow \quad Stat11:\ b \in \{[x,y] : x \in o_1, y \in o_2\}$
$\langle c, d \rangle \hookrightarrow Stat11 \Rightarrow \quad Stat12:\ c \in o_1 \ \& \ d \in o_2 \ \& \ b = [c,d]$
$\langle o_1, c \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(c)$
$\langle o_2, d \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(d)$
$\langle c, d \rangle \hookrightarrow T27 \Rightarrow \quad Stat13:\ \mathcal{O}(c \cup d)$

-- We shall now show that there can exist no $[c_1, d_1]$ in $\mathbf{range}(f_{|w})$ such that c_1 or d_1 is greater than $c \cup d$. For otherwise there would be an $x \in f$ with $x^{[1]} \in w$ such that $x^{[2]} = [c_1, d_1]$ where $c_1 \cup d_1$ is greater than $c \cup d$, and so, since f is an order-preserving map of $\mathbf{domain}(f)$ onto $o_1 \times o_2$, we would have $e^{[1]} \in w$, contradicting $b \notin \mathbf{range}(f_{|w})$.

Use_def (1–1) $\Rightarrow \quad Svm(f)$
Suppose $\Rightarrow \quad Stat14:\ \neg\langle \forall x \in f \,|\, x^{[1]} \in w \rightarrow c \cup d \notin x^{[2]^{[1]}} \ \& \ c \cup d \notin x^{[2]^{[2]}} \rangle$
$\langle x \rangle \hookrightarrow Stat14([Stat14, \cap]) \Rightarrow \quad Stat15:\ x \in f \ \& \ x^{[1]} \in w \ \& \ c \cup d \in x^{[2]^{[1]}} \vee c \cup d \in x^{[2]^{[2]}}$
$\langle e \rangle \hookrightarrow T55 \Rightarrow \quad e^{[1]} \in \mathbf{domain}(f)$
$\langle x \rangle \hookrightarrow T55 \Rightarrow \quad x^{[1]} \in \mathbf{domain}(f)$
$\langle Stat15 \rangle$ ELEM $\Rightarrow \quad Stat16:\ c \cup d \in x^{[2]^{[1]}} \cup x^{[2]^{[2]}}$
$\langle Stat16, Stat10, Stat12 \rangle$ ELEM $\Rightarrow \quad e^{[2]^{[1]}} \cup e^{[2]^{[2]}} \in x^{[2]^{[1]}} \cup x^{[2]^{[2]}}$
$\langle f, e \rangle \hookrightarrow T67 \Rightarrow \quad e^{[2]} = f\restriction e^{[1]}$
$\langle f, x \rangle \hookrightarrow T67 \Rightarrow \quad x^{[2]} = f\restriction x^{[1]}$
EQUAL $\Rightarrow \quad f\restriction e^{[1]^{[1]}} \cup f\restriction e^{[1]^{[2]}} \in f\restriction x^{[1]^{[1]}} \cup f\restriction x^{[1]^{[2]}}$
Suppose $\Rightarrow \quad x^{[1]} = e^{[1]}$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad x^{[1]} \neq e^{[1]}$
$\langle e^{[1]}, x^{[1]} \rangle \hookrightarrow Stat4 \Rightarrow \quad x^{[1]} \notin e^{[1]}$
$\langle \mathbf{domain}(f), e^{[1]} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(e^{[1]})$
$\langle \mathbf{domain}(f), x^{[1]} \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(x^{[1]})$
$\langle e^{[1]}, x^{[1]} \rangle \hookrightarrow T28 \Rightarrow \quad e^{[1]} \in x^{[1]}$
$\langle w, x^{[1]} \rangle \hookrightarrow T12 \Rightarrow \quad x^{[1]} \subseteq w$
ELEM $\Rightarrow \quad e^{[1]} \in w$
Suppose $\Rightarrow \quad e^{[2]} \notin \mathbf{range}(f_{|w})$
ELEM $\Rightarrow \quad Stat17:\ e^{[2]} \notin \{p^{[2]} : p \in f \,|\, p^{[1]} \in w\}$
$\langle e \rangle \hookrightarrow Stat17 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad e^{[2]} \in \mathbf{range}(f_{|w})$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat18:\ \langle \forall x \in f \,|\, x^{[1]} \in w \rightarrow c \cup d \notin x^{[2]^{[1]}} \ \& \ c \cup d \notin x^{[2]^{[2]}} \rangle$

-- It follows that $\mathbf{range}(f_{|w})$ is included in $\mathsf{next}(c \cup d) \times \mathsf{next}(c \cup d)$.

Suppose $\Rightarrow$ $\mathit{Stat19}:\ \mathbf{range}(f_{|w}) \not\subseteq \mathsf{next}(c \cup d) \times \mathsf{next}(c \cup d)$
$\langle u \rangle \hookrightarrow \mathit{Stat19} \Rightarrow$ $u \in \mathbf{range}(f_{|w})\ \&\ u \notin \mathsf{next}(c \cup d) \times \mathsf{next}(c \cup d)$
Use_def($\times$) $\Rightarrow$ $\mathit{Stat20}:\ u \notin \{[x, y] : x \in \mathsf{next}(c \cup d), y \in \mathsf{next}(c \cup d)\}$
ELEM $\Rightarrow$ $\mathit{Stat21}:\ u \in \left\{p^{[2]} : p \in f \mid p^{[1]} \in w\right\}$
$\langle p \rangle \hookrightarrow \mathit{Stat21} \Rightarrow$ $\mathit{Stat22}:\ p \in f\ \&\ u = p^{[2]}\ \&\ p^{[1]} \in w$
$\langle p \rangle \hookrightarrow \mathit{Stat18} \Rightarrow$ $\mathit{Stat23}:\ c \cup d \notin p^{[2]^{[1]}}\ \&\ c \cup d \notin p^{[2]^{[2]}}$
$\langle p, f \rangle \hookrightarrow \mathit{T56} \Rightarrow$ $\mathit{Stat24}:\ p^{[2]} \in \mathbf{range}(f)$
$\langle \mathit{Stat24}, \mathit{Stat3} \rangle$ ELEM $\Rightarrow$ $p^{[2]} \in o_1 \times o_2$
Use_def($\times$) $\Rightarrow$ $\mathit{Stat25}:\ p^{[2]} \in \{[x, y] : x \in o_1, y \in o_2\}$
$\langle c_2, d_2 \rangle \hookrightarrow \mathit{Stat25} \Rightarrow$ $\mathit{Stat26}:\ c_2 \in o_1\ \&\ d_2 \in o_2\ \&\ p^{[2]} = [c_2, d_2]$
$\langle \mathit{Stat23}, \mathit{Stat26} \rangle$ ELEM $\Rightarrow$ $c \cup d \notin c_2\ \&\ c \cup d \notin d_2$
$\langle o_1, c_2 \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(c_2)$
$\langle o_2, d_2 \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(d_2)$
$\langle c_2, c \cup d \rangle \hookrightarrow \mathit{T28} \Rightarrow$ $\mathit{Stat27}:\ c_2 \in c \cup d \vee c_2 = c \cup d$
$\langle d_2, c \cup d \rangle \hookrightarrow \mathit{T28} \Rightarrow$ $d_2 \in c \cup d \vee d_2 = c \cup d$
$\langle \mathit{Stat27} \rangle$ ELEM $\Rightarrow$ $c_2, d_2 \in c \cup d \cup \{c \cup d\}$
Use_def(next) $\Rightarrow$ $\mathit{Stat28}:\ c_2, d_2 \in \mathsf{next}(c \cup d)$
$\langle c_2, d_2 \rangle \hookrightarrow \mathit{Stat20}(\langle \mathit{Stat28} \rangle) \Rightarrow$ $\mathit{Stat29}:\ u \neq [c_2, d_2]$
$\langle \mathit{Stat29}, \mathit{Stat22}, \mathit{Stat26} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat30}:\ \mathbf{range}(f_{|w}) \subseteq \mathsf{next}(c \cup d) \times \mathsf{next}(c \cup d)$
$\langle o_1 \cup o_2, o_1 \rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\neg\mathsf{Finite}(o_1 \cup o_2)$
$\langle o_1, o_2 \rangle \hookrightarrow \mathit{T26} \Rightarrow$ $\mathit{Stat31}:\ o_1 \subseteq o_2 \vee o_2 \subseteq o_1$
$\langle \mathit{Stat31} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat32}:\ o_1 \cup o_2 = o_1 \vee o_1 \cup o_2 = o_2$
$\langle c, d \rangle \hookrightarrow \mathit{T26} \Rightarrow$ $\mathit{Stat33}:\ c \subseteq d \vee d \subseteq c$
$\langle \mathit{Stat33} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat34}:\ c \cup d = c \vee c \cup d = d$
$\langle \mathit{Stat32}, \mathit{Stat34}, \mathit{Stat12} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat35}:\ c \cup d \in o_1 \cup o_2$

-- It is easily seen that $o_1 \cup o_2$, which is one of o_1 and o_2, must be a cardinal.

Suppose $\Rightarrow$ $\neg\mathsf{Card}(o_1 \cup o_2)$
Suppose $\Rightarrow$ $o_1 \cup o_2 = o_1$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $o_1 \cup o_2 = o_2$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Card}(o_1 \cup o_2)$
Use_def(Card) $\Rightarrow$ $\mathcal{O}(o_1 \cup o_2)$
$\langle c \cup d \rangle \hookrightarrow \mathit{T29} \Rightarrow$ $\mathcal{O}(\mathsf{next}(c \cup d))$
Suppose $\Rightarrow$ $\mathsf{next}(c \cup d) \notin o_1 \cup o_2$

-- But since o_1 and o_2, are cardinals, we have $\mathsf{next}(c \cup d) \in o_1 \cup o_2$, so that we can take $y = \mathsf{next}(c \cup d)$ as the element whose existence is asserted by our theorem.

$\langle o_1 \cup o_2, \mathsf{next}(c \cup d) \rangle \hookrightarrow \mathit{T32} \Rightarrow$ $\mathit{Stat36}:\ \mathsf{next}(c \cup d) \supseteq o_1 \cup o_2$

$\langle \mathsf{next}(\mathsf{c}\cup\mathsf{d}),\mathsf{o}_1\cup\mathsf{o}_2\rangle \hookrightarrow T162 \Rightarrow \quad \neg\mathsf{Finite}(\mathsf{next}(\mathsf{c}\cup\mathsf{d}))$
$\langle \mathsf{c}\cup\mathsf{d}\rangle \hookrightarrow T173 \Rightarrow \quad \neg\mathsf{Finite}(\mathsf{c}\cup\mathsf{d})$
$\mathsf{Use\_def}(\mathsf{next}) \Rightarrow \quad \mathsf{next}(\mathsf{c}\cup\mathsf{d}) = \mathsf{c}\cup\mathsf{d}\cup\{\mathsf{c}\cup\mathsf{d}\}$
$\mathsf{EQUAL} \Rightarrow \quad \#\mathsf{next}(\mathsf{c}\cup\mathsf{d}) = \#(\mathsf{c}\cup\mathsf{d}\cup\{\mathsf{c}\cup\mathsf{d}\})$
$\langle \mathsf{o}_1\cup\mathsf{o}_2,\mathsf{next}(\mathsf{c}\cup\mathsf{d})\rangle \hookrightarrow T144(\langle Stat36\rangle) \Rightarrow \quad Stat37:\ \#\mathsf{next}(\mathsf{c}\cup\mathsf{d}) \supseteq \#(\mathsf{o}_1\cup\mathsf{o}_2)$
$\langle \mathsf{c}\cup\mathsf{d},\mathsf{c}\cup\mathsf{d}\rangle \hookrightarrow T278 \Rightarrow \quad \#\mathsf{next}(\mathsf{c}\cup\mathsf{d}) = \#(\mathsf{c}\cup\mathsf{d})$
$\langle Stat37\rangle\ \mathsf{ELEM} \Rightarrow \quad Stat38:\ \#(\mathsf{c}\cup\mathsf{d}) \supseteq \#(\mathsf{o}_1\cup\mathsf{o}_2)$
$\langle \mathsf{c}\cup\mathsf{d},\mathsf{c}\cup\mathsf{d}\rangle \hookrightarrow T143([Stat38, Stat13]) \Rightarrow \quad \mathsf{c}\cup\mathsf{d} \supseteq \#(\mathsf{o}_1\cup\mathsf{o}_2)$
$\langle \mathsf{o}_1\cup\mathsf{o}_2\rangle \hookrightarrow T138 \Rightarrow \quad Stat39:\ \mathsf{c}\cup\mathsf{d} \supseteq \mathsf{o}_1\cup\mathsf{o}_2$
$\langle Stat39, Stat35\rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat40:\ \mathsf{next}(\mathsf{c}\cup\mathsf{d}) \in \mathsf{o}_1\cup\mathsf{o}_2$
$\langle \mathsf{next}(\mathsf{c}\cup\mathsf{d})\rangle \hookrightarrow Stat6([Stat40, Stat30]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- The following theorem, whose proof we have already described, asserts that any infinite set can be divided in half, and is in fact in 1-1 correspondence with $\mathsf{s}\times\{\emptyset,1\}$.

-- Cardinal Doubling Theorem

Theorem 368 (285) $\neg\mathsf{Finite}(\mathsf{S}) \rightarrow \#(\mathsf{S}\times\{\emptyset,1\}) = \#\mathsf{S}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s}) \Rightarrow \quad \neg\mathsf{Finite}(\mathsf{s})\ \&\ \#(\mathsf{s}\times\{\emptyset,1\}) \neq \#\mathsf{s}$

-- If we suppose the contrary, then clearly s is an infinite set and $\#(\mathsf{s}\times\{\emptyset,1\}) \neq \#\mathsf{s}$. Since $\#(\mathsf{s}\times\{\emptyset,1\})$ is obviously at least as large as $\#(\mathsf{s}\times\{\emptyset\}) = \#\mathsf{s}$, $\#(\mathsf{s}\times\{\emptyset,1\})$ must be larger than $\#\mathsf{s}$, so that $\#\mathsf{s} \in \#(\mathsf{s}\times\{\emptyset,1\})$. I. e. there exists an infinite ordinal such that x is less than $\#(\mathsf{x}\times\{\emptyset,1\})$, and so by the principle of transfinite induction there exists a least such x.

$\mathsf{Use\_def}(*) \Rightarrow \quad \mathsf{s} * \{\emptyset,1\} \neq \#\mathsf{s}$
$\langle \{\emptyset,1\},\mathsf{s}\rangle \hookrightarrow T217 \Rightarrow \quad \{\emptyset,1\} * \mathsf{s} \neq \#\mathsf{s}$
$\langle \{\emptyset,1\},\mathsf{s}\rangle \hookrightarrow T200 \Rightarrow \quad \{\emptyset,1\} * \#\mathsf{s} \neq \#\mathsf{s}$
$\langle \{\emptyset,1\},\#\mathsf{s}\rangle \hookrightarrow T217 \Rightarrow \quad \#\mathsf{s} * \{\emptyset,1\} \neq \#\mathsf{s}$
$\mathsf{Use\_def}(*) \Rightarrow \quad \#(\#\mathsf{s}\times\{\emptyset,1\}) \neq \#\mathsf{s}$
$\mathsf{ELEM} \Rightarrow \quad \{\emptyset,1\} \neq \emptyset\ \&\ \{\emptyset,1\} = 2$
$T183 \Rightarrow \quad \{\emptyset,1\} \in \mathbb{N}$
$T181 \Rightarrow \quad \mathsf{Card}(\mathbb{N})$
$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N},\{\emptyset,1\}\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(\{\emptyset,1\})$
$\langle \emptyset,1\rangle \hookrightarrow T175 \Rightarrow \quad \mathsf{Finite}(\{\emptyset,1\})$
$\langle \{\emptyset,1\}\rangle \hookrightarrow T178 \Rightarrow \quad \mathsf{Card}(\{\emptyset,1\})$
$\langle \{\emptyset,1\},\#\mathsf{s}\rangle \hookrightarrow T282 \Rightarrow \quad \#\#\mathsf{s} \subseteq \#(\#\mathsf{s}\times\{\emptyset,1\})$
$\langle \mathsf{s}\rangle \hookrightarrow T140 \Rightarrow \quad \#\#\mathsf{s} \subseteq \#(\#\mathsf{s}\times\{\emptyset,1\})\ \&\ \#(\#\mathsf{s}\times\{\emptyset,1\}) \neq \#\#\mathsf{s}$

$\langle \#s \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#\#s)$
$\langle \#s \times \{\emptyset, 1\} \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#(\#s \times \{\emptyset, 1\}))$
$\langle \#(\#s \times \{\emptyset, 1\}), \#\#s \rangle \hookrightarrow T31 \Rightarrow \quad \#\#s \in \#(\#s \times \{\emptyset, 1\})$
$\langle s \rangle \hookrightarrow T166 \Rightarrow \quad \neg \mathsf{Finite}(\#s)$
$\langle s \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#s) \,\&\, \mathcal{O}(\#s)$
ELEM $\Rightarrow \quad \mathcal{O}(\#s) \,\&\, \neg \mathsf{Finite}(\#s) \,\&\, \#\#s \in \#(\#s \times \{\emptyset, 1\})$
APPLY $\langle mt_\Theta : x_2 \rangle$ transfinite_induction $(n \mapsto \#s, P(y) \mapsto \mathcal{O}(y) \,\&\, \neg \mathsf{Finite}(y) \,\&\, \#y \in \#(y \times \{\emptyset, 1\})) \Rightarrow$

$Stat2: \langle \forall k \mid (\mathcal{O}(x_2) \,\&\, \neg \mathsf{Finite}(x_2) \,\&\, \#x_2 \in \#(x_2 \times \{\emptyset, 1\})) \,\&\, \big(k \in x_2 \rightarrow \neg(\mathcal{O}(k) \,\&\, \neg \mathsf{Finite}(k) \,\&\, \#k \in \#(k \times \{\emptyset, 1\}))\big) \rangle$

-- it is easy to see that the minimality of x_2 implies that $x_2 = \#x_2$, i. e. that the ordinal x_2 is a cardinal.

$\langle a \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat1: \mathcal{O}(x_2) \,\&\, \neg \mathsf{Finite}(x_2) \,\&\, \#x_2 \in \#(x_2 \times \{\emptyset, 1\})$
$\langle x_2 \rangle \hookrightarrow T122 \Rightarrow \quad x_2 \notin \#x_2$
$\langle x_2 \rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#x_2)$
$\langle x_2 \rangle \hookrightarrow T166 \Rightarrow \quad \neg \mathsf{Finite}(\#x_2)$
$\langle \#x_2, x_2 \rangle \hookrightarrow T32 \Rightarrow \quad x_2 \supseteq \#x_2$
Suppose $\Rightarrow \quad x_2 \neq \#x_2$
$\langle x_2, \#x_2 \rangle \hookrightarrow T31 \Rightarrow \quad \#x_2 \in x_2$
$\langle \#x_2 \rangle \hookrightarrow Stat2 \Rightarrow \quad \#\#x_2 \notin \#(\#x_2 \times \{\emptyset, 1\})$
$\langle x_2 \rangle \hookrightarrow T140 \Rightarrow \quad \#\#x_2 = \#x_2$
$\langle x_2, \{\emptyset, 1\} \rangle \hookrightarrow T203 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad Stat3: x_2 = \#x_2$
$\langle x_2 \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#x_2)$
EQUAL $\Rightarrow \quad \mathsf{Card}(x_2)$
$\langle \{\emptyset, 1\}, x_2 \rangle \hookrightarrow T162 \Rightarrow \quad x_2 \not\subseteq \{\emptyset, 1\}$
$\langle \{\emptyset, 1\}, x_2 \rangle \hookrightarrow T26 \Rightarrow \quad x_2 \supseteq \{\emptyset, 1\}$
ELEM $\Rightarrow \quad Stat3a: x_2 = x_2 \cup \{\emptyset, 1\}$
EQUAL $\Rightarrow \quad \neg \mathsf{Finite}(x_2 \cup \{\emptyset, 1\})$

-- Since x_2 and $\{\emptyset, 1\}$ are both cardinals, we can apply Theorem 284 to put $x_2 \times \{\emptyset, 1\}$ into a 1-1 monotone correspondence f with some ordinal $o = \mathbf{domain}(f)$.

$\langle x_2, \{\emptyset, 1\} \rangle \hookrightarrow T284 \Rightarrow \quad Stat4:$
$\langle \exists f \mid 1\text{–}1(f) \,\&\, \mathcal{O}(\mathbf{domain}(f)) \,\&\, \mathbf{range}(f) = x_2 \times \{\emptyset, 1\} \,\&\, \langle \forall x \in \mathbf{domain}(f), \exists y \in x_2 \cup \{\emptyset, 1\} \mid \mathbf{range}(f_{|x}) \subseteq y \times y \rangle \rangle$
$\langle f \rangle \hookrightarrow Stat4 \Rightarrow \quad Stat5: 1\text{–}1(f) \,\&\, \mathcal{O}(\mathbf{domain}(f)) \,\&\, \mathbf{range}(f) = x_2 \times \{\emptyset, 1\} \,\&\, Stat6: \langle \forall x \in \mathbf{domain}(f), \exists y \in x_2 \cup \{\emptyset, 1\} \mid \mathbf{range}(f_{|x}) \subseteq y \times y \rangle$
$\langle f, x_2 \rangle \hookrightarrow T72 \Rightarrow \quad Stat7: \mathbf{range}(f_{|x_2}) \subseteq x_2 \times \{\emptyset, 1\}$

-- Our next aim is to show that $\#\mathbf{domain}(f)$ is no more than x_2. This will follow from the fact that for each x in $\mathbf{domain}(f)$, $\#x$ is less than x_2.

Suppose $\Rightarrow$ $\#\mathbf{domain}(\mathsf{f}) \not\subseteq \mathsf{x}_2$

$\langle \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#\mathbf{domain}(\mathsf{f}))$

$\langle \#\mathbf{domain}(\mathsf{f}), \mathsf{x}_2 \rangle \hookrightarrow T32 \Rightarrow$ $Stat7a:\ \mathsf{x}_2 \in \#\mathbf{domain}(\mathsf{f})$

$\langle \mathbf{domain}(\mathsf{f}), \mathbf{domain}(\mathsf{f}) \rangle \hookrightarrow T143(\langle Stat5, Stat7a, * \rangle) \Rightarrow$ $\mathsf{x}_2 \in \mathbf{domain}(\mathsf{f})$

$\langle \mathsf{x}_2 \rangle \hookrightarrow Stat6 \Rightarrow$ $Stat8:\ \langle \exists \mathsf{y} \in \mathsf{x}_2 \cup \{\emptyset, 1\} \mid \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \times \mathsf{y} \rangle$

$\langle \mathsf{y} \rangle \hookrightarrow Stat8([Stat3a, Stat8]) \Rightarrow$ $Stat9:\ \mathsf{y} \in \mathsf{x}_2\ \&\ \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \times \mathsf{y}$

$\langle Stat9, Stat7 \rangle$ ELEM $\Rightarrow$ $\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \times \mathsf{y} \cap (\mathsf{x}_2 \times \{\emptyset, 1\})$

$\langle \mathsf{y}, \mathsf{y}, \mathsf{x}_2, \{\emptyset, 1\} \rangle \hookrightarrow T220 \Rightarrow$ $Stat9a:\ \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \cap \mathsf{x}_2 \times (\mathsf{y} \cap \{\emptyset, 1\})$

$\langle \mathsf{y} \cap \mathsf{x}_2, \mathsf{y}, \mathsf{y} \cap \{\emptyset, 1\}, \{\emptyset, 1\} \rangle \hookrightarrow T219(\langle Stat9a \rangle) \Rightarrow$

$\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \times \{\emptyset, 1\}$

$\langle \mathsf{x}_2, \mathsf{y} \rangle \hookrightarrow T11 \Rightarrow$ $Stat10:\ \mathcal{O}(\mathsf{y})$

$\langle \mathbf{domain}(\mathsf{f}), \mathsf{x}_2 \rangle \hookrightarrow T12 \Rightarrow$ $\mathsf{x}_2 \subseteq \mathbf{domain}(\mathsf{f})$

$\langle \mathsf{f}, \mathsf{x}_2 \rangle \hookrightarrow T84 \Rightarrow$ $\mathbf{domain}(\mathsf{f}_{|\mathsf{x}_2}) = \mathsf{x}_2$

EQUAL $\Rightarrow$ $\neg\mathsf{Finite}(\mathbf{domain}(\mathsf{f}_{|\mathsf{x}_2}))$

$\langle \mathsf{f}, \mathsf{x}_2 \rangle \hookrightarrow T53 \Rightarrow$ $1\text{–}1(\mathsf{f}_{|\mathsf{x}_2})$

$\langle \mathsf{f}_{|\mathsf{x}_2} \rangle \hookrightarrow T164 \Rightarrow$ $\neg\mathsf{Finite}(\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}))$

$\langle \mathsf{y} \times \{\emptyset, 1\}, \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \rangle \hookrightarrow T162 \Rightarrow$ $\neg\mathsf{Finite}(\mathsf{y} \times \{\emptyset, 1\})$

$\langle \mathsf{y}, \{\emptyset, 1\} \rangle \hookrightarrow T225 \Rightarrow$ $\neg\mathsf{Finite}(\mathsf{y})$

$\langle \mathsf{y} \rangle \hookrightarrow Stat2 \Rightarrow$ $\#\mathsf{y} \notin \#(\mathsf{y} \times \{\emptyset, 1\})$

$\langle \mathsf{y} \rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#\mathsf{y})$

$\langle \mathsf{y} \times \{\emptyset, 1\} \rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#(\mathsf{y} \times \{\emptyset, 1\}))$

$\langle \#(\mathsf{y} \times \{\emptyset, 1\}), \#\mathsf{y} \rangle \hookrightarrow T32 \Rightarrow$ $\#(\mathsf{y} \times \{\emptyset, 1\}) \subseteq \#\mathsf{y}$

$\langle \{\emptyset, 1\}, \mathsf{y} \rangle \hookrightarrow T214 \Rightarrow$ $\#\mathsf{y} = \#(\mathsf{y} \times \{\emptyset, 1\})$

$\langle \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}), \mathsf{y} \times \{\emptyset, 1\} \rangle \hookrightarrow T144 \Rightarrow$ $\#\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \#\mathsf{y}$

$\langle \mathsf{f}_{|\mathsf{x}_2} \rangle \hookrightarrow T131 \Rightarrow$ $\#\mathbf{domain}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \#\mathsf{y}$

-- And now we have a contradiction with the fact that $\mathsf{y} \in \mathsf{x}_2$, thereby proving our theorem.

EQUAL $\Rightarrow$ $Stat11:\ \mathsf{x}_2 \subseteq \#\mathsf{y}$

$\langle \mathsf{y}, \mathsf{y} \rangle \hookrightarrow T143(\langle Stat10, Stat11, * \rangle) \Rightarrow$ $\mathsf{x}_2 \subseteq \mathsf{y}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\#\mathbf{domain}(\mathsf{f}) \subseteq \mathsf{x}_2$

$\langle \mathsf{f} \rangle \hookrightarrow T131 \Rightarrow$ $\#\mathbf{domain}(\mathsf{f}) = \#\mathbf{range}(\mathsf{f})$

EQUAL $\Rightarrow$ $\#(\mathsf{x}_2 \times \{\emptyset, 1\}) \subseteq \mathsf{x}_2$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that if one of the two sets s, t in a union is infinite, the the cardinal sum of the two sets is the union (i. e. maximum) m of the cardinalities $\#\mathsf{s}$, $\#\mathsf{t}$.

Theorem 369 (286) $\neg\mathsf{Finite}(\mathsf{S}) \rightarrow \mathsf{S} + T = \#\mathsf{S} \cup \# T$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{s},\mathsf{t}) \Rightarrow \quad \neg\mathsf{Finite}(\mathsf{s})\ \&\ \mathsf{s} + \mathsf{t} \neq \#\mathsf{s} \cup \#\mathsf{t}$

-- For let s, t be a counterexample to our assertion, so than either $\mathsf{s} + \mathsf{t} \vee \#(\mathsf{s} \cup \mathsf{t})$ differs from the larger of $\#\mathsf{s}$ and $\#\mathsf{t}$. One of the two cardinals $\#\mathsf{s}$, $\#\mathsf{t}$ must be a subset of the other. If $\#\mathsf{s}$ is a subset of $\#\mathsf{t}$, then both these cardinals are infinite, and the maximum $\#\mathsf{s} \cup \#\mathsf{t}$ is $\#\mathsf{t}$, so $\#\mathsf{s} + \#\mathsf{t} \neq \#\mathsf{t}$. Plainly $\#\mathsf{s} + \#\mathsf{t}$ is no less than $\emptyset + \#\mathsf{t} = \#\mathsf{t}$, so $\#\mathsf{s} + \#\mathsf{t}$ must be greater than $\#\mathsf{t}$.

$\langle \mathsf{s},\mathsf{t} \rangle \hookrightarrow T190 \Rightarrow \quad \#\mathsf{s} + \#\mathsf{t} \neq \#\mathsf{s} \cup \#\mathsf{t}$
$\langle \mathsf{s} \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#\mathsf{s})\ \&\ \mathsf{Card}(\#\mathsf{s})$
$\langle \mathsf{t} \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#\mathsf{t})\ \&\ \mathsf{Card}(\#\mathsf{t})$
$\langle \#\mathsf{s}, \#\mathsf{t} \rangle \hookrightarrow T26 \Rightarrow \quad \#\mathsf{s} \subseteq \#\mathsf{t} \vee \#\mathsf{t} \subseteq \#\mathsf{s}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{s} + \mathsf{t} \neq \#\mathsf{s} \cup \#\mathsf{t}$
$\langle \mathsf{s},\mathsf{t} \rangle \hookrightarrow T190 \Rightarrow \quad \#\mathsf{s} + \#\mathsf{t} \neq \#\mathsf{s} \cup \#\mathsf{t}$
$\langle \mathsf{s} \rangle \hookrightarrow T166 \Rightarrow \quad \neg\mathsf{Finite}(\#\mathsf{s})$
$\mathsf{Suppose} \Rightarrow \quad \#\mathsf{s} \subseteq \#\mathsf{t}$
$\mathsf{ELEM} \Rightarrow \quad \#\mathsf{s} + \#\mathsf{t} \neq \#\mathsf{t}$
$\langle \#\mathsf{t}, \#\mathsf{s} \rangle \hookrightarrow T162 \Rightarrow \quad \neg\mathsf{Finite}(\#\mathsf{t})$
$\langle \emptyset, \#\mathsf{s}, \#\mathsf{t} \rangle \hookrightarrow T244 \Rightarrow \quad \emptyset + \#\mathsf{t} \subseteq \#\mathsf{s} + \#\mathsf{t}$
$\langle \mathsf{t} \rangle \hookrightarrow T211 \Rightarrow \quad \#\mathsf{t} + \emptyset = \#\mathsf{t}$
$\langle \emptyset, \#\mathsf{t} \rangle \hookrightarrow T216 \Rightarrow \quad \emptyset + \#\mathsf{t} = \#\mathsf{t}$

-- But by monotonicity $\#\mathsf{s} + \#\mathsf{t}$ is no greater than $\#\mathsf{t} + \#\mathsf{t}$, which by Theorem 285 is $\#\mathsf{t}$. This contradiction rules out the possibility that $\#\mathsf{t} \supseteq \#\mathsf{s}$. Thus it follows that $\#\mathsf{s}$ must be the larger of $\#\mathsf{t}$ and $\#\mathsf{s}$.

$\mathsf{Set\_monot} \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{s}\} \subseteq \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{t}\}$
$\mathsf{ELEM} \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{s}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\} \subseteq \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{t}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\}$
$\mathsf{ELEM} \Rightarrow \quad \{\emptyset\} \cup \{1\} = \{\emptyset, 1\}$
$\mathsf{Set\_monot} \Rightarrow \quad \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \#\mathsf{t}, \mathsf{y} \in \{\emptyset\}\} \cup \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \#\mathsf{t}, \mathsf{y} \in \{1\}\} = \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \#\mathsf{t}, \mathsf{y} \in \{\emptyset\} \cup \{1\}\}$
$\mathsf{SIMPLF} \Rightarrow \quad \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \#\mathsf{t}, \mathsf{y} \in \{\emptyset\}\} \cup \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \#\mathsf{t}, \mathsf{y} \in \{1\}\} =$
$\{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{t}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\}$
$\mathsf{EQUAL} \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{t}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\} = \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \#\mathsf{t}, \mathsf{y} \in \{\emptyset, 1\}\}$
$\mathsf{Use\_def}(\times) \Rightarrow \quad \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{s}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\} \subseteq \#\mathsf{t} \times \{\emptyset, 1\}$
$\langle \{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{s}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\}, \#\mathsf{t} \times \{\emptyset, 1\} \rangle \hookrightarrow T144 \Rightarrow$
$\#(\{[\mathsf{x},\emptyset] : \mathsf{x} \in \#\mathsf{s}\} \cup \{[\mathsf{x},1] : \mathsf{x} \in \#\mathsf{t}\}) \subseteq \#(\#\mathsf{t} \times \{\emptyset, 1\})$
$\mathsf{Use\_def}(+) \Rightarrow \quad \#\mathsf{s} + \#\mathsf{t} \subseteq \#(\#\mathsf{t} \times \{\emptyset, 1\})$
$\langle \#\mathsf{t} \rangle \hookrightarrow T285 \Rightarrow \quad \#\mathsf{s} + \#\mathsf{t} \subseteq \#\#\mathsf{t}$
$\langle \#\mathsf{t} \rangle \hookrightarrow T138 \Rightarrow \quad \#\mathsf{s} + \#\mathsf{t} \subseteq \#\mathsf{t}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\#t \subseteq \#s \,\&\, \#s + \#t \neq \#s$

-- Arguing in the same way once more, it follows that $\#s + \#t$ must be $\#s$, and hence must be the maximum $\#s \cup \#t$ in this case also. Thus our theorem is verified in all cases.

Set_monot $\Rightarrow$ $\{[x, 1] : x \in \#t\} \subseteq \{[x, 1] : x \in \#s\}$
ELEM $\Rightarrow$ $\{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#t\} \subseteq \{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#s\}$
ELEM $\Rightarrow$ $\{\emptyset\} \cup \{1\} = \{\emptyset, 1\}$
Set_monot $\Rightarrow$ $\{[x, y] : x \in \#s, y \in \{\emptyset\}\} \cup \{[x, y] : x \in \#s, y \in \{1\}\} = \{[x, y] : x \in \#s, y \in \{\emptyset\} \cup \{1\}\}$
SIMPLF $\Rightarrow$ $\{[x, y] : x \in \#s, y \in \{\emptyset\}\} \cup \{[x, y] : x \in \#s, y \in \{1\}\} =$
$\{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#s\}$
EQUAL $\Rightarrow$ $\{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#s\} = \{[x, y] : x \in \#s, y \in \{\emptyset, 1\}\}$
Use_def($\times$) $\Rightarrow$ $\{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#t\} \subseteq \#s \times \{\emptyset, 1\}$
$\langle \{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#t\}, \#s \times \{\emptyset, 1\} \rangle \hookrightarrow T144 \Rightarrow$
$\#(\{[x, \emptyset] : x \in \#s\} \cup \{[x, 1] : x \in \#t\}) \subseteq \#(\#s \times \{\emptyset, 1\})$
Use_def($+$) $\Rightarrow$ $\#s + \#t \subseteq \#(\#s \times \{\emptyset, 1\})$
$\langle \#s \rangle \hookrightarrow T285 \Rightarrow$ $\#s + \#t \subseteq \#\#s$
$\langle \#t \rangle \hookrightarrow T138 \Rightarrow$ $\#s + \#t \subseteq \#s$
$\langle \emptyset, \#t, \#s \rangle \hookrightarrow T244 \Rightarrow$ $\emptyset + \#s \subseteq \#t + \#s$
$\langle s \rangle \hookrightarrow T211 \Rightarrow$ $\#s + \emptyset = \#s$
$\langle \emptyset, \#s \rangle \hookrightarrow T216 \Rightarrow$ $\emptyset + \#s = \#s$
$\langle \#t, \#s \rangle \hookrightarrow T216 \Rightarrow$ $\#t + \#s = \#s + \#t$

-- The possibility that $s + t \neq \#s \cup \#t$ now having been ruled out, it follows that $\#(s \cup t) \neq \#s \cup \#t$. However, Theorem 191 tells us that $\#(s \cup t) = \#s + \#(t \backslash s)$. Since $s + t = \#s + \#t = \#s \cup \#t$ also by Theorem 190,

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We can also prove that if one of the two sets s, t in a union is infinite, the cardinality of $s \cup t$ is the union (i. e. maximum) of the cardinalities $\#s, \#t$.

Theorem 370 (287) $\neg\mathsf{Finite}(S) \to \#(S \cup T) = \#S \cup \#T$. PROOF:

Suppose_not(s, t) $\Rightarrow$ $\neg\mathsf{Finite}(s) \,\&\, \#(s \cup t) \neq \#s \cup \#t$

-- For one of the two ordinals $\#s$, $\#t$ must be included in the other. If $\#t$ is included in $\#s$, we would have $\#(s \cup t) \neq \#s$ and so $\#s \in \#(s \cup t)$, contradicting the fact that $\#(s \cup t)$ can be no larger than $\#(s + s) = \#s$ by the preceding results.

$\langle \mathsf{t} \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#\mathsf{t})$
$\langle \mathsf{s} \rangle \hookrightarrow T130 \Rightarrow \quad \mathcal{O}(\#\mathsf{s})$
$\langle \#\mathsf{s}, \#\mathsf{t} \rangle \hookrightarrow T26 \Rightarrow \quad \#\mathsf{s} \subseteq \#\mathsf{t} \vee \#\mathsf{t} \subseteq \#\mathsf{s}$
$\langle \mathsf{s} \rangle \hookrightarrow T166 \Rightarrow \quad \neg\mathsf{Finite}(\#\mathsf{s})$
Suppose $\Rightarrow \quad \#\mathsf{t} \subseteq \#\mathsf{s}$
ELEM $\Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) \neq \#\mathsf{s}$
$\langle \mathsf{t}\backslash\mathsf{s}, \mathsf{t} \rangle \hookrightarrow T144 \Rightarrow \quad \#(\mathsf{t}\backslash\mathsf{s}) \subseteq \#\mathsf{s}$
$\langle \mathsf{s}, \mathsf{t}\backslash\mathsf{s} \rangle \hookrightarrow T191 \Rightarrow \quad \#\big(\mathsf{s} \cup (\mathsf{t}\backslash\mathsf{s})\big) = \#\mathsf{s} + \#(\mathsf{t}\backslash\mathsf{s})$
ELEM $\Rightarrow \quad \mathsf{s} \cup (\mathsf{t}\backslash\mathsf{s}) = \mathsf{s} \cup \mathsf{t}$
EQUAL $\Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\mathsf{s} + \#(\mathsf{t}\backslash\mathsf{s})$
$\langle \#\mathsf{s}, \#(\mathsf{t}\backslash\mathsf{s}) \rangle \hookrightarrow T286 \Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\#\mathsf{s} \cup \#\#(\mathsf{t}\backslash\mathsf{s})$
$\langle \mathsf{s} \rangle \hookrightarrow T140 \Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\mathsf{s} \cup \#\#(\mathsf{t}\backslash\mathsf{s})$
$\langle \mathsf{t}\backslash\mathsf{s} \rangle \hookrightarrow T140 \Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\mathsf{s}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad \#\mathsf{s} \subseteq \#\mathsf{t}$

-- But by symmetry we can we argue in exactly the same way with s and t reversed, excluding this case also and so proving our theorem.

$\langle \#\mathsf{t}, \#\mathsf{s} \rangle \hookrightarrow T162 \Rightarrow \quad \neg\mathsf{Finite}(\#\mathsf{t})$
ELEM $\Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) \neq \#\mathsf{t}$
$\langle \mathsf{s}\backslash\mathsf{t}, \mathsf{s} \rangle \hookrightarrow T144 \Rightarrow \quad \#(\mathsf{s}\backslash\mathsf{t}) \subseteq \#\mathsf{t}$
$\langle \mathsf{t}, \mathsf{s}\backslash\mathsf{t} \rangle \hookrightarrow T191 \Rightarrow \quad \#\big(\mathsf{t} \cup (\mathsf{s}\backslash\mathsf{t})\big) = \#\mathsf{t} + \#(\mathsf{s}\backslash\mathsf{t})$
ELEM $\Rightarrow \quad \mathsf{t} \cup (\mathsf{s}\backslash\mathsf{t}) = \mathsf{s} \cup \mathsf{t}$
EQUAL $\Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\mathsf{t} + \#(\mathsf{s}\backslash\mathsf{t})$
$\langle \#\mathsf{t}, \#(\mathsf{s}\backslash\mathsf{t}) \rangle \hookrightarrow T286 \Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\#\mathsf{t} \cup \#\#(\mathsf{s}\backslash\mathsf{t})$
$\langle \mathsf{t} \rangle \hookrightarrow T140 \Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\mathsf{t} \cup \#\#(\mathsf{s}\backslash\mathsf{t})$
$\langle \mathsf{s}\backslash\mathsf{t} \rangle \hookrightarrow T140 \Rightarrow \quad \#(\mathsf{s} \cup \mathsf{t}) = \#\mathsf{t}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We are now ready to prove the Cardinal Square Theorem, but again give an informal outline of its proof before entering upon its formal details. The theorem asserts that for every infinite set s, $\#s = \#(s \times s)$. If this is false, then $\#s \neq \#(s \times s) = s * s = \#s * \#s = \#(\#s \times \#s)$, so our assertion is also false for the infinite cardinal $\#s$, and hence for some infinite ordinal. It follows by the axiom of choice that our assertion is also false for some smallest infinite ordinal s_1. s1 must be a cardinal, since otherwise $\#s_1 \in s_1$, but also $s_1 \in \#(s_1 \times s_1) = \#(\#s_1 \times \#s_1)$, so we would have $\#s_1 \in \#(\#s_1 \times \#s_1)$ also, contradicting the minimality of s1. Order $s_1 \times s_1$ by the product ordering described above. The theory we have just established tells us that this is a well-ordering, so that by our previous theory of well-orderings there exists a 1-1 map f from some ordinal o onto $s_1 \times s_1$ which is monotone increasing if o is given its standard ordering and $s_1 \times s_1$ is given its product ordering. If o_1 is a finite member of the ordinal o, then obviously $\#o_1$ is a member of the infinite cardinal s_1. If o_1 is an infinite member of the ordinal o, then it is a proper subset of o; so $\mathbf{range}(f_{|o_1})$ is a proper subset t of $s_1 \times s_1$. As in the proof of theorem 285 we show that there exists an n in s_1 such that $\mathbf{range}(f_{|o_1})$ is a subset of $n \times n$, and so it follows from the minimality of s_1 that $\#t \subseteq \#(n \times n) = n \in s_1$. Hence $\#o_1$ is a member of s_1 for each $o_1 \in o$, proving that $\#o \subseteq s_1$ in this case also. Since o is in 1-1 correspondence with $s_1 \times s_1$ it follows that $\#(s_1 \times s_1) \subseteq s_1$, contrary to our assumption. This contradiction proves our desired theorem.

-- Cardinal Square Theorem

Theorem 371 (288) $\neg\mathsf{Finite}(S) \rightarrow \#(S \times S) = \#S$. Proof:

Suppose_not(s) $\Rightarrow$ $\neg\mathsf{Finite}(s)\ \&\ \#(s \times s) \neq \#s$

-- If we suppose the contrary, then clearly s is an infinite set and $\#(s \times s) \neq \#s$. Since $\#(s \times s)$ is obviously at least as large as $\#(s \times \{\emptyset\}) = \#s$, $\#(s \times s)$ must be larger than $\#s$, so that $\#s \in \#(s \times s)$. I. e. there exists an infinite ordinal such that x is less than $\#(x \times x)$, and so by the principle of transfinite induction there exists a least such x.

Use_def($*$) $\Rightarrow$ $s * s \neq \#s$
$\langle s, s\rangle \hookrightarrow T199 \Rightarrow$ $\#s * \#s \neq \#s$
Use_def($*$) $\Rightarrow$ $\#(\#s \times \#s) \neq \#s$
$T161 \Rightarrow$ $\mathsf{Finite}(\emptyset)$
Suppose $\Rightarrow$ $s = \emptyset$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $s \neq \emptyset$
$\langle s\rangle \hookrightarrow T136 \Rightarrow$ $\#s \neq \emptyset$
$\langle \#s, \#s\rangle \hookrightarrow T282 \Rightarrow$ $\#s \subseteq \#(\#s \times \#s)$
$\langle \#s\rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}(\#s)$
$\langle \#s \times \#s\rangle \hookrightarrow T130 \Rightarrow$ $\mathcal{O}\big(\#(\#s \times \#s)\big)$

$\langle \#(\#s \times \#s), \#s \rangle \hookrightarrow T31 \Rightarrow \quad \#s \in \#(\#s \times \#s)$
$\langle s \rangle \hookrightarrow T166 \Rightarrow \quad \neg \mathsf{Finite}(\#s)$
$\langle s \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#s) \;\&\; \mathcal{O}(\#s)$
APPLY $\langle \mathsf{mt}_\Theta : x_2 \rangle$ transfinite_induction$\left(n \mapsto \#s, P(y) \mapsto \mathcal{O}(y) \;\&\; \neg \mathsf{Finite}(y) \;\&\; \#y \in \#(y \times y)\right) \Rightarrow$

$$Stat2: \left\langle \forall k \mid \left(\mathcal{O}(x_2) \;\&\; \neg \mathsf{Finite}(x_2) \;\&\; \#x_2 \in \#(x_2 \times x_2)\right) \;\&\; \left(k \in x_2 \rightarrow \neg\left(\mathcal{O}(k) \;\&\; \neg \mathsf{Finite}(k) \;\&\; \#k \in \#(k \times k)\right)\right)\right\rangle$$

$\langle a \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat1: \mathcal{O}(x_2) \;\&\; \neg \mathsf{Finite}(x_2) \;\&\; \#x_2 \in \#(x_2 \times x_2)$

-- it is easy to see that the minimality of x_2 implies that $x_2 = \#x_2$, i. e. that the ordinal x_2 is a cardinal.

$\langle x_2 \rangle \hookrightarrow T122 \Rightarrow \quad x_2 \notin \#x_2$
$\langle x_2 \rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#x_2)$
$\langle x_2 \rangle \hookrightarrow T166 \Rightarrow \quad \neg \mathsf{Finite}(\#x_2)$
$\langle \#x_2, x_2 \rangle \hookrightarrow T32 \Rightarrow \quad x_2 \supseteq \#x_2$
Suppose $\Rightarrow \quad x_2 \neq \#x_2$
$\langle x_2, \#x_2 \rangle \hookrightarrow T31 \Rightarrow \quad \#x_2 \in x_2$
$\langle \#x_2 \rangle \hookrightarrow Stat2 \Rightarrow \quad \#\#x_2 \notin \#(\#x_2 \times \#x_2)$
$\langle x_2 \rangle \hookrightarrow T140 \Rightarrow \quad \#\#x_2 = \#x_2$
$\langle x_2, x_2 \rangle \hookrightarrow T201 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat3: x_2 = \#x_2$
$\langle x_2 \rangle \hookrightarrow T130 \Rightarrow \quad \mathsf{Card}(\#x_2)$
EQUAL $\Rightarrow \quad \mathsf{Card}(x_2)$
TELEM $\Rightarrow \quad x_2 = x_2 \cup x_2$
EQUAL $\Rightarrow \quad \neg \mathsf{Finite}(x_2 \cup x_2)$

-- Since x_2 is a cardinal, we can apply Theorem 284 to put $x_2 \times x_2$ into a 1-1 monotone correspondence f with some ordinal $o = \mathbf{domain}(f)$.

$\langle x_2, x_2 \rangle \hookrightarrow T284 \Rightarrow \quad Stat4: \left\langle \exists f \mid 1\text{–}1(f) \;\&\; \mathcal{O}(\mathbf{domain}(f)) \;\&\; \mathbf{range}(f) = x_2 \times x_2 \;\&\; \left\langle \forall x \in \mathbf{domain}(f), \exists y \in x_2 \cup x_2 \mid \mathbf{range}(f_{|x}) \subseteq y \times y \right\rangle \right\rangle$
$\langle f \rangle \hookrightarrow Stat4([]) \Rightarrow \quad Stat5: 1\text{–}1(f) \;\&\; \mathcal{O}(\mathbf{domain}(f)) \;\&\; \mathbf{range}(f) = x_2 \times x_2 \;\&\; Stat6: \left\langle \forall x \in \mathbf{domain}(f), \exists y \in x_2 \cup x_2 \mid \mathbf{range}(f_{|x}) \subseteq y \times y \right\rangle$
$\langle f, x_2 \rangle \hookrightarrow T72 \Rightarrow \quad Stat7: \mathbf{range}(f_{|x_2}) \subseteq x_2 \times x_2$

-- Our next aim is to show that $\#\mathbf{domain}(f)$ is no more than x_2. This will follow from the fact that for each x in $\mathbf{domain}(f)$, $\#x$ is less than x_2.

Suppose $\Rightarrow \quad \#\mathbf{domain}(f) \not\subseteq x_2$
$\langle \mathbf{domain}(f) \rangle \hookrightarrow T130([]) \Rightarrow \quad \mathcal{O}(\#\mathbf{domain}(f))$
$\langle \#\mathbf{domain}(f), x_2 \rangle \hookrightarrow T32 \Rightarrow \quad x_2 \in \#\mathbf{domain}(f)$
$\langle \mathbf{domain}(f), \mathbf{domain}(f) \rangle \hookrightarrow T143(\langle Stat1 \rangle) \Rightarrow \quad Stat7a: x_2 \in \mathbf{domain}(f)$
$\langle x_2 \rangle \hookrightarrow Stat6(\langle Stat7a \rangle) \Rightarrow \quad Stat8: \left\langle \exists y \in x_2 \cup x_2 \mid \mathbf{range}(f_{|x_2}) \subseteq y \times y \right\rangle$
$\langle y \rangle \hookrightarrow Stat8(\langle Stat8 \rangle) \Rightarrow \quad Stat9: y \in x_2 \;\&\; \mathbf{range}(f_{|x_2}) \subseteq y \times y$

$\langle \mathit{Stat9}, \mathit{Stat7}, * \rangle$ ELEM $\Rightarrow$ $\mathit{Stat9a}:$ $\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \times \mathsf{y} \cap (\mathsf{x}_2 \times \mathsf{x}_2)$
$\langle \mathsf{y}, \mathsf{y}, \mathsf{x}_2, \mathsf{x}_2 \rangle \hookrightarrow \mathit{T220}(\langle \mathit{Stat9a} \rangle) \Rightarrow$ $\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \cap \mathsf{x}_2 \times (\mathsf{y} \cap \mathsf{x}_2)$
$\langle \mathsf{y} \cap \mathsf{x}_2, \mathsf{y}, \mathsf{y} \cap \mathsf{x}_2, \mathsf{x}_2 \rangle \hookrightarrow \mathit{T219} \Rightarrow$ $\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \mathsf{y} \times \mathsf{y}$
$\langle \mathsf{x}_2, \mathsf{y} \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathit{Stat10}:$ $\mathcal{O}(\mathsf{y})$
$\langle \mathbf{domain}(\mathsf{f}), \mathsf{x}_2 \rangle \hookrightarrow \mathit{T12} \Rightarrow$ $\mathsf{x}_2 \subseteq \mathbf{domain}(\mathsf{f})$
$\langle \mathsf{f}, \mathsf{x}_2 \rangle \hookrightarrow \mathit{T84} \Rightarrow$ $\mathbf{domain}(\mathsf{f}_{|\mathsf{x}_2}) = \mathsf{x}_2$
EQUAL $\Rightarrow$ $\neg\mathsf{Finite}(\mathbf{domain}(\mathsf{f}_{|\mathsf{x}_2}))$
$\langle \mathsf{f}, \mathsf{x}_2 \rangle \hookrightarrow \mathit{T53} \Rightarrow$ $1\text{--}1(\mathsf{f}_{|\mathsf{x}_2})$
$\langle \mathsf{f}_{|\mathsf{x}_2} \rangle \hookrightarrow \mathit{T164} \Rightarrow$ $\neg\mathsf{Finite}(\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}))$
$\langle \mathsf{y} \times \mathsf{y}, \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\neg\mathsf{Finite}(\mathsf{y} \times \mathsf{y})$
$\langle \mathsf{y}, \mathsf{y} \rangle \hookrightarrow \mathit{T225} \Rightarrow$ $\neg\mathsf{Finite}(\mathsf{y})$
Suppose $\Rightarrow$ $\mathsf{y} = \emptyset$
EQUAL $\Rightarrow$ $\neg\mathsf{Finite}(\emptyset)$
$\mathit{T161} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{y} \neq \emptyset$
$\langle \mathsf{y} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\#\mathsf{y} \notin \#(\mathsf{y} \times \mathsf{y})$
$\langle \mathsf{y} \rangle \hookrightarrow \mathit{T130} \Rightarrow$ $\mathcal{O}(\#\mathsf{y})$
$\langle \mathsf{y} \times \mathsf{y} \rangle \hookrightarrow \mathit{T130} \Rightarrow$ $\mathcal{O}(\#(\mathsf{y} \times \mathsf{y}))$
$\langle \#(\mathsf{y} \times \mathsf{y}), \#\mathsf{y} \rangle \hookrightarrow \mathit{T32} \Rightarrow$ $\#(\mathsf{y} \times \mathsf{y}) \subseteq \#\mathsf{y}$
$\langle \mathsf{y}, \mathsf{y} \rangle \hookrightarrow \mathit{T214} \Rightarrow$ $\#\mathsf{y} = \#(\mathsf{y} \times \mathsf{y})$
$\langle \mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}), \mathsf{y} \times \mathsf{y} \rangle \hookrightarrow \mathit{T144} \Rightarrow$ $\#\mathbf{range}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \#\mathsf{y}$
$\langle \mathsf{f}_{|\mathsf{x}_2} \rangle \hookrightarrow \mathit{T131} \Rightarrow$ $\#\mathbf{domain}(\mathsf{f}_{|\mathsf{x}_2}) \subseteq \#\mathsf{y}$

-- And now we have a contradiction with the fact that $\mathsf{y} \in \mathsf{x}_2$, thereby proving our theorem.

EQUAL $\Rightarrow$ $\mathit{Stat11}:$ $\mathsf{x}_2 \subseteq \#\mathsf{y}$
$\langle \mathsf{y}, \mathsf{y} \rangle \hookrightarrow \mathit{T143}(\langle \mathit{Stat10}, \mathit{Stat11}, * \rangle) \Rightarrow$ $\mathsf{x}_2 \subseteq \mathsf{y}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\#\mathbf{domain}(\mathsf{f}) \subseteq \mathsf{x}_2$
$\langle \mathsf{f} \rangle \hookrightarrow \mathit{T131} \Rightarrow$ $\#\mathbf{domain}(\mathsf{f}) = \#\mathbf{range}(\mathsf{f})$
EQUAL $\Rightarrow$ $\#(\mathsf{x}_2 \times \mathsf{x}_2) \subseteq \mathsf{x}_2$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

10 The signed integers

-- After the preceding digression to prove the Cardinal Product theorem, we now return to our main task: developing the standard foundations of analysis. Our first step is to define signed integers and prove their properties. Signed integers and the arithmetic operations on them are defined as follows. Note that we choose to define a signed integer as a pair $[x, y]$ of ordinary integers, one or both of which is always zero. The idea is that $[x, \emptyset]$ represents the positive integer x, while $[\emptyset, x]$ represents the negative integer $\backslash(x)$.

-- Signed Integers

DEF 26. $\mathbb{Z} \;=_{\text{Def}}\; \{[x, y] : x \in \mathbb{N}, y \in \mathbb{N} \,|\, x = \emptyset \vee y = \emptyset\}$

-- Signed Integer Reduction to Normal Form

DEF 27. $\mathsf{Red}(X) \;=_{\text{Def}}\; \left[X^{[1]} - X^{[1]} \cap X^{[2]}, X^{[2]} - X^{[1]} \cap X^{[2]}\right]$

-- Signed Sum

DEF 28. $X +_{\mathbb{Z}} Y \;=_{\text{Def}}\; \mathsf{Red}(\left[X^{[1]} + Y^{[1]}, X^{[2]} + Y^{[2]}\right])$

-- Absolute value

DEF 28*a*. $\#X_{\mathbb{Z}} \;=_{\text{Def}}\; X^{[1]} \cup X^{[2]}$

-- Negative

DEF 28*b*. $\mathsf{Rev}_{\mathbb{Z}}(X) \;=_{\text{Def}}\; \left[X^{[2]}, X^{[1]}\right]$

-- Signed Product

DEF 29. $X *_{\mathbb{Z}} Y \;=_{\text{Def}}\; \mathsf{Red}(\left[X^{[1]} * Y^{[1]} + X^{[2]} * Y^{[2]}, X^{[1]} * Y^{[2]} + Y^{[1]} * X^{[2]}\right])$

-- Signed Difference

DEF 32. $X -_{\mathbb{Z}} Y \;=_{\text{Def}}\; \mathsf{Red}(\left[Y^{[2]} + X^{[1]}, Y^{[1]} + X^{[2]}\right])$

-- Sign of a signed integer

DEF 33. $\mathsf{is\_nonneg}_{\mathbb{N}}(X) \;\leftrightarrow_{\text{Def}}\; X^{[1]} \supseteq X^{[2]}$

-- Our first result concerning the small family of operations just introduced is that the reduction $\mathsf{Red}([m, n])$ of any pair of positive integers is a signed integer, and that $m \cap n$ is always $\emptyset$.

Theorem 372 (289) $M, N \in \mathbb{N} \rightarrow \mathsf{Red}([M, N]) \in \mathbb{Z} \;\&\; M \cap N \in \mathbb{N}$. PROOF:

$\mathsf{Suppose\_not}(m, n) \Rightarrow \quad m, n \in \mathbb{N} \;\&\; \mathsf{Red}([m, n]) \notin \mathbb{Z} \vee m \cap n \notin \mathbb{N}$

-- For suppose that there is a counterexample m, n to our assertion. m and n are plainly finite ordinals, and so their intersection is also a finite ordinal and therefore an integer. Hence the differences $m - m \cap n$ and $n - m \cap n$ are integers also.

$\langle m \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Card}(m) \,\&\, \mathsf{Finite}(m)$
$\langle n \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Card}(n) \,\&\, \mathsf{Finite}(n)$
$\mathsf{Use\_def}(\mathsf{Card}) \Rightarrow \quad \mathcal{O}(m) \,\&\, \mathcal{O}(n)$
$\langle m, n \rangle \hookrightarrow T25 \Rightarrow \quad \mathcal{O}(m \cap n)$
$\langle m, m \cap n \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{Finite}(m \cap n)$
$\langle m \cap n \rangle \hookrightarrow T178 \Rightarrow \quad \mathsf{Card}(m \cap n)$
$\langle m \cap n \rangle \hookrightarrow T179 \Rightarrow \quad m \cap n \in \mathbb{N}$
$\langle m, m \cap n \rangle \hookrightarrow T239 \Rightarrow \quad m - m \cap n \in \mathbb{N}$
$\langle n, m \cap n \rangle \hookrightarrow T239 \Rightarrow \quad n - m \cap n \in \mathbb{N}$

-- Since we have seen that $m \cap n$ is an integer, only the second conclusion of our theorem can fail. But one of m and n must be no larger than the other, and if, e. g. this is m we have $m \cap n = m$ and so $\mathsf{Red}([m, n]) = m - m$ by definition, implying $\mathsf{Red}([m, n]) = \emptyset$ and so proving that $\mathsf{Red}([m, n]) \in \mathbb{Z}$. The proof in case n is no larger than m is equally trivial.

$\langle m, n \rangle \hookrightarrow T26 \Rightarrow \quad m \subseteq n \vee n \subseteq m$
$\mathsf{ELEM} \Rightarrow \quad m \cap n = m \vee m \cap n = n \,\&\, \mathsf{Red}([m, n]) \notin \mathbb{Z}$
$\mathsf{Use\_def}(\mathsf{Red}) \Rightarrow \quad [m - m \cap n, n - m \cap n] \notin \mathbb{Z}$
$\mathsf{Use\_def}(\mathbb{Z}) \Rightarrow \quad Stat1 : \; [m - m \cap n, n - m \cap n] \notin \{[x, y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle m - m \cap n, n - m \cap n \rangle \hookrightarrow Stat1 \Rightarrow \quad m - m \cap n \neq \emptyset \,\&\, n - m \cap n \neq \emptyset$
$\mathsf{Suppose} \Rightarrow \quad m \cap n = m$
$\mathsf{EQUAL} \Rightarrow \quad m - m \neq \emptyset$
$\langle m \rangle \hookrightarrow T229 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad m \cap n = n$
$\mathsf{EQUAL} \Rightarrow \quad n - n \neq \emptyset$
$\langle n \rangle \hookrightarrow T229 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- Next we note an entirely trivial consequence of our definition of signed integers: if n is and unsigned integer, then $[n, \emptyset]$ (the signed version of n) and $[\emptyset, n]$ (corresponding to n) are signed integers.

Theorem 373 (290) $N \in \mathbb{N} \rightarrow [N, \emptyset], [\emptyset, N] \in \mathbb{Z}$. PROOF:

$\mathsf{Suppose\_not}(n) \Rightarrow \quad n \in \mathbb{N} \,\&\, [n, \emptyset] \notin \mathbb{Z} \vee [\emptyset, n] \notin \mathbb{Z}$
$T182 \Rightarrow \quad \emptyset \in \mathbb{N}$
$\mathsf{Use\_def}(\mathbb{Z}) \Rightarrow \quad \mathbb{Z} = \{[x, y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\mathsf{Suppose} \Rightarrow \quad Stat1 : \; [n, \emptyset] \notin \{[x, y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle n, \emptyset \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat2 : \; [\emptyset, n] \notin \{[x, y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle \emptyset, n \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- It is also trivial that $[\emptyset,\emptyset]$ (the signed $\emptyset$), $[\emptyset,1]$ (the signed 1), and $[\emptyset,1]$ (corresponding to \(1)) are all signed integers.

Theorem 374 (291) $[\emptyset,\emptyset], [1,\emptyset], [\emptyset,1] \in \mathbb{Z} \,\&\, [1,\emptyset] \neq [\emptyset,\emptyset] \,\&\, [\emptyset,1] \neq [\emptyset,\emptyset] \,\&\, [1,\emptyset] \neq [\emptyset,1]$. PROOF:

Suppose_not $\Rightarrow$
$\neg([\emptyset,\emptyset], [1,\emptyset], [\emptyset,1] \in \mathbb{Z} \,\&\, [1,\emptyset] \neq [\emptyset,\emptyset] \,\&\, [\emptyset,1] \neq [\emptyset,\emptyset] \,\&\, [1,\emptyset] \neq [\emptyset,1])$
T183 $\Rightarrow$ $\neg\,[\emptyset,\emptyset], [1,\emptyset], [\emptyset,1] \in \mathbb{Z}$
T182 $\Rightarrow$ $\emptyset, 1 \in \mathbb{N}$
$\langle 1 \rangle \hookrightarrow$ *T290* $\Rightarrow$ $[\emptyset,\emptyset] \notin \mathbb{Z}$
$\langle \emptyset \rangle \hookrightarrow$ *T290* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we note various elementary facts concerning signed integers: they are always pairs of unsigned integers, one of which is zero; signed integers are invariant under the reduction operator Red; and the minimum of the two components of a signed integer is always $\emptyset$.

Theorem 375 (292) $\mathsf{N} \in \mathbb{Z} \rightarrow \mathsf{N} = \left[\mathsf{N}^{[1]}, \mathsf{N}^{[2]}\right] \,\&\, \mathsf{N}^{[1]} = \emptyset \vee \mathsf{N}^{[2]} = \emptyset \,\&\, \mathsf{N}^{[1]}, \mathsf{N}^{[2]} \in \mathbb{N} \,\&\, \mathsf{Red}(\mathsf{N}) = \mathsf{N} \,\&\, \mathsf{N}^{[1]} \cap \mathsf{N}^{[2]} = \emptyset$. PROOF:

Suppose_not(n) $\Rightarrow$ *Stat1*: $\mathsf{n} \in \mathbb{Z} \,\&\, \mathsf{n} \neq \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right] \vee (\mathsf{n}^{[1]} \neq \emptyset \,\&\, \mathsf{n}^{[2]} \neq \emptyset) \vee \mathsf{n}^{[1]} \notin \mathbb{N} \vee \mathsf{n}^{[2]} \notin \mathbb{N} \vee \mathsf{Red}(\mathsf{n}) \neq \mathsf{n} \vee \mathsf{n}^{[1]} \cap \mathsf{n}^{[2]} \neq \emptyset$

-- Since n is a signed integer, it has the form $\mathsf{n} = [\mathsf{i},\mathsf{j}]$ where either i or j is $\emptyset$. Hence our assertion can only be false if $\mathsf{Red}(\mathsf{n}) \neq \mathsf{n}$. But by definition $\mathsf{Red}(\mathsf{n}) = [\mathsf{i} - \mathsf{i} \cap \mathsf{j}, \mathsf{j} - \mathsf{i} \cap \mathsf{j}]$, so our assertion follows immediately from Theorems 118, 92, and 158.

Use_def($\mathbb{Z}$) $\Rightarrow$ *Stat2*: $\mathsf{n} \in \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$
$\langle \mathsf{i},\mathsf{j} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3*: $\mathsf{n} = [\mathsf{i},\mathsf{j}] \,\&\, \mathsf{i},\mathsf{j} \in \mathbb{N} \,\&\, \mathsf{i} = \emptyset \vee \mathsf{j} = \emptyset$
ELEM $\Rightarrow$ *Stat4*: $\mathsf{i} \cap \mathsf{j} = \emptyset$
$\langle$ *Stat3* $\rangle$ ELEM $\Rightarrow$ *Stat5*: $\mathsf{n}^{[1]} = \emptyset \vee \mathsf{n}^{[2]} = \emptyset \,\&\, \mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{N} \,\&\, \mathsf{n}^{[1]} \cap \mathsf{n}^{[2]} = \emptyset$
$\langle$ *Stat3, Stat4* $\rangle$ ELEM $\Rightarrow$ *Stat6*: $\mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right]$
$\langle$ *Stat1, Stat5, Stat6*, $*$ $\rangle$ ELEM $\Rightarrow$ *Stat7*: $\mathsf{Red}(\mathsf{n}) \neq \mathsf{n}$
Use_def(Red) $\Rightarrow$ *Stat8*: $\mathsf{Red}([\mathsf{i},\mathsf{j}]) = [\mathsf{i} - \mathsf{i} \cap \mathsf{j}, \mathsf{j} - \mathsf{i} \cap \mathsf{j}]$
EQUAL $\Rightarrow$ $\mathsf{Red}([\mathsf{i},\mathsf{j}]) = \mathsf{Red}(\mathsf{n})$
EQUAL $\Rightarrow$ $\mathsf{i} - \mathsf{i} \cap \mathsf{j} = \mathsf{i} - \emptyset$
EQUAL $\Rightarrow$ $\mathsf{j} - \mathsf{i} \cap \mathsf{j} = \mathsf{j} - \emptyset$
$\langle \mathsf{i} \rangle \hookrightarrow$ *T230* $\Rightarrow$ $\mathsf{i} - \emptyset = \#\mathsf{i}$
$\langle \mathsf{j} \rangle \hookrightarrow$ *T230* $\Rightarrow$ $\mathsf{j} - \emptyset = \#\mathsf{j}$
$\langle$ *Stat8* $\rangle$ ELEM $\Rightarrow$ *Stat9*: $\mathsf{Red}(\mathsf{n}) = [\#\mathsf{i}, \#\mathsf{j}]$
$\langle \mathsf{i} \rangle \hookrightarrow$ *T180* $\Rightarrow$ $\mathsf{i} = \#\mathsf{i}$

$\langle j\rangle \hookrightarrow T180 \Rightarrow \quad j = \#j$
$\langle Stat9\rangle$ ELEM $\Rightarrow \quad Stat10:\ \mathrm{Red}(n) = [i, j]$
$\langle Stat3, Stat7, Stat10, *\rangle$ ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- The following variant form of the preceding theorem is often a bit more useful.

Theorem 376 (293) $\quad N \in \mathbb{Z} \rightarrow N = \left[N^{[1]}, \emptyset\right] \vee N = \left[\emptyset, N^{[2]}\right] \ \&\ N^{[1]} = \emptyset \vee N^{[2]} = \emptyset \ \&\ N^{[1]}, N^{[2]} \in \mathbb{N} \ \&\ \mathrm{Red}(N) = N \ \&\ N^{[1]} \cap N^{[2]} = \emptyset$. PROOF:

Suppose_not$(n) \Rightarrow \quad Stat1:$
$\quad n \in \mathbb{Z} \ \&\ \neg\left(n = \left[n^{[1]}, \emptyset\right] \vee n = \left[\emptyset, n^{[2]}\right] \ \&\ n^{[1]} = \emptyset \vee n^{[2]} = \emptyset \ \&\ n^{[1]}, n^{[2]} \in \mathbb{N} \ \&\ \mathrm{Red}(n) = n \ \&\ n^{[1]} \cap n^{[2]} = \emptyset\right)$
$\langle n\rangle \hookrightarrow T292 \Rightarrow \quad n = \left[n^{[1]}, n^{[2]}\right] \ \&\ \neg\left(n = \left[n^{[1]}, \emptyset\right] \vee n = \left[\emptyset, n^{[2]}\right]\right) \ \&\ n^{[1]} = \emptyset \vee n^{[2]} = \emptyset$
Suppose $\Rightarrow \quad n^{[1]} = \emptyset$
ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad n^{[2]} = \emptyset$
ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- The following theorem states that the set of signed integers is closed under addition and multiplication.

Theorem 377 (294) $\quad N, M \in \mathbb{Z} \rightarrow N +_{\mathbb{Z}} M, N *_{\mathbb{Z}} M \in \mathbb{Z}$. PROOF:

-- The proof is elementary. We just use the definitions of the operators involved and the information concerning the form of signed integers provided by Theorem 292.

Suppose_not$(n, m) \Rightarrow \quad n, m \in \mathbb{Z} \ \&\ \neg n +_{\mathbb{Z}} m, n *_{\mathbb{Z}} m \in \mathbb{Z}$
$\langle n\rangle \hookrightarrow T292 \Rightarrow \quad n^{[1]}, n^{[2]} \in \mathbb{N}$
$\langle m\rangle \hookrightarrow T292 \Rightarrow \quad m^{[1]}, m^{[2]} \in \mathbb{N}$
Use_def$(+_{\mathbb{Z}}) \Rightarrow \quad n +_{\mathbb{Z}} m = \mathrm{Red}\left(\left[n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]}\right]\right)$
ALGEBRA $\Rightarrow \quad n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]} \in \mathbb{N}$
$\langle n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]}\rangle \hookrightarrow T289 \Rightarrow \quad n +_{\mathbb{Z}} m \in \mathbb{Z}$
Use_def$(*_{\mathbb{Z}}) \Rightarrow \quad n *_{\mathbb{Z}} m = \mathrm{Red}\left(\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\right]\right)$
ALGEBRA $\Rightarrow \quad n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]} \in \mathbb{N}$
$\langle n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\rangle \hookrightarrow T289 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- Our next theorem asserts that the negative $\mathrm{Rev}_{\mathbb{Z}}(n)$ of a signed integer, and the difference $n -_{\mathbb{Z}} m$ of two signed integers, are always signed integers.

Theorem 378 (295) $\quad N, M \in \mathbb{Z} \rightarrow \mathrm{Rev}_{\mathbb{Z}}(M), N -_{\mathbb{Z}} M \in \mathbb{Z}$. PROOF:

Suppose_not$(n,m) \Rightarrow \quad n,m \in \mathbb{Z} \;\&\; \neg \mathrm{Rev}_{\mathbb{Z}}(m), n -_{\mathbb{Z}} m \in \mathbb{Z}$

-- For a signed integer n is simply a pair [a, b] of ordinary integers one of which is zero, so plainly its reverse [b, a] has the same property.

$\langle n \rangle \hookrightarrow T292 \Rightarrow \quad n = [n^{[1]}, n^{[2]}] \;\&\; n^{[1]} = \emptyset \vee n^{[2]} = \emptyset \;\&\; n^{[1]}, n^{[2]} \in \mathbb{N}$
$\langle m \rangle \hookrightarrow T292 \Rightarrow \quad m = [m^{[1]}, m^{[2]}] \;\&\; m^{[1]} = \emptyset \vee m^{[2]} = \emptyset \;\&\; m^{[1]}, m^{[2]} \in \mathbb{N}$
Use_def$(\mathrm{Rev}_{\mathbb{Z}}) \Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(m) = [m^{[2]}, m^{[1]}]$
ELEM $\Rightarrow \quad [m^{[2]}, m^{[1]}]^{[1]} = m^{[2]} \;\&\; [m^{[2]}, m^{[1]}]^{[2]} = m^{[1]}$
EQUAL $\Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}^{[1]}(m) = m^{[2]} \;\&\; \mathrm{Rev}_{\mathbb{Z}}^{[2]}(m) = m^{[1]}$
Suppose $\Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(m) \notin \mathbb{Z}$
Use_def$(\mathbb{Z}) \Rightarrow \quad Stat1: \; \mathrm{Rev}_{\mathbb{Z}}(m) \notin \{[x,y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle m^{[2]}, m^{[1]} \rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(m) \in \mathbb{Z}$
ELEM $\Rightarrow \quad n -_{\mathbb{Z}} m \notin \mathbb{Z}$

-- Moreover the difference of two signed integers is by definition the reduction of a pair of unsigned integers, and so is a signed integer by definition.

Use_def$(-_{\mathbb{Z}}) \Rightarrow \quad \mathrm{Red}([m^{[2]} + n^{[1]}, m^{[1]} + n^{[2]}]) \notin \mathbb{Z}$
ALGEBRA $\Rightarrow \quad m^{[2]} + n^{[1]}, m^{[1]} + n^{[2]} \in \mathbb{N}$
$\langle m^{[2]} + n^{[1]}, m^{[1]} + n^{[2]} \rangle \hookrightarrow T289 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Our next lemma states the elementary fact that any pair [n, n] of unsigned integers reduces to the pair [0, 0].

Theorem 379 (296) $N \in \mathbb{N} \rightarrow \mathrm{Red}([N,N]) = [\emptyset, \emptyset]$. PROOF:

Suppose_not$(n) \Rightarrow \quad n \in \mathbb{N} \;\&\; \mathrm{Red}([n,n]) \neq [\emptyset, \emptyset]$
Use_def$(\mathrm{Red}) \Rightarrow \quad \mathrm{Red}([n,n]) = [n - n \cap n, n - n \cap n]$
ELEM $\Rightarrow \quad n \cap n = n$
EQUAL $\Rightarrow \quad \mathrm{Red}([n,n]) = [n - n, n - n]$
$\langle n \rangle \hookrightarrow T229 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The following theorem, which generalizes the Lemma just noted, states that the value to which any pair [j, k] of integers reduces is unchanged if a common integer m is added to both i and j.

Theorem 380 (297) $J, K, M \in \mathbb{N} \rightarrow \mathrm{Red}([J+M, K+M]) = \mathrm{Red}([J,K])$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{j},\mathsf{k},\mathsf{m}) \Rightarrow \quad \mathsf{j},\mathsf{k},\mathsf{m} \in \mathbb{N} \,\&\, \mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) \neq \mathsf{Red}([\mathsf{j},\mathsf{k}])$

-- Suppose that $\mathsf{j},\mathsf{k},\mathsf{m}$ form a counterexample to our theorem. Of the two integers j, k one is smaller. Suppose for the moment that this is j, so that j is a subset of k. It follows that $\mathsf{j} \cap \mathsf{k} = \mathsf{j}$, so by definition of the reduction operator, $\mathsf{Red}([\mathsf{j},\mathsf{k}]) = [\emptyset, \mathsf{k}-\mathsf{j}]$.

$\mathsf{Use\_def}(\mathsf{Red}) \Rightarrow \quad \mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) = [\mathsf{j}+\mathsf{m}-(\mathsf{j}+\mathsf{m})\cap(\mathsf{k}+\mathsf{m}), \mathsf{k}+\mathsf{m}-(\mathsf{j}+\mathsf{m})\cap(\mathsf{k}+\mathsf{m})]$
$\mathsf{Use\_def}(\mathsf{Red}) \Rightarrow \quad \mathsf{Red}([\mathsf{j},\mathsf{k}]) = [\mathsf{j}-\mathsf{j}\cap\mathsf{k}, \mathsf{k}-\mathsf{j}\cap\mathsf{k}]$
$\langle \mathsf{j} \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(\mathsf{j})$
$\langle \mathsf{k} \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(\mathsf{k})$
$\langle \mathsf{j},\mathsf{k} \rangle \hookrightarrow T26 \Rightarrow \quad \mathsf{j} \subseteq \mathsf{k} \vee \mathsf{k} \subseteq \mathsf{j}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{j} \subseteq \mathsf{k}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{j} \cap \mathsf{k} = \mathsf{j}$
$\mathsf{EQUAL} \Rightarrow \quad Stat0: \ \mathsf{Red}([\mathsf{j},\mathsf{k}]) = [\mathsf{j}-\mathsf{j}, \mathsf{k}-\mathsf{j}]$
$\langle \mathsf{j} \rangle \hookrightarrow T229(\langle Stat0 \rangle) \Rightarrow \quad \mathsf{Red}([\mathsf{j},\mathsf{k}]) = [\emptyset, \mathsf{k}-\mathsf{j}]$

-- By the monotonicity of addition, $\mathsf{j}+\mathsf{m}$ is no greater than $\mathsf{k}+\mathsf{m}$, so it follows by the same argument as above that $\mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) = [\emptyset, \mathsf{k}+\mathsf{m}-(\mathsf{j}+\mathsf{m})]$. Using Theorem 229 we now see that $\mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) = \mathsf{Red}([\mathsf{j},\mathsf{k}])$, a contradiction which excludes the possibility that j in less than k.

$\langle \mathsf{j},\mathsf{k},\mathsf{m} \rangle \hookrightarrow T244 \Rightarrow \quad \mathsf{j}+\mathsf{m} \subseteq \mathsf{k}+\mathsf{m}$
$\mathsf{ELEM} \Rightarrow \quad (\mathsf{j}+\mathsf{m}) \cap (\mathsf{k}+\mathsf{m}) = \mathsf{j}+\mathsf{m}$
$\mathsf{EQUAL} \Rightarrow \quad Stat1: \ \mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) = [\mathsf{j}+\mathsf{m}-(\mathsf{j}+\mathsf{m}), \mathsf{k}+\mathsf{m}-(\mathsf{j}+\mathsf{m})]$
$\langle \mathsf{j}+\mathsf{m} \rangle \hookrightarrow T229(\langle Stat1 \rangle) \Rightarrow \quad \mathsf{j}+\mathsf{m}-(\mathsf{j}+\mathsf{m}) = \emptyset$
$\langle Stat1 \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) = [\emptyset, \mathsf{k}+\mathsf{m}-(\mathsf{j}+\mathsf{m})]$
$\langle \mathsf{k},\mathsf{j},\mathsf{m} \rangle \hookrightarrow T260 \Rightarrow \quad \mathsf{k}+\mathsf{m}-(\mathsf{j}+\mathsf{m}) = \mathsf{k}-\mathsf{j}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{k} \subseteq \mathsf{j}$

-- It follows that k is less than j. But in this case an argument exactly symmetric to that just given leads to a contradiction, which proves our theorem.

$\mathsf{ELEM} \Rightarrow \quad \mathsf{j} \cap \mathsf{k} = \mathsf{k}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Red}([\mathsf{j},\mathsf{k}]) = [\mathsf{j}-\mathsf{k}, \mathsf{k}-\mathsf{k}]$
$\langle \mathsf{k} \rangle \hookrightarrow T229 \Rightarrow \quad \mathsf{k}-\mathsf{k} = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Red}([\mathsf{j},\mathsf{k}]) = [\mathsf{j}-\mathsf{k}, \emptyset]$
$\langle \mathsf{k},\mathsf{j},\mathsf{m} \rangle \hookrightarrow T244 \Rightarrow \quad \mathsf{k}+\mathsf{m} \subseteq \mathsf{j}+\mathsf{m}$
$\mathsf{ELEM} \Rightarrow \quad (\mathsf{k}+\mathsf{m}) \cap (\mathsf{j}+\mathsf{m}) = \mathsf{k}+\mathsf{m}$
$\mathsf{Use\_def}(\mathsf{Red}) \Rightarrow$

$$\mathsf{Red}([\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]) = \Big[\mathsf{j}+\mathsf{m}-[\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]^{[1]} \cap [\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]^{[2]}, \mathsf{k}+\mathsf{m}-[\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]^{[1]} \cap [\mathsf{j}+\mathsf{m},\mathsf{k}+\mathsf{m}]^{[2]}\Big]$$

ELEM $\Rightarrow$ $[j+m,k+m]^{[1]} \cap [j+m,k+m]^{[2]} = k+m$
EQUAL $\Rightarrow$ $\mathsf{Red}([j+m,k+m]) = [j+m-(k+m), k+m-(k+m)]$
$\langle k+m\rangle \hookrightarrow T229 \Rightarrow$ $k+m-(k+m) = \emptyset$
EQUAL $\Rightarrow$ $\mathsf{Red}([j+m,k+m]) = [j+m-(k+m), \emptyset]$
$\langle j,k,m\rangle \hookrightarrow T260 \Rightarrow$ $j+m-(k+m) = j-k$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the signed integer sum of any two pairs of integers is the sum of the first with the reduction of the second.

Theorem 381 (298) $J,K,N,M \in \mathbb{N} \rightarrow [J,K] +_{\mathbb{Z}} [N,M] = [J,K] +_{\mathbb{Z}} \mathsf{Red}([N,M])$. PROOF:

Suppose_not$(j,k,n,m) \Rightarrow$ $j,k,n,m \in \mathbb{N}$ & $[j,k] +_{\mathbb{Z}} [n,m] \neq [j,k] +_{\mathbb{Z}} \mathsf{Red}([n,m])$

-- For suppose that there is a counterexample j,k,n,m to our assertion. Of the two integers n, m, one is smaller, so that $n \cap m$ is an integer in any case.

$\langle n\rangle \hookrightarrow T180 \Rightarrow$ $\mathcal{O}(n)$ & $n = \#n$
$\langle m\rangle \hookrightarrow T180 \Rightarrow$ $\mathcal{O}(m)$ & $m = \#m$
$\langle n,m\rangle \hookrightarrow T26 \Rightarrow$ $n \subseteq m \vee m \subseteq n$
ELEM $\Rightarrow$ $n \cap m \in \mathbb{N}$ & $n \cap m = n \vee n \cap m = m$
$\langle n, n\cap m\rangle \hookrightarrow T239 \Rightarrow$ $n - n\cap m \in \mathbb{N}$
$\langle m, n\cap m\rangle \hookrightarrow T239 \Rightarrow$ $m - n\cap m \in \mathbb{N}$

-- Use of the definitions of the $+_{\mathbb{Z}}$ and Red operators converts the negative of our theorem into the inequality just below.

Use_def$(+_{\mathbb{Z}}) \Rightarrow$ $\mathsf{Red}([j+n,k+m]) \neq \mathsf{Red}\left(\left[j+\mathsf{Red}^{[1]}([n,m]), k+\mathsf{Red}^{[2]}([n,m])\right]\right)$
Use_def(Red) $\Rightarrow$ *Stat1*: $\mathsf{Red}([n,m]) = [n - n\cap m, m - n\cap m]$
$\langle Stat1\rangle$ ELEM $\Rightarrow$ $n \supseteq n\cap m$ & $m \supseteq n\cap m$ & $\mathsf{Red}^{[1]}([n,m]) = n - n\cap m$ & $\mathsf{Red}^{[2]}([n,m]) = m - n\cap m$
EQUAL $\Rightarrow$ $\mathsf{Red}([j+n,k+m]) \neq \mathsf{Red}\left([j+(n-n\cap m), k+(m-n\cap m)]\right)$

-- But we can add $n \cap m$ to both components of the pair seen on the right-hand side of this last inequality without changing its reduction, and so after a bit more algebraic manipulation derive a contradiction which proves the present theorem.

ALGEBRA $\Rightarrow$ $j+(n-n\cap m), k+(m-n\cap m) \in \mathbb{N}$
$\langle j+(n-n\cap m), k+(m-n\cap m), n\cap m\rangle \hookrightarrow T297 \Rightarrow$ $\mathsf{Red}\left([j+(n-n\cap m), k+(m-n\cap m)]\right) =$
$\mathsf{Red}\left([j+(n-n\cap m)+n\cap m, k+(m-n\cap m)+n\cap m]\right)$
ALGEBRA $\Rightarrow$ $\mathsf{Red}\left([j+(n-n\cap m), k+(m-n\cap m)]\right) = \mathsf{Red}\left([j+(n-n\cap m+n\cap m), k+(m-n\cap m+n\cap m)]\right)$

$\langle n \cap m, n \rangle \hookrightarrow T233 \Rightarrow \quad n - n \cap m + n \cap m = n$
$\langle n \cap m, m \rangle \hookrightarrow T233 \Rightarrow \quad m - n \cap m + n \cap m = m$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following elemenary corollary to the theorem just proved tells us that addition of any pair p of integers to a signed integer x produces the same result as addition of the reduced form of p to x.

Theorem 382 (299) $K \in \mathbb{Z} \,\&\, N, M \in \mathbb{N} \rightarrow K +_{\mathbb{Z}} [N, M] = K +_{\mathbb{Z}} \mathrm{Red}([N, M])$. PROOF:

Suppose_not$(k, n, m) \Rightarrow \quad k \in \mathbb{Z} \,\&\, n, m \in \mathbb{N} \,\&\, k +_{\mathbb{Z}} [n, m] \neq k +_{\mathbb{Z}} \mathrm{Red}([n, m])$
$\langle k \rangle \hookrightarrow T292 \Rightarrow \quad k = \left[k^{[1]}, k^{[2]}\right] \,\&\, k^{[1]} = \emptyset \vee k^{[2]} = \emptyset \,\&\, k^{[1]}, k^{[2]} \in \mathbb{N} \,\&\, \mathrm{Red}(k) = k$
$\langle k^{[1]}, k^{[2]}, n, m \rangle \hookrightarrow T298 \Rightarrow \quad \left[k^{[1]}, k^{[2]}\right] +_{\mathbb{Z}} [n, m] = \left[k^{[1]}, k^{[2]}\right] +_{\mathbb{Z}} \mathrm{Red}([n, m])$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following 'multiplication' analog to Theorem 299 states that multiplication of a signed integer k by any pair p of integers produces the same result as multiplication of k by the reduced form of p.

Theorem 383 (300) $K \in \mathbb{Z} \,\&\, N, M \in \mathbb{N} \rightarrow K *_{\mathbb{Z}} [N, M] = K *_{\mathbb{Z}} \mathrm{Red}([N, M])$. PROOF:

Suppose_not$(k, n, m) \Rightarrow \quad k \in \mathbb{Z} \,\&\, n, m \in \mathbb{N} \,\&\, k *_{\mathbb{Z}} [n, m] \neq k *_{\mathbb{Z}} \mathrm{Red}([n, m])$

-- For suppose that $k = [cak, cdk]$, n, m comprise a counterexample to our theorem, where plainly all of cak, cdk, n, m must be integers, and so must all the other quantities formed from them in the argument which follows. Since $n \cap m$ is the minimum of n and m and so is no more than either, we have $n - n \cap m + n \cap m = n$, and $m - n \cap m + n \cap m = m$, by theorems on unsigned integer subtractioand addition proved earlier.

$\langle k \rangle \hookrightarrow T292 \Rightarrow \quad Stat1: \; k = \left[k^{[1]}, k^{[2]}\right] \,\&\, k^{[1]} = \emptyset \vee k^{[2]} = \emptyset \,\&\, k^{[1]}, k^{[2]} \in \mathbb{N} \,\&\, \mathrm{Red}(k) = k$
$\langle n \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(n) \,\&\, n = \#n$
$\langle m \rangle \hookrightarrow T180 \Rightarrow \quad \mathcal{O}(m) \,\&\, m = \#m$
$\langle n, m \rangle \hookrightarrow T26 \Rightarrow \quad n \subseteq m \vee m \subseteq n$
ELEM $\Rightarrow \quad n \cap m \in \mathbb{N}$
$\langle n \cap m \rangle \hookrightarrow T180 \Rightarrow \quad n \cap m = \#(n \cap m)$
$\langle n, n \cap m \rangle \hookrightarrow T239 \Rightarrow \quad n - n \cap m \in \mathbb{N}$
$\langle m, n \cap m \rangle \hookrightarrow T239 \Rightarrow \quad m - n \cap m \in \mathbb{N}$
$\langle n \cap m, n \rangle \hookrightarrow T232 \Rightarrow \quad \#(n \cap m) + (n - n \cap m) = \#n$

$\langle n \cap m, m \rangle \hookrightarrow T232 \Rightarrow \quad \#(n \cap m) + (m - n \cap m) = \#m$
EQUAL $\Rightarrow \quad n \cap m + (n - n \cap m) = n$
EQUAL $\Rightarrow \quad n \cap m + (m - n \cap m) = m$
$\langle n - n \cap m, n \cap m \rangle \hookrightarrow T216 \Rightarrow \quad Stat2: \; n - n \cap m + n \cap m = n$
$\langle m - n \cap m, n \cap m \rangle \hookrightarrow T216 \Rightarrow \quad Stat3: \; m - n \cap m + n \cap m = m$

-- Using the definitions of signed integer multiplication and of reduction we can expand the negative of our theorem into the inequality between reductions seen below.

Use_def$(*_{\mathbb{Z}}) \Rightarrow$
$$\mathrm{Red}\left(\left[k^{[1]} * n + k^{[2]} * m, k^{[1]} * m + n * k^{[2]}\right]\right) \neq$$
$$\mathrm{Red}\left(\left[k^{[1]} * \mathrm{Red}^{[1]}([n,m]) + k^{[2]} * \mathrm{Red}^{[2]}([n,m]), k^{[1]} * \mathrm{Red}^{[2]}([n,m]) + \mathrm{Red}^{[1]}([n,m]) * k^{[2]}\right]\right)$$
Loc_def $\Rightarrow \quad Stat4: \; cak = k^{[1]}$
Loc_def $\Rightarrow \quad Stat5: \; cdk = k^{[2]}$
$\langle Stat1, Stat4, Stat5 \rangle$ ELEM $\Rightarrow \quad k = [cak, cdk] \;\&\; cak, cdk \in \mathbb{N}$
Use_def(Red) $\Rightarrow \quad Stat6: \; \mathrm{Red}([n,m]) = [n - n \cap m, m - n \cap m]$
EQUAL $\Rightarrow$
$$\mathrm{Red}([cak * n + cdk * m, cak * m + n * cdk]) \neq$$
$$\mathrm{Red}\left(\left[cak * \mathrm{Red}^{[1]}([n,m]) + cdk * \mathrm{Red}^{[2]}([n,m]), cak * \mathrm{Red}^{[2]}([n,m]) + \mathrm{Red}^{[1]}([n,m]) * cdk\right]\right)$$
$\langle Stat6 \rangle$ ELEM $\Rightarrow \quad n \supseteq n \cap m \;\&\; m \supseteq n \cap m \;\&\; \mathrm{Red}^{[1]}([n,m]) = n - n \cap m \;\&\; \mathrm{Red}^{[2]}([n,m]) = m - n \cap m$
EQUAL $\Rightarrow \quad \mathrm{Red}([cak * n + cdk * m, cak * m + n * cdk]) \neq \mathrm{Red}\left([cak * (n - n \cap m) + cdk * (m - n \cap m), cak * (m - n \cap m) + (n - n \cap m) * cdk]\right)$
ALGEBRA $\Rightarrow \quad cak * (n - n \cap m) + cdk * (m - n \cap m) \in \mathbb{N}$
ALGEBRA $\Rightarrow \quad cak * (m - n \cap m) + (n - n \cap m) * cdk \in \mathbb{N}$
ALGEBRA $\Rightarrow \quad cak * (n \cap m) + cdk * (n \cap m) \in \mathbb{N}$

-- But now adding the quantity $cak * (n \cap m) + cdk * (n \cap m)$ to both components of the right hand side of this last inequality allows us to reduce it, via a series of elementary algebraic transformations, to $\mathrm{Red}([cak * n + cdk * m, cak * m + cdk * n])$,

$\langle cak * (n - n \cap m) + cdk * (m - n \cap m), cak * (m - n \cap m) + (n - n \cap m) * cdk, cak * (n \cap m) + cdk * (n \cap m) \rangle \hookrightarrow T297 \Rightarrow$
$$\mathrm{Red}\left([cak * (n - n \cap m) + cdk * (m - n \cap m), cak * (m - n \cap m) + (n - n \cap m) * cdk]\right) =$$
$$\mathrm{Red}\Big(\left[cak * (n - n \cap m) + cdk * (m - n \cap m) + \big(cak * (n \cap m) + cdk * (n \cap m)\big), cak * (m - n \cap m) + (n - n \cap m) * cdk + \big(cak * (n \cap m) + cdk * (n \cap m)\big)\right]\Big)$$
ALGEBRA $\Rightarrow \quad cak * (n - n \cap m) + cdk * (m - n \cap m) + \big(cak * (n \cap m) + cdk * (n \cap m)\big) =$
$$cak * (n - n \cap m) + cdk * (m - n \cap m) + cak * (n \cap m) + cdk * (n \cap m)$$
ALGEBRA $\Rightarrow \quad cak * (m - n \cap m) + (n - n \cap m) * cdk + \big(cak * (n \cap m) + cdk * (n \cap m)\big) =$
$$cak * (m - n \cap m) + (n - n \cap m) * cdk + cak * (n \cap m) + cdk * (n \cap m)$$
EQUAL $\Rightarrow$
$$\mathrm{Red}\left([cak * (n - n \cap m) + cdk * (m - n \cap m), cak * (m - n \cap m) + (n - n \cap m) * cdk]\right) =$$

$\mathsf{Red}\big(\,[\mathsf{cak} * (\mathsf{n} - \mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m}) + \mathsf{cak} * (\mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{n} \cap \mathsf{m}), \mathsf{cak} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m}) + (\mathsf{n} - \mathsf{n} \cap \mathsf{m}) * \mathsf{cdk} + \mathsf{cak} * (\mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{n} \cap \mathsf{m})]\,\big)$

ALGEBRA $\Rightarrow$ $\quad Stat7:$

$\mathsf{Red}\big(\,[\mathsf{cak} * (\mathsf{n} - \mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m}), \mathsf{cak} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m}) + (\mathsf{n} - \mathsf{n} \cap \mathsf{m}) * \mathsf{cdk}]\,\big) =$

$\mathsf{Red}\big(\,[\mathsf{cak} * (\mathsf{n} - \mathsf{n} \cap \mathsf{m} + \mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m} + \mathsf{n} \cap \mathsf{m}), \mathsf{cak} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m} + \mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{n} - \mathsf{n} \cap \mathsf{m} + \mathsf{n} \cap \mathsf{m})]\,\big)$

EQUAL $\langle Stat2, Stat3, Stat7\rangle \Rightarrow$ $\quad \mathsf{Red}\big(\,[\mathsf{cak} * (\mathsf{n} - \mathsf{n} \cap \mathsf{m}) + \mathsf{cdk} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m}), \mathsf{cak} * (\mathsf{m} - \mathsf{n} \cap \mathsf{m}) + (\mathsf{n} - \mathsf{n} \cap \mathsf{m}) * \mathsf{cdk}]\,\big) =$
$\mathsf{Red}([\mathsf{cak} * \mathsf{n} + \mathsf{cdk} * \mathsf{m}, \mathsf{cak} * \mathsf{m} + \mathsf{cdk} * \mathsf{n}])$

-- and now a contraction results immediately by one final algebraic step, completing the proof of the present theorem.

ALGEBRA $\Rightarrow$ $\quad \mathsf{Red}([\mathsf{cak} * \mathsf{n} + \mathsf{cdk} * \mathsf{m}, \mathsf{cak} * \mathsf{m} + \mathsf{cdk} * \mathsf{n}]) = \mathsf{Red}([\mathsf{cak} * \mathsf{n} + \mathsf{cdk} * \mathsf{m}, \mathsf{cak} * \mathsf{m} + \mathsf{n} * \mathsf{cdk}])$

ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- The two following elementary lemmas prepare for proof of the commutativity of signed integer addition. We first prove commutativity for the sum of a signed integer with an arbitrary pair of unsigned integers.

-- Commutativity Lemma

Theorem 384 (301) $\quad \mathsf{k} \in \mathbb{Z} \,\&\, \mathsf{n}, \mathsf{m} \in \mathbb{N} \rightarrow \mathsf{k} +_{\mathbb{Z}} [\mathsf{n}, \mathsf{m}] = [\mathsf{n}, \mathsf{m}] +_{\mathbb{Z}} \mathsf{k}$. PROOF:

Suppose_not$(\mathsf{k}, \mathsf{n}, \mathsf{m}) \Rightarrow$ $\quad \mathsf{k} \in \mathbb{Z} \,\&\, \mathsf{n}, \mathsf{m} \in \mathbb{N} \,\&\, \mathsf{k} +_{\mathbb{Z}} [\mathsf{n}, \mathsf{m}] \neq [\mathsf{n}, \mathsf{m}] +_{\mathbb{Z}} \mathsf{k}$

$\langle \mathsf{k}\rangle \hookrightarrow T292 \Rightarrow$ $\quad \mathsf{k} = \left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right] \,\&\, \mathsf{k}^{[1]} = \emptyset \vee \mathsf{k}^{[2]} = \emptyset \,\&\, \mathsf{k}^{[1]}, \mathsf{k}^{[2]} \in \mathbb{N}$

Use_def$(+_{\mathbb{Z}}) \Rightarrow$ $\quad \mathsf{Red}\left(\left[\mathsf{k}^{[1]} + [\mathsf{n}, \mathsf{m}]^{[1]}, \mathsf{k}^{[2]} + [\mathsf{n}, \mathsf{m}]^{[2]}\right]\right) \neq \mathsf{Red}\left(\left[[\mathsf{n}, \mathsf{m}]^{[1]} + \mathsf{k}^{[1]}, [\mathsf{n}, \mathsf{m}]^{[2]} + \mathsf{k}^{[2]}\right]\right)$

ELEM $\Rightarrow$ $\quad \mathsf{Red}\left(\left[\mathsf{k}^{[1]} + \mathsf{n}, \mathsf{k}^{[2]} + \mathsf{m}\right]\right) \neq \mathsf{Red}\left(\left[\mathsf{n} + \mathsf{k}^{[1]}, \mathsf{m} + \mathsf{k}^{[2]}\right]\right)$

ALGEBRA $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Next we prove commutativity for the sum of two pairs of unsigned integers.

-- Commutativity Lemma

Theorem 385 (302) $\quad \mathsf{j}, \mathsf{k}, \mathsf{n}, \mathsf{m} \in \mathbb{N} \rightarrow [\mathsf{j}, \mathsf{k}] +_{\mathbb{Z}} [\mathsf{n}, \mathsf{m}] = [\mathsf{n}, \mathsf{m}] +_{\mathbb{Z}} [\mathsf{j}, \mathsf{k}]$. PROOF:

Suppose_not$(\mathsf{j}, \mathsf{k}, \mathsf{n}, \mathsf{m}) \Rightarrow$ $\quad \mathsf{j}, \mathsf{k}, \mathsf{n}, \mathsf{m} \in \mathbb{N} \,\&\, [\mathsf{j}, \mathsf{k}] +_{\mathbb{Z}} [\mathsf{n}, \mathsf{m}] \neq [\mathsf{n}, \mathsf{m}] +_{\mathbb{Z}} [\mathsf{j}, \mathsf{k}]$

Use_def$(+_{\mathbb{Z}}) \Rightarrow$

$\mathsf{Red}\left(\left[[\mathsf{j}, \mathsf{k}]^{[1]} + [\mathsf{n}, \mathsf{m}]^{[1]}, [\mathsf{j}, \mathsf{k}]^{[2]} + [\mathsf{n}, \mathsf{m}]^{[2]}\right]\right) \neq$

$\mathsf{Red}\left(\left[[\mathsf{n}, \mathsf{m}]^{[1]} + [\mathsf{j}, \mathsf{k}]^{[1]}, [\mathsf{n}, \mathsf{m}]^{[2]} + [\mathsf{j}, \mathsf{k}]^{[2]}\right]\right)$

ELEM $\Rightarrow$ $\quad \mathsf{Red}([\mathsf{j} + \mathsf{n}, \mathsf{k} + \mathsf{m}]) \neq \mathsf{Red}([\mathsf{n} + \mathsf{j}, \mathsf{m} + \mathsf{k}])$

ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Using the lemmas just proved, we can show immediately that signed integer addition is commutative.

-- Commutative Law for Addition

Theorem 386 (303) $n, m \in \mathbb{Z} \rightarrow n +_{\mathbb{Z}} m = m +_{\mathbb{Z}} n$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n, m \in \mathbb{Z} \,\&\, n +_{\mathbb{Z}} m \neq m +_{\mathbb{Z}} n$
$\langle m \rangle \hookrightarrow T292 \Rightarrow$ $m = [m^{[1]}, m^{[2]}] \,\&\, m^{[1]} = \emptyset \vee m^{[2]} = \emptyset \,\&\, m^{[1]}, m^{[2]} \in \mathbb{N}$
$\langle n, m^{[1]}, m^{[2]} \rangle \hookrightarrow T301 \Rightarrow$ $n +_{\mathbb{Z}} [m^{[1]}, m^{[2]}] = [m^{[1]}, m^{[2]}] +_{\mathbb{Z}} n$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Using commutativity we can prove the following occasionally useful variants of Theorem 298, the first stating that the sum of two pairs of unsigned integers is the sum of the reduction of the first pair with the second pair, while the next states that the same sum is is the sum of the first pair with the reduction of the second pair. Both proofs are elementary applicaions of fromulae already established.

Theorem 387 (304) $J, K, N, M \in \mathbb{N} \rightarrow [J, K] +_{\mathbb{Z}} [N, M] = \text{Red}([J, K]) +_{\mathbb{Z}} [N, M]$. PROOF:

Suppose_not$(j, k, n, m) \Rightarrow$ $j, k, n, m \in \mathbb{N} \,\&\, [j, k] +_{\mathbb{Z}} [n, m] \neq \text{Red}([j, k]) +_{\mathbb{Z}} [n, m]$
$\langle j, k, n, m \rangle \hookrightarrow T302 \Rightarrow$ $[j, k] +_{\mathbb{Z}} [n, m] = [n, m] +_{\mathbb{Z}} [j, k]$
$\langle j, k \rangle \hookrightarrow T289 \Rightarrow$ $\text{Red}([j, k]) \in \mathbb{Z}$
$\langle \text{Red}([j, k]), n, m \rangle \hookrightarrow T301 \Rightarrow$ $\text{Red}([j, k]) +_{\mathbb{Z}} [n, m] = [n, m] +_{\mathbb{Z}} \text{Red}([j, k])$
$\langle n, m, j, k \rangle \hookrightarrow T298 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 388 (305) $J, K, N, M \in \mathbb{N} \rightarrow [J, K] +_{\mathbb{Z}} [N, M] = \text{Red}([J, K]) +_{\mathbb{Z}} \text{Red}([N, M])$. PROOF:

Suppose_not$(j, k, n, m) \Rightarrow$ $j, k, n, m \in \mathbb{N} \,\&\, [j, k] +_{\mathbb{Z}} [n, m] \neq \text{Red}([j, k]) +_{\mathbb{Z}} \text{Red}([n, m])$
$\langle j, k, n, m \rangle \hookrightarrow T304 \Rightarrow$ $[j, k] +_{\mathbb{Z}} [n, m] = \text{Red}([j, k]) +_{\mathbb{Z}} [n, m]$
$\langle j, k \rangle \hookrightarrow T289 \Rightarrow$ $\text{Red}([j, k]) \in \mathbb{Z}$
$\langle \text{Red}([j, k]) \rangle \hookrightarrow T292 \Rightarrow$ $\text{Red}([j, k]) = \left[\text{Red}^{[1]}([j, k]), \text{Red}^{[2]}([j, k])\right] \,\&\, \text{Red}^{[1]}([j, k]) \in \mathbb{N} \,\&$
$\text{Red}^{[2]}([j, k]) \in \mathbb{N}$
EQUAL $\Rightarrow$ $[j, k] +_{\mathbb{Z}} [n, m] = \left[\text{Red}^{[1]}([j, k]), \text{Red}^{[2]}([j, k])\right] +_{\mathbb{Z}} [n, m]$
$\langle \text{Red}^{[1]}([j, k]), \text{Red}^{[2]}([j, k]), n, m \rangle \hookrightarrow T298 \Rightarrow$ $[j, k] +_{\mathbb{Z}} [n, m] =$

$\left[\mathsf{Red}^{[1]}([j,k]),\mathsf{Red}^{[2]}([j,k])\right] +_{\mathbb{Z}} \mathsf{Red}([n,m])$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is equally easy to show that the sum of two pairs of unsigned integers is the sum of their reductions.

Theorem 389 (306) $j,k,N,M \in \mathbb{N} \to [j,k] +_{\mathbb{Z}} [N,M] = \mathsf{Red}([j,k]) +_{\mathbb{Z}} \mathsf{Red}([N,M])$. PROOF:

Suppose_not$(j,k,n,m) \Rightarrow$ $j,k,n,m \in \mathbb{N}\ \&\ [j,k] +_{\mathbb{Z}} [n,m] \neq \mathsf{Red}([j,k]) +_{\mathbb{Z}} \mathsf{Red}([n,m])$

$\langle j,k\rangle \hookrightarrow T289 \Rightarrow$ $\mathsf{Red}([j,k]) \in \mathbb{Z}$

$\langle j,k,n,m\rangle \hookrightarrow T304 \Rightarrow$ $[j,k] +_{\mathbb{Z}} [n,m] = \mathsf{Red}([j,k]) +_{\mathbb{Z}} [n,m]$

$\langle \mathsf{Red}([j,k]),n,m\rangle \hookrightarrow T299 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next result gives the commutative law for signed integer multiplication.

-- Commutative Law for multiplication

Theorem 390 (307) $N,M \in \mathbb{Z} \to N *_{\mathbb{Z}} M = M *_{\mathbb{Z}} N$. PROOF:

Suppose_not$(n,m) \Rightarrow$ $n,m \in \mathbb{Z}\ \&\ n *_{\mathbb{Z}} m \neq m *_{\mathbb{Z}} n\ \&\ n *_{\mathbb{Z}} m \neq m *_{\mathbb{Z}} n$

-- The proof results immediately by definition of the operators involved and by use of the elementary algebraic properties of unsigned integers.

$\langle n\rangle \hookrightarrow T292 \Rightarrow$ $n = \left[n^{[1]}, n^{[2]}\right]\ \&\ n^{[1]} = \emptyset \vee n^{[2]} = \emptyset\ \&\ n^{[1]}, n^{[2]} \in \mathbb{N}$

$\langle m\rangle \hookrightarrow T292 \Rightarrow$ $m = \left[m^{[1]}, m^{[2]}\right]\ \&\ m^{[1]} = \emptyset \vee m^{[2]} = \emptyset\ \&\ m^{[1]}, m^{[2]} \in \mathbb{N}$

ALGEBRA $\Rightarrow$ $m^{[1]} * n^{[1]} + m^{[2]} * n^{[2]} = n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}\ \&\ n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]} = m^{[1]} * n^{[2]} + n^{[1]} * m^{[2]}$

Use_def$(*_{\mathbb{Z}}) \Rightarrow$ $\mathsf{Red}\left(\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\right]\right) \neq$
$\mathsf{Red}\left(\left[m^{[1]} * n^{[1]} + m^{[2]} * n^{[2]}, m^{[1]} * n^{[2]} + n^{[1]} * m^{[2]}\right]\right)$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next goal is to prove the associative lases of signed integers. We begin by proving the associative law for signed integer addition.

-- Associative Law

Theorem 391 (308) $K,N,M \in \mathbb{Z} \to N +_{\mathbb{Z}} (M +_{\mathbb{Z}} K) = (N +_{\mathbb{Z}} M) +_{\mathbb{Z}} K$. PROOF:

-- Supposing that k, n, m form a counterexample to our assertion, we begin by expanding the inner $+_{\mathbb{Z}}$ operators using their definition, and then using Theorem 282 to eliminate the unnecessary reduction operators Red that appear.

$\mathsf{Suppose\_not}(k,n,m) \Rightarrow \quad k,n,m \in \mathbb{Z} \;\&\; n +_{\mathbb{Z}} (m +_{\mathbb{Z}} k) \neq n +_{\mathbb{Z}} m +_{\mathbb{Z}} k$

$\langle n \rangle \hookrightarrow T292 \Rightarrow \quad n = \left[n^{[1]}, n^{[2]}\right] \;\&\; n^{[1]} = \emptyset \vee n^{[2]} = \emptyset \;\&\; n^{[1]}, n^{[2]} \in \mathbb{N}$

$\langle m \rangle \hookrightarrow T292 \Rightarrow \quad m = \left[m^{[1]}, m^{[2]}\right] \;\&\; m^{[1]} = \emptyset \vee m^{[2]} = \emptyset \;\&\; m^{[1]}, m^{[2]} \in \mathbb{N}$

$\langle k \rangle \hookrightarrow T292 \Rightarrow \quad k = \left[k^{[1]}, k^{[2]}\right] \;\&\; k^{[1]} = \emptyset \vee k^{[2]} = \emptyset \;\&\; k^{[1]}, k^{[2]} \in \mathbb{N}$

$\langle n, m \rangle \hookrightarrow T294 \Rightarrow \quad n +_{\mathbb{Z}} m \in \mathbb{Z}$

$\langle k, n +_{\mathbb{Z}} m \rangle \hookrightarrow T303 \Rightarrow \quad n +_{\mathbb{Z}} m +_{\mathbb{Z}} k = k +_{\mathbb{Z}} (n +_{\mathbb{Z}} m)$

$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad n +_{\mathbb{Z}} (m +_{\mathbb{Z}} k) = n +_{\mathbb{Z}} \mathsf{Red}(\left[m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]}\right])$

$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad k +_{\mathbb{Z}} (n +_{\mathbb{Z}} m) = k +_{\mathbb{Z}} \mathsf{Red}(\left[n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]}\right])$

$\mathsf{ALGEBRA} \Rightarrow \quad m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]} \in \mathbb{N}$

$\langle n, m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]} \rangle \hookrightarrow T299 \Rightarrow \quad n +_{\mathbb{Z}} \mathsf{Red}(\left[m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]}\right]) =$
$n +_{\mathbb{Z}} \left[m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]}\right]$

$\mathsf{ALGEBRA} \Rightarrow \quad n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]} \in \mathbb{N}$

$\langle k, n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]} \rangle \hookrightarrow T299 \Rightarrow \quad k +_{\mathbb{Z}} \mathsf{Red}(\left[n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]}\right]) =$
$k +_{\mathbb{Z}} \left[n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]}\right]$

-- Next we expand the outer $+_{\mathbb{Z}}$ operators using their definition, and note that by the known algebraic properties of positive integer addition it follows that the two resulting expressions are equal, a contradiction which proves our theorem.

$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad n +_{\mathbb{Z}} \left[m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]}\right] = \mathsf{Red}(\left[n^{[1]} + (m^{[1]} + k^{[1]}), n^{[2]} + (m^{[2]} + k^{[2]})\right])$

$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad k +_{\mathbb{Z}} \left[n^{[1]} + m^{[1]}, n^{[2]} + m^{[2]}\right] = \mathsf{Red}(\left[k^{[1]} + (n^{[1]} + m^{[1]}), k^{[2]} + (n^{[2]} + m^{[2]})\right])$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{Red}(\left[n^{[1]} + (m^{[1]} + k^{[1]}), n^{[2]} + (m^{[2]} + k^{[2]})\right]) = \mathsf{Red}(\left[k^{[1]} + (n^{[1]} + m^{[1]}), k^{[2]} + (n^{[2]} + m^{[2]})\right])$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Our next theorem gives the distributive law for (signed) integer multiplication over addition.
-- Distributive Law

Theorem 392 (309) $k,n,m \in \mathbb{Z} \rightarrow n *_{\mathbb{Z}} (m +_{\mathbb{Z}} k) = n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} k$. PROOF:

-- As in all the proofs of the present group, we begin by expanding the operators involved into their definitions, and removing all the redundant reduction operators Red that appear. This is first done for the left-hand side of our assertion.

$\mathsf{Suppose\_not}(k,n,m) \Rightarrow \quad k,n,m \in \mathbb{Z} \,\&\, n *_{\mathbb{Z}} (m +_{\mathbb{Z}} k) \neq n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} k$

$\langle n \rangle \hookrightarrow T292 \Rightarrow \quad n = \left[n^{[1]}, n^{[2]}\right] \,\&\, n^{[1]} = \emptyset \vee n^{[2]} = \emptyset \,\&\, n^{[1]}, n^{[2]} \in \mathbb{N}$

$\langle m \rangle \hookrightarrow T292 \Rightarrow \quad m = \left[m^{[1]}, m^{[2]}\right] \,\&\, m^{[1]} = \emptyset \vee m^{[2]} = \emptyset \,\&\, m^{[1]}, m^{[2]} \in \mathbb{N}$

$\langle k \rangle \hookrightarrow T292 \Rightarrow \quad k = \left[k^{[1]}, k^{[2]}\right] \,\&\, k^{[1]} = \emptyset \vee k^{[2]} = \emptyset \,\&\, k^{[1]}, k^{[2]} \in \mathbb{N}$

$\langle m,k \rangle \hookrightarrow T294 \Rightarrow \quad m +_{\mathbb{Z}} k \in \mathbb{Z}$

$\langle n,m \rangle \hookrightarrow T294 \Rightarrow \quad n *_{\mathbb{Z}} m \in \mathbb{Z}$

$\langle n,k \rangle \hookrightarrow T294 \Rightarrow \quad n *_{\mathbb{Z}} k \in \mathbb{Z}$

$\mathsf{ALGEBRA} \Rightarrow \quad m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]} \in \mathbb{N}$

$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad n *_{\mathbb{Z}} (m +_{\mathbb{Z}} k) = n *_{\mathbb{Z}} \mathsf{Red}\left(\left[m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]}\right]\right)$

$\langle n, m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]} \rangle \hookrightarrow T300 \Rightarrow \quad n *_{\mathbb{Z}} (m +_{\mathbb{Z}} k) = n *_{\mathbb{Z}} \left[m^{[1]} + k^{[1]}, m^{[2]} + k^{[2]}\right]$

$\mathsf{Use\_def}(*_{\mathbb{Z}}) \Rightarrow \quad n *_{\mathbb{Z}} (m +_{\mathbb{Z}} k) = \mathsf{Red}\left(\left[n^{[1]} * (m^{[1]} + k^{[1]}) + n^{[2]} * (m^{[2]} + k^{[2]}), n^{[1]} * (m^{[2]} + k^{[2]}) + (m^{[1]} + k^{[1]}) * n^{[2]}\right]\right)$

-- Next we expand and simplify the right-hand side of our assertion in the same way.

$\mathsf{Use\_def}(*_{\mathbb{Z}}) \Rightarrow$

$n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} k =$

$\mathsf{Red}\left(\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\right]\right) +_{\mathbb{Z}} \mathsf{Red}\left(\left[n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}, n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]}\right]\right)$

$\mathsf{ALGEBRA} \Rightarrow \quad n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]} \in \mathbb{N} \,\&$
$n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}, n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]} \in \mathbb{N}$

$\langle n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}, n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}, n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]} \rangle \hookrightarrow T305 \Rightarrow \quad Stat1:$

$n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} k =$

$\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\right] +_{\mathbb{Z}} \left[n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}, n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]}\right]$

-- We complete our expansion of the left side of our assertion by expanding the central signed addition operator which appears in it.

$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad Stat2:$

$\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\right] +_{\mathbb{Z}} \left[n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}, n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]}\right] =$

$\mathsf{Red}\left(\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]} + (n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}), n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]} + (n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]})\right]\right)$

$\langle Stat1, Stat2, * \rangle\ \mathsf{ELEM} \Rightarrow \quad n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} k =$

$\mathsf{Red}\left(\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]} + (n^{[1]} * k^{[1]} + n^{[2]} * k^{[2]}), n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]} + (n^{[1]} * k^{[2]} + k^{[1]} * n^{[2]})\right]\right)$

-- Now rearraangement of thems using the algebraic properties of signed integers brings us to the desired result.

$\mathsf{ALGEBRA} \Rightarrow \quad n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} k = \mathsf{Red}\left(\left[n^{[1]} * (m^{[1]} + k^{[1]}) + n^{[2]} * (m^{[2]} + k^{[2]}), n^{[1]} * (m^{[2]} + k^{[2]}) + (m^{[1]} + k^{[1]}) * n^{[2]}\right]\right)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following easy lemma shows that pairs of the form $[n, \emptyset]$ are invariant under the reduction operator 'Red'.

Theorem 393 (310) $N \in \mathbb{N} \to \mathrm{Red}([N, \emptyset]) = [N, \emptyset]$. PROOF:

-- The result follows trivially from the definition of reduction, since for all such pairs the minimum of the two components is clearly 0.

Suppose_not(n) $\Rightarrow$ *Stat0* : $n \in \mathbb{N}\ \&\ \mathrm{Red}([n, \emptyset]) \neq [n, \emptyset]$
Use_def(Red) $\Rightarrow$ $\mathrm{Red}([n, \emptyset]) = [n - n \cap \emptyset, \emptyset - n \cap \emptyset]$
TELEM $\Rightarrow$ $n \cap \emptyset = \emptyset$
EQUAL $\Rightarrow$ *Stat1* : $\mathrm{Red}([n, \emptyset]) = [n - \emptyset, \emptyset - \emptyset]$
$\langle n \rangle \hookrightarrow T230(\langle Stat1 \rangle) \Rightarrow$ $\mathrm{Red}([n, \emptyset]) = [\#n, \emptyset - \emptyset]$
$\langle n \rangle \hookrightarrow T180(\langle Stat0 \rangle) \Rightarrow$ $n = \#n$
EQUAL $\Rightarrow$ $\mathrm{Red}([n, \emptyset]) = [n, \emptyset - \emptyset]$
$\langle \emptyset \rangle \hookrightarrow T230([]) \Rightarrow$ $\emptyset - \emptyset = \#\emptyset$
T161 $\Rightarrow$ $\mathrm{Card}(\emptyset)$
$\langle \emptyset \rangle \hookrightarrow T138([]) \Rightarrow$ $\emptyset = \#\emptyset$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next result tells us that the embedding $n \mapsto [n, \emptyset]$ of integers into signed integers is an algebraic isomorphism.

-- Embedding of Integers in Signed Integers

Theorem 394 (311) $N, M \in \mathbb{N} \to$
$([N + M, \emptyset] = [N, \emptyset] +_{\mathbb{Z}} [M, \emptyset]\ \&\ [N * M, \emptyset] = [N, \emptyset] *_{\mathbb{Z}} [M, \emptyset])\ \&\ (N \supseteq M \to [N, \emptyset] -_{\mathbb{Z}} [M, \emptyset] = [N - M, \emptyset])$. PROOF:

Suppose_not(n, m) $\Rightarrow$
$n, m \in \mathbb{N}\ \&$
$[n + m, \emptyset] \neq [n, \emptyset] +_{\mathbb{Z}} [m, \emptyset] \vee [n * m, \emptyset] \neq [n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] \vee (n \supseteq m\ \&\ [n, \emptyset] -_{\mathbb{Z}} [m, \emptyset] \neq [n - m, \emptyset])$

-- For signed addition and multiplication, our assertion follows immediately from their definitions.

Use_def($+_{\mathbb{Z}}$) $\Rightarrow$ $[n, \emptyset] +_{\mathbb{Z}} [m, \emptyset] = \mathrm{Red}([n + m, \emptyset + \emptyset])$
ALGEBRA $\Rightarrow$ $[n, \emptyset] +_{\mathbb{Z}} [m, \emptyset] = \mathrm{Red}([n + m, \emptyset])\ \&\ n + m \in \mathbb{N}$
$\langle n + m \rangle \hookrightarrow T310 \Rightarrow$ $[n, \emptyset] +_{\mathbb{Z}} [m, \emptyset] = [n + m, \emptyset]$
Use_def($*_{\mathbb{Z}}$) $\Rightarrow$ $[n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] = \mathrm{Red}([n * m + \emptyset * \emptyset, n * \emptyset + m * \emptyset])$

ALGEBRA ⇒ $n * m + \emptyset * \emptyset = n * m \;\&\; n * \emptyset + m * \emptyset = \emptyset \;\&\; n * m \in \mathbb{N}$
EQUAL ⇒ $[n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] = \mathrm{Red}([n * m, \emptyset])$
$\langle n * m \rangle \hookrightarrow T310$ ⇒ $[n * m, \emptyset] = [n, \emptyset] *_{\mathbb{Z}} [m, \emptyset]$

-- Thus only the final clause of our theorem can be false. But in this case our assertion follows immediately from the definition and fact that the unsigned integer difference $m - m$ is zero.

ELEM ⇒ $n \supseteq m \;\&\; [n, \emptyset] -_{\mathbb{Z}} [m, \emptyset] \neq [n - m, \emptyset]$
Use_def$(-_{\mathbb{Z}})$ ⇒ $[n, \emptyset] -_{\mathbb{Z}} [m, \emptyset] = \mathrm{Red}([\emptyset + n, m + \emptyset])$
ALGEBRA ⇒ $\emptyset + n = n \;\&\; m + \emptyset = m$
EQUAL ⇒ $[n - m, \emptyset] \neq \mathrm{Red}([n, m])$
Use_def(Red) ⇒ $[n - m, \emptyset] \neq [n - n \cap m, m - n \cap m]$
ELEM ⇒ $n \cap m = m$
EQUAL ⇒ $\mathit{Stat1}: [n - m, \emptyset] \neq [n - m, m - m]$
$\langle \mathit{Stat1} \rangle$ ELEM ⇒ $m - m \neq \emptyset$
$\langle m \rangle \hookrightarrow T229$ ⇒ false; Discharge ⇒ QED

-- The trivial lemma which now follows states that sign-reversal for signed integers corresponds to interchange of their unsigned integer components.

Theorem 395 (312) $N, M \in \mathbb{N} \rightarrow \mathrm{Rev}_{\mathbb{Z}}(\mathrm{Red}([M, N])) = \mathrm{Red}([N, M])$. PROOF:

Suppose_not(n, m) ⇒ $n, m \in \mathbb{N} \;\&\; \mathrm{Rev}_{\mathbb{Z}}(\mathrm{Red}([m, n])) \neq \mathrm{Red}([n, m])$
Use_def(Red) ⇒ $\mathrm{Rev}_{\mathbb{Z}}([m - m \cap n, n - m \cap n]) \neq [n - m \cap n, m - m \cap n]$
Use_def$(\mathrm{Rev}_{\mathbb{Z}})$ ⇒ false; Discharge ⇒ QED

-- Next we show that the negative of the product of two signed integers is the product of the first by the negative of the second.

Theorem 396 (313) $N, M \in \mathbb{Z} \rightarrow N *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(M) = \mathrm{Rev}_{\mathbb{Z}}(N *_{\mathbb{Z}} M)$. PROOF:

Suppose_not(n, m) ⇒ $n, m \in \mathbb{Z} \;\&\; n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m) \neq \mathrm{Rev}_{\mathbb{Z}}(n *_{\mathbb{Z}} m)$

-- For let n, m form a counterexample to our assertion, and use the definitions of the operators involved to express $n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m)$ as the reduction of a pair formed algebraically from the components of n and m.

$\langle n \rangle \hookrightarrow T292$ ⇒ $n = [n^{[1]}, n^{[2]}] \;\&\; n^{[1]}, n^{[2]} \in \mathbb{N}$
$\langle m \rangle \hookrightarrow T292$ ⇒ $m = [m^{[1]}, m^{[2]}] \;\&\; m^{[1]}, m^{[2]} \in \mathbb{N}$

Use_def $(*_{\mathbb{Z}}) \Rightarrow$ $n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m) = \mathrm{Red}\left(\left[n^{[1]} * \mathrm{Rev}_{\mathbb{Z}}{}^{[1]}(m) + n^{[2]} * \mathrm{Rev}_{\mathbb{Z}}{}^{[2]}(m), n^{[1]} * \mathrm{Rev}_{\mathbb{Z}}{}^{[2]}(m) + \mathrm{Rev}_{\mathbb{Z}}{}^{[1]}(m) * n^{[2]}\right]\right)$
Use_def $(\mathrm{Rev}_{\mathbb{Z}}) \Rightarrow$ $Stat1: \mathrm{Rev}_{\mathbb{Z}}(m) = \left[m^{[2]}, m^{[1]}\right]$
$\langle Stat1 \rangle$ ELEM $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}{}^{[1]}(m) = m^{[2]}$
$\langle Stat1, Stat1 \rangle$ ELEM $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}{}^{[2]}(m) = m^{[1]}$
EQUAL $\Rightarrow$ $n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m) = \mathrm{Red}(\left[n^{[1]} * m^{[2]} + n^{[2]} * m^{[1]}, n^{[1]} * m^{[1]} + m^{[2]} * n^{[2]}\right])$

-- Now use these definitions once more to express $\mathrm{Rev}_{\mathbb{Z}}(n *_{\mathbb{Z}} m)$ in the same way. Then a final bit of algebra on the positive-integer components of the two resulting pairs shows that they are equal, ad so proves our theorem.

Use_def $(*_{\mathbb{Z}}) \Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(n *_{\mathbb{Z}} m) = \mathrm{Rev}_{\mathbb{Z}}\left(\mathrm{Red}(\left[n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}\right])\right)$
ALGEBRA $\Rightarrow$ $n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}, n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]} \in \mathbb{N}$
$\langle n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}, n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]} \rangle \hookrightarrow T312 \Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(n *_{\mathbb{Z}} m) =$
$\mathrm{Red}(\left[n^{[1]} * m^{[2]} + m^{[1]} * n^{[2]}, n^{[1]} * m^{[1]} + n^{[2]} * m^{[2]}\right])$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem asserts that the reverse of any signed integer n is also a signed integer, and that the sum of n and its reverse is zero.

Theorem 397 (314) $N \in \mathbb{Z} \to \mathrm{Rev}_{\mathbb{Z}}(N) \in \mathbb{Z} \ \& \ \mathrm{Rev}_{\mathbb{Z}}(N) +_{\mathbb{Z}} N = [\emptyset, \emptyset] \ \& \ \mathrm{Rev}_{\mathbb{Z}}(\mathrm{Rev}_{\mathbb{Z}}(N)) = N$. PROOF:

Suppose_not(n) $\Rightarrow$ $Stat0: n \in \mathbb{Z} \ \& \ \mathrm{Rev}_{\mathbb{Z}}(n) \notin \mathbb{Z} \lor \mathrm{Rev}_{\mathbb{Z}}(n) +_{\mathbb{Z}} n \neq [\emptyset, \emptyset] \lor \mathrm{Rev}_{\mathbb{Z}}(\mathrm{Rev}_{\mathbb{Z}}(n)) \neq n$

-- That $\mathrm{Rev}_{\mathbb{Z}}(n)$ is a signed integer follows trivially fro its definition.

$\langle n \rangle \hookrightarrow T292 \Rightarrow$ $Stat1: n = \left[n^{[1]}, n^{[2]}\right] \ \& \ n^{[1]} = \emptyset \lor n^{[2]} = \emptyset \ \& \ n^{[1]}, n^{[2]} \in \mathbb{N}$
ALGEBRA $\Rightarrow$ $n^{[1]} + n^{[2]} \in \mathbb{N}$
Use_def $(\mathrm{Rev}_{\mathbb{Z}}) \Rightarrow$ $Stat2: \mathrm{Rev}_{\mathbb{Z}}(n) = \left[n^{[2]}, n^{[1]}\right]$
$\langle Stat2 \rangle$ ELEM $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}{}^{[1]}(n) = n^{[2]} \ \& \ \mathrm{Rev}_{\mathbb{Z}}{}^{[2]}(n) = n^{[1]}$
Suppose $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(n) \notin \mathbb{Z}$
Use_def $(\mathbb{Z}) \Rightarrow$ $Stat3: \left[n^{[2]}, n^{[1]}\right] \notin \{[x, y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \lor y = \emptyset\}$
$\langle n^{[2]}, n^{[1]} \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat4: \mathrm{Rev}_{\mathbb{Z}}(n) \in \mathbb{Z}$

-- Moreover, $\mathrm{Rev}_{\mathbb{Z}}(n) +_{\mathbb{Z}} n$ is $\mathrm{Red}(\left[n^{[1]} + n^{[2]}, n^{[1]} + n^{[2]}\right])$, and so reduces to $[\emptyset, \emptyset]$, completing our proof

EQUAL $\Rightarrow$ $Stat5: \mathrm{Rev}_{\mathbb{Z}}(\mathrm{Rev}_{\mathbb{Z}}(n)) = \mathrm{Rev}_{\mathbb{Z}}(\left[n^{[2]}, n^{[1]}\right])$
Use_def $(\mathrm{Rev}_{\mathbb{Z}}) \Rightarrow$ $Stat6: \mathrm{Rev}_{\mathbb{Z}}(\left[n^{[2]}, n^{[1]}\right]) = \left[\left[n^{[2]}, n^{[1]}\right]^{[2]}, \left[n^{[2]}, n^{[1]}\right]^{[1]}\right]$

$\langle \mathit{Stat6} \rangle$ ELEM $\Rightarrow$ $\quad \mathit{Stat7}:\ \mathsf{Rev}_{\mathbb{Z}}([\mathsf{n}^{[2]},\mathsf{n}^{[1]}]) = [\mathsf{n}^{[1]},\mathsf{n}^{[2]}]$
$\langle \mathit{Stat4, Stat5, Stat7, Stat1}, * \rangle$ ELEM $\Rightarrow$ $\quad \mathit{Stat7a}:\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \in \mathbb{Z}\ \&\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n})) = \mathsf{n}$
$\langle \mathit{Stat0, Stat4, Stat7a}, * \rangle$ ELEM $\Rightarrow$ $\quad \mathit{Stat5a}:\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{n} \neq [\emptyset,\emptyset]$
Use_def$(+_{\mathbb{Z}})$ $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{n} = \mathsf{Red}\left(\left[\mathsf{Rev}_{\mathbb{Z}}{}^{[1]}(\mathsf{n}) + \mathsf{n}^{[1]}, \mathsf{Rev}_{\mathbb{Z}}{}^{[2]}(\mathsf{n}) + \mathsf{n}^{[2]}\right]\right)$
EQUAL $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{n} = \mathsf{Red}([\mathsf{n}^{[2]} + \mathsf{n}^{[1]}, \mathsf{n}^{[1]} + \mathsf{n}^{[2]}])$
ALGEBRA $\Rightarrow$ $\quad \mathit{Stat8}:\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{n} = \mathsf{Red}([\mathsf{n}^{[1]} + \mathsf{n}^{[2]}, \mathsf{n}^{[1]} + \mathsf{n}^{[2]}])$
$\langle \mathsf{n}^{[1]} + \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T296}(\langle \cap \rangle)$ $\Rightarrow$ $\quad \mathit{Stat9}:\ \mathsf{Red}([\mathsf{n}^{[1]} + \mathsf{n}^{[2]}, \mathsf{n}^{[1]} + \mathsf{n}^{[2]}]) = [\emptyset,\emptyset]$
$\langle \mathit{Stat5a, Stat8, Stat9}, * \rangle$ ELEM $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- Using commutativity, it is trivial to generalize Theorem 313 by showing that the negative of the product of two signed integers is the product of either by the negative of the other.

-- Inversion Lemma

Theorem 398 (315) $\mathsf{N},\mathsf{M} \in \mathbb{Z} \rightarrow \mathsf{Rev}_{\mathbb{Z}}(\mathsf{N} *_{\mathbb{Z}} \mathsf{M}) = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{N}) *_{\mathbb{Z}} \mathsf{M}\ \&\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{N} *_{\mathbb{Z}} \mathsf{M}) = \mathsf{N} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{M})$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\quad \mathsf{n},\mathsf{m} \in \mathbb{Z}\ \&\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}) \neq \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} \mathsf{m} \vee \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}) \neq \mathsf{n} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m})$
$\langle \mathsf{n},\mathsf{m} \rangle \hookrightarrow \mathit{T313}$ $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}) \neq \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} \mathsf{m}$
$\langle \mathsf{m},\mathsf{n} \rangle \hookrightarrow \mathit{T313}$ $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m} *_{\mathbb{Z}} \mathsf{n}) = \mathsf{m} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n})$
$\langle \mathsf{m},\mathsf{n} \rangle \hookrightarrow \mathit{T307}$ $\Rightarrow$ $\quad \mathsf{n} *_{\mathbb{Z}} \mathsf{m} = \mathsf{m} *_{\mathbb{Z}} \mathsf{n}$
EQUAL $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}) = \mathsf{m} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow \mathit{T314}$ $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \in \mathbb{Z}$
$\langle \mathsf{m}, \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \rangle \hookrightarrow \mathit{T307}$ $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- Next we not that for any signed integer n the 'double negative' $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}))$ is n. The proof follows from thae fact that a double reversal of any pair yields the original pair.

-- inversion Lemma II

Theorem 399 (316) $\mathsf{N} \in \mathbb{Z} \rightarrow \mathsf{Rev}_{\mathbb{Z}}(\mathsf{Rev}_{\mathbb{Z}}(\mathsf{N})) = \mathsf{N}$. PROOF:

Suppose_not(n) $\Rightarrow$ $\quad \mathsf{n} \in \mathbb{Z}\ \&\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n})) \neq \mathsf{n}$
$\langle \mathsf{n} \rangle \hookrightarrow \mathit{T292}$ $\Rightarrow$ $\quad \mathsf{n} = [\mathsf{n}^{[1]},\mathsf{n}^{[2]}]\ \&\ \mathsf{n}^{[1]} = \emptyset \vee \mathsf{n}^{[2]} = \emptyset\ \&\ \mathsf{n}^{[1]},\mathsf{n}^{[2]} \in \mathbb{N}$
Use_def$(\mathsf{Rev}_{\mathbb{Z}})$ $\Rightarrow$ $\quad \mathit{Stat1}:\ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) = [\mathsf{n}^{[2]},\mathsf{n}^{[1]}]$
EQUAL $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}([\mathsf{n}^{[2]},\mathsf{n}^{[1]}]) \neq \mathsf{n}$
Use_def$(\mathsf{Rev}_{\mathbb{Z}})$ $\Rightarrow$ $\quad \left[[\mathsf{n}^{[2]},\mathsf{n}^{[1]}]^{[2]}, [\mathsf{n}^{[2]},\mathsf{n}^{[1]}]^{[1]}\right] \neq \mathsf{n}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next aim is to prove the associative law for signed integer multiplication. We approach this via a sequence of steps: first the case of three non-negative signed integers is considered; next the case of one general signed integer and two non-negative signed integers; next the case one two signed integers and one non-negative signed integer; and then finally the general case. Our first lemma states the associativity rule for multiplication of three non-negative signed integers.

-- Associativity Lemma

Theorem 400 (317) $N, M, K \in \mathbb{N} \rightarrow [N, \emptyset] *_{\mathbb{Z}} ([M, \emptyset] *_{\mathbb{Z}} [K, \emptyset]) = ([N, \emptyset] *_{\mathbb{Z}} [M, \emptyset]) *_{\mathbb{Z}} [K, \emptyset]$. PROOF:

Suppose_not$(n, m, k) \Rightarrow$ $n, m, k \in \mathbb{N}$ & $[n, \emptyset] *_{\mathbb{Z}} ([m, \emptyset] *_{\mathbb{Z}} [k, \emptyset]) \neq [n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] *_{\mathbb{Z}} [k, \emptyset]$

-- In this case the proof follows directly from the definition of signed integer multiplication and the associative law of unsigned integer multiplication.

$\langle m, k \rangle \hookrightarrow T311 \Rightarrow$ $[m, \emptyset] *_{\mathbb{Z}} [k, \emptyset] = [m * k, \emptyset]$

$\langle n, m \rangle \hookrightarrow T311 \Rightarrow$ $[n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] = [n * m, \emptyset]$

ALGEBRA $\Rightarrow$ $m * k, n * m \in \mathbb{N}$

EQUAL $\Rightarrow$

$[n, \emptyset] *_{\mathbb{Z}} ([m, \emptyset] *_{\mathbb{Z}} [k, \emptyset]) = [n, \emptyset] *_{\mathbb{Z}} [m * k, \emptyset]$ &

$[n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] *_{\mathbb{Z}} [k, \emptyset] = [n * m, \emptyset] *_{\mathbb{Z}} [k, \emptyset]$

$\langle n * m, k \rangle \hookrightarrow T311 \Rightarrow$ $[n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] *_{\mathbb{Z}} [k, \emptyset] = [n * m * k, \emptyset]$

$\langle n, m * k \rangle \hookrightarrow T311 \Rightarrow$ $[n, \emptyset] *_{\mathbb{Z}} ([m, \emptyset] *_{\mathbb{Z}} [k, \emptyset]) = [n * (m * k), \emptyset]$

ALGEBRA $\Rightarrow$ $n * (m * k) = (n * m) * k$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove the associativity rule for multiplication of two non-negative signed integers and one general signed integer.

-- Associativity Lemma

Theorem 401 (318) $K \in \mathbb{Z}$ & $N, M \in \mathbb{N} \rightarrow [N, \emptyset] *_{\mathbb{Z}} ([M, \emptyset] *_{\mathbb{Z}} K) = ([N, \emptyset] *_{\mathbb{Z}} [M, \emptyset]) *_{\mathbb{Z}} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow$ $k \in \mathbb{Z}$ & $n, m \in \mathbb{N}$ & $[n, \emptyset] *_{\mathbb{Z}} ([m, \emptyset] *_{\mathbb{Z}} k) \neq [n, \emptyset] *_{\mathbb{Z}} [m, \emptyset] *_{\mathbb{Z}} k$

-- Consider a counterexample k, n, m. The case in which $k = \left[k^{[1]}, \emptyset\right]$ is nonnegative is covered by Theorem 317, so we have only to consider the case in which $k = \left[\emptyset, k^{[2]}\right]$ is negative, and thus $\mathrm{Rev}_{\mathbb{Z}}(k) = \left[k^{[2]}, \emptyset\right]$ is non-negative.

$\langle \mathsf{k} \rangle \hookrightarrow T292 \Rightarrow \quad \mathsf{k} = \left[\mathsf{k}^{[1]}, \emptyset\right] \vee \mathsf{k} = \left[\emptyset, \mathsf{k}^{[2]}\right] \;\&\; \mathsf{k}^{[1]}, \mathsf{k}^{[2]} \in \mathbb{N}$

Suppose $\Rightarrow \quad \mathsf{k} = \left[\mathsf{k}^{[1]}, \emptyset\right]$

EQUAL $\Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \left[\mathsf{k}^{[1]}, \emptyset\right]) \neq [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \left[\mathsf{k}^{[1]}, \emptyset\right]$

$\langle \mathsf{n}, \mathsf{m}, \mathsf{k}^{[1]} \rangle \hookrightarrow T317 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathsf{k} = \left[\emptyset, \mathsf{k}^{[2]}\right]$

Use_def $(\mathsf{Rev}_{\mathbb{Z}}) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k}) = \left[\mathsf{k}^{[2]}, \emptyset\right]$

$T182 \Rightarrow \quad \emptyset \in \mathbb{N}$

Suppose $\Rightarrow \quad [\mathsf{n}, \emptyset] \notin \mathbb{Z}$

Use_def $(\mathbb{Z}) \Rightarrow \quad Stat1: \; [\mathsf{n}, \emptyset] \notin \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$

$\langle \mathsf{n}, \emptyset \rangle \hookrightarrow Stat1 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad [\mathsf{n}, \emptyset] \in \mathbb{Z}$

Suppose $\Rightarrow \quad [\mathsf{m}, \emptyset] \notin \mathbb{Z}$

Use_def $(\mathbb{Z}) \Rightarrow \quad Stat2: \; [\mathsf{m}, \emptyset] \notin \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$

$\langle \mathsf{m}, \emptyset \rangle \hookrightarrow Stat2 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad [\mathsf{m}, \emptyset] \in \mathbb{Z}$

Suppose $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k}) \notin \mathbb{Z}$

Use_def $(\mathbb{Z}) \Rightarrow \quad Stat3: \; \left[\mathsf{k}^{[2]}, \emptyset\right] \notin \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$

$\langle \mathsf{k}^{[2]}, \emptyset \rangle \hookrightarrow Stat3 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k}) \in \mathbb{Z}$

$\langle [\mathsf{n}, \emptyset], [\mathsf{m}, \emptyset] \rangle \hookrightarrow T294 \Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] \in \mathbb{Z}$

$\langle [\mathsf{m}, \emptyset], \mathsf{k} \rangle \hookrightarrow T294 \Rightarrow \quad [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k} \in \mathbb{Z}$

-- In this case associativity applies to the product $[\mathsf{n}, \emptyset] *_{\mathbb{Z}} \big([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k})\big)$, and so several uses of theorems 216 and of the associative law for unsigned integers brings us to the desired result.

$\langle \mathsf{n}, \mathsf{m}, \mathsf{k}^{[2]} \rangle \hookrightarrow T317 \Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \left[\mathsf{k}^{[2]}, \emptyset\right]) =$
$\quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \left[\mathsf{k}^{[2]}, \emptyset\right]$

EQUAL $\Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k})) = ([\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset]) *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k})$

$\langle [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset], \mathsf{k} \rangle \hookrightarrow T315 \Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k}) =$
$\quad \mathsf{Rev}_{\mathbb{Z}}([\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})$

$\langle [\mathsf{m}, \emptyset], \mathsf{k} \rangle \hookrightarrow T315 \Rightarrow \quad [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k}) = \mathsf{Rev}_{\mathbb{Z}}([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})$

$\langle [\mathsf{n}, \emptyset], [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k} \rangle \hookrightarrow T315 \Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k}) =$
$\quad \mathsf{Rev}_{\mathbb{Z}}\big([\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})\big)$

EQUAL $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}\Big(\mathsf{Rev}_{\mathbb{Z}}\big([\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})\big)\Big) = \mathsf{Rev}_{\mathbb{Z}}\big(\mathsf{Rev}_{\mathbb{Z}}([\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})\big)$

$\langle [\mathsf{n}, \emptyset], [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k} \rangle \hookrightarrow T294 \Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k}) \in \mathbb{Z}$

$\langle [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset], \mathsf{k} \rangle \hookrightarrow T294 \Rightarrow \quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} [\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k} \in \mathbb{Z}$

$\langle [\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k}) \rangle \hookrightarrow T314 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}\Big(\mathsf{Rev}_{\mathbb{Z}}\big([\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})\big)\Big) =$
$\quad [\mathsf{n}, \emptyset] *_{\mathbb{Z}} ([\mathsf{m}, \emptyset] *_{\mathbb{Z}} \mathsf{k})$

$\langle [n,\emptyset] *_{\mathbb{Z}} [m,\emptyset] *_{\mathbb{Z}} k \rangle \hookrightarrow T314 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove the associativity rule for multiplication of a non-negative signed integer and two general signed integers.

-- Associativity Lemma

Theorem 402 (319) $K \in \mathbb{Z} \,\&\, N \in \mathbb{N} \,\&\, M \in \mathbb{Z} \rightarrow [N,\emptyset] *_{\mathbb{Z}} (M *_{\mathbb{Z}} K) = ([N,\emptyset] *_{\mathbb{Z}} M) *_{\mathbb{Z}} K$. PROOF:

Suppose_not$(k,n,m) \Rightarrow$ $k \in \mathbb{Z} \,\&\, n \in \mathbb{N} \,\&\, m \in \mathbb{Z} \,\&\, [n,\emptyset] *_{\mathbb{Z}} (m *_{\mathbb{Z}} k) \neq [n,\emptyset] *_{\mathbb{Z}} m *_{\mathbb{Z}} k$

-- Consider a counterexample k, n, m. The case in which $m = [m^{[1]},\emptyset]$ is nonnegative is covered by Theorem 317, so we have only to consider the case in which $m = [\emptyset, m^{[2]}]$ is negative, and thus $\mathrm{Rev}_{\mathbb{Z}}(m) = [m^{[2]},\emptyset]$ is non-negative.

$T182 \Rightarrow$ $\emptyset \in \mathbb{N}$
$\langle m \rangle \hookrightarrow T292 \Rightarrow$ $m = [m^{[1]},\emptyset] \vee m = [\emptyset, m^{[2]}] \,\&\, m^{[1]}, m^{[2]} \in \mathbb{N}$
Suppose $\Rightarrow$ $m = [m^{[1]},\emptyset]$
EQUAL $\Rightarrow$ $[n,\emptyset] *_{\mathbb{Z}} ([m^{[1]},\emptyset] *_{\mathbb{Z}} k) \neq [n,\emptyset] *_{\mathbb{Z}} [m^{[1]},\emptyset] *_{\mathbb{Z}} k$
$\langle k, n, m^{[1]} \rangle \hookrightarrow T318 \Rightarrow$ false; Discharge $\Rightarrow$ $m = [\emptyset, m^{[2]}]$
Use_def$(\mathrm{Rev}_{\mathbb{Z}}) \Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(m) = [m^{[2]},\emptyset]$
Suppose $\Rightarrow$ $[n,\emptyset] \notin \mathbb{Z}$
Use_def$(\mathbb{Z}) \Rightarrow$ $Stat1: [n,\emptyset] \notin \{[x,y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle n, \emptyset \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $[n,\emptyset] \in \mathbb{Z}$
Suppose $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(m) \notin \mathbb{Z}$
Use_def$(\mathbb{Z}) \Rightarrow$ $Stat2: [m^{[2]},\emptyset] \notin \{[x,y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle m^{[2]}, \emptyset \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(m) \in \mathbb{Z}$
$\langle [n,\emptyset], m \rangle \hookrightarrow T294 \Rightarrow$ $[n,\emptyset] *_{\mathbb{Z}} m \in \mathbb{Z}$
$\langle m, k \rangle \hookrightarrow T294 \Rightarrow$ $m *_{\mathbb{Z}} k \in \mathbb{Z}$

-- In this case associativity applies to the product $[N,\emptyset] *_{\mathbb{Z}} ([m^{[2]},\emptyset] *_{\mathbb{Z}} k)$, leading directly via manipulation of the reversal operator $\mathrm{Rev}_{\mathbb{Z}}$ to the formula we need.

$\langle k, n, m^{[2]} \rangle \hookrightarrow T318 \Rightarrow$ $[n,\emptyset] *_{\mathbb{Z}} ([m^{[2]},\emptyset] *_{\mathbb{Z}} k) = ([n,\emptyset] *_{\mathbb{Z}} [m^{[2]},\emptyset]) *_{\mathbb{Z}} k$
EQUAL $\Rightarrow$ $[n,\emptyset] *_{\mathbb{Z}} (\mathrm{Rev}_{\mathbb{Z}}(m) *_{\mathbb{Z}} k) = ([n,\emptyset] *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m)) *_{\mathbb{Z}} k$
$\langle [n,\emptyset], m \rangle \hookrightarrow T315 \Rightarrow$ $[n,\emptyset] *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m) = \mathrm{Rev}_{\mathbb{Z}}([n,\emptyset] *_{\mathbb{Z}} m)$
$\langle m, k \rangle \hookrightarrow T315 \Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}(m) *_{\mathbb{Z}} k = \mathrm{Rev}_{\mathbb{Z}}(m *_{\mathbb{Z}} k)$
EQUAL $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}([n,\emptyset] *_{\mathbb{Z}} m) *_{\mathbb{Z}} k = [n,\emptyset] *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m *_{\mathbb{Z}} k)$
$\langle [n,\emptyset] *_{\mathbb{Z}} m, k \rangle \hookrightarrow T315 \Rightarrow$ $\mathrm{Rev}_{\mathbb{Z}}([n,\emptyset] *_{\mathbb{Z}} m) *_{\mathbb{Z}} k = \mathrm{Rev}_{\mathbb{Z}}([n,\emptyset] *_{\mathbb{Z}} m *_{\mathbb{Z}} k)$
$\langle [n,\emptyset], m *_{\mathbb{Z}} k \rangle \hookrightarrow T315 \Rightarrow$ $[n,\emptyset] *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(m *_{\mathbb{Z}} k) = \mathrm{Rev}_{\mathbb{Z}}\left([n,\emptyset] *_{\mathbb{Z}} (m *_{\mathbb{Z}} k)\right)$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{Rev}_{\mathbb{Z}}\left([\mathsf{n},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)\right) = \mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{Rev}_{\mathbb{Z}}([\mathsf{n},\emptyset] *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)$

$\langle [\mathsf{n},\emptyset], \mathsf{m} *_{\mathbb{Z}} \mathsf{k}\rangle \hookrightarrow T294 \Rightarrow \quad [\mathsf{n},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) \in \mathbb{Z}$

$\langle [\mathsf{n},\emptyset] *_{\mathbb{Z}} \mathsf{m}, \mathsf{k}\rangle \hookrightarrow T294 \Rightarrow \quad [\mathsf{n},\emptyset] *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k} \in \mathbb{Z}$

$\langle [\mathsf{n},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\rangle \hookrightarrow T314 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{Rev}_{\mathbb{Z}}\left([\mathsf{n},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)\right) = [\mathsf{n},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})$

$\langle [\mathsf{n},\emptyset] *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k}\rangle \hookrightarrow T314 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Finally we prove the associativity rule for multiplication of signed integers in the general case.

-- Associative Law

Theorem 403 (320) $\mathsf{K}, \mathsf{N}, \mathsf{M} \in \mathbb{Z} \rightarrow \mathsf{N} *_{\mathbb{Z}} (\mathsf{M} *_{\mathbb{Z}} \mathsf{K}) = (\mathsf{N} *_{\mathbb{Z}} \mathsf{M}) *_{\mathbb{Z}} \mathsf{K}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{k},\mathsf{n},\mathsf{m}) \Rightarrow \quad \mathsf{k},\mathsf{n},\mathsf{m} \in \mathbb{Z} \;\&\; \mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) \neq \mathsf{n} *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k}$

-- Consider a counterexample $\mathsf{k},\mathsf{n},\mathsf{m}$. The case in which $\mathsf{n} = [\mathsf{n}^{[1]},\emptyset]$ is nonnegative is covered by Theorem 317, so we have only to consider the case in which $\mathsf{n} = [\emptyset, \mathsf{n}^{[2]}]$ is negative, and thus $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) = [\mathsf{n}^{[2]},\emptyset]$ is non-negative.

$\langle \mathsf{n}\rangle \hookrightarrow T292 \Rightarrow \quad \mathsf{n} = [\mathsf{n}^{[1]},\emptyset] \vee \mathsf{n} = [\emptyset,\mathsf{n}^{[2]}] \;\&\; \mathsf{n}^{[1]},\mathsf{n}^{[2]} \in \mathbb{N}$

$T182 \Rightarrow \quad \emptyset \in \mathbb{N}$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{n} = [\mathsf{n}^{[1]},\emptyset]$

$\mathsf{EQUAL} \Rightarrow \quad [\mathsf{n}^{[1]},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) \neq [\mathsf{n}^{[1]},\emptyset] *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k}$

$\langle \mathsf{k},\mathsf{n}^{[1]},\mathsf{m}\rangle \hookrightarrow T319 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{n} = [\emptyset,\mathsf{n}^{[2]}]$

$\mathsf{Use\_def}(\mathsf{Rev}_{\mathbb{Z}}) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) = [\mathsf{n}^{[2]},\emptyset]$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \notin \mathbb{Z}$

$\mathsf{Use\_def}(\mathbb{Z}) \Rightarrow \quad Stat1: \; [\mathsf{n}^{[2]},\emptyset] \notin \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$

$\langle \mathsf{n}^{[2]},\emptyset\rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \in \mathbb{Z}$

-- But in this case the preceding theorem tells us that associativity applies to the product $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})$, and so we can derive the required conclusion by an easy manipulation of the reversal operator $\mathsf{Rev}_{\mathbb{Z}}$.

$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T294 \Rightarrow \quad \mathsf{n} *_{\mathbb{Z}} \mathsf{m} \in \mathbb{Z}$

$\langle \mathsf{m},\mathsf{k}\rangle \hookrightarrow T294 \Rightarrow \quad \mathsf{m} *_{\mathbb{Z}} \mathsf{k} \in \mathbb{Z}$

$\langle \mathsf{k},\mathsf{n}^{[2]},\mathsf{m}\rangle \hookrightarrow T319 \Rightarrow \quad [\mathsf{n}^{[2]},\emptyset] *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) = ([\mathsf{n}^{[2]},\emptyset] *_{\mathbb{Z}} \mathsf{m}) *_{\mathbb{Z}} \mathsf{k}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) = (\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} \mathsf{m}) *_{\mathbb{Z}} \mathsf{k}$

$\langle \mathsf{n},\mathsf{m} *_{\mathbb{Z}} \mathsf{k}\rangle \hookrightarrow T315 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}))$

$\langle \mathsf{n},\mathsf{m}\rangle \hookrightarrow T315 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) *_{\mathbb{Z}} \mathsf{m} = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m})$

EQUAL $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}) *_{\mathbb{Z}} \mathsf{k} = \mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)$
$\langle \mathsf{n} *_{\mathbb{Z}} \mathsf{m}, \mathsf{k}\rangle \hookrightarrow$ *T315* $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}) *_{\mathbb{Z}} \mathsf{k} = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k})$
$\langle \mathsf{n}, \mathsf{m} *_{\mathbb{Z}} \mathsf{k}\rangle \hookrightarrow$ *T315* $\Rightarrow$ $\mathsf{n} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) = \mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)$
EQUAL $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)\right) = \mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)$
$\langle \mathsf{n}, \mathsf{m} *_{\mathbb{Z}} \mathsf{k}\rangle \hookrightarrow$ *T294* $\Rightarrow$ $\mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k}) \in \mathbb{Z}$
$\langle \mathsf{n} *_{\mathbb{Z}} \mathsf{m}, \mathsf{k}\rangle \hookrightarrow$ *T294* $\Rightarrow$ $\mathsf{n} *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k} \in \mathbb{Z}$
$\langle \mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\rangle \hookrightarrow$ *T314* $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{Rev}_{\mathbb{Z}}\left(\mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})\right)\right) = \mathsf{n} *_{\mathbb{Z}} (\mathsf{m} *_{\mathbb{Z}} \mathsf{k})$
$\langle \mathsf{n} *_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} \mathsf{k}\rangle \hookrightarrow$ *T314* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that subtraction of one signed integer from another is equivalent to addition of the negative of the first to the second.

Theorem 404 (321) $\mathsf{N}, \mathsf{M} \in \mathbb{Z} \rightarrow \mathsf{N} -_{\mathbb{Z}} \mathsf{M} = \mathsf{N} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{M})$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathsf{n}, \mathsf{m} \in \mathbb{Z}\ \&\ \mathsf{n} -_{\mathbb{Z}} \mathsf{m} \neq \mathsf{n} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m})$

-- The proof results easily by expanding the definitions of the operators involved and a bit of unsigned integer arithmetic.

Use_def($\mathbb{Z}$) $\Rightarrow$ *Stat1* : $\mathsf{m} \in \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$
$\langle \mathsf{m}_1, \mathsf{m}_2\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathsf{m} = [\mathsf{m}_1, \mathsf{m}_2]\ \&\ \mathsf{m}_1, \mathsf{m}_2 \in \mathbb{N}$
ELEM $\Rightarrow$ $\mathsf{m} = \left[\mathsf{m}^{[1]}, \mathsf{m}^{[2]}\right]\ \&\ \mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{N}$
Use_def($\mathbb{Z}$) $\Rightarrow$ *Stat2* : $\mathsf{n} \in \{[\mathsf{x}, \mathsf{y}] : \mathsf{x} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{x} = \emptyset \vee \mathsf{y} = \emptyset\}$
$\langle \mathsf{n}_1, \mathsf{n}_2\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat3* : $\mathsf{n} = [\mathsf{n}_1, \mathsf{n}_2]\ \&\ \mathsf{n}_1, \mathsf{n}_2 \in \mathbb{N}$
$\langle$ *Stat3* $\rangle$ ELEM $\Rightarrow$ $\mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right]\ \&\ \mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{N}$
Use_def($-_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{n} -_{\mathbb{Z}} \mathsf{m} = \mathsf{Red}\left(\left[\mathsf{m}^{[2]} + \mathsf{n}^{[1]}, \mathsf{m}^{[1]} + \mathsf{n}^{[2]}\right]\right)$
Use_def($\mathsf{Rev}_{\mathbb{Z}}$) $\Rightarrow$ *Stat4* : $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = \left[\mathsf{m}^{[2]}, \mathsf{m}^{[1]}\right]$
$\langle$ *Stat4* $\rangle$ ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}^{[1]}(\mathsf{m}) = \mathsf{m}^{[2]}\ \&\ \mathsf{Rev}_{\mathbb{Z}}^{[2]}(\mathsf{m}) = \mathsf{m}^{[1]}$
Use_def($+_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{n} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = \mathsf{Red}\left(\left[\mathsf{n}^{[1]} + \mathsf{Rev}_{\mathbb{Z}}^{[1]}(\mathsf{m}), \mathsf{n}^{[2]} + \mathsf{Rev}_{\mathbb{Z}}^{[2]}(\mathsf{m})\right]\right)$
EQUAL $\Rightarrow$ $\mathsf{n} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = \mathsf{Red}\left(\left[\mathsf{n}^{[1]} + \mathsf{m}^{[2]}, \mathsf{n}^{[2]} + \mathsf{m}^{[1]}\right]\right)$
ALGEBRA $\Rightarrow$ $\mathsf{m}^{[2]} + \mathsf{n}^{[1]} = \mathsf{n}^{[1]} + \mathsf{m}^{[2]}\ \&\ \mathsf{m}^{[1]} + \mathsf{n}^{[2]} = \mathsf{n}^{[2]} + \mathsf{m}^{[1]}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem tells us that for signed integers subtraction is always the reverse of addition. It is this generalization of the corresponding but more restricted rule for unsigned integers that justifies the introduction of the signed integers.

Theorem 405 (322) $\mathsf{N},\mathsf{M}\in\mathbb{Z}\rightarrow\mathsf{N}=\mathsf{M}+_{\mathbb{Z}}(\mathsf{N}-_{\mathbb{Z}}\mathsf{M})$. Proof:

$\mathsf{Suppose\_not}(n,m)\Rightarrow\quad n,m\in\mathbb{Z}\ \&\ n\neq m+_{\mathbb{Z}}(n-_{\mathbb{Z}}m)$

-- By expanding the definitions of the operators involved, a bit of unsigned integer arithmetic, and elimination of superfluous reduction operators, we can readily reduce the negative of our assertion to the inequality seen below.

$\langle n\rangle\hookrightarrow T292\Rightarrow\quad n=\left[n^{[1]},n^{[2]}\right]\ \&\ n^{[1]}=\emptyset\vee n^{[2]}=\emptyset\ \&\ n^{[1]},n^{[2]}\in\mathbb{N}$
$\mathsf{ELEM}\Rightarrow\quad n^{[1]}\cap n^{[2]}=\emptyset$
$\langle m\rangle\hookrightarrow T292\Rightarrow\quad m=\left[m^{[1]},m^{[2]}\right]\ \&\ m^{[1]}=\emptyset\vee m^{[2]}=\emptyset\ \&\ m^{[1]},m^{[2]}\in\mathbb{N}$
$\mathsf{Use\_def}(-_{\mathbb{Z}})\Rightarrow\quad n\neq m+_{\mathbb{Z}}\mathsf{Red}(\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right])$
$\mathsf{ALGEBRA}\Rightarrow\quad m^{[2]}+n^{[1]},m^{[1]}+n^{[2]},m^{[1]}+m^{[2]}\in\mathbb{N}$
$\langle m,m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\rangle\hookrightarrow T299\Rightarrow\quad m+_{\mathbb{Z}}\mathsf{Red}(\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right])=$
$m+_{\mathbb{Z}}\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]$
$\mathsf{Use\_def}(+_{\mathbb{Z}})\Rightarrow\quad m+_{\mathbb{Z}}\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]=$
$\mathsf{Red}(\left[m^{[1]}+\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]^{[1]},m^{[2]}+\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]^{[2]}\right])$
$\mathsf{ELEM}\Rightarrow\quad \left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]^{[1]}=m^{[2]}+n^{[1]}$
$\mathsf{ELEM}\Rightarrow\quad \left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]^{[2]}=m^{[1]}+n^{[2]}$
$\mathsf{EQUAL}\Rightarrow\quad m+_{\mathbb{Z}}\left[m^{[2]}+n^{[1]},m^{[1]}+n^{[2]}\right]=\mathsf{Red}(\left[m^{[1]}+(m^{[2]}+n^{[1]}),m^{[2]}+(m^{[1]}+n^{[2]})\right])$
$\mathsf{ELEM}\Rightarrow\quad n\neq\mathsf{Red}(\left[m^{[1]}+(m^{[2]}+n^{[1]}),m^{[2]}+(m^{[1]}+n^{[2]})\right])$

-- But since the same quantity $m^{[1]}+m^{[2]}$ is added to both components of the pair appearing on the right of this last equality, it reduces readily to the contradictory form $n\neq\left[n^{[1]},n^{[2]}\right]$, thereby proving our assertion.

$\mathsf{ALGEBRA}\Rightarrow\quad n\neq\mathsf{Red}(\left[n^{[1]}+(m^{[1]}+m^{[2]}),n^{[2]}+(m^{[1]}+m^{[2]})\right])$
$\langle n^{[1]},n^{[2]},m^{[1]}+m^{[2]}\rangle\hookrightarrow T297\Rightarrow\quad n\neq\mathsf{Red}(\left[n^{[1]},n^{[2]}\right])$
$\mathsf{Use\_def}(\mathsf{Red})\Rightarrow\quad n\neq\left[n^{[1]}-n^{[1]}\cap n^{[2]},n^{[2]}-n^{[1]}\cap n^{[2]}\right]$
$\mathsf{EQUAL}\Rightarrow\quad n\neq\left[n^{[1]}-\emptyset,n^{[2]}-\emptyset\right]$
$\langle n^{[1]}\rangle\hookrightarrow T230\Rightarrow\quad n^{[1]}-\emptyset=\#n^{[1]}$
$\langle n^{[2]}\rangle\hookrightarrow T230\Rightarrow\quad n^{[2]}-\emptyset=\#n^{[2]}$
$\langle n^{[1]}\rangle\hookrightarrow T179\Rightarrow\quad \mathsf{Card}(n^{[1]})$
$\langle n^{[2]}\rangle\hookrightarrow T179\Rightarrow\quad \mathsf{Card}(n^{[2]})$
$\langle n^{[1]}\rangle\hookrightarrow T138\Rightarrow\quad n^{[1]}-\emptyset=n^{[1]}$
$\langle n^{[2]}\rangle\hookrightarrow T138\Rightarrow\quad n^{[2]}-\emptyset=n^{[2]}$
$\mathsf{EQUAL}\Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge}\Rightarrow$ QED

-- Next we prove that the negative of $\mathsf{m} +_{\mathbb{Z}} \mathsf{n}$ is the sum of $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n})$ and $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m})$.

Theorem 406 (323) $\mathsf{N}, \mathsf{M} \in \mathbb{Z} \rightarrow \mathsf{Rev}_{\mathbb{Z}}(\mathsf{N} +_{\mathbb{Z}} \mathsf{M}) = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{N}) +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{M})$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathsf{n}, \mathsf{m} \in \mathbb{Z}$ & $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} +_{\mathbb{Z}} \mathsf{m}) \neq \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m})$

$\langle \mathsf{n} \rangle \hookrightarrow T292 \Rightarrow$ $\mathsf{n} = [\mathsf{n}^{[1]}, \mathsf{n}^{[2]}]$ & $\mathsf{n}^{[1]} = \emptyset \vee \mathsf{n}^{[2]} = \emptyset$ & $\mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{N}$
Use_def(Rev$_{\mathbb{Z}}$) $\Rightarrow$ *Stat1*: $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) = [\mathsf{n}^{[2]}, \mathsf{n}^{[1]}]$
$\langle$*Stat1*$\rangle$ ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}{}^{[1]}(\mathsf{n}) = \mathsf{n}^{[2]}$
$\langle$*Stat1, Stat1*$\rangle$ ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}{}^{[2]}(\mathsf{n}) = \mathsf{n}^{[1]}$
Use_def(Rev$_{\mathbb{Z}}$) $\Rightarrow$ *Stat2*: $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = [\mathsf{m}^{[2]}, \mathsf{m}^{[1]}]$
$\langle \mathsf{m} \rangle \hookrightarrow T292 \Rightarrow$ $\mathsf{m} = [\mathsf{m}^{[1]}, \mathsf{m}^{[2]}]$ & $\mathsf{m}^{[1]} = \emptyset \vee \mathsf{m}^{[2]} = \emptyset$ & $\mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{N}$
$\langle$*Stat2, Stat2*$\rangle$ ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}{}^{[1]}(\mathsf{m}) = \mathsf{m}^{[2]}$
$\langle$*Stat2, Stat2*$\rangle$ ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}{}^{[2]}(\mathsf{m}) = \mathsf{m}^{[1]}$

Use_def($+_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{n} +_{\mathbb{Z}} \mathsf{m} = \mathsf{Red}([\mathsf{n}^{[1]} + \mathsf{m}^{[1]}, \mathsf{n}^{[2]} + \mathsf{m}^{[2]}])$
Use_def($+_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = \mathsf{Red}\left(\left[\mathsf{Rev}_{\mathbb{Z}}{}^{[1]}(\mathsf{n}) + \mathsf{Rev}_{\mathbb{Z}}{}^{[1]}(\mathsf{m}), \mathsf{Rev}_{\mathbb{Z}}{}^{[2]}(\mathsf{n}) + \mathsf{Rev}_{\mathbb{Z}}{}^{[2]}(\mathsf{m})\right]\right)$
EQUAL $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = \mathsf{Red}([\mathsf{n}^{[2]} + \mathsf{m}^{[2]}, \mathsf{n}^{[1]} + \mathsf{m}^{[1]}])$
EQUAL $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{N} +_{\mathbb{Z}} \mathsf{M}) = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{Red}([\mathsf{n}^{[1]} + \mathsf{m}^{[1]}, \mathsf{n}^{[2]} + \mathsf{m}^{[2]}]))$
ALGEBRA $\Rightarrow$ $\mathsf{n}^{[1]} + \mathsf{m}^{[1]}, \mathsf{n}^{[2]} + \mathsf{m}^{[2]} \in \mathbb{N}$
$\langle \mathsf{n}^{[2]} + \mathsf{m}^{[2]}, \mathsf{n}^{[1]} + \mathsf{m}^{[1]} \rangle \hookrightarrow T312 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We now go on to establish multiplication rules for the signed integer constants zero and one, proving that zero times anything is zero, while one times any signed integer n is n. Moreover the product of the reverse $[\emptyset, 1]$ of $[1, \emptyset]$ by itself is $[1, \emptyset]$.

Theorem 407 (324) $[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] = [1, \emptyset]$ & $(\mathsf{X} \in \mathbb{Z} \longrightarrow [1, \emptyset] *_{\mathbb{Z}} \mathsf{X} = \mathsf{X}$ & $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{X} = [\emptyset, \emptyset])$. PROOF:

Suppose_not(x) $\Rightarrow$ $[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] \neq [1, \emptyset] \vee (\mathsf{x} \in \mathbb{Z}$ & $[1, \emptyset] *_{\mathbb{Z}} \mathsf{x} \neq \mathsf{x} \vee [\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{x} \neq [\emptyset, \emptyset])$

Use_def($*_{\mathbb{Z}}$) $\Rightarrow$
$[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] =$
$\mathsf{Red}\left(\left[[\emptyset, 1]^{[1]} * [\emptyset, 1]^{[1]} + [\emptyset, 1]^{[2]} * [\emptyset, 1]^{[2]}, [\emptyset, 1]^{[1]} * [\emptyset, 1]^{[2]} + [\emptyset, 1]^{[1]} * [\emptyset, 1]^{[2]}\right]\right)$
ELEM $\Rightarrow$ $[\emptyset, 1]^{[1]} = \emptyset$ & $[\emptyset, 1]^{[2]} = 1$ & $1 \cap \emptyset = \emptyset$
$T182 \Rightarrow$ *Stat0*: $1, \emptyset \in \mathbb{N}$
EQUAL $\Rightarrow$ $[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{Red}([\emptyset * \emptyset + 1 * 1, \emptyset * 1 + \emptyset * 1])$
ALGEBRA $\Rightarrow$ $[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{Red}([1, \emptyset])$

$T182 \Rightarrow$ $\mathsf{Card}(1)$
$T182 \Rightarrow$ $\mathsf{Card}(\emptyset)$
$\langle 1 \rangle \hookrightarrow T138 \Rightarrow$ $Stat1a:\ 1 = \#1$
$\langle \emptyset \rangle \hookrightarrow T138 \Rightarrow$ $\emptyset = \#\emptyset$
$\mathsf{Use\_def(Red)} \Rightarrow$ $[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] = [1 - 1 \cap \emptyset, \emptyset - 1 \cap \emptyset]$
$\mathsf{EQUAL} \Rightarrow$ $Stat1:\ [\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] = [1 - \emptyset, \emptyset - \emptyset]$
$\langle 1 \rangle \hookrightarrow T230([Stat1, Stat1a]) \Rightarrow$ $[\emptyset, 1] *_{\mathbb{Z}} [\emptyset, 1] = [1, \emptyset - \emptyset]$
$\langle \emptyset \rangle \hookrightarrow T230 \Rightarrow$ $\emptyset - \emptyset = \#\emptyset$
$\mathsf{EQUAL} \Rightarrow$ $Stat2:\ \mathsf{x} \in \mathbb{Z}\ \&\ [1, \emptyset] *_{\mathbb{Z}} \mathsf{x} \neq \mathsf{x} \vee [\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{x} \neq [\emptyset, \emptyset]$
$\mathsf{Use\_def}(*_{\mathbb{Z}}) \Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}\left(\left[[\emptyset, \emptyset]^{[1]} * \mathsf{x}^{[1]} + [\emptyset, \emptyset]^{[2]} * \mathsf{x}^{[2]}, [\emptyset, \emptyset]^{[1]} * \mathsf{x}^{[2]} + \mathsf{x}^{[1]} * [\emptyset, \emptyset]^{[2]}\right]\right)$
$TELEM \Rightarrow$ $[\emptyset, \emptyset]^{[1]} = \emptyset\ \&\ [\emptyset, \emptyset]^{[2]} = \emptyset\ \&\ [1, \emptyset]^{[1]} = 1\ \&\ [1, \emptyset]^{[2]} = \emptyset$
$\mathsf{EQUAL} \Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}\left(\left[\emptyset * \mathsf{x}^{[1]} + \emptyset * \mathsf{x}^{[2]}, \emptyset * \mathsf{x}^{[2]} + \mathsf{x}^{[1]} * \emptyset\right]\right)$
$\mathsf{Use\_def}(\mathbb{Z}) \Rightarrow$ $Stat3:\ \mathsf{x} \in \{[\mathsf{u}, \mathsf{y}] :\ \mathsf{u} \in \mathbb{N}, \mathsf{y} \in \mathbb{N} \mid \mathsf{u} = \emptyset \vee \mathsf{y} = \emptyset\}$
$\langle \mathsf{u}, \mathsf{y} \rangle \hookrightarrow Stat3 \Rightarrow$ $Stat3a:\ \mathsf{x} = [\mathsf{u}, \mathsf{y}]\ \&\ \mathsf{u}, \mathsf{y} \in \mathbb{N}\ \&\ \mathsf{u} = \emptyset \vee \mathsf{y} = \emptyset$
$\langle Stat3a, * \rangle\ \mathsf{ELEM} \Rightarrow$ $Stat4:\ \mathsf{x} = [\mathsf{u}, \mathsf{y}]$
$\langle Stat4 \rangle\ \mathsf{ELEM} \Rightarrow$ $Stat4a:\ \mathsf{x}^{[1]} = \mathsf{u}\ \&\ \mathsf{x}^{[2]} = \mathsf{y}\ \&\ \mathsf{x} = \left[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}\right]$
$\langle Stat3a, Stat4a, * \rangle\ \mathsf{ELEM} \Rightarrow$ $Stat5:\ \mathsf{x}^{[1]}, \mathsf{x}^{[2]} \in \mathbb{N}\ \&\ \mathsf{x}^{[1]} \cap \mathsf{x}^{[2]} = \emptyset$
$\langle \mathsf{x}^{[1]} \rangle \hookrightarrow T180([Stat5, \cap]) \Rightarrow$ $\mathsf{x}^{[1]} = \#\mathsf{x}^{[1]}$
$\langle \mathsf{x}^{[2]} \rangle \hookrightarrow T180([Stat5, \cap]) \Rightarrow$ $\mathsf{x}^{[2]} = \#\mathsf{x}^{[2]}$
$\mathsf{ALGEBRA} \Rightarrow$ $Stat6:\ [\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}([\emptyset, \emptyset])$
$\langle \emptyset \rangle \hookrightarrow T296([Stat6, Stat0]) \Rightarrow$ $Stat7:\ [\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{x} = [\emptyset, \emptyset]$
$\mathsf{Use\_def}(*_{\mathbb{Z}}) \Rightarrow$ $[1, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}\left(\left[[1, \emptyset]^{[1]} * \mathsf{x}^{[1]} + [1, \emptyset]^{[2]} * \mathsf{x}^{[2]}, [1, \emptyset]^{[1]} * \mathsf{x}^{[2]} + \mathsf{x}^{[1]} * [1, \emptyset]^{[2]}\right]\right)$
$\mathsf{EQUAL} \Rightarrow$ $[1, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}\left(\left[1 * \mathsf{x}^{[1]} + \emptyset * \mathsf{x}^{[2]}, 1 * \mathsf{x}^{[2]} + \mathsf{x}^{[1]} * \emptyset\right]\right)$
$\mathsf{ALGEBRA} \Rightarrow$ $[1, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}\left(\left[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}\right]\right)$
$\mathsf{EQUAL} \Rightarrow$ $Stat8:\ [1, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{Red}(\mathsf{x})$
$\mathsf{Use\_def(Red)} \Rightarrow$ $\mathsf{Red}(\mathsf{x}) = \left[\mathsf{x}^{[1]} - \mathsf{x}^{[1]} \cap \mathsf{x}^{[2]}, \mathsf{x}^{[2]} - \mathsf{x}^{[1]} \cap \mathsf{x}^{[2]}\right]$
$\mathsf{EQUAL} \Rightarrow$ $Stat9:\ \mathsf{Red}(\mathsf{x}) = \left[\mathsf{x}^{[1]} - \emptyset, \mathsf{x}^{[2]} - \emptyset\right]$
$\langle \mathsf{x}^{[1]} \rangle \hookrightarrow T230 \Rightarrow$ $\mathsf{x}^{[1]} - \emptyset = \#\mathsf{x}^{[1]}$
$\langle \mathsf{x}^{[2]} \rangle \hookrightarrow T230 \Rightarrow$ $\mathsf{x}^{[2]} - \emptyset = \#\mathsf{x}^{[2]}$
$\mathsf{EQUAL} \Rightarrow$ $Stat10:\ \mathsf{Red}(\mathsf{x}) = \left[\mathsf{x}^{[1]}, \mathsf{x}^{[2]}\right]$
$\langle Stat8, Stat10, Stat4a, Stat5, * \rangle\ \mathsf{ELEM} \Rightarrow$ $Stat11:\ [1, \emptyset] *_{\mathbb{Z}} \mathsf{x} = \mathsf{x}$
$\langle Stat2, Stat11, Stat7, * \rangle\ \mathsf{ELEM} \Rightarrow$ $\mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ QED

-- It follows trially from the preceding theorem that the product of any signed integer n by one is n.

Theorem 408 (325) $\mathsf{K} \in \mathbb{Z} \to \mathsf{K} *_{\mathbb{Z}} [1, \emptyset] = \mathsf{K}$. PROOF:

Suppose_not(k) $\Rightarrow$ $\mathsf{k} \in \mathbb{Z} \,\&\, \mathsf{k} *_{\mathbb{Z}} [1, \emptyset] \neq \mathsf{k}$

$\langle \mathsf{k} \rangle \hookrightarrow T292 \Rightarrow$ $\mathsf{k} = \left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right] \,\&\, \mathsf{k}^{[1]}, \mathsf{k}^{[2]} \in \mathbb{N} \,\&\, \mathsf{k}^{[1]} = \emptyset \vee \mathsf{k}^{[2]} = \emptyset$

$T182 \Rightarrow$ $1, \emptyset \in \mathbb{N}$

Use_def($*_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} [1, \emptyset] = \mathsf{Red}\left(\left[\mathsf{k}^{[1]} * [1,\emptyset]^{[1]} + \mathsf{k}^{[2]} * [1,\emptyset]^{[2]}, \mathsf{k}^{[1]} * [1,\emptyset]^{[2]} + [1,\emptyset]^{[1]} * \mathsf{k}^{[2]}\right]\right)$

ELEM $\Rightarrow$ $[1,\emptyset]^{[1]} = 1 \,\&\, [1,\emptyset]^{[2]} = \emptyset \,\&\, 1 \cap \emptyset = \emptyset$

EQUAL $\Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} [1, \emptyset] = \mathsf{Red}\left(\left[\mathsf{k}^{[1]} * 1 + \mathsf{k}^{[2]} * \emptyset, \mathsf{k}^{[1]} * \emptyset + 1 * \mathsf{k}^{[2]}\right]\right)$

ALGEBRA $\Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} [1, \emptyset] = \mathsf{Red}\left(\left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right]\right)$

Use_def(Red) $\Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} [1, \emptyset] = \left[\mathsf{k}^{[1]} - \mathsf{k}^{[1]} \cap \mathsf{k}^{[2]}, \mathsf{k}^{[2]} - \mathsf{k}^{[1]} \cap \mathsf{k}^{[2]}\right]$

ELEM $\Rightarrow$ $\mathsf{k}^{[1]} \cap \mathsf{k}^{[2]} = \emptyset$

EQUAL $\Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} [1, \emptyset] = \left[\mathsf{k}^{[1]} - \emptyset, \mathsf{k}^{[2]} - \emptyset\right]$

$\langle \mathsf{k}^{[1]} \rangle \hookrightarrow T230 \Rightarrow$ $\mathsf{k}^{[1]} - \emptyset = \#\mathsf{k}^{[1]}$

$\langle \mathsf{k}^{[2]} \rangle \hookrightarrow T230 \Rightarrow$ $\mathsf{k}^{[2]} - \emptyset = \#\mathsf{k}^{[2]}$

$\langle \mathsf{k}^{[1]} \rangle \hookrightarrow T180 \Rightarrow$ $\mathsf{k}^{[1]} = \#\mathsf{k}^{[1]}$

$\langle \mathsf{k}^{[2]} \rangle \hookrightarrow T180 \Rightarrow$ $\mathsf{k}^{[2]} = \#\mathsf{k}^{[2]}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 409 (326) $\mathsf{K}, \mathsf{M} \in \mathbb{Z} \to \mathsf{K} -_{\mathbb{Z}} \mathsf{M} = \mathsf{K} +_{\mathbb{Z}} \mathsf{M} *_{\mathbb{Z}} [\emptyset, 1]$. PROOF:

Suppose_not(k, m) $\Rightarrow$ $\mathsf{k}, \mathsf{m} \in \mathbb{Z} \,\&\, \mathsf{k} -_{\mathbb{Z}} \mathsf{m} \neq \mathsf{k} +_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} [\emptyset, 1]$

$\langle \mathsf{k} \rangle \hookrightarrow T292 \Rightarrow$ $\mathsf{k} = \left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right] \,\&\, \mathsf{k}^{[1]}, \mathsf{k}^{[2]} \in \mathbb{N} \,\&\, \mathsf{k}^{[1]} = \emptyset \vee \mathsf{k}^{[2]} = \emptyset \,\&\, \mathsf{Red}(\mathsf{k}) = \mathsf{k}$

$\langle \mathsf{m} \rangle \hookrightarrow T292 \Rightarrow$ $\mathsf{m} = \left[\mathsf{m}^{[1]}, \mathsf{m}^{[2]}\right] \,\&\, \mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{N} \,\&\, \mathsf{m}^{[1]} = \emptyset \vee \mathsf{m}^{[2]} = \emptyset \,\&\, \mathsf{Red}(\mathsf{m}) = \mathsf{m}$

Use_def($*_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{m} *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{Red}\left(\left[\mathsf{m}^{[1]} * [\emptyset,1]^{[1]} + \mathsf{m}^{[2]} * [\emptyset,1]^{[2]}, \mathsf{m}^{[1]} * [\emptyset,1]^{[2]} + [\emptyset,1]^{[1]} * \mathsf{m}^{[2]}\right]\right)$

ELEM $\Rightarrow$ $[\emptyset,1]^{[1]} = \emptyset \,\&\, [\emptyset,1]^{[2]} = 1$

EQUAL $\Rightarrow$ $\mathsf{m} *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{Red}\left(\left[\mathsf{m}^{[1]} * \emptyset + \mathsf{m}^{[2]} * 1, \mathsf{m}^{[1]} * 1 + \emptyset * \mathsf{m}^{[2]}\right]\right)$

ALGEBRA $\Rightarrow$ $\mathsf{m} *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{Red}\left(\left[\mathsf{m}^{[2]}, \mathsf{m}^{[1]}\right]\right)$

Use_def($-_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{k} -_{\mathbb{Z}} \mathsf{m} = \mathsf{Red}\left(\left[\mathsf{m}^{[2]} + \mathsf{k}^{[1]}, \mathsf{m}^{[1]} + \mathsf{k}^{[2]}\right]\right)$

EQUAL $\Rightarrow$ $\mathsf{k} +_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{k} +_{\mathbb{Z}} \mathsf{Red}\left(\left[\mathsf{m}^{[2]}, \mathsf{m}^{[1]}\right]\right)$

$\langle \mathsf{k}, \mathsf{m}^{[2]}, \mathsf{m}^{[1]} \rangle \hookrightarrow T299 \Rightarrow$ $\mathsf{k} +_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{k} +_{\mathbb{Z}} \left[\mathsf{m}^{[2]}, \mathsf{m}^{[1]}\right]$

Use_def($+_{\mathbb{Z}}$) $\Rightarrow$ $\mathsf{k} +_{\mathbb{Z}} \mathsf{m} *_{\mathbb{Z}} [\emptyset, 1] = \mathsf{Red}\left(\left[\mathsf{k}^{[1]} + \mathsf{m}^{[2]}, \mathsf{k}^{[2]} + \mathsf{m}^{[1]}\right]\right)$

ELEM $\Rightarrow$ $\mathsf{Red}\left(\left[\mathsf{m}^{[2]} + \mathsf{k}^{[1]}, \mathsf{m}^{[1]} + \mathsf{k}^{[2]}\right]\right) \neq \mathsf{Red}\left(\left[\mathsf{k}^{[1]} + \mathsf{m}^{[2]}, \mathsf{k}^{[2]} + \mathsf{m}^{[1]}\right]\right)$

ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we note that for any singed integer k, $\mathsf{k} -_{\mathbb{Z}} \mathsf{k}$ is zero. The proof results trivially from the definition of the operators involved.

Theorem 410 (327) $\mathsf{K} \in \mathbb{Z} \rightarrow \mathsf{K} -_{\mathbb{Z}} \mathsf{K} = [\emptyset, \emptyset]$. PROOF:

Suppose_not$(\mathsf{k}) \Rightarrow \quad \mathsf{k} \in \mathbb{Z} \,\&\, \mathsf{k} -_{\mathbb{Z}} \mathsf{k} \neq [\emptyset, \emptyset]$
$\langle \mathsf{k} \rangle \hookrightarrow T292 \Rightarrow \quad \mathsf{k} = \left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right] \,\&\, \mathsf{k}^{[1]}, \mathsf{k}^{[2]} \in \mathbb{N} \,\&\, \mathsf{k}^{[1]} = \emptyset \vee \mathsf{k}^{[2]} = \emptyset \,\&\, \mathsf{Red}(\mathsf{k}) = \mathsf{k}$
Use_def$(-_{\mathbb{Z}}) \Rightarrow \quad \mathsf{k} -_{\mathbb{Z}} \mathsf{k} = \mathsf{Red}(\left[\mathsf{k}^{[2]} + \mathsf{k}^{[1]}, \mathsf{k}^{[1]} + \mathsf{k}^{[2]}\right])$
ALGEBRA $\Rightarrow \quad \mathsf{Red}(\left[\mathsf{k}^{[1]} + \mathsf{k}^{[2]}, \mathsf{k}^{[1]} + \mathsf{k}^{[2]}\right]) \neq [\emptyset, \emptyset]$
ALGEBRA $\Rightarrow \quad \mathsf{k}^{[1]} + \mathsf{k}^{[2]} \in \mathbb{N}$
$\langle \mathsf{k}^{[1]} + \mathsf{k}^{[2]} \rangle \hookrightarrow T296 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- It follows equally trivially that the sum of any signed integer k and zero is k.

Theorem 411 (328) $\mathsf{K} \in \mathbb{Z} \rightarrow \mathsf{K} +_{\mathbb{Z}} [\emptyset, \emptyset] = \mathsf{K}$. PROOF:

Suppose_not$(\mathsf{k}) \Rightarrow \quad \mathsf{k} \in \mathbb{Z} \,\&\, \mathsf{k} +_{\mathbb{Z}} [\emptyset, \emptyset] \neq \mathsf{k}$
$\langle \mathsf{k} \rangle \hookrightarrow T292 \Rightarrow \quad \mathsf{k} = \left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right] \,\&\, \mathsf{k}^{[1]}, \mathsf{k}^{[2]} \in \mathbb{N} \,\&\, \mathsf{Red}(\mathsf{k}) = \mathsf{k}$
Use_def$(+_{\mathbb{Z}}) \Rightarrow \quad \mathsf{k} +_{\mathbb{Z}} [\emptyset, \emptyset] = \mathsf{Red}(\left[\mathsf{k}^{[1]} + \emptyset, \mathsf{k}^{[2]} + \emptyset\right])$
ALGEBRA $\Rightarrow \quad \mathsf{k} +_{\mathbb{Z}} [\emptyset, \emptyset] = \mathsf{Red}(\left[\mathsf{k}^{[1]}, \mathsf{k}^{[2]}\right])$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- And so it follows equally trivially by commutativity that the sum of zero and any signed integer k is k.

Theorem 412 (329) $\mathsf{K} \in \mathbb{Z} \rightarrow [\emptyset, \emptyset] +_{\mathbb{Z}} \mathsf{K} = \mathsf{K}$. PROOF:

Suppose_not$(\mathsf{k}) \Rightarrow \quad \mathsf{k} \in \mathbb{Z} \,\&\, [\emptyset, \emptyset] +_{\mathbb{Z}} \mathsf{k} \neq \mathsf{k}$
$T291 \Rightarrow \quad [\emptyset, \emptyset] \in \mathbb{Z}$
$\langle [\emptyset, \emptyset], \mathsf{k} \rangle \hookrightarrow T303 \Rightarrow \quad [\emptyset, \emptyset] +_{\mathbb{Z}} \mathsf{k} = \mathsf{k} +_{\mathbb{Z}} [\emptyset, \emptyset]$
$\langle \mathsf{k} \rangle \hookrightarrow T328 \Rightarrow \quad \mathsf{k} +_{\mathbb{Z}} [\emptyset, \emptyset] = \mathsf{k}$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The following easy theorem gives the very important cancellation rule for signed integer multiplication: if the product of two signed integers is zero, one of them must be zero. This fact is central to the discussion of rational numbers which follows subsequently.

-- Si is an Integral Domain

Theorem 413 (330) $\mathsf{N},\mathsf{M}\in\mathbb{Z}\ \&\ \mathsf{M}*_{\mathbb{Z}}\mathsf{N}=[\emptyset,\emptyset]\rightarrow\mathsf{M}=[\emptyset,\emptyset]\vee\mathsf{N}=[\emptyset,\emptyset]$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n},\mathsf{m})\Rightarrow\quad Stat1:\ \mathsf{n},\mathsf{m}\in\mathbb{Z}\ \&\ \mathsf{m}*_{\mathbb{Z}}\mathsf{n}=[\emptyset,\emptyset]\ \&\ \mathsf{m}\neq[\emptyset,\emptyset]\ \&\ \mathsf{n}\neq[\emptyset,\emptyset]$

$\langle\mathsf{m}\rangle\hookrightarrow T292\Rightarrow\quad Stat2:\ \mathsf{m}=\left[\mathsf{m}^{[1]},\mathsf{m}^{[2]}\right]\ \&\ \mathsf{m}^{[1]}=\emptyset\vee\mathsf{m}^{[2]}=\emptyset\ \&\ \mathsf{m}^{[1]},\mathsf{m}^{[2]}\in\mathbb{N}$

$\langle\mathsf{n}\rangle\hookrightarrow T292\Rightarrow\quad Stat3:\ \mathsf{n}=\left[\mathsf{n}^{[1]},\mathsf{n}^{[2]}\right]\ \&\ \mathsf{n}^{[1]}=\emptyset\vee\mathsf{n}^{[2]}=\emptyset\ \&\ \mathsf{n}^{[1]},\mathsf{n}^{[2]}\in\mathbb{N}$

$\mathsf{Use\_def}(*_{\mathbb{Z}})\Rightarrow\quad Stat4:\ \mathsf{Red}(\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]}+\mathsf{m}^{[2]}*\mathsf{n}^{[2]},\mathsf{m}^{[1]}*\mathsf{n}^{[2]}+\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\right])=[\emptyset,\emptyset]$

$\mathsf{Suppose}\Rightarrow\quad Stat5:\ \mathsf{n}^{[1]}=\emptyset\ \&\ \mathsf{n}^{[2]}=\emptyset$

$\mathsf{EQUAL}\ \langle Stat1, Stat3, Stat5\rangle\Rightarrow\quad\mathsf{false};\qquad\mathsf{Discharge}\Rightarrow\quad Stat6:\ \mathsf{n}^{[1]}\neq\emptyset\vee\mathsf{n}^{[2]}\neq\emptyset$

$\langle Stat6\rangle\ \mathsf{ELEM}\Rightarrow\quad\mathsf{n}^{[1]}\neq\emptyset\vee\mathsf{n}^{[2]}\neq\emptyset$

$\mathsf{Suppose}\Rightarrow\quad Stat7:\ \mathsf{m}^{[1]}=\emptyset\ \&\ \mathsf{m}^{[2]}=\emptyset$

$\mathsf{EQUAL}\ \langle Stat1, Stat2, Stat7\rangle\Rightarrow\quad\mathsf{false};\qquad\mathsf{Discharge}\Rightarrow\quad Stat8:\ \mathsf{m}^{[1]}\neq\emptyset\vee\mathsf{m}^{[2]}\neq\emptyset$

$\langle Stat8\rangle\ \mathsf{ELEM}\Rightarrow\quad\mathsf{m}^{[1]}\neq\emptyset\vee\mathsf{m}^{[2]}\neq\emptyset$

$\mathsf{Suppose}\Rightarrow\quad Stat9:\ \mathsf{n}^{[1]}\neq\emptyset$

$\langle Stat3\rangle\ \mathsf{ELEM}\Rightarrow\quad\mathsf{n}^{[2]}=\emptyset$

$\mathsf{EQUAL}\Rightarrow\quad\mathsf{Red}(\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]}+\mathsf{m}^{[2]}*\emptyset,\mathsf{m}^{[1]}*\emptyset+\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\right])=[\emptyset,\emptyset]$

$\mathsf{ALGEBRA}\Rightarrow\quad Stat10:\ [\emptyset,\emptyset]=\mathsf{Red}(\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]},\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\right])$

$\mathsf{Use\_def}(\mathsf{Red})\Rightarrow\quad\mathsf{Red}(\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]},\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\right])=\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]}-\mathsf{m}^{[1]}*\mathsf{n}^{[1]}\cap(\mathsf{n}^{[1]}*\mathsf{m}^{[2]}),\mathsf{n}^{[1]}*\mathsf{m}^{[2]}-\mathsf{m}^{[1]}*\mathsf{n}^{[1]}\cap(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})\right]$

$\langle\mathsf{n}^{[1]}\rangle\hookrightarrow T209\Rightarrow\quad Stat11:\ \mathsf{n}^{[1]}*\emptyset=\emptyset$

$\mathsf{Suppose}\Rightarrow\quad Stat12:\ \mathsf{m}^{[1]}*\mathsf{n}^{[1]}\cap(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})\neq\emptyset$

$\langle Stat12\rangle\ \mathsf{ELEM}\Rightarrow\quad\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\neq\emptyset\ \&\ \mathsf{m}^{[1]}*\mathsf{n}^{[1]}\neq\emptyset$

$\mathsf{Suppose}\Rightarrow\quad\mathsf{m}^{[2]}=\emptyset$

$\mathsf{EQUAL}\Rightarrow\quad\mathsf{false};\qquad\mathsf{Discharge}\Rightarrow\quad\mathsf{m}^{[1]}=\emptyset$

$\mathsf{EQUAL}\Rightarrow\quad\emptyset*\mathsf{n}^{[1]}\neq\emptyset$

$\langle\emptyset,\mathsf{n}^{[1]}\rangle\hookrightarrow T217\Rightarrow\quad Stat13:\ \mathsf{n}^{[1]}*\emptyset\neq\emptyset$

$\langle Stat11, Stat13\rangle\ \mathsf{ELEM}\Rightarrow\quad\mathsf{false};\qquad\mathsf{Discharge}\Rightarrow\quad\mathsf{m}^{[1]}*\mathsf{n}^{[1]}\cap(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})=\emptyset$

$\mathsf{EQUAL}\Rightarrow\quad\mathsf{Red}(\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]},\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\right])=\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]}-\emptyset,\mathsf{n}^{[1]}*\mathsf{m}^{[2]}-\emptyset\right]$

$\langle\mathsf{m}^{[1]}*\mathsf{n}^{[1]}\rangle\hookrightarrow T230\Rightarrow\quad\mathsf{m}^{[1]}*\mathsf{n}^{[1]}-\emptyset=\#(\mathsf{m}^{[1]}*\mathsf{n}^{[1]})$

$\langle\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\rangle\hookrightarrow T230\Rightarrow\quad\mathsf{n}^{[1]}*\mathsf{m}^{[2]}-\emptyset=\#(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})$

$\mathsf{EQUAL}\Rightarrow\quad Stat14:\ \mathsf{Red}(\left[\mathsf{m}^{[1]}*\mathsf{n}^{[1]},\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\right])=\left[\#(\mathsf{m}^{[1]}*\mathsf{n}^{[1]}),\#(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})\right]$

$\langle Stat14, Stat10\rangle\ \mathsf{ELEM}\Rightarrow\quad Stat15:\ \left[\#(\mathsf{m}^{[1]}*\mathsf{n}^{[1]}),\#(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})\right]=[\emptyset,\emptyset]$

$\langle Stat15\rangle\ \mathsf{ELEM}\Rightarrow\quad\#(\mathsf{m}^{[1]}*\mathsf{n}^{[1]})=\emptyset\ \&\ \#(\mathsf{n}^{[1]}*\mathsf{m}^{[2]})=\emptyset$

$\langle\mathsf{m}^{[1]}*\mathsf{n}^{[1]}\rangle\hookrightarrow T136\Rightarrow\quad Stat16:\ \mathsf{m}^{[1]}*\mathsf{n}^{[1]}=\emptyset$

$\langle\mathsf{n}^{[1]}*\mathsf{m}^{[2]}\rangle\hookrightarrow T136\Rightarrow\quad Stat17:\ \mathsf{n}^{[1]}*\mathsf{m}^{[2]}=\emptyset$

$\langle\mathsf{m}^{[1]},\mathsf{n}^{[1]}\rangle\hookrightarrow T254([Stat16, Stat9])\Rightarrow\quad\mathsf{m}^{[1]}=\emptyset$

$\langle\mathsf{n}^{[1]},\mathsf{m}^{[2]}\rangle\hookrightarrow T254([Stat17, Stat9])\Rightarrow\quad\mathsf{m}^{[2]}=\emptyset$

$\mathsf{ELEM}\Rightarrow\quad\mathsf{false};\qquad\mathsf{Discharge}\Rightarrow\quad Stat18:\ \mathsf{n}^{[1]}=\emptyset\ \&\ \mathsf{n}^{[2]}\neq\emptyset$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Red}([\mathsf{m}^{[1]} * \emptyset + \mathsf{m}^{[2]} * \mathsf{n}^{[2]}, \mathsf{m}^{[1]} * \mathsf{n}^{[2]} + \emptyset * \mathsf{m}^{[2]}]) = [\emptyset, \emptyset]$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathit{Stat19}: \ [\emptyset, \emptyset] = \mathsf{Red}([\mathsf{m}^{[2]} * \mathsf{n}^{[2]}, \mathsf{m}^{[1]} * \mathsf{n}^{[2]}])$
$\mathsf{Use\_def(Red)} \Rightarrow \quad \mathsf{Red}([\mathsf{m}^{[2]} * \mathsf{n}^{[2]}, \mathsf{m}^{[1]} * \mathsf{n}^{[2]}]) = [\mathsf{m}^{[2]} * \mathsf{n}^{[2]} - \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \cap (\mathsf{m}^{[1]} * \mathsf{n}^{[2]}), \mathsf{m}^{[1]} * \mathsf{n}^{[2]} - \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \cap (\mathsf{m}^{[1]} * \mathsf{n}^{[2]})]$
$\langle \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T209} \Rightarrow \quad \mathit{Stat20}: \ \mathsf{n}^{[2]} * \emptyset = \emptyset$
$\langle \mathsf{n}^{[2]}, \emptyset \rangle \hookrightarrow \mathit{T217}(\langle \mathit{Stat20} \rangle) \Rightarrow \quad \mathit{Stat21}: \ \emptyset * \mathsf{n}^{[2]} = \emptyset$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat22}: \ \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \cap (\mathsf{m}^{[1]} * \mathsf{n}^{[2]}) \neq \emptyset$
$\langle \mathit{Stat22} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \neq \emptyset \ \& \ \mathsf{m}^{[1]} * \mathsf{n}^{[2]} \neq \emptyset$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{m}^{[2]} = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{m}^{[1]} = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \emptyset * \mathsf{n}^{[2]} \neq \emptyset$
$\langle \emptyset, \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T217} \Rightarrow \quad \mathit{Stat23}: \ \mathsf{n}^{[2]} * \emptyset \neq \emptyset$
$\langle \mathit{Stat20}, \mathit{Stat23} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \cap (\mathsf{m}^{[1]} * \mathsf{n}^{[2]}) = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Red}([\mathsf{m}^{[2]} * \mathsf{n}^{[2]}, \mathsf{m}^{[1]} * \mathsf{n}^{[2]}]) = [\mathsf{m}^{[2]} * \mathsf{n}^{[2]} - \emptyset, \mathsf{m}^{[1]} * \mathsf{n}^{[2]} - \emptyset]$
$\langle \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T230} \Rightarrow \quad \mathsf{m}^{[2]} * \mathsf{n}^{[2]} - \emptyset = \#(\mathsf{m}^{[2]} * \mathsf{n}^{[2]})$
$\langle \mathsf{m}^{[1]} * \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T230} \Rightarrow \quad \mathsf{m}^{[1]} * \mathsf{n}^{[2]} - \emptyset = \#(\mathsf{m}^{[1]} * \mathsf{n}^{[2]})$
$\mathsf{EQUAL} \Rightarrow \quad \mathit{Stat24}: \ \mathsf{Red}([\mathsf{m}^{[2]} * \mathsf{n}^{[2]}, \mathsf{m}^{[1]} * \mathsf{n}^{[2]}]) = [\#(\mathsf{m}^{[2]} * \mathsf{n}^{[2]}), \#(\mathsf{m}^{[1]} * \mathsf{n}^{[2]})]$
$\langle \mathit{Stat24}, \mathit{Stat19} \rangle \ \mathsf{ELEM} \Rightarrow \quad \mathit{Stat25}: \ [\#(\mathsf{m}^{[2]} * \mathsf{n}^{[2]}), \#(\mathsf{m}^{[1]} * \mathsf{n}^{[2]})] = [\emptyset, \emptyset]$
$\langle \mathit{Stat25} \rangle \ \mathsf{ELEM} \Rightarrow \quad \#(\mathsf{m}^{[2]} * \mathsf{n}^{[2]}) = \emptyset \ \& \ \#(\mathsf{m}^{[1]} * \mathsf{n}^{[2]}) = \emptyset$
$\langle \mathsf{m}^{[2]} * \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T136} \Rightarrow \quad \mathit{Stat26}: \ \mathsf{m}^{[2]} * \mathsf{n}^{[2]} = \emptyset$
$\langle \mathsf{m}^{[1]} * \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T136} \Rightarrow \quad \mathit{Stat27}: \ \mathsf{m}^{[1]} * \mathsf{n}^{[2]} = \emptyset$
$\langle \mathsf{m}^{[1]}, \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T254}([\mathit{Stat27}, \mathit{Stat18}]) \Rightarrow \quad \mathsf{m}^{[1]} = \emptyset$
$\langle \mathsf{m}^{[2]}, \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T254}([\mathit{Stat26}, \mathit{Stat18}]) \Rightarrow \quad \mathsf{m}^{[2]} = \emptyset$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we prove the distributivity of multiplication over subtraction.

-- Distributivity of multiplication over subtraction

Theorem 414 (331) $\mathsf{N}, \mathsf{M}, \mathsf{K} \in \mathbb{Z} \rightarrow \mathsf{m} *_{\mathbb{Z}} \mathsf{n} -_{\mathbb{Z}} \mathsf{k} *_{\mathbb{Z}} \mathsf{n} = (\mathsf{m} -_{\mathbb{Z}} \mathsf{k}) *_{\mathbb{Z}} \mathsf{n}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}, \mathsf{m}, \mathsf{k}) \Rightarrow \quad \mathsf{n}, \mathsf{m}, \mathsf{k} \in \mathbb{Z} \ \& \ \mathsf{m} *_{\mathbb{Z}} \mathsf{n} -_{\mathbb{Z}} \mathsf{k} *_{\mathbb{Z}} \mathsf{n} \neq (\mathsf{m} -_{\mathbb{Z}} \mathsf{k}) *_{\mathbb{Z}} \mathsf{n}$

-- For suppose that signed integers n, m, k are a counterexample to our assertion. Using Theorems 289 and T315 we can rewrite the negative of our assertion (in a form using $+_{\mathbb{Z}}$ instead of $-_{\mathbb{Z}}$) as $\mathsf{m} *_{\mathbb{Z}} \mathsf{n} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k}) *_{\mathbb{Z}} \mathsf{n} \neq (\mathsf{m} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k})) *_{\mathbb{Z}} \mathsf{n}$

$\langle \mathsf{m}, \mathsf{n} \rangle \hookrightarrow \mathit{T294} \Rightarrow \quad \mathsf{m} *_{\mathbb{Z}} \mathsf{n} \in \mathbb{Z}$
$\langle \mathsf{k}, \mathsf{n} \rangle \hookrightarrow \mathit{T294} \Rightarrow \quad \mathsf{k} *_{\mathbb{Z}} \mathsf{n} \in \mathbb{Z}$
$\langle \mathsf{m} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{n} \rangle \hookrightarrow \mathit{T321} \Rightarrow \quad \mathsf{m} *_{\mathbb{Z}} \mathsf{n} -_{\mathbb{Z}} \mathsf{k} *_{\mathbb{Z}} \mathsf{n} = \mathsf{m} *_{\mathbb{Z}} \mathsf{n} +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{k} *_{\mathbb{Z}} \mathsf{n})$

$\langle m, k \rangle \hookrightarrow T321 \Rightarrow \quad m -_{\mathbb{Z}} k = m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)$
$\langle k, n \rangle \hookrightarrow T315 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(k *_{\mathbb{Z}} n) = \mathrm{Rev}_{\mathbb{Z}}(k) *_{\mathbb{Z}} n$
$\mathrm{EQUAL} \Rightarrow \quad m *_{\mathbb{Z}} n +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k) *_{\mathbb{Z}} n \neq \left(m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)\right) *_{\mathbb{Z}} n$

-- Then, using the commutativity of the $*_{\mathbb{Z}}$ operator, we can rewrite the above inequality as $n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k) \neq n *_{\mathbb{Z}} \left(m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)\right)$ which stands in contradiction to the distributivity rule for signed integer addition, and so proves our theorem.

$\langle k \rangle \hookrightarrow T314 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(k) \in \mathbb{Z}$
$\langle m, \mathrm{Rev}_{\mathbb{Z}}(k) \rangle \hookrightarrow T294 \Rightarrow \quad m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k) \in \mathbb{Z}$
$\langle m, n \rangle \hookrightarrow T307 \Rightarrow \quad m *_{\mathbb{Z}} n = n *_{\mathbb{Z}} m$
$\langle \mathrm{Rev}_{\mathbb{Z}}(k), n \rangle \hookrightarrow T307 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(k) *_{\mathbb{Z}} n = n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)$
$\langle m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k), n \rangle \hookrightarrow T307 \Rightarrow \quad \left(m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)\right) *_{\mathbb{Z}} n = n *_{\mathbb{Z}} \left(m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)\right)$
$\mathrm{EQUAL} \Rightarrow \quad n *_{\mathbb{Z}} m +_{\mathbb{Z}} n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k) \neq n *_{\mathbb{Z}} \left(m +_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)\right)$
$\langle \mathrm{Rev}_{\mathbb{Z}}(k), n \rangle \hookrightarrow T307 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Z}}(k) *_{\mathbb{Z}} n = n *_{\mathbb{Z}} \mathrm{Rev}_{\mathbb{Z}}(k)$
$\langle \mathrm{Rev}_{\mathbb{Z}}(k), n, m \rangle \hookrightarrow T309 \Rightarrow \quad \mathrm{false}; \qquad \mathrm{Discharge} \Rightarrow$ QED

-- Our next theorem states the principle of cancellation for the ring of signed integers. The proof is straightforward and algebraic: from $m *_{\mathbb{Z}} n = k *_{\mathbb{Z}} n$ deduce $(m -_{\mathbb{Z}} k) *_{\mathbb{Z}} n = \emptyset$, and then apply theorem 330.

-- Si Cancellation

Theorem 415 (332) $N, M, K \in \mathbb{Z} \;\&\; M *_{\mathbb{Z}} N = K *_{\mathbb{Z}} N \;\&\; N \neq [\emptyset, \emptyset] \rightarrow M = K$. PROOF:

$\mathrm{Suppose\_not}(n, m, k) \Rightarrow \quad n, m, k \in \mathbb{Z} \;\&\; m *_{\mathbb{Z}} n = k *_{\mathbb{Z}} n \;\&\; n \neq [\emptyset, \emptyset] \;\&\; m \neq k$
$\mathrm{EQUAL} \Rightarrow \quad m *_{\mathbb{Z}} n -_{\mathbb{Z}} k *_{\mathbb{Z}} n = k *_{\mathbb{Z}} n -_{\mathbb{Z}} k *_{\mathbb{Z}} n$
$\langle k, n \rangle \hookrightarrow T294 \Rightarrow \quad k *_{\mathbb{Z}} n \in \mathbb{Z}$
$\langle k *_{\mathbb{Z}} n \rangle \hookrightarrow T327 \Rightarrow \quad m *_{\mathbb{Z}} n -_{\mathbb{Z}} k *_{\mathbb{Z}} n = [\emptyset, \emptyset]$
$\langle m, k \rangle \hookrightarrow T295 \Rightarrow \quad m -_{\mathbb{Z}} k \in \mathbb{Z}$
$\langle n, m, k \rangle \hookrightarrow T331 \Rightarrow \quad (m -_{\mathbb{Z}} k) *_{\mathbb{Z}} n = [\emptyset, \emptyset]$
$\langle n, m -_{\mathbb{Z}} k \rangle \hookrightarrow T330 \Rightarrow \quad m -_{\mathbb{Z}} k = [\emptyset, \emptyset]$
$\mathrm{EQUAL} \Rightarrow \quad k +_{\mathbb{Z}} (m -_{\mathbb{Z}} k) = k +_{\mathbb{Z}} [\emptyset, \emptyset]$
$\langle m, k \rangle \hookrightarrow T322 \Rightarrow \quad m = k +_{\mathbb{Z}} [\emptyset, \emptyset]$
$\langle k \rangle \hookrightarrow T328 \Rightarrow \quad \mathrm{false}; \qquad \mathrm{Discharge} \Rightarrow$ QED

-- Next we prove that the product of any signed integer n by my minus 0n e is the reverse of n.

-- Multiplication by - 1

Theorem 416 ($\textcolor{magenta}{333}$) $\mathsf{N} \in \mathbb{Z} \rightarrow \mathsf{Rev}_{\mathbb{Z}}(\mathsf{N}) = [\emptyset, 1] *_{\mathbb{Z}} \mathsf{N}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}) \Rightarrow \quad \mathsf{n} \in \mathbb{Z} \ \& \ \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \neq [\emptyset, 1] *_{\mathbb{Z}} \mathsf{n}$

$\langle \mathsf{n} \rangle \hookrightarrow T292 \Rightarrow \quad \mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right] \ \& \ \mathsf{n}^{[1]} = \emptyset \vee \mathsf{n}^{[2]} = \emptyset \ \& \ \mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{N}$

$\mathsf{Use\_def}(*_{\mathbb{Z}}) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \neq \mathsf{Red}\left(\left[[\emptyset, 1]^{[1]} * \mathsf{n}^{[1]} + [\emptyset, 1]^{[2]} * \mathsf{n}^{[2]}, [\emptyset, 1]^{[1]} * \mathsf{n}^{[2]} + \mathsf{n}^{[1]} * [\emptyset, 1]^{[2]}\right]\right)$

$\mathsf{ELEM} \Rightarrow \quad [\emptyset, 1]^{[1]} = \emptyset \ \& \ [\emptyset, 1]^{[2]} = 1$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \neq \mathsf{Red}\left(\left[\emptyset * \mathsf{n}^{[1]} + 1 * \mathsf{n}^{[2]}, \emptyset * \mathsf{n}^{[2]} + \mathsf{n}^{[1]} * 1\right]\right)$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) \neq \mathsf{Red}\left(\left[\mathsf{n}^{[2]}, \mathsf{n}^{[1]}\right]\right)$

$\mathsf{Use\_def}(\mathsf{Red}) \Rightarrow \quad \mathsf{Red}\left(\left[\mathsf{n}^{[2]}, \mathsf{n}^{[1]}\right]\right) = \left[\mathsf{n}^{[2]} - \mathsf{n}^{[2]} \cap \mathsf{n}^{[1]}, \mathsf{n}^{[1]} - \mathsf{n}^{[2]} \cap \mathsf{n}^{[1]}\right]$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{n}^{[2]} \cap \mathsf{n}^{[1]} = \emptyset$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Red}\left(\left[\mathsf{n}^{[2]}, \mathsf{n}^{[1]}\right]\right) = \left[\mathsf{n}^{[2]} - \emptyset, \mathsf{n}^{[1]} - \emptyset\right]$

$\left\langle \mathsf{n}^{[2]} \right\rangle \hookrightarrow T230 \Rightarrow \quad \mathsf{n}^{[2]} - \emptyset = \#\mathsf{n}^{[2]}$

$\left\langle \mathsf{n}^{[1]} \right\rangle \hookrightarrow T230 \Rightarrow \quad \mathsf{n}^{[1]} - \emptyset = \#\mathsf{n}^{[1]}$

$\left\langle \mathsf{n}^{[2]} \right\rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Card}(\mathsf{n}^{[2]})$

$\left\langle \mathsf{n}^{[1]} \right\rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Card}(\mathsf{n}^{[1]})$

$\left\langle \mathsf{n}^{[2]} \right\rangle \hookrightarrow T138 \Rightarrow \quad \#\mathsf{n}^{[2]} = \mathsf{n}^{[2]}$

$\left\langle \mathsf{n}^{[1]} \right\rangle \hookrightarrow T138 \Rightarrow \quad \#\mathsf{n}^{[1]} = \mathsf{n}^{[1]}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Red}\left(\left[\mathsf{n}^{[2]}, \mathsf{n}^{[1]}\right]\right) = \left[\mathsf{n}^{[2]}, \mathsf{n}^{[1]}\right]$

$\mathsf{Use\_def}(\mathsf{Rev}_{\mathbb{Z}}) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}) = \left[\mathsf{n}^{[2]}, \mathsf{n}^{[1]}\right]$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

11 Mathematical induction for integers; the general summation operator

-- We now develop the standard theory of mathematical induction (for integers), which tells us that if there exists an integer n having some property $\mathsf{P}(\mathsf{n})$, there exists a smallest integer m having the property $\mathsf{P}(\mathsf{m})$.

THEORY mathematical_induction$(\mathsf{n}, \mathsf{P}(\mathsf{x}))$
$\quad \mathsf{n} \in \mathbb{N} \ \& \ \mathsf{P}(\mathsf{n})$
END mathematical_induction

ENTER_THEORY mathematical_induction

-- We begin with two small 'glue' theorems, the first of which merely resates the assumptionof the present theory, therby making it available to the theorem-level APPLY inference which follows.

Theorem 417 (mathematical_induction$_{00}$) $\mathsf{n} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{n})$. PROOF:

Suppose_not $\Rightarrow$ $\neg\big(\mathsf{n} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{n})\big)$
Assump $\Rightarrow$ $\mathsf{n} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{n})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we APPLY transfinite_induction, to get a conclusion close to that which we desire.

APPLY $\langle \mathsf{mt}_\Theta : \mathsf{m}_\Theta \rangle$ transfinite_induction$\Big(\mathsf{n} \mapsto \mathsf{n}, \mathsf{P}(\mathsf{x}) \mapsto \big(\mathsf{x} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{x})\big)\Big) \Rightarrow$

Theorem 418 (mathematical_induction$_0$) $\big\langle \forall \mathsf{k} \,|\, \big(\mathsf{m}_\Theta \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{m}_\Theta)\big) \,\&\, \Big(\mathsf{k} \in \mathsf{m}_\Theta \rightarrow \neg\big(\mathsf{k} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{k})\big)\Big)\big\rangle$.

-- The following result, which is the sole externally useful theorem of the present theory, shows that the quantity m_Θ supplied by applying standard transfinite induction to the predicate $\mathsf{n} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{n})$ has the minimality property we desire.

Theorem 419 (mathematical_induction$_1$) $\mathsf{m}_\Theta \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{m}_\Theta) \,\&\, \big\langle \forall \mathsf{k} \in \mathsf{m}_\Theta \,|\, \neg \mathsf{P}(\mathsf{k})\big\rangle$. PROOF:

Suppose_not $\Rightarrow$ *Stat1* : $\neg\big(\mathsf{m}_\Theta \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{m}_\Theta) \,\&\, \big\langle \forall \mathsf{k} \in \mathsf{m}_\Theta \,|\, \neg \mathsf{P}(\mathsf{k})\big\rangle\big)$

-- Unwrap part of quantified statement which is not affected by the quantifier:

Tmathematical_induction0 $\Rightarrow$ *Stat2* : $\big\langle \forall \mathsf{k} \,|\, \big(\mathsf{m}_\Theta \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{m}_\Theta)\big) \,\&\, \Big(\mathsf{k} \in \mathsf{m}_\Theta \rightarrow \neg\big(\mathsf{k} \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{k})\big)\Big)\big\rangle$
$\langle \emptyset \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{m}_\Theta \in \mathbb{N} \,\&\, \mathsf{P}(\mathsf{m}_\Theta)$

-- For if not there would be some member m of m_Θ having the property $\mathsf{P}(\mathsf{m})$, and since this m would necessarily be an integer we have a contradiction which proves our assertion.

ELEM $\Rightarrow$ *Stat3* : $\neg\big\langle \forall \mathsf{n} \in \mathsf{m}_\Theta \,|\, \neg \mathsf{P}(\mathsf{n})\big\rangle$
$\langle \mathsf{m} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\mathsf{m} \in \mathsf{m}_\Theta \,\&\, \mathsf{P}(\mathsf{m})$
$\langle \mathsf{m} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathsf{m} \notin \mathbb{N}$
$\langle \mathsf{junk} \rangle \hookrightarrow$ *T179* $\Rightarrow$ $\mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, \mathsf{m}_\Theta \rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathsf{m}_\Theta)$
$\langle \mathsf{m}_\Theta, \mathsf{m} \rangle \hookrightarrow$ *T11* $\Rightarrow$ $\mathcal{O}(\mathsf{m})$

$\langle\mathbb{N},\mathsf{m}\rangle\hookrightarrow T32 \Rightarrow \quad \mathbb{N}\subseteq\mathsf{m}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY mathematical_induction

THEORY mathematical_induction$\big(\mathsf{P}(\mathsf{x})\big)$
$\langle\exists\mathsf{n}\in\mathbb{N}\,|\,\mathsf{P}(\mathsf{n})\rangle$
$\Rightarrow (\mathsf{m}_\Theta)$
$\mathsf{m}_\Theta\in\mathbb{N}\ \&\ \mathsf{P}(\mathsf{m}_\Theta)\ \&\ \langle\forall\mathsf{n}\in\mathsf{m}_\Theta\,|\,\neg\mathsf{P}(\mathsf{n})\rangle$
END mathematical_induction

-- One sometimes needs to give proofs by 'double induction', which use the fact that if there exist any n and k satisfying a 2-variable predicate $\mathsf{R}(\mathsf{n},\mathsf{k})$, then there exist m and j satisfying this same predicate, which are minimal in the sense that $\mathsf{R}(\mathsf{k},\mathsf{i})$ must be false for any k less than m and any i at all, while $\mathsf{R}(\mathsf{m},\mathsf{i})$ must also be false for any i less than j. We give theories capable of supplying this fact in two variants: the first in which n and k are general sets, the second in which all the quantities involved are integers.

THEORY double_transfinite_induction$\big(\mathsf{n},\mathsf{k},\mathsf{R}(\mathsf{x},\mathsf{y})\big)$
$\mathsf{R}(\mathsf{n},\mathsf{k})$
END double_transfinite_induction

ENTER_THEORY double_transfinite_induction

Theorem 420 (double_transfinite_induction · 0) $\langle\exists\mathsf{i}\,|\,\mathsf{R}(\mathsf{n},\mathsf{i})\rangle$. PROOF:

Suppose_not(n) $\Rightarrow$ *Stat0*: $\neg\langle\exists\mathsf{i}\,|\,\mathsf{R}(\mathsf{n},\mathsf{i})\rangle$
Assump $\Rightarrow$ $\mathsf{R}(\mathsf{n},\mathsf{k})$
$\langle\mathsf{k}\rangle\hookrightarrow Stat0 \Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle\mathsf{mt}_\Theta : \mathsf{m}_\Theta\rangle$ transfinite_induction$\big(\mathsf{n}\mapsto\mathsf{n},\mathsf{P}(\mathsf{x})\mapsto\langle\exists\mathsf{i}\,|\,\mathsf{R}(\mathsf{x},\mathsf{i})\rangle\big)\Rightarrow$

Theorem 421 (double_transfinite_induction · 1) $\langle\forall\mathsf{k}\,|\,\langle\exists\mathsf{i}\,|\,\mathsf{R}(\mathsf{m}_\Theta,\mathsf{i})\rangle\ \&\ \big(\mathsf{k}\in\mathsf{m}_\Theta\rightarrow\neg\langle\exists\mathsf{i}\,|\,\mathsf{R}(\mathsf{k},\mathsf{i})\rangle\big)\rangle$.

Theorem 422 (double_transfinite_induction · 2) $\langle\exists\mathsf{i}\,|\,\mathsf{R}(\mathsf{m}_\Theta,\mathsf{i})\rangle$. PROOF:

$\mathsf{Suppose\_not}(m_\Theta) \Rightarrow \quad \mathit{Stat0}: \neg\langle\exists i \mid R(m_\Theta, i)\rangle$
$\mathit{Tdouble\_transfinite\_induction} \cdot 1 \Rightarrow \quad \mathit{Stat1}: \langle\forall k \mid \langle\exists i \mid R(m_\Theta, i)\rangle \,\&\, (k \in m_\Theta \rightarrow \neg\langle\exists i \mid R(k, i)\rangle)\rangle$
$\langle\emptyset\rangle\hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathit{Stat2}: \langle\exists i \mid R(m_\Theta, i)\rangle$
$\langle \mathit{Stat0}, \mathit{Stat2}\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle v1_\Theta : ei\rangle$ Skolem$\Rightarrow$

Theorem 423 (double_transfinite_induction · 3) $R(m_\Theta, ei)$.

APPLY $\langle mt_\Theta : j_\Theta\rangle$ transfinite_induction$(n \mapsto ei, P(i) \mapsto R(m_\Theta, i))\Rightarrow$

Theorem 424 (double_transfinite_induction · 4) $\langle\forall i \mid R(m_\Theta, j_\Theta) \,\&\, (i \in j_\Theta \rightarrow \neg R(m_\Theta, i))\rangle$.

-- As an obvious corollary of the statements proved by the preceding two applications of transfinite induction, we get the following statement, which will be externalized by the present THEORY.

Theorem 425 (double_transfinite_induction · 5) $R(m_\Theta, j_\Theta) \,\&\, (K \in m_\Theta \rightarrow \neg R(K, I)) \,\&\, (I \in j_\Theta \rightarrow \neg R(m_\Theta, I))$. PROOF:

$\mathsf{Suppose\_not}(k, i) \Rightarrow \quad \neg R(m_\Theta, j_\Theta) \vee (k \in m_\Theta \,\&\, R(k, i)) \vee (i \in j_\Theta \,\&\, R(m_\Theta, i))$
Suppose $\Rightarrow \quad \mathit{Stat0}: k \in m_\Theta \,\&\, R(k, i)$
$\mathit{Tdouble\_transfinite\_induction} \cdot 1 \Rightarrow \quad \mathit{Stat1}: \langle\forall k \mid \langle\exists i \mid R(m_\Theta, i)\rangle \,\&\, (k \in m_\Theta \rightarrow \neg\langle\exists i \mid R(k, i)\rangle)\rangle$
$\langle k\rangle\hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathit{Stat3}: \neg\langle\exists i \mid R(k, i)\rangle$
$\langle i\rangle\hookrightarrow \mathit{Stat3} \Rightarrow$ false; Discharge $\Rightarrow \quad \neg R(m_\Theta, j_\Theta) \vee (i \in j_\Theta \,\&\, R(m_\Theta, i))$
$\mathit{Tdouble\_transfinite\_induction} \cdot 4 \Rightarrow \quad \mathit{Stat4}: \langle\forall i \mid R(m_\Theta, j_\Theta) \,\&\, (i \in j_\Theta \rightarrow \neg R(m_\Theta, i))\rangle$
$\langle i\rangle\hookrightarrow \mathit{Stat4} \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY double_transfinite_induction

THEORY double_transfinite_induction$(R(x, y))$
$\langle\exists n, k \mid R(n, k)\rangle$
$\Rightarrow (m_\Theta, j_\Theta)$

$R(m_\Theta, j_\Theta)$ & $\langle \forall k \in m_\Theta, i \mid \neg R(k,i)\rangle$ & $\langle \forall i \in j_\Theta \mid \neg R(m_\Theta, i)\rangle$

END double_transfinite_induction

-- The following simple variant of the preceding theory tells us that if there exist any integers n and k satisfying a 2-variable predicate $R(n,k)$, then there exist integers m and j satisfyfing this same predicate, which are minimal in the sense that $R(k,i)$ must be false for any integer k less than m and any integer i at all, while $R(m,i)$ must also be false for any integer i less than j.

THEORY double_mathematical_induction$(n, k, R(x,y))$

$n, k \in \mathbb{N}$ & $R(n,k)$

END double_mathematical_induction

ENTER_THEORY double_mathematical_induction

Theorem 426 (double_mathematical_induction · 0) $\langle \exists i \mid i \in \mathbb{N}$ & $R(n,i)\rangle$. PROOF:

Suppose_not(n) ⇒ *Stat0*: $\neg\langle \exists i \mid i \in \mathbb{N}$ & $R(n,i)\rangle$

Assump ⇒ $k \in \mathbb{N}$ & $R(n,k)$

$\langle k\rangle \hookrightarrow$ *Stat0* ⇒ false; Discharge ⇒ QED

APPLY $\langle m_\Theta : mm_\Theta\rangle$ mathematical_induction$(n \mapsto n, P(x) \mapsto \langle \exists i \mid i \in \mathbb{N}$ & $R(x,i)\rangle)$⇒

Theorem 427 (double_mathematical_induction · 1) $mm_\Theta \in \mathbb{N}$ & $\langle \exists i \mid i \in \mathbb{N}$ & $R(mm_\Theta, i)\rangle$ & $\langle \forall k \in mm_\Theta \mid \neg\langle \exists i \mid i \in \mathbb{N}$ & $R(k,i)\rangle\rangle$.

Theorem 428 (double_mathematical_induction · 2) $\langle \exists i \mid i \in \mathbb{N}$ & $R(mm_\Theta, i)\rangle$. PROOF:

Suppose_not(mm_Θ) ⇒ *Stat0*: $\neg\langle \exists i \mid i \in \mathbb{N}$ & $R(mm_\Theta, i)\rangle$

Tdouble_mathematical_induction · 1 ⇒ *Stat1*: $\langle \forall k \mid mm_\Theta \in \mathbb{N}$ & $\langle \exists i \mid i \in \mathbb{N}$ & $R(mm_\Theta, i)\rangle$ & $(k \in mm_\Theta \rightarrow \neg\langle \exists i \mid i \in \mathbb{N}$ & $R(k,i)\rangle)\rangle$

$\langle \emptyset\rangle \hookrightarrow$ *Stat1* ⇒ *Stat2*: $\langle \exists i \mid i \in \mathbb{N}$ & $R(mm_\Theta, i)\rangle$

$\langle$ *Stat0, Stat2* $\rangle$ ELEM ⇒ false; Discharge ⇒ QED

APPLY $\langle v1_\Theta : ei\rangle$ Skolem⇒

Theorem 429 (double_mathematical_induction · 3) $ei \in \mathbb{N}$ & $R(mm_\Theta, ei)$.

APPLY $\langle m_\Theta : j_\Theta \rangle$ mathematical_induction$\big(n \mapsto ei, P(x) \mapsto R(mm_\Theta, x)\big) \Rightarrow$

Theorem 430 (double_mathematical_induction · 4) $\langle \forall i \mid j_\Theta \in \mathbb{N} \,\&\, R(mm_\Theta, j_\Theta) \,\&\, \Big(i \in j_\Theta \rightarrow \neg\big(i \in \mathbb{N} \,\&\, R(mm_\Theta, i)\big)\Big)\rangle$.

-- As a straightforward corollary of the statements proved by the preceding two applications of mathematical induction, we get the following statement, which will be externalized by the present THEORY.

Theorem 431 (double_mathematical_induction · 5) $mm_\Theta, j_\Theta \in \mathbb{N} \,\&\, R(mm_\Theta, j_\Theta) \,\&\, \big(K \in mm_\Theta \,\&\, I \in \mathbb{N} \rightarrow \neg R(K, I)\big) \,\&\, \big(I \in j_\Theta \rightarrow \neg R(mm_\Theta, I)\big)$. Proof:

Suppose_not$(k, i) \Rightarrow$ $mm_\Theta \notin \mathbb{N} \vee j_\Theta \notin \mathbb{N} \vee \neg R(mm_\Theta, j_\Theta) \vee \big(k \in mm_\Theta \,\&\, i \in \mathbb{N} \,\&\, R(k, i)\big) \vee \big(i \in j_\Theta \,\&\, R(mm_\Theta, i)\big)$
Suppose $\Rightarrow$ $mm_\Theta \notin \mathbb{N} \vee \big(k \in mm_\Theta \,\&\, i \in \mathbb{N} \,\&\, R(k, i)\big)$
Tdouble_mathematical_induction · 1 $\Rightarrow$ $mm_\Theta \in \mathbb{N} \,\&\, \langle \exists i \mid i \in \mathbb{N} \,\&\, R(mm_\Theta, i)\rangle \,\&\, Stat1 : \langle \forall k \in mm_\Theta \mid \neg\langle \exists i \mid i \in \mathbb{N} \,\&\, R(k, i)\rangle\rangle$
$\langle k \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat3 : \neg\langle \exists i \mid i \in \mathbb{N} \,\&\, R(k, i)\rangle$
$\langle i \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $j_\Theta \notin \mathbb{N} \vee \neg R(mm_\Theta, j_\Theta) \vee \big(i \in j_\Theta \,\&\, R(mm_\Theta, i)\big)$
Tdouble_mathematical_induction · 4 $\Rightarrow$ $Stat4 : \langle \forall i \mid j_\Theta \in \mathbb{N} \,\&\, R(mm_\Theta, j_\Theta) \,\&\, \Big(i \in j_\Theta \rightarrow \neg\big(i \in \mathbb{N} \,\&\, R(mm_\Theta, i)\big)\Big)\rangle$
$\langle i \rangle \hookrightarrow Stat4 \Rightarrow$ $i \in j_\Theta \,\&\, j_\Theta \in \mathbb{N} \,\&\, i \notin \mathbb{N}$
$\langle junk \rangle \hookrightarrow T179 \Rightarrow$ $\mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, j_\Theta \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(j_\Theta)$
$\langle j_\Theta, i \rangle \hookrightarrow T11 \Rightarrow$ $\mathcal{O}(i)$
$\langle \mathbb{N}, i \rangle \hookrightarrow T32 \Rightarrow$ $\mathbb{N} \subseteq i$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ Qed

Enter_theory Set_theory

Display double_mathematical_induction

Theory double_mathematical_induction$\big(R(x, y)\big)$
$\langle \exists n \in \mathbb{N}, k \in \mathbb{N} \mid R(n, k)\rangle$
$\Rightarrow (mm_\Theta, j_\Theta)$
$mm_\Theta, j_\Theta \in \mathbb{N} \,\&\, R(mm_\Theta, j_\Theta) \,\&\, \langle \forall k \in mm_\Theta, i \in \mathbb{N} \mid \neg R(k, i)\rangle \,\&\, \langle \forall i \in j_\Theta \mid \neg R(mm_\Theta, i)\rangle$
End double_mathematical_induction

-- Next we prove a general version of the principle that recursive definitions of functions h over any well-founded set are valid if the definition of $\mathsf{h}(\mathsf{x})$ involves only values y which precede y in the well-founded relation of the set. Given a well-ordering relationship $\lhd$, and given any functions $\mathsf{f}(\mathsf{x})$, $\mathsf{g}(\mathsf{x},\mathsf{y},\mathsf{u},\mathsf{v})$, and $\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{u})$ our aim is to prove that therese exists a function h which satisfies the identity (FORALL x in s, t in OM |h_thryvar (x, s, t) = f ({g (h_thryvar (y, s, t), y, x, t): y in s |arg1_bef_arg2 (y, x) & P (h_thryvar (y, s, t), s, y, x, t)}, s, x, t)). This is not an immediate consequence of our principle of recursive definition, which insists on using the membership relator in place of the general well-foundedness relator $\lhd$ which we now consider. We therefore proceed by introducng an enumerator which relates $\lhd$ closely enough to memebership for our principle of recursive definition to be used.

THEORY $\mathsf{wellfounded\_recursive\_fcn}\big(\mathsf{s}, \mathsf{y} \lhd \mathsf{x}, \mathsf{f}(\mathsf{b},\mathsf{x},\mathsf{t}), \mathsf{g}(\mathsf{a},\mathsf{y},\mathsf{x},\mathsf{t}), \mathsf{P}(\mathsf{a},\mathsf{y},\mathsf{x},\mathsf{t})\big)$
$\big\langle \forall \mathsf{t} \,|\, \mathsf{t} \subseteq \mathsf{s} \,\&\, \mathsf{t} \neq \emptyset \longrightarrow \big\langle \exists \mathsf{x} \in \mathsf{t}, \forall \mathsf{y} \in \mathsf{t} \,|\, \neg \mathsf{y} \lhd \mathsf{x} \big\rangle\big\rangle$
END wellfounded_recursive_fcn

ENTER_THEORY wellfounded_recursive_fcn

-- We begin by importing a theorem of the theory well_founded_set into the present theory.

APPLY $\big\langle \mathsf{Minrel}_\Theta : \mathsf{Minrel}_\Theta, \mathsf{orden}_\Theta : \mathsf{orden}, \mathsf{ord}_\Theta : \mathsf{o} \big\rangle$ $\mathsf{well\_founded\_set}(\mathsf{s} \mapsto \mathsf{s}, \mathsf{x} \lhd \mathsf{y} \mapsto \mathsf{x} \lhd \mathsf{y}) \Rightarrow$

Theorem 432 (wellfounded_recursive_fcn · 100) $\big\langle \forall \mathsf{u}, \mathsf{v} \,|\, \mathcal{O}(\mathsf{u}) \,\&\, \mathcal{O}(\mathsf{v}) \,\&\, \mathsf{orden}(\mathsf{u}) \neq \mathsf{s} \,\&\, \mathsf{orden}(\mathsf{u}) \lhd \mathsf{orden}(\mathsf{v}) \longrightarrow \mathsf{u} \in \mathsf{v} \big\rangle \,\&\,$ $\big\langle \exists \mathsf{o} \in \mathsf{next}(\#\mathcal{P}\mathsf{s}) \,|\, \mathcal{O}(\mathsf{o}) \,\&\, \mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\} \,\&\, \big\langle \forall \mathsf{x} \in \mathsf{o} \,|\, \mathsf{orden}(\mathsf{x}) \neq \mathsf{s} \big\rangle \,\&\, 1\text{–}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\})\big\rangle$.

-- To draw the conclusion seen a bit below we must siplify the quantifier which appears in the preceding statement.

Theorem 433 (wellfounded_recursive_fcn · 100a) $\big\langle \exists \mathsf{o} \,|\, \mathcal{O}(\mathsf{o}) \,\&\, \mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\} \,\&\, \big\langle \forall \mathsf{x} \in \mathsf{o} \,|\, \mathsf{orden}(\mathsf{x}) \neq \mathsf{s} \big\rangle \,\&\, 1\text{–}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\})\big\rangle$. PROOF:

Suppose_not(v) $\Rightarrow$ *Stat1* : $\neg\big\langle \exists \mathsf{o} \,|\, \mathcal{O}(\mathsf{o}) \,\&\, \mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\} \,\&\, \big\langle \forall \mathsf{x} \in \mathsf{o} \,|\, \mathsf{orden}(\mathsf{x}) \neq \mathsf{s} \big\rangle \,\&\, 1\text{–}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\})\big\rangle$
Twellfounded_recursive_fcn · 100 $\Rightarrow$ *Stat2* : $\big\langle \exists \mathsf{o} \in \mathsf{next}(\#\mathcal{P}\mathsf{s}) \,|\, \mathcal{O}(\mathsf{o}) \,\&\, \mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\} \,\&\, \big\langle \forall \mathsf{x} \in \mathsf{o} \,|\, \mathsf{orden}(\mathsf{x}) \neq \mathsf{s} \big\rangle \,\&\, 1\text{–}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\})\big\rangle$
$\big\langle \mathsf{o} \big\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\mathcal{O}(\mathsf{o}) \,\&\, \mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\} \,\&\, \big\langle \forall \mathsf{x} \in \mathsf{o} \,|\, \mathsf{orden}(\mathsf{x}) \neq \mathsf{s} \big\rangle \,\&\, 1\text{–}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\})$
$\big\langle \mathsf{o} \big\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We can now introduce a specific constant o by Skolemizing the existential statement seen just above.

APPLY $\langle \mathsf{v1}_\Theta : \mathsf{o} \rangle$ Skolem$\Rightarrow$

Theorem 434 (wellfounded_recursive_fcn · 101) $\mathcal{O}(\mathsf{o})$ & $\mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\}$ & $\langle \forall \mathsf{x} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{x}) \neq \mathsf{s} \rangle$ & $1\text{--}1(\{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\})$.

-- We also introduce an indexing function which will turn out to be the inverse of this one-to-one mapping and to assign consecutive ordinals to the elements of s.

Def 00*r*. $\mathsf{index}(\mathsf{X}) =_{\mathrm{Def}} \mathbf{arb}(\{\mathsf{j} : \mathsf{j} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{j}) = \mathsf{X}\})$

-- The desired function h is not directly definable by a recursive rule based on the well-founded relation. On the other hand, we can define an auxiliary function hh by means of a recursive rule based on membership; and then define h in terms of hh and the indexing function.

Def 00*s*. $\mathsf{hh}(\mathsf{X}, \mathsf{Y}) =_{\mathrm{Def}} \mathsf{f}\Big(\{\mathsf{g}\big(\mathsf{hh}(\mathsf{j}, \mathsf{Y}), \mathsf{orden}(\mathsf{j}), \mathsf{orden}(\mathsf{X}), \mathsf{Y}\big) : \mathsf{j} \in \mathsf{X} \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{orden}(\mathsf{X})\ \&\ \mathsf{P}\big(\mathsf{hh}(\mathsf{j}, \mathsf{Y}), \mathsf{orden}(\mathsf{j}), \mathsf{orden}(\mathsf{X}), \mathsf{Y}\big)\}, \mathsf{orden}(\mathsf{X}), \mathsf{Y}\Big)$

Def 00*t*. $\mathsf{h}_\Theta(\mathsf{X}, \mathsf{Y}) =_{\mathrm{Def}} \mathsf{hh}\big(\mathsf{index}(\mathsf{X}), \mathsf{Y}\big)$

-- We first prove that the 'index' function defined above assigns an ordinal preceding o to every element of s and is a partial inverse of the enumerator orden.

Theorem 435 (wellfounded_recursive_fcn · 1) $\mathsf{V} \in \mathsf{s} \rightarrow \mathsf{index}(\mathsf{V}) \in \mathsf{o}$ & $\mathcal{O}\big(\mathsf{index}(\mathsf{V})\big)$ & $\mathsf{orden}\big(\mathsf{index}(\mathsf{V})\big) = \mathsf{V}$. Proof:

Suppose_not(v) $\Rightarrow$ $\mathsf{v} \in \mathsf{s}$ & $\mathsf{index}(\mathsf{v}) \notin \mathsf{o} \vee \neg\mathcal{O}\big(\mathsf{index}(\mathsf{v})\big) \vee \mathsf{orden}\big(\mathsf{index}(\mathsf{v})\big) \neq \mathsf{v}$

-- For if v is a counterexample to our theorem, it must be the image under the orden of some element $\mathsf{x} \in \mathsf{o}$. hence the set $\{\mathsf{j} : \mathsf{j} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{j}) = \mathsf{v}\}$ cannot be empty, and so the index of v must belong to this set.

Twellfounded_recursive_fcn · 101 $\Rightarrow$ $\mathcal{O}(\mathsf{o})$ & $\mathsf{s} = \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\}$
EQUAL $\Rightarrow$ *Stat1*: $\mathsf{v} \in \{\mathsf{orden}(\mathsf{x}) : \mathsf{x} \in \mathsf{o}\}$
$\langle \mathsf{x} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathsf{v} = \mathsf{orden}(\mathsf{x})$ & $\mathsf{x} \in \mathsf{o}$
Suppose $\Rightarrow$ *Stat2*: $\{\mathsf{j} : \mathsf{j} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{j}) = \mathsf{v}\} = \emptyset$
$\langle \mathsf{x} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{arb}(\{\mathsf{j} : \mathsf{j} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{j}) = \mathsf{v}\}) \in \{\mathsf{j} : \mathsf{j} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{j}) = \mathsf{v}\}$
Use_def(index) $\Rightarrow$ *Stat3*: $\mathsf{index}(\mathsf{v}) \in \{\mathsf{j} : \mathsf{j} \in \mathsf{o} \mid \mathsf{orden}(\mathsf{j}) = \mathsf{v}\}$

-- $\mathsf{index}(\mathsf{v})$ must therefore be an element j of o whose image under orden is v. By our initial assumption this implies, that it cannot be an ordinal. Since o is an ordinal, so that its elements are also ordinals, this leads to an immediate ontradiction.

$\langle j\rangle\hookrightarrow Stat3 \Rightarrow \quad Stat4: \ \mathsf{index}(v) = j \ \& \ j \in o \ \& \ \mathsf{orden}(j) = v$
EQUAL $\Rightarrow \quad \mathsf{orden}\big(\mathsf{index}(v)\big) = v$
$\langle o, \mathsf{index}(v)\rangle\hookrightarrow T11 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Next we prove that for every element v of the set s of the well-founded relation is defined, the subset $\{j : j \in \mathsf{index}(v) \mid \mathsf{orden}(j) \lhd v\}$ can be written as $\{\mathsf{index}(w) : w \in s \mid w \lhd v\}$.

Theorem 436 (wellfounded_recursive_fcn · 2) $V \in s \rightarrow \{j : j \in \mathsf{index}(V) \mid \mathsf{orden}(j) \lhd V\} = \{\mathsf{index}(w) : w \in s \mid w \lhd V\}$. PROOF:

Suppose_not(v) $\Rightarrow \quad v \in s \ \& \ Stat1: \ \{j : j \in \mathsf{index}(v) \mid \mathsf{orden}(j) \lhd v\} \neq \{\mathsf{index}(w) : w \in s \mid w \lhd v\}$

-- Suppose that v is a counterexample v to our assertion, and let c be an element which belongs to one of these two setsbut not the other. If c belongs to $\{\mathsf{index}(w) : w \in s \mid w \lhd v\}$, then c is the index of some $w \in s$. in which case the preceding Theorem tells us that v and w have indices which are ordinals whose images under the enumerator enum are v and w respectively. Theorem wellfounded_recursive_fcn. 100 then tells us that the index of w precedes that of v in the standard ordering of ordinals, which leads easily to a contradiction. in this case.

$\langle v\rangle\hookrightarrow Twellfounded\_recursive\_fcn \cdot 1 \Rightarrow \quad \mathsf{index}(v) \in o \ \& \ \mathcal{O}\big(\mathsf{index}(v)\big) \ \& \ \mathsf{orden}\big(\mathsf{index}(v)\big) = v$
$\langle c\rangle\hookrightarrow Stat1 \Rightarrow \quad c \in \{j : j \in \mathsf{index}(v) \mid \mathsf{orden}(j) \lhd v\} \leftrightarrow c \notin \{\mathsf{index}(w) : w \in s \mid w \lhd v\}$
Suppose $\Rightarrow \quad Stat2: \ c \in \{\mathsf{index}(w) : w \in s \mid w \lhd v\} \ \& \ c \notin \{j : j \in \mathsf{index}(v) \mid \mathsf{orden}(j) \lhd v\}$
$\langle w, \mathsf{index}(w)\rangle\hookrightarrow Stat2 \Rightarrow \quad c = \mathsf{index}(w) \ \& \ w \in s \ \& \ w \lhd v \ \& \ \mathsf{index}(w) \notin \mathsf{index}(v) \vee \neg \mathsf{orden}\big(\mathsf{index}(w)\big) \lhd v$
$\langle w\rangle\hookrightarrow Twellfounded\_recursive\_fcn \cdot 1 \Rightarrow \quad \mathcal{O}\big(\mathsf{index}(w)\big) \ \& \ \mathsf{orden}\big(\mathsf{index}(w)\big) = w$
EQUAL $\Rightarrow \quad \mathsf{orden}\big(\mathsf{index}(w)\big) \lhd v \ \& \ \mathsf{orden}\big(\mathsf{index}(w)\big) \lhd \mathsf{orden}\big(\mathsf{index}(v)\big)$
$Twellfounded\_recursive\_fcn \cdot 100 \Rightarrow \quad Stat3: \ \big\langle \forall u, v \mid \mathcal{O}(u) \ \& \ \mathcal{O}(v) \ \& \ \mathsf{orden}(u) \neq s \ \& \ \mathsf{orden}(u) \lhd \mathsf{orden}(v) \longrightarrow u \in v \big\rangle$
$\langle \mathsf{index}(w), \mathsf{index}(v)\rangle\hookrightarrow Stat3 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat4: \ c \in \{j : j \in \mathsf{index}(v) \mid \mathsf{orden}(j) \lhd v\} \ \& \ c \notin \{\mathsf{index}(w) : w \in s \mid w \lhd v\}$

-- Hence c must belong to $\{j : j \in \mathsf{index}(v) \mid \mathsf{orden}(j) \lhd v\}$; and thus $\mathsf{orden}(c)$ must precede v in our well-founded relation. Moreover, it is readily seen that c belongs to o and that $\mathsf{orden}(c)$ in s.

$\langle j, \mathsf{orden}(c)\rangle\hookrightarrow Stat4 \Rightarrow \quad c = j \ \& \ c \in \mathsf{index}(v) \ \& \ \mathsf{orden}(j) \lhd v \ \& \ c \neq \mathsf{index}\big(\mathsf{orden}(c)\big) \vee \mathsf{orden}(c) \notin s \vee \neg \mathsf{orden}(c) \lhd v$
EQUAL $\Rightarrow \quad \mathsf{orden}(c) \lhd v$
$Twellfounded\_recursive\_fcn \cdot 101 \Rightarrow \quad \mathcal{O}(o) \ \& \ s = \{\mathsf{orden}(x) : x \in o\} \ \& \ 1\text{–}1(\{[x, \mathsf{orden}(x)] : x \in o\})$
$\langle o, \mathsf{index}(v)\rangle\hookrightarrow T12 \Rightarrow \quad c \in o$
Suppose $\Rightarrow \quad \mathsf{orden}(c) \notin s$
EQUAL $\Rightarrow \quad Stat5: \ \mathsf{orden}(c) \notin \{\mathsf{orden}(x) : x \in o\}$
$\langle c\rangle\hookrightarrow Stat5 \Rightarrow \quad \mathsf{orden}(c) \in s \ \& \ c \neq \mathsf{index}\big(\mathsf{orden}(c)\big)$

-- Using Theorem wellfounded_recursive_fcn. 1 again, we find that the index i of $\mathsf{orden}(\mathsf{c})$ belongs to o and has $\mathsf{orden}(\mathsf{i}) = \mathsf{orden}(\mathsf{c})$. Since orden is one-to-one over s, we get a contradiction in this case too, proving our theorem.

$\langle \mathsf{orden}(\mathsf{c}) \rangle \hookrightarrow Twellfounded\_recursive\_fcn \cdot 1 \Rightarrow \quad \mathsf{index}(\mathsf{orden}(\mathsf{c})) \in \mathsf{o} \;\&\; \mathsf{orden}\Big(\mathsf{index}(\mathsf{orden}(\mathsf{c}))\Big) = \mathsf{orden}(\mathsf{c})$

Suppose $\Rightarrow \quad Stat6: \; [\mathsf{c}, \mathsf{orden}(\mathsf{c})] \notin \{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\}$

$\langle \mathsf{c} \rangle \hookrightarrow Stat6 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad [\mathsf{c}, \mathsf{orden}(\mathsf{c})] \in \{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\}$

Suppose $\Rightarrow \quad Stat7: \; \Big[\mathsf{index}(\mathsf{orden}(\mathsf{c})), \mathsf{orden}\Big(\mathsf{index}(\mathsf{orden}(\mathsf{c}))\Big)\Big] \notin \{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\}$

$\langle \mathsf{index}(\mathsf{orden}(\mathsf{c})) \rangle \hookrightarrow Stat7 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \Big[\mathsf{index}(\mathsf{orden}(\mathsf{c})), \mathsf{orden}\Big(\mathsf{index}(\mathsf{orden}(\mathsf{c}))\Big)\Big] \in \{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\}$

Use_def(1–1) $\Rightarrow \quad Stat8: \; \langle \forall \mathsf{x} \in \{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\}, \mathsf{y} \in \{[\mathsf{x}, \mathsf{orden}(\mathsf{x})] : \mathsf{x} \in \mathsf{o}\} \mid \mathsf{x}^{[2]} = \mathsf{y}^{[2]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$

$\langle \Big[\mathsf{index}(\mathsf{orden}(\mathsf{c})), \mathsf{orden}\Big(\mathsf{index}(\mathsf{orden}(\mathsf{c}))\Big)\Big], [\mathsf{c}, \mathsf{orden}(\mathsf{c})] \rangle \hookrightarrow Stat8 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- Now we show that the function h, although defined indirectly in the manner seen above, nevertheless satisfies the recursive relationship that we have defined.

Theorem 437 (wellfounded_recursive_fcn · 3) $\mathsf{X} \in \mathsf{s} \rightarrow \mathsf{h}_\Theta(\mathsf{X}, \mathcal{T}) = \mathsf{f}\Big(\{\mathsf{g}(\mathsf{h}_\Theta(\mathsf{y}, \mathcal{T}), \mathsf{y}, \mathsf{X}, \mathcal{T}) : \mathsf{y} \in \mathsf{s} \mid \mathsf{y} \lhd \mathsf{X} \;\&\; \mathsf{P}(\mathsf{h}_\Theta(\mathsf{y}, \mathcal{T}), \mathsf{y}, \mathsf{X}, \mathcal{T})\}, \mathsf{X}, \mathcal{T}\Big)$. PROOF:

Suppose_not(v, t) $\Rightarrow \quad \mathsf{v} \in \mathsf{s} \;\&\; \mathsf{h}_\Theta(\mathsf{v}, \mathsf{t}) \neq \mathsf{f}\Big(\{\mathsf{g}(\mathsf{h}_\Theta(\mathsf{y}, \mathsf{t}), \mathsf{y}, \mathsf{v}, \mathsf{t}) : \mathsf{y} \in \mathsf{s} \mid \mathsf{y} \lhd \mathsf{v} \;\&\; \mathsf{P}(\mathsf{h}_\Theta(\mathsf{y}, \mathsf{t}), \mathsf{y}, \mathsf{v}, \mathsf{t})\}, \mathsf{v}, \mathsf{t}\Big)$

-- For suppose that some v in the domain s of the well_founded relation h fails to satisfy this recursion. Expand the definition of h and then simplify the resulting expression, using Theorems wellfounded_recursive_fcn. 1 and 2 in the following way to derive a contradiction.

Use_def(h_Θ) $\Rightarrow \quad \mathsf{h}_\Theta(\mathsf{v}, \mathsf{t}) = \mathsf{hh}(\mathsf{index}(\mathsf{v}), \mathsf{t})$

Use_def(hh) $\Rightarrow \quad \mathsf{hh}(\mathsf{index}(\mathsf{v}), \mathsf{t}) = \mathsf{f}\Bigg(\Big\{\mathsf{g}\Big(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{orden}(\mathsf{index}(\mathsf{v})), \mathsf{t}\Big) : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{orden}(\mathsf{index}(\mathsf{v})) \;\&\; \mathsf{P}\Big(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{orden}(\mathsf{index}(\mathsf{v})), \mathsf{t}\Big)\Big\}, \mathsf{orden}(\mathsf{ind}$

$\langle \mathsf{v} \rangle \hookrightarrow Twellfounded\_recursive\_fcn \cdot 1 \Rightarrow \quad \mathsf{orden}(\mathsf{index}(\mathsf{v})) = \mathsf{v}$

EQUAL $\Rightarrow \quad \mathsf{h}_\Theta(\mathsf{v}, \mathsf{t}) = \mathsf{f}\Big(\{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{v} \;\&\; \mathsf{P}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t})\}, \mathsf{v}, \mathsf{t}\Big)$

Suppose $\Rightarrow$

$\{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{v} \;\&\; \mathsf{P}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t})\} \neq$
$\{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \{\mathsf{j} : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{v}\} \mid \mathsf{P}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t})\}$

SIMPLF $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{v} \;\&\; \mathsf{P}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t})\} = \{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \{\mathsf{j} : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orde}$

$\langle \mathsf{v} \rangle \hookrightarrow Twellfounded\_recursive\_fcn \cdot 2 \Rightarrow \quad \{\mathsf{j} : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{v}\} = \{\mathsf{index}(\mathsf{y}) : \mathsf{y} \in \mathsf{s} \mid \mathsf{y} \lhd \mathsf{v}\}$

EQUAL $\Rightarrow$

$\{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \{\mathsf{j} : \mathsf{j} \in \mathsf{index}(\mathsf{v}) \mid \mathsf{orden}(\mathsf{j}) \lhd \mathsf{v}\} \mid \mathsf{P}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t})\} =$
$\{\mathsf{g}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t}) : \mathsf{j} \in \{\mathsf{index}(\mathsf{y}) : \mathsf{y} \in \mathsf{s} \mid \mathsf{y} \lhd \mathsf{v}\} \mid \mathsf{P}(\mathsf{hh}(\mathsf{j}, \mathsf{t}), \mathsf{orden}(\mathsf{j}), \mathsf{v}, \mathsf{t})\}$

SIMPLF $\Rightarrow$
$\{g(hh(j,t), orden(j), v, t) : j \in \{index(y) : y \in s \mid y \lhd v\} \mid P(hh(j,t), orden(j), v, t)\} =$
$\left\{g\left(hh(index(y),t), orden(index(y)), v, t\right) : y \in s \mid y \lhd v \,\&\, P\left(hh(index(y),t), orden(index(y)), v, t\right)\right\}$

Suppose $\Rightarrow$ *Stat1* : $\left\{g\left(hh(index(y),t), orden(index(y)), v, t\right) : y \in s \mid y \lhd v \,\&\, P\left(hh(index(y),t), orden(index(y)), v, t\right)\right\} \neq$
$\{g(h_\Theta(y,t), y, v, t) : y \in s \mid y \lhd v \,\&\, P(h_\Theta(y,t), y, v, t)\}$

$\langle w \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $w \in s \,\&\, \left(g\left(hh(index(w),t), orden(index(w)), v, t\right) \neq g(h_\Theta(w,t), w, v, t) \lor \neg P\left(hh(index(w),t), orden(index(w)), v, t\right) \leftrightarrow P(h_\Theta(w,t), w, v, t)\right)$

Use_def (h_Θ) $\Rightarrow$ $h_\Theta(w,t) = hh(index(w), t)$

$\langle w \rangle \hookrightarrow$ *Twellfounded_recursive_fcn* $\cdot 1 \Rightarrow$ $orden(index(w)) = w$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\left\{g\left(hh(index(y),t), orden(index(y)), v, t\right) : y \in s \mid y \lhd v \,\&\, P\left(hh(index(y),t), orden(index(y)), v, t\right)\right\} = \{g(h_\Theta(y,t), y, v, t) : y \in s \mid y \lhd v$

EQUAL $\Rightarrow$ $h_\Theta(v,t) = f\left(\{g(h_\Theta(y,t), y, v, t) : y \in s \mid y \lhd v \,\&\, P(h_\Theta(y,t), y, v, t)\}, v, t\right)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The theory just derived can be summarized as follows.

DISPLAY wellfounded_recursive_fcn

THEORY wellfounded_recursive_fcn $(s, y \lhd x, f(b,x,t), g(a,y,x,t), P(a,y,x,t))$
$\langle \forall t \subseteq s \mid t \neq \emptyset \longrightarrow \langle \exists x \in t, \forall y \in t \mid \neg y \lhd x \rangle \rangle$
$\Rightarrow (h_\Theta)$
$\left\langle \forall x \in s, t \mid h_\Theta(x,t) = f\left(\{g(h_\Theta(y,t), y, x, t) : y \in s \mid y \lhd x \,\&\, P(h_\Theta(y,t), y, x, t)\}, x, t\right)\right\rangle$
END wellfounded_recursive_fcn

-- The following theorem states that inclusion is a well-founded relation on each family of finite sets.

Theorem 438 (334) $\langle \forall v \in X \mid Finite(v) \rangle \,\&\, U \subseteq X \,\&\, U \neq \emptyset \rightarrow \langle \exists w \in U, \forall y \in U \mid \neg(y \subseteq w \,\&\, y \neq w) \rangle$. PROOF:

Suppose_not(x, u) $\Rightarrow$ *Stat1* : $\langle \forall w \in u, \exists y \in u \mid y \subseteq w \,\&\, y \neq w \rangle \,\&\,$ *Stat2* : $\langle \forall v \in x \mid Finite(v) \rangle \,\&\, u \subseteq x \,\&\, u \neq \emptyset$

-- For, assuming the contrary, there must exist a finite non-null set u none of whose elements is inclusion minimal. Apply the theory of finite induction to **arb**(u), thereby obtaining a set m having strict subset k belonging to u but which is such that no strict subset of any such k belongs to u. Consideration of m and such a k leads immediatiately to a contradiction which proves our theorem.

ELEM $\Rightarrow$ $\mathbf{arb}(u) \in u$
$\langle \mathbf{arb}(u) \rangle \hookrightarrow Stat1 \Rightarrow$ $\langle \exists y \in u \mid y \subseteq \mathbf{arb}(u) \ \& \ y \neq \mathbf{arb}(u) \rangle$
$\langle \mathbf{arb}(u) \rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{Finite}(\mathbf{arb}(u))$
APPLY $\langle m_\Theta : m \rangle$ finite_induction$(n \mapsto \mathbf{arb}(u), P(x) \mapsto \langle \exists y \in u \mid y \subseteq x \ \& \ y \neq x \rangle) \Rightarrow$
$Stat3 : \langle \exists y \in u \mid y \subseteq m \ \& \ y \neq m \rangle \ \& \ Stat4 : \langle \forall k \subseteq m \mid k \neq m \rightarrow \neg \langle \exists y \in u \mid y \subseteq k \ \& \ y \neq k \rangle \rangle$
$\langle k \rangle \hookrightarrow Stat3 \Rightarrow$ $k \in u \ \& \ k \subseteq m \ \& \ k \neq m$
$\langle k \rangle \hookrightarrow Stat4 \Rightarrow$ $\neg \langle \exists y \in u \mid y \subseteq k \ \& \ y \neq k \rangle$
$\langle k \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our main aim in the theorems which now follow is to prove that every one of a wide class of recursions specifies a function defined everywhere over the integers, or more generally over various families of finite sets. As a first step in this direction, we consider a pair of functions h_q and h_r, both of which satisfy the same recursive relationship on their domains d, which are assumed to be such that every subset of a member of of d is finite and belongs to d. (For example, d might be $\{x : x \subseteq s \mid \mathsf{Finite}(x)\}$). We show that h_q and h_r necessarily agree on the intersection of their domains, and so have a common single-valued extension.

THEORY finite_recursion_coherence$(q, r, h_q(x,t), h_r(x,t), f(b,x,t), g(a,y,x,t), P(a,y,x,t))$
$\langle \forall x \in q, y \subseteq x \mid \mathsf{Finite}(x) \ \& \ y \in q \rangle$
$\langle \forall x \in r, y \subseteq x \mid \mathsf{Finite}(x) \ \& \ y \in r \rangle$
$\langle \forall x \in q, t \mid h\_q(x,t) = f\Big(\{ g(h\_q(y,t), y, x, t) : y \in q \mid y \subseteq x \ \& \ y \neq x \ \& \ P(h\_q(y,t), y, x, t) \}, x, t \Big) \rangle$
$\langle \forall x \in r, t \mid h\_r(x,t) = f\Big(\{ g(h\_r(y,t), y, x, t) : y \in r \mid y \subseteq x \ \& \ y \neq x \ \& \ P(h\_r(y,t), y, x, t) \}, x, t \Big) \rangle$
END finite_recursion_coherence

ENTER_THEORY finite_recursion_coherence

Theorem 439 (finite_recursion_coherence · 1) $X \in q \ \& \ X \in r \rightarrow h\_q(X, T) = h\_r(X, T)$. PROOF:

Suppose_not$(x, t) \Rightarrow$ $x \in q \ \& \ x \in r \ \& \ h\_q(x,t) \neq h\_r(x,t)$

-- For suppose that m, q, r, and t form a counterexample to our assertion, where by the principle of finite induction we can assume that no proper subset of m is also a counterexample.

Assump $\Rightarrow$ $Stat1 : \langle \forall x \in q, y \subseteq x \mid \mathsf{Finite}(x) \ \& \ y \in q \rangle$
$\langle x, x \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{Finite}(x)$
APPLY $\langle m_\Theta : m \rangle$ finite_induction$(n \mapsto x, P(v) \mapsto h_q(v,t) \neq h_r(v,t)) \Rightarrow$

$m \subseteq x \,\&\, h\_q(m,t) \neq h\_r(m,t) \,\&\,$ *Stat2*: $\langle \forall k \subseteq m \mid k \neq m \rightarrow \neg h\_q(k,t) \neq h\_r(k,t) \rangle$

$\langle x, m \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $m \in q$

Assump $\Rightarrow$ *Stat3*: $\langle \forall x \in r, y \subseteq x \mid \mathrm{Finite}(x) \,\&\, y \in r \rangle$

$\langle x, m \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $m \in r$

-- Since $h\_q(m,t)$ and $h\_r(m,t)$ difer, the recursive expressions for these quantities must also differ, and so there must exist a c belonging to one, but not the other, of of the two sets displayed below.

Assump $\Rightarrow$ *Stat4*: $\langle \forall x \in q, t \mid h\_q(x,t) = f\Big(\{g(h\_q(y,t),y,x,t) : y \in q \mid y \subseteq x \,\&\, y \neq x \,\&\, P(h\_q(y,t),y,x,t)\}, x, t\Big)\rangle$

$\langle m, t \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $h\_q(m,t) = f\Big(\{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\}, m, t\Big)$

Assump $\Rightarrow$ *Stat5*: $\langle \forall x \in r, t \mid h\_r(x,t) = f\Big(\{g(h\_r(y,t),y,x,t) : y \in r \mid y \subseteq x \,\&\, y \neq x \,\&\, P(h\_r(y,t),y,x,t)\}, x, t\Big)\rangle$

$\langle m, t \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $h\_r(m,t) = f\Big(\{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t),y,m,t)\}, m, t\Big)$

EQUAL $\Rightarrow$

$f\Big(\{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\}, m, t\Big) \neq$

$f\Big(\{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t),y,m,t)\}, m, t\Big)$

Suppose $\Rightarrow$ $\{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\} =$

$\{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t),y,m,t)\}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6*: $\{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\} \neq \{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t)$

$\langle c \rangle \hookrightarrow$ *Stat6* $\Rightarrow$

$c \in \{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\} \leftrightarrow c \notin \{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t),y,m,t)\}$

-- Suppose that c belongs to the first, but not the second, of these sets. Then m must have a proper subset k for which $h\_q(k,t)$ and $h\_q(k,t)$ differ, which is impossible by the minimality of m

Suppose $\Rightarrow$ *Stat7*:

$c \in \{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\} \,\&\,$

$c \notin \{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t),y,m,t)\}$

$\langle k, k \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $k \in q \,\&\, k \subseteq m \,\&\, k \neq m \,\&\, P(h\_q(k,t),k,m,t) \,\&\,$

$g(h\_q(k,t),k,m,t) \neq g(h\_r(k,t),k,m,t) \vee k \notin r \vee \neg P(h\_r(k,t),k,m,t)$

$\langle m, k \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $k \in r$

$\langle k \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $h\_q(k,t) = h\_r(k,t)$

EQUAL $\Rightarrow$ $g(h\_q(k,t),k,m,t) = g(h\_r(k,t),k,m,t) \,\&\, P(h\_r(k,t),k,m,t)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$

Stat8: $c \in \{g(h\_r(y,t),y,m,t) : y \in r \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_r(y,t),y,m,t)\} \,\&\, c \notin \{g(h\_q(y,t),y,m,t) : y \in q \mid y \subseteq m \,\&\, y \neq m \,\&\, P(h\_q(y,t),y,m,t)\}$

-- But exactly the same argument applies if c belongs to the second, but not the first, of these sets, and so our theorem is proved.

$\langle k', k' \rangle \hookrightarrow Stat8 \Rightarrow \quad Stat9:$
$k' \in r \;\&\; k' \subseteq m \;\&\; k' \neq m \;\&\; P\big(h\_r(k', t), k', m, t\big) \;\&\; g\big(h\_r(k', t), k', m, t\big) \neq g\big(h\_q(k', t), k', m, t\big) \lor k' \notin q \lor \neg P\big(h\_q(k', t), k', m, t\big)$

$\langle m, k' \rangle \hookrightarrow Stat1 \Rightarrow \quad k' \in q$

$\langle k' \rangle \hookrightarrow Stat2 \Rightarrow \quad h\_q(k', t) = h\_r(k', t)$

EQUAL $\Rightarrow \quad g\big(h\_q(k', t), k', m, t\big) = g\big(h\_r(k', t), k', m, t\big) \;\&\; P\big(h\_q(k', t), k', m, t\big)$

$\langle Stat9 \rangle$ ELEM $\Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad$ QED

ENTER_THEORY Set_theory

-- The theory just established can be summarized as follows.

DISPLAY finite_recursion_coherence

THEORY finite_recursion_coherence$\big(q, r, h_q(x, t), h_r(x, t), f(b, x, t), g(a, y, x, t), P(a, y, x, t)\big)$

$\langle \forall x \in q, y \subseteq x \mid \mathrm{Finite}(x) \;\&\; y \in q \rangle$

$\langle \forall x \in r, y \subseteq x \mid \mathrm{Finite}(x) \;\&\; y \in r \rangle$

$\langle \forall x \in q, t \mid h\_q(x, t) = f\Big(\big\{g\big(h\_q(y, t), y, x, t\big) : y \in q \mid y \subseteq x \;\&\; y \neq x \;\&\; P\big(h\_q(y, t), y, x, t\big)\big\}, x, t\Big) \rangle$

$\langle \forall x \in r, t \mid h\_r(x, t) = f\Big(\big\{g\big(h\_r(y, t), y, x, t\big) : y \in r \mid y \subseteq x \;\&\; y \neq x \;\&\; P\big(h\_r(y, t), y, x, t\big)\big\}, x, t\Big) \rangle$

$\Rightarrow$

$\langle \forall x, t \mid x \in q \;\&\; x \in r \rightarrow h\_q(x, t) = h\_r(x, t) \rangle$

END finite_recursion_coherence

-- In further preparation for the theory of recursive function defined over finite sets, we show that any relation devoid of cycles, when restricted to a finite set, turns out to be well-founded over such domain. To make matters simple (since defining acyclic relations would be a relatively complicated matter, we assume the relation to be transitive (this covers, for example, the significant case of strict inclusion).

THEORY fin_well_founded$(s, y \lhd x)$

$\langle \forall x, y, zz \mid x \lhd y \;\&\; y \lhd zz \rightarrow x \lhd zz \rangle$

$\langle \forall x \mid \neg x \lhd x \rangle$

$\mathrm{Finite}(s)$

END fin_well_founded

ENTER_THEORY fin_well_founded

Theorem 440 (fin_well_founded · 1) $V \subseteq s \,\&\, V \neq \emptyset \rightarrow \langle \exists m \in V, \forall y \in V \mid \neg y \lhd m \rangle$. PROOF:

Suppose_not$(v) \Rightarrow \quad v \subseteq s \,\&\, v \neq \emptyset \,\&\, \neg\langle \exists m \in v, \forall y \in v \mid \neg y \lhd m \rangle$

-- Assuming by contradiction that the relation $\lhd$ is not well-founded on the finite set s, there would be a non-null subset v of s which has no minimal element. By application of the THEORY finite_induction, we can choose a smallest subset w of s which has no minimal element.

Assump $\Rightarrow \quad$ Finite(s)
Assump $\Rightarrow \quad$ *Stat0* : $\langle \forall x \mid \neg x \lhd x \rangle$
$\langle s, v \rangle \hookrightarrow T162 \Rightarrow \quad$ Finite(v)
APPLY $\langle m_\Theta : w \rangle$ finite_induction$\left(n \mapsto v, P(x) \mapsto (x \neq \emptyset \,\&\, \neg\langle \exists m \in x, \forall y \in x \mid \neg y \lhd m \rangle)\right) \Rightarrow$
$w \subseteq v \,\&\, w \neq \emptyset \,\&\, Stat1 : \neg\langle \exists m \in w, \forall y \in w \mid \neg y \lhd m \rangle \,\&\, Stat2 : \langle \forall k \subseteq w \mid k \neq w \longrightarrow \neg(k \neq \emptyset \,\&\, \neg\langle \exists m \in k, \forall y \in k \mid \neg y \lhd m \rangle)\rangle$
$\langle \mathbf{arb}(w) \rangle \hookrightarrow Stat0 \Rightarrow \quad \neg \mathbf{arb}(w) \lhd \mathbf{arb}(w)$

-- Clearly, w cannot be a singleton (else $\mathbf{arb}(w)$ would be its minimum). We can hence find a minimal element m in $w \setminus \{\mathbf{arb}(w)\}$.

Suppose $\Rightarrow \quad w = \{\mathbf{arb}(w)\}$
$\langle \mathbf{arb}(w) \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat3 : \neg\langle \forall y \in w \mid \neg y \lhd \mathbf{arb}(w) \rangle$
$\langle y \rangle \hookrightarrow Stat3 \Rightarrow \quad y \in w \,\&\, y \lhd \mathbf{arb}(w)$
ELEM $\Rightarrow \quad y = \mathbf{arb}(w)$
EQUAL $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad w \setminus \{\mathbf{arb}(w)\} \subseteq w \,\&\, w \setminus \{\mathbf{arb}(w)\} \neq w \,\&\, w \setminus \{\mathbf{arb}(w)\} \neq \emptyset$
$\langle w \setminus \{\mathbf{arb}(w)\} \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat4 : \langle \exists m \in w \setminus \{\mathbf{arb}(w)\}, \forall y \in w \setminus \{\mathbf{arb}(w)\} \mid \neg y \lhd m \rangle$
$\langle m \rangle \hookrightarrow Stat4 \Rightarrow \quad m, m \in w \setminus \{\mathbf{arb}(w)\} \,\&\, Stat5 : \langle \forall y \in w \setminus \{\mathbf{arb}(w)\} \mid \neg y \lhd m \rangle$
$\langle m \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat6 : \neg\langle \forall y \in w \mid \neg y \lhd m \rangle$

-- If $\mathbf{arb}(w)$ did not precede m in the relation $\lhd$, then it would readily follow that m would be minimal in w; to avoid this contradiction, we must assume that $\mathbf{arb}(w)$ precedes m.

Suppose $\Rightarrow \quad \neg \mathbf{arb}(w) \lhd m$
$\langle y' \rangle \hookrightarrow Stat6 \Rightarrow \quad y' \in w \,\&\, y' \lhd m$
Suppose $\Rightarrow \quad y' = \mathbf{arb}(w)$
EQUAL $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad y' \neq \mathbf{arb}(w)$
$\langle y' \rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \mathbf{arb}(w) \lhd m$

-- Then x cannot precede $\mathbf{arb}(w)$ in the relation $\lhd$, for any $w \in w$ (else we would have $x \neq \mathbf{arb}(w)$ and x before m in the relation $\lhd$). Consequently, $\mathbf{arb}(w)$ is a minimal element of w; this new contradiction enables us to conclude with the desired statement.

$\langle \mathbf{arb}(w) \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat7: \neg\langle \forall y \in w \mid \neg y \lhd \mathbf{arb}(w) \rangle$
$\langle yq \rangle \hookrightarrow Stat7 \Rightarrow$ $yq \in w \;\&\; yq \lhd \mathbf{arb}(w)$
Assump $\Rightarrow$ $Stat8: \langle \forall x, y, zz \mid x \lhd y \;\&\; y \lhd zz \rightarrow x \lhd zz \rangle$
$\langle yq, \mathbf{arb}(w), m \rangle \hookrightarrow Stat8 \Rightarrow$ $yq \lhd m$
Suppose $\Rightarrow$ $\mathbf{arb}(w) = yq$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $yq \in w \setminus \{\mathbf{arb}(w)\}$
$\langle yq \rangle \hookrightarrow Stat5 \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- The theory just established can be summarized as follows.

DISPLAY fin_well_founded

THEORY fin_well_founded$(s, x \lhd y)$
$\langle \forall x, y, z \mid x \lhd y \;\&\; y \lhd z \rightarrow x \lhd z \rangle$
$\langle \forall x \mid \neg x \lhd x \rangle$
Finite(s)
$\Rightarrow$
$V \subseteq s \;\&\; V \neq \emptyset \rightarrow \langle \exists m \in V, \forall x \in V \mid \neg x \lhd m \rangle$
END fin_well_founded

-- Our next aim is to prove that any recursion which determines a value $h_2(s, t)$ dependent on two parameters, the first of these being a finite set, in terms of the values $h_2(s', t)$ for which s' is a proper subset of s, actually defines a function $h_2(s, t)$ single-valued for all finite s and all t. We state this result by defining an auxiliary theory whose first theorem, seen just below, has a form allowing immediate Skolemization. By Skolemizing this we will derive a second result in the convenient form desired.

THEORY finite_recursive_fcn$\big(f(b, x, t), g(a, y, x, t), P(a, y, x, t)\big)$
END finite_recursive_fcn

ENTER_THEORY finite_recursive_fcn

-- We now show that the finite subsets of any set s are well-ordered by strict inclusion.

Theorem 441 (finite_recursive_fcn · 0) $T \subseteq \{y : y \subseteq S \mid \text{Finite}(y)\} \;\&\; T \neq \emptyset \rightarrow \langle \exists x \in T, \forall y \in T \mid \neg(y \subseteq x \;\&\; y \neq x) \rangle$. PROOF:

-- As a matter of fact, if we consider any non-null subset t of the family of all finite subsets of s, and then take an element r of t, then the set mm of all minorants of r in t turns out to be finite (inasmuch as a subset of $\mathcal{P}$r, which is finite). Hence it will have an inclusion-minimal element m, which will also be minimal in t.

Suppose_not(t, s) ⇒ $\mathit{Stat0}: \mathsf{t} \neq \emptyset \;\&\; \mathsf{t} \subseteq \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{s} \,|\, \mathsf{Finite}(\mathsf{y})\} \;\&\; \mathit{Stat1}: \neg\langle \exists \mathsf{x} \in \mathsf{t}, \forall \mathsf{y} \in \mathsf{t} \,|\, \neg(\mathsf{y} \subseteq \mathsf{x} \;\&\; \mathsf{y} \neq \mathsf{x})\rangle$
$\langle \mathsf{r} \rangle \hookrightarrow \mathit{Stat0} \Rightarrow$ $\mathsf{r} \in \mathsf{t} \;\&\; \mathit{Stat2}: \mathsf{r} \in \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{s} \,|\, \mathsf{Finite}(\mathsf{y})\}$
$\langle \mathsf{a} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathsf{r} = \mathsf{a} \;\&\; \mathsf{Finite}(\mathsf{a})$
EQUAL ⇒ $\mathsf{Finite}(\mathsf{r})$
Loc_def ⇒ $\mathsf{mm} = \{\mathsf{y} : \mathsf{y} \in \mathsf{t} \,|\, \mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r}\}$
Suppose ⇒ $\mathit{Stat3}: \mathsf{mm} \not\subseteq \mathcal{P}\mathsf{r}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $\mathsf{c} \in \mathsf{mm} \;\&\; \mathsf{c} \notin \mathcal{P}\mathsf{r}$
Use_def($\mathcal{P}$) ⇒ $\mathit{Stat4}: \mathsf{c} \in \{\mathsf{y} : \mathsf{y} \in \mathsf{t} \,|\, \mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r}\} \;\&\; \mathsf{c} \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{r}\}$
$\langle \mathsf{y}', \mathsf{y}' \rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ false; Discharge ⇒ $\mathsf{mm} \subseteq \mathcal{P}\mathsf{r}$
$\langle \mathsf{r} \rangle \hookrightarrow \mathit{T227} \Rightarrow$ $\mathsf{Finite}(\mathcal{P}\mathsf{r})$
$\langle \mathcal{P}\mathsf{r}, \mathsf{mm} \rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\mathsf{Finite}(\mathsf{mm})$
$\langle \mathsf{r} \rangle \hookrightarrow \mathit{T227} \Rightarrow$ $\mathsf{Finite}(\mathcal{P}\mathsf{r})$
$\langle \mathcal{P}\mathsf{r}, \mathsf{mm} \rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\mathsf{Finite}(\mathsf{mm})$
Suppose ⇒ $\mathit{Stat5}: \{\mathsf{y} : \mathsf{y} \in \mathsf{t} \,|\, \mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r}\} = \emptyset$
$\langle \mathsf{r} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat6}: \neg\langle \forall \mathsf{y} \in \mathsf{t} \,|\, \neg(\mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r})\rangle$
$\langle \mathsf{y} \rangle \hookrightarrow \mathit{Stat6} \Rightarrow$ $\mathsf{y} \in \mathsf{t} \;\&\; \mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r}$
$\langle \mathsf{y} \rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ false; Discharge ⇒ $\mathsf{mm} \neq \emptyset$
Suppose ⇒ $\mathit{Stat7}: \neg\langle \forall \mathsf{x} \,|\, \neg(\mathsf{x} \subseteq \mathsf{x} \;\&\; \mathsf{x} \neq \mathsf{x})\rangle$
$\langle \mathsf{x}' \rangle \hookrightarrow \mathit{Stat7} \Rightarrow$ false; Discharge ⇒ $\langle \forall \mathsf{x} \,|\, \neg(\mathsf{x} \subseteq \mathsf{x} \;\&\; \mathsf{x} \neq \mathsf{x})\rangle$
Suppose ⇒ $\mathit{Stat8}: \neg\langle \forall \mathsf{xx}, \mathsf{yy}, \mathsf{zz} \,|\, (\mathsf{xx} \subseteq \mathsf{yy} \;\&\; \mathsf{xx} \neq \mathsf{yy}) \;\&\; \mathsf{yy} \subseteq \mathsf{zz} \;\&\; \mathsf{yy} \neq \mathsf{zz} \rightarrow \mathsf{xx} \subseteq \mathsf{zz} \;\&\; \mathsf{xx} \neq \mathsf{zz}\rangle$
$\langle \mathsf{xx}, \mathsf{yy}, \mathsf{zz} \rangle \hookrightarrow \mathit{Stat8}(\langle \cap \rangle) \Rightarrow$ false; Discharge ⇒ $\langle \forall \mathsf{xx}, \mathsf{yy}, \mathsf{zz} \,|\, (\mathsf{xx} \subseteq \mathsf{yy} \;\&\; \mathsf{xx} \neq \mathsf{yy}) \;\&\; \mathsf{yy} \subseteq \mathsf{zz} \;\&\; \mathsf{yy} \neq \mathsf{zz} \rightarrow \mathsf{xx} \subseteq \mathsf{zz} \;\&\; \mathsf{xx} \neq \mathsf{zz}\rangle$
APPLY $\langle\ \rangle$ fin_well_founded$(\mathsf{s} \mapsto \mathcal{P}\mathsf{r}, \mathsf{y} \lhd \mathsf{x} \mapsto (\mathsf{y} \subseteq \mathsf{x} \;\&\; \mathsf{y} \neq \mathsf{x}))$ ⇒
$\mathit{Stat9}: \langle \forall \mathsf{v} \,|\, \mathsf{v} \subseteq \mathcal{P}\mathsf{r} \;\&\; \mathsf{v} \neq \emptyset \rightarrow \langle \exists \mathsf{m} \in \mathsf{v}, \forall \mathsf{x} \in \mathsf{v} \,|\, \neg(\mathsf{x} \subseteq \mathsf{m} \;\&\; \mathsf{x} \neq \mathsf{m})\rangle\rangle$
$\langle \mathsf{mm} \rangle \hookrightarrow \mathit{Stat9}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathit{Stat10}: \langle \exists \mathsf{m} \in \mathsf{mm}, \forall \mathsf{x} \in \mathsf{mm} \,|\, \neg(\mathsf{x} \subseteq \mathsf{m} \;\&\; \mathsf{x} \neq \mathsf{m})\rangle$
$\langle \mathsf{m} \rangle \hookrightarrow \mathit{Stat10}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathit{Stat11}: \mathsf{m} \in \{\mathsf{y} : \mathsf{y} \in \mathsf{t} \,|\, \mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r}\} \;\&\; \mathit{Stat12}: \langle \forall \mathsf{x} \in \mathsf{mm} \,|\, \neg(\mathsf{x} \subseteq \mathsf{m} \;\&\; \mathsf{x} \neq \mathsf{m})\rangle$
$\langle \mathsf{yq} \rangle \hookrightarrow \mathit{Stat11}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathsf{m} \in \mathsf{t} \;\&\; \mathsf{m} \subseteq \mathsf{r} \;\&\; \mathsf{m} \neq \mathsf{r}$
$\langle \mathsf{m} \rangle \hookrightarrow \mathit{Stat1}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathit{Stat13}: \neg\langle \forall \mathsf{y} \in \mathsf{t} \,|\, \neg(\mathsf{y} \subseteq \mathsf{m} \;\&\; \mathsf{y} \neq \mathsf{m})\rangle$
$\langle \mathsf{u} \rangle \hookrightarrow \mathit{Stat13}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathsf{u} \in \mathsf{t} \;\&\; \mathsf{u} \subseteq \mathsf{m} \;\&\; \mathsf{u} \neq \mathsf{m}$
Suppose ⇒ $\mathit{Stat14}: \mathsf{u} \notin \{\mathsf{y} : \mathsf{y} \in \mathsf{t} \,|\, \mathsf{y} \subseteq \mathsf{r} \;\&\; \mathsf{y} \neq \mathsf{r}\}$
$\langle \mathsf{u} \rangle \hookrightarrow \mathit{Stat14}([\mathit{Stat0}, \cap]) \Rightarrow$ false; Discharge ⇒ $\mathsf{u} \in \mathsf{mm}$
$\langle \mathsf{u} \rangle \hookrightarrow \mathit{Stat12}([\mathit{Stat0}, \cap]) \Rightarrow$ false; Discharge ⇒ QED

Theorem 442 (finite_recursive_fcn · 1)
$\langle \forall \mathsf{s}, \mathsf{t}, \exists \mathsf{h}, \mathsf{x} \,|\, \mathsf{x} \subseteq \mathsf{s} \;\&\; \mathsf{Finite}(\mathsf{x}) \rightarrow \mathsf{h}\restriction\mathsf{x} = \mathsf{f}(\{\mathsf{g}(\mathsf{h}\restriction\mathsf{y}, \mathsf{y}, \mathsf{x}, \mathsf{t}) : \mathsf{y} \subseteq \mathsf{x} \,|\, \mathsf{y} \neq \mathsf{x} \;\&\; \mathsf{P}(\mathsf{h}\restriction\mathsf{y}, \mathsf{y}, \mathsf{x}, \mathsf{t})\}, \mathsf{x}, \mathsf{t})\rangle$. PROOF:

Suppose_not $\Rightarrow$ $\quad Stat1: \neg$
$\langle \forall s, t, \exists h, x \mid x \subseteq s \,\&\, \mathsf{Finite}(x) \rightarrow h{\restriction}x = f(\{g(h{\restriction}y, y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(h{\restriction}y, y, x, t)\}, x, t)\rangle$

-- For if we let s, t be a counterexample to our assertion and apply our well-founded_recursive_fcn THEORY, a contradiction results easily. Indeed, the well-founded_recursive_fcn Theory defined previously gives us a function hh satisfying the recursive relationship seen just below.

$\langle s, t\rangle \hookrightarrow Stat1 \Rightarrow$ $\quad Stat2: \neg$
$\langle \exists h, \forall x \mid x \subseteq s \,\&\, \mathsf{Finite}(x) \rightarrow h{\restriction}x = f(\{g(h{\restriction}y, y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(h{\restriction}y, y, x, t)\}, x, t)\rangle$
Suppose $\Rightarrow$ $\quad Stat2a: \neg\langle \forall t \mid t \subseteq \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \,\&\, t \neq \emptyset \rightarrow \langle \exists x \in t, \forall y \in t \mid \neg(y \subseteq x \,\&\, y \neq x)\rangle\rangle$
$\langle d\rangle \hookrightarrow Stat2a \Rightarrow$ $\quad (d \subseteq \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \,\&\, d \neq \emptyset) \,\&\, \neg\langle \exists x \in d, \forall y \in d \mid \neg(y \subseteq x \,\&\, y \neq x)\rangle$
$\langle d, s\rangle \hookrightarrow Tfinite\_recursive\_fcn \cdot 0 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad \langle \forall t \mid t \subseteq \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \,\&\, t \neq \emptyset \rightarrow \langle \exists x \in t, \forall y \in t \mid \neg(y \subseteq x \,\&\, y \neq x)\rangle\rangle$
APPLY $\langle h_\Theta : hh\rangle$ wellfounded_recursive_fcn $(s \mapsto \{y : y \subseteq s \mid \mathsf{Finite}(y)\}, y \lhd x \mapsto (y \subseteq x \,\&\, y \neq x), f(b, x, t) \mapsto f(b, x, t), g(a, y, x, t) \mapsto g(a, y, x, t), P(a, y, x, t) \mapsto P(a, y, x, t)) \Rightarrow$

$Stat3: \langle \forall x, t \mid x \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \rightarrow hh(x, t) = f\Big(\{g(hh(y, t), y, x, t) : y \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \mid y \subseteq x \,\&\, y \neq x \,\&\, P(hh(y, t), y, x, t)\}, x, t\Big)\rangle$

-- But this recursive relationship can be rewritten in the simplified form seen in the following. For were this not the case, the two sets seen below woul necesarily differ.

Suppose $\Rightarrow$ $\quad Stat4: \neg\langle \forall x, t \mid x \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \rightarrow hh(x, t) = f\Big(\{g(hh(y, t), y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hh(y, t), y, x, t)\}, x, t\Big)\rangle$
$\langle x', t'\rangle \hookrightarrow Stat4 \Rightarrow$ $\quad Stat5:$
$x' \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \,\&\, hh(x', t') \neq f\Big(\{g(hh(y, t'), y, x', t') : y \subseteq x' \mid y \neq x' \,\&\, P(hh(y, t'), y, x', t')\}, x', t'\Big)$
$\langle x\rangle \hookrightarrow Stat5 \Rightarrow$ $\quad x = x' \,\&\, x' \subseteq s \,\&\, \mathsf{Finite}(x)$
EQUAL $\Rightarrow$ $\quad \mathsf{Finite}(x')$
$\langle x, t\rangle \hookrightarrow Stat3 \Rightarrow$ $\quad hh(x, t) =$
$f\Big(\{g(hh(y, t), y, x, t) : y \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \mid y \subseteq x \,\&\, y \neq x \,\&\, P(hh(y, t), y, x, t)\}, x, t\Big)$
Suppose $\Rightarrow$
$\{g(hh(y, t), y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hh(y, t), y, x, t)\} =$
$\{g(hh(y, t), y, x, t) : y \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \mid y \subseteq x \,\&\, y \neq x \,\&\, P(hh(y, t), y, x, t)\}$
EQUAL $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$
$Stat6: \{g(hh(y, t), y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hh(y, t), y, x, t)\} \neq \{g(hh(y, t), y, x, t) : y \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \mid y \subseteq x \,\&\, y \neq x \,\&\, P(hh(y, t), y, x, t)\}$

-- And it is easily seen that these two sets must be equal.

$\langle c\rangle \hookrightarrow Stat6 \Rightarrow$
$c \in \{g(hh(y, t), y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hh(y, t), y, x, t)\} \leftrightarrow c \notin \{g(hh(y, t), y, x, t) : y \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\} \mid y \subseteq x \,\&\, y \neq x \,\&\, P(hh(y, t), y, x, t)\}$
Suppose $\Rightarrow$ $\quad Stat7:$
$c \in \{g(hh(y, t), y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hh(y, t), y, x, t)\} \,\&\, Stat8:$

$c \notin \{g(hh(y,t),y,x,t) : y \in \{y : y \subseteq s \mid Finite(y)\} \mid y \subseteq x \;\&\; y \neq x \;\&\; P(hh(y,t),y,x,t)\}$

$\langle y \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $c = g(hh(y,t),y,x,t) \;\&\; y \subseteq x \;\&\; y \neq x \;\&\; P(hh(y,t),y,x,t)$

$\langle y \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ *Stat9*: $y \notin \{y : y \subseteq s \mid Finite(y)\}$

$\langle x, y \rangle \hookrightarrow$ *T162* $\Rightarrow$ $Finite(y)$

$\langle y \rangle \hookrightarrow$ *Stat9* $\Rightarrow$ false; Discharge $\Rightarrow$

Stat10: $c \notin \{g(hh(y,t),y,x,t) : y \subseteq x \mid y \neq x \;\&\; P(hh(y,t),y,x,t)\}$ & *Stat11*: $c \in \{g(hh(y,t),y,x,t) : y \in \{y : y \subseteq s \mid Finite(y)\} \mid y \subseteq x \;\&\; y \neq x \;\&\; P(hh(y,t),y,x,t)\}$

$\langle y' \rangle \hookrightarrow$ *Stat11* $\Rightarrow$ $c = g(hh(y',t),y',x,t) \;\&\; y' \subseteq x \;\&\; y' \neq x \;\&\; P(hh(y',t),y',x,t) \;\&\; y' \in \{y : y \subseteq s \mid Finite(y)\}$

$\langle y' \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat12*: $\langle \forall x, t \mid x \in \{y : y \subseteq s \mid Finite(y)\} \rightarrow hh(x,t) = f(\{g(hh(y,t),y,x,t) : y \subseteq x \mid y \neq x \;\&\; P(hh(y,t),y,x,t)\}, x, t)\rangle$

-- Holding t fixed, we can now view $hh(x,t)$ as a map h_2 satisfying $h_2 \restriction x = \mathbf{if}\ x \in \{y : y \subseteq s\}\ \mathbf{then}\ hh(x,t)\ \mathbf{else}\ \emptyset\ \mathbf{fi}$ for all x.

Loc_def $\Rightarrow$ $h_2 = \{[x, hh(x,t)] : x \in \{y : y \subseteq s \mid Finite(y)\}\}$

APPLY $\langle\,\rangle$ fcn_symbol $(f(x) \mapsto hh(x,t), g \mapsto h_2, s \mapsto \{y : y \subseteq s \mid Finite(y)\})$ $\Rightarrow$

$Svm(h_2)$ & *Stat13*: $\langle \forall x \mid h_2 \restriction x = \mathbf{if}\ x \in \{y : y \subseteq s \mid Finite(y)\}\ \mathbf{then}\ hh(x,t)\ \mathbf{else}\ \emptyset\ \mathbf{fi}\rangle$

SIMPLF $\Rightarrow$ $h_2 = \{[x, hh(x,t)] : x \subseteq s \mid Finite(x)\}$

-- But now h_2 is easily seen to satify the recursive relationship seen below, and so can serve as the h whose existence our theorem asserts.

Suppose $\Rightarrow$ *Stat14*: $\neg\langle \forall x \mid x \subseteq s \;\&\; Finite(x) \rightarrow h_2 \restriction x = f(\{g(h_2 \restriction y, y, x, t) : y \subseteq x \mid y \neq x \;\&\; P(h_2 \restriction y, y, x, t)\}, x, t)\rangle$

$\langle x_2 \rangle \hookrightarrow$ *Stat14* $\Rightarrow$ $x_2 \subseteq s \;\&\; Finite(x_2) \;\&\; h_2 \restriction x_2 \neq f(\{g(h_2 \restriction y, y, x_2, t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(h_2 \restriction y, y, x_2, t)\}, x_2, t)$

$\langle x_2 \rangle \hookrightarrow$ *Stat13* $\Rightarrow$ $h_2 \restriction x_2 = \mathbf{if}\ x_2 \in \{y : y \subseteq s \mid Finite(y)\}\ \mathbf{then}\ hh(x_2,t)\ \mathbf{else}\ \emptyset\ \mathbf{fi}$

Suppose $\Rightarrow$ *Stat15*: $x_2 \notin \{y : y \subseteq s \mid Finite(y)\}$

$\langle x_2 \rangle \hookrightarrow$ *Stat15* $\Rightarrow$ false; Discharge $\Rightarrow$ $x_2 \in \{y : y \subseteq s \mid Finite(y)\}$

ELEM $\Rightarrow$ $h_2 \restriction x_2 = hh(x_2,t)$

$\langle x_2, t \rangle \hookrightarrow$ *Stat12* $\Rightarrow$ $hh(x_2,t) = f(\{g(hh(y,t),y,x_2,t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(hh(y,t),y,x_2,t)\}, x_2, t)$

ELEM $\Rightarrow$ $f(\{g(h_2 \restriction y, y, x_2, t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(h_2 \restriction y, y, x_2, t)\}, x_2, t) \neq$
$f(\{g(hh(y,t),y,x_2,t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(hh(y,t),y,x_2,t)\}, x_2, t)$

Suppose $\Rightarrow$ $\{g(h_2 \restriction y, y, x_2, t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(h_2 \restriction y, y, x_2, t)\} = \{g(hh(y,t),y,x_2,t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(hh(y,t),y,x_2,t)\}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat16*: $\{g(h_2 \restriction y, y, x_2, t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(h_2 \restriction y, y, x_2, t)\} \neq \{g(hh(y,t),y,x_2,t) : y \subseteq x_2 \mid y \neq x_2 \;\&\; P(hh(y,t),y,x_2,t)\}$

$\langle d' \rangle \hookrightarrow$ *Stat16* $\Rightarrow$

$d' \subseteq x_2 \;\&$

$g(h_2 \restriction d', d', x_2, t) \neq g(hh(d',t), d', x_2, t) \vee \left(d' \neq x_2 \;\&\; P(h_2 \restriction d', d', x_2, t) \;\&\; \neg\left(d' \neq x_2 \;\&\; P(hh(d',t), d', x_2, t)\right)\right) \vee \left(\neg(d' \neq x_2 \;\&\; P(h_2 \restriction d', d', x_2, t)) \;\&\; d' \neq x_2 \;\&\; P(hh(d',t)\right.$

ELEM $\Rightarrow$ $g(h_2 \restriction d', d', x_2, t) \neq g(hh(d',t), d', x_2, t) \vee \neg\left(P(hh(d',t), d', x_2, t) \leftrightarrow P(h_2 \restriction d', d', x_2, t)\right)$

Suppose $\Rightarrow$ $h_2 \restriction d' = hh(d',t)$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $h_2 \restriction d' \neq hh(d',t)$

$\langle d' \rangle \hookrightarrow Stat13 \Rightarrow$ $h_2 \restriction d' = \mathbf{if}\ d' \in \{y : y \subseteq s \mid \mathsf{Finite}(y)\}\ \mathbf{then}\ hh(d', t)\ \mathbf{else}\ \emptyset\ \mathbf{fi}$
Suppose $\Rightarrow$ $Stat17 : d' \notin \{y : y \subseteq s \mid \mathsf{Finite}(y)\}$
$\langle x_2, d' \rangle \hookrightarrow T162 \Rightarrow$ $\mathsf{Finite}(d')$
$\langle d' \rangle \hookrightarrow Stat17 \Rightarrow$ false; Discharge $\Rightarrow$ $\langle \forall x \mid x \subseteq s \,\&\, \mathsf{Finite}(x) \rightarrow h_2 \restriction x = f(\{g(h_2 \restriction y, y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(h_2 \restriction y, y, x, t)\}, x, t)\rangle$
$\langle h_2 \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We now define an auxiliary function $hsko(s, t)$ by Skolemizing the preceding theorem. The formal definition is as follows.

APPLY $\langle v1_\Theta : hsko \rangle$ Skolem$\Rightarrow$

Theorem 443 (finite_recursive_fcn · a)
$\langle \forall s, t, x \mid x \subseteq s \,\&\, \mathsf{Finite}(x) \rightarrow hsko(s, t) \restriction x = f\Big(\{g(hsko(s, t) \restriction y, y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hsko(s, t) \restriction y, y, x, t)\}, x, t\Big)\rangle.$

-- The following theorem simply transforms the preceding into a more readily usable form.

Theorem 444 (finite_recursive_fcn · 2) $\langle \forall x \mid x \subseteq S \,\&\, \mathsf{Finite}(x) \rightarrow hsko(S, T) \restriction x = f\Big(\{g(hsko(S, T) \restriction y, y, x, T) : y \subseteq x \mid y \neq x \,\&\, P(hsko(S, T) \restriction y, y, x, T)\}, x, T\Big)\rangle$. Proof:

Suppose_not$(s, t) \Rightarrow$
$\neg\langle \forall x \mid x \subseteq s \,\&\, \mathsf{Finite}(x) \rightarrow hsko(s, t) \restriction x = f\Big(\{g(hsko(s, t) \restriction y, y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hsko(s, t) \restriction y, y, x, t)\}, x, t\Big)\rangle$
Tfinite_recursive_fcn · a $\Rightarrow$ $Stat1 :$
$\langle \forall s, t, x \mid x \subseteq s \,\&\, \mathsf{Finite}(x) \rightarrow hsko(s, t) \restriction x = f\Big(\{g(hsko(s, t) \restriction y, y, x, t) : y \subseteq x \mid y \neq x \,\&\, P(hsko(s, t) \restriction y, y, x, t)\}, x, t\Big)\rangle$
$\langle s, t \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The function h_thryvar at which the present theory aims is defined as follows in terms of hsko.

Def finite_recursive_fcn · b. $h_\Theta(X, Y) =_{\mathrm{Def}} hsko(X, Y) \restriction X$

-- Our next theorem states the property of h_Θ at which we aim.

Theorem 445 (finite_recursive_fcn · 3) $\mathsf{Finite}(S) \rightarrow h_\Theta(S, T) = f\Big(\{g(h_\Theta(y, T), y, S, T) : y \subseteq S \mid y \neq S \,\&\, P(h_\Theta(y, T), y, S, T)\}, S, T\Big)$. Proof:

Suppose_not$(s,t) \Rightarrow$ $\quad \mathsf{Finite}(s) \;\&\; h_\Theta(s,t) \neq f\Big(\,\{g\big(h_\Theta(y,t),y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(h_\Theta(y,t),y,s,t\big)\}\,,s,t\Big)$

Use_def$(h_\Theta) \Rightarrow$ $\quad \mathsf{hsko}(s,t)\restriction s \neq f\Big(\,\{g\big(\mathsf{hsko}(y,t)\restriction y,y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(\mathsf{hsko}(y,t)\restriction y,y,s,t\big)\}\,,s,t\Big)$

Tfinite_recursive_fcn $\cdot\, 2 \Rightarrow$ $\quad$ *Stat1* :

$\big\langle \forall x \mid x \subseteq s \;\&\; \mathsf{Finite}(x) \rightarrow \mathsf{hsko}(s,t)\restriction x = f\Big(\,\{g\big(\mathsf{hsko}(s,t)\restriction y,y,x,t\big) :\; y \subseteq x \mid y \neq x \;\&\; P\big(\mathsf{hsko}(s,t)\restriction y,y,x,t\big)\}\,,x,t\Big)\big\rangle$

$\langle s\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad \mathsf{hsko}(s,t)\restriction s = f\Big(\,\{g\big(\mathsf{hsko}(s,t)\restriction y,y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(\mathsf{hsko}(s,t)\restriction y,y,s,t\big)\}\,,s,t\Big)$

Suppose $\Rightarrow$

$\{g\big(\mathsf{hsko}(s,t)\restriction y,y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(\mathsf{hsko}(s,t)\restriction y,y,s,t\big)\} =$
$\quad \{g\big(\mathsf{hsko}(y,t)\restriction y,y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(\mathsf{hsko}(y,t)\restriction y,y,s,t\big)\}$

EQUAL $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$

Stat2 : $\{g\big(\mathsf{hsko}(s,t)\restriction y,y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(\mathsf{hsko}(s,t)\restriction y,y,s,t\big)\} \neq \{g\big(\mathsf{hsko}(y,t)\restriction y,y,s,t\big) :\; y \subseteq s \mid y \neq s \;\&\; P\big(\mathsf{hsko}(y,t)\restriction y,y,s,t\big)\}$

$\langle c\rangle \hookrightarrow$ *Stat2* $\Rightarrow$

$c \subseteq s \;\&$

$\quad g\big(\mathsf{hsko}(s,t)\restriction c,c,s,t\big) \neq g\big(\mathsf{hsko}(c,t)\restriction c,c,s,t\big) \vee \Big(c \neq s \;\&\; P\big(\mathsf{hsko}(s,t)\restriction c,c,s,t\big) \;\&\; \neg\big(c \neq s \;\&\; P\big(\mathsf{hsko}(c,t)\restriction c,c,s,t\big)\big)\Big) \vee \Big(\neg\big(c \neq s \;\&\; P\big(\mathsf{hsko}(s,t)\restriction c,c,s,t\big)\big) \;\&\; c \neq s \;\&$

$\langle s,c\rangle \hookrightarrow$ *T162* $\Rightarrow$ $\quad \mathsf{Finite}(c)$

ELEM $\Rightarrow$ $\quad g\big(\mathsf{hsko}(s,t)\restriction c,c,s,t\big) \neq g\big(\mathsf{hsko}(c,t)\restriction c,c,s,t\big) \vee \neg\Big(P\big(\mathsf{hsko}(s,t)\restriction c,c,s,t\big) \leftrightarrow P\big(\mathsf{hsko}(c,t)\restriction c,c,s,t\big)\Big)$

Suppose $\Rightarrow$ $\quad \mathsf{hsko}(c,t)\restriction c = \mathsf{hsko}(s,t)\restriction c$

EQUAL $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \mathsf{hsko}(c,t)\restriction c \neq \mathsf{hsko}(s,t)\restriction c$

-- Next we show that the pair $\mathsf{hsko}(c,t)\restriction y$ and $\mathsf{hsko}(s,t)\restriction y$ of functions have the four properties needed to allow application of the finite_recursion_coherence THEORY derived above. This is done for each of the four necessary statements in turn by showing that the opposite supposition leads to a contradiction.

Suppose $\Rightarrow$ $\quad$ *Stat6* : $\neg\big\langle \forall x \in \mathcal{P}c, y \subseteq x \mid \mathsf{Finite}(x) \;\&\; y \in \mathcal{P}c\big\rangle$

$\langle a_2, b_2\rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $\quad a_2 \in \mathcal{P}c \;\&\; b_2 \subseteq a_2 \;\&\; \neg\mathsf{Finite}(a_2) \vee b_2 \notin \mathcal{P}c$

Use_def$(\mathcal{P}) \Rightarrow$ $\quad$ *Stat7* : $a_2 \in \{x :\; x \subseteq c\}$

$\langle a2p\rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $\quad a_2 \subseteq c$

$\langle c, a_2\rangle \hookrightarrow$ *T162* $\Rightarrow$ $\quad b_2 \notin \mathcal{P}c$

Use_def$(\mathcal{P}) \Rightarrow$ $\quad$ *Stat8* : $b_2 \notin \{x :\; x \subseteq c\}$

$\langle b_2\rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \big\langle \forall x \in \mathcal{P}c, y \subseteq x \mid \mathsf{Finite}(x) \;\&\; y \in \mathcal{P}c\big\rangle$

-- Next we consider the second of the four necessary properties.

Suppose $\Rightarrow$ $\quad$ *Stat3* : $\neg\big\langle \forall x \in \mathcal{P}s, y \subseteq x \mid \mathsf{Finite}(x) \;\&\; y \in \mathcal{P}s\big\rangle$

$\langle a, b\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad a \in \mathcal{P}s \;\&\; b \subseteq a \;\&\; \neg\mathsf{Finite}(a) \vee b \notin \mathcal{P}s$

Use_def$(\mathcal{P}) \Rightarrow$ $\quad$ *Stat4* : $a \in \{x :\; x \subseteq s\}$

$\langle a' \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad a \subseteq s$

$\langle s, a \rangle \hookrightarrow \mathit{T162} \Rightarrow \quad b \notin \mathcal{P}s$

$\text{Use\_def}(\mathcal{P}) \Rightarrow \quad \mathit{Stat5}:\ b \notin \{x :\ x \subseteq s\}$

$\langle b \rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \langle \forall x \in \mathcal{P}s, y \subseteq x \mid \text{Finite}(x) \ \&\ y \in \mathcal{P}s \rangle$

-- Having now established the first two of the four necessary properties we consider the third.

$\text{Suppose} \Rightarrow \quad \mathit{Stat9}:\ \neg$

$\quad \langle \forall x \in \mathcal{P}c, t \mid \text{hsko}(c,t){\upharpoonright}x = f\Big(\ \{g(\text{hsko}(c,t){\upharpoonright}y, y, x, t) :\ y \in \mathcal{P}c \mid y \subseteq x \ \&\ y \neq x \ \&\ P(\text{hsko}(c,t){\upharpoonright}y, y, x, t)\}, x, t\Big)\rangle$

$\langle a_3, tq \rangle \hookrightarrow \mathit{Stat9}(\langle \mathit{Stat9} \rangle) \Rightarrow \quad a_3 \in \mathcal{P}c \ \&$

$\quad \text{hsko}(c,tq){\upharpoonright}a_3 \neq f\Big(\ \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \in \mathcal{P}c \mid y \subseteq a_3 \ \&\ y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\}, a_3, tq\Big)$

$\text{Use\_def}(\mathcal{P}) \Rightarrow \quad \mathit{Stat10}:\ a_3 \in \{x :\ x \subseteq c\}$

$\langle d_3 \rangle \hookrightarrow \mathit{Stat10} \Rightarrow \quad a_3 \subseteq c$

$\langle c, a_3 \rangle \hookrightarrow \mathit{T162} \Rightarrow \quad \text{Finite}(a_3)$

$\langle c, tq \rangle \hookrightarrow \mathit{Tfinite\_recursive\_fcn} \cdot 2 \Rightarrow \quad \mathit{Stat11}:$

$\quad \langle \forall x \mid x \subseteq c \ \&\ \text{Finite}(x) \rightarrow \text{hsko}(c,tq){\upharpoonright}x = f\Big(\ \{g(\text{hsko}(c,tq){\upharpoonright}y, y, x, tq) :\ y \subseteq x \mid y \neq x \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, x, tq)\}, x, tq\Big)\rangle$

$\langle a_3 \rangle \hookrightarrow \mathit{Stat11} \Rightarrow \quad \text{hsko}(c,tq){\upharpoonright}a_3 = f\Big(\ \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \subseteq a_3 \mid y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\}, a_3, tq\Big)$

$\text{Suppose} \Rightarrow$

$\quad \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \in \mathcal{P}c \mid y \subseteq a_3 \ \&\ y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\} =$

$\quad\quad \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \subseteq a_3 \mid y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\}$

$\text{EQUAL} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$

$\quad \mathit{Stat12}:\ \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \in \mathcal{P}c \mid y \subseteq a_3 \ \&\ y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\} \neq \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \subseteq a_3 \mid y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\}$

$\langle c_2 \rangle \hookrightarrow \mathit{Stat12} \Rightarrow$

$\quad\quad c_2 \in \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \in \mathcal{P}c \mid y \subseteq a_3 \ \&\ y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\} \leftrightarrow c_2 \notin \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \subseteq a_3 \mid y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq$

$\text{Suppose} \Rightarrow \quad \mathit{Stat13}:$

$\quad c_2 \in \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \in \mathcal{P}c \mid y \subseteq a_3 \ \&\ y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\} \ \&\ \mathit{Stat14}:$

$\quad\quad c_2 \notin \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \subseteq a_3 \mid y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\}$

$\langle d_2 \rangle \hookrightarrow \mathit{Stat13} \Rightarrow \quad d_2 \in \mathcal{P}c \ \&\ c_2 = g(\text{hsko}(c,tq){\upharpoonright}d_2, d_2, a_3, tq) \ \&\ d_2 \subseteq a_3 \ \&\ d_2 \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}d_2, d_2, a_3, tq)$

$\langle d_2 \rangle \hookrightarrow \mathit{Stat14} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$

$\quad \mathit{Stat15}:$

$\quad\quad c_2 \notin \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \in \mathcal{P}c \mid y \subseteq a_3 \ \&\ y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq)\} \ \&\ \mathit{Stat16}:\ c_2 \in \{g(\text{hsko}(c,tq){\upharpoonright}y, y, a_3, tq) :\ y \subseteq a_3 \mid y \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}y, y$

$\langle c_3 \rangle \hookrightarrow \mathit{Stat16} \Rightarrow \quad c_3 \subseteq a_3 \ \&\ c_2 = g(\text{hsko}(c,tq){\upharpoonright}c_3, c_3, a_3, tq) \ \&\ c_3 \neq a_3 \ \&\ P(\text{hsko}(c,tq){\upharpoonright}c_3, c_3, a_3, tq)$

$\langle c_3 \rangle \hookrightarrow \mathit{Stat15} \Rightarrow \quad c_3 \notin \mathcal{P}c$

$\text{Use\_def}(\mathcal{P}) \Rightarrow \quad \mathit{Stat17}:\ c_3 \notin \{x :\ x \subseteq c\}$

$\langle c_3 \rangle \hookrightarrow \mathit{Stat17} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \langle \forall x \in \mathcal{P}c, t \mid \text{hsko}(c,t){\upharpoonright}x = f\Big(\ \{g(\text{hsko}(c,t){\upharpoonright}y, y, x, t) :\ y \in \mathcal{P}c \mid y \subseteq x \ \&\ y \neq x \ \&\ P(\text{hsko}(c,t){\upharpoonright}y, y, x, t)\}, x, t\Big)\rangle$

-- Having now established the first three of the four necessary properties it only remains to consider the fourth and last. This can be handled in a manner almost identical to that of the third case considered above.

Suppose $\Rightarrow$ $\quad Stat18: \neg$

$\langle \forall x \in \mathcal{P}s, t \mid \mathrm{hsko}(s,t)\lceil x = f\Big(\{g(\mathrm{hsko}(s,t)\lceil y, y, x, t) : y \in \mathcal{P}s \mid y \subseteq x \,\&\, y \neq x \,\&\, P(\mathrm{hsko}(s,t)\lceil y, y, x, t)\}, x, t\Big)\rangle$

$\langle a_4, t' \rangle \hookrightarrow Stat18(\langle Stat18 \rangle) \Rightarrow \quad a_4 \in \mathcal{P}s \,\&$

$\mathrm{hsko}(s,t')\lceil a_4 \neq f\Big(\{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \in \mathcal{P}s \mid y \subseteq a_4 \,\&\, y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\}, a_4, t'\Big)$

Use_def $(\mathcal{P}) \Rightarrow \quad Stat19: a_4 \in \{x : x \subseteq s\}$

$\langle d_4 \rangle \hookrightarrow Stat19 \Rightarrow \quad Stat20: a_4 \subseteq s$

$\langle s, a_4 \rangle \hookrightarrow T162 \Rightarrow \quad \mathrm{Finite}(a_4)$

$\langle s, t' \rangle \hookrightarrow Tfinite\_recursive\_fcn \cdot 2 \Rightarrow \quad Stat21:$

$\langle \forall x \mid x \subseteq s \,\&\, \mathrm{Finite}(x) \rightarrow \mathrm{hsko}(s,t')\lceil x = f\Big(\{g(\mathrm{hsko}(s,t')\lceil y, y, x, t') : y \subseteq x \mid y \neq x \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, x, t')\}, x, t'\Big)\rangle$

$\langle a_4 \rangle \hookrightarrow Stat21(\langle Stat20 \rangle) \Rightarrow \quad \mathrm{hsko}(s,t')\lceil a_4 = f\Big(\{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \subseteq a_4 \mid y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\}, a_4, t'\Big)$

Suppose $\Rightarrow$

$\{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \in \mathcal{P}s \mid y \subseteq a_4 \,\&\, y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\} =$
$\quad \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \subseteq a_4 \mid y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\}$

EQUAL $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$

$Stat22: \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \in \mathcal{P}s \mid y \subseteq a_4 \,\&\, y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\} \neq \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \subseteq a_4 \mid y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\}$

$\langle c_4 \rangle \hookrightarrow Stat22 \Rightarrow$

$c_4 \in \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \in \mathcal{P}s \mid y \subseteq a_4 \,\&\, y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\} \leftrightarrow c_4 \notin \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \subseteq a_4 \mid y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\}$

Suppose $\Rightarrow$ $\quad Stat23:$

$c_4 \in \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \in \mathcal{P}s \mid y \subseteq a_4 \,\&\, y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\} \,\&\, Stat24:$
$\quad c_4 \notin \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \subseteq a_4 \mid y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\}$

$\langle d_5 \rangle \hookrightarrow Stat23 \Rightarrow \quad Stat25: d_5 \in \mathcal{P}s \,\&\, d_5 \subseteq a_4 \,\&\, c_4 = g(\mathrm{hsko}(s,t')\lceil d_5, d_5, a_4, t') \,\&\, d_5 \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil d_5, d_5, a_4, t')$

$\langle d_5 \rangle \hookrightarrow Stat24(\langle Stat25 \rangle) \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow$

$Stat26:$

$c_4 \notin \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \in \mathcal{P}s \mid y \subseteq a_4 \,\&\, y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4, t')\} \,\&\, Stat27: c_4 \in \{g(\mathrm{hsko}(s,t')\lceil y, y, a_4, t') : y \subseteq a_4 \mid y \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil y, y, a_4,$

$\langle d_6 \rangle \hookrightarrow Stat27 \Rightarrow \quad d_6 \subseteq a_4 \,\&\, c_4 = g(\mathrm{hsko}(s,t')\lceil d_6, d_6, a_4, t') \,\&\, d_6 \neq a_4 \,\&\, P(\mathrm{hsko}(s,t')\lceil d_6, d_6, a_4, t')$

$\langle d_6 \rangle \hookrightarrow Stat26 \Rightarrow \quad d_6 \notin \mathcal{P}s$

Use_def $(\mathcal{P}) \Rightarrow \quad Stat28: d_6 \notin \{x : x \subseteq s\}$

$\langle d_6 \rangle \hookrightarrow Stat28 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \langle \forall x \in \mathcal{P}s, t \mid \mathrm{hsko}(s,t)\lceil x = f\Big(\{g(\mathrm{hsko}(s,t)\lceil y, y, x, t) : y \in \mathcal{P}s \mid y \subseteq x \,\&\, y \neq x \,\&\, P(\mathrm{hsko}(s,t)\lceil y, y, x, t)\}, x, t\Big)\rangle$

-- We now have everything needed to apply the finite_recursion_coherence THEORY developed above. This gives a direct contradiction with the inequality $\mathrm{hsko}(c,t)\lceil c \neq \mathrm{hsko}(s,t)\lceil c$ proved earlier, and so completes the proof of the present theorem.

APPLY $\langle\,\rangle$ finite_recursion_coherence$(q \mapsto \mathcal{P}c, r \mapsto \mathcal{P}s, h_q(x,t) \mapsto \mathrm{hsko}(c,t)\lceil x, h_r(x,t) \mapsto \mathrm{hsko}(s,t)\lceil x, f(b,x,t) \mapsto f(b,x,t), g(a,y,x,t) \mapsto g(a,y,x,t), P(a,y,x,t) \mapsto P(a,y,x,t)) \Rightarrow$

Stat29 : $\langle\forall x, t \mid x \in \mathcal{P}c \;\&\; x \in \mathcal{P}s \rightarrow \mathsf{hsko}(c,t)|x = \mathsf{hsko}(s,t)|x\rangle$

Suppose $\Rightarrow$ $c \notin \mathcal{P}s$

Use_def $(\mathcal{P}) \Rightarrow$ *Stat30* : $c \notin \{x : x \subseteq s\}$

$\langle c\rangle \hookrightarrow$ *Stat30* $\Rightarrow$ false; Discharge $\Rightarrow$ $c \in \mathcal{P}s$

Suppose $\Rightarrow$ $c \notin \mathcal{P}c$

Use_def $(\mathcal{P}) \Rightarrow$ *Stat31* : $c \notin \{x : x \subseteq c\}$

$\langle c\rangle \hookrightarrow$ *Stat31* $\Rightarrow$ false; Discharge $\Rightarrow$ $c \in \mathcal{P}c$

$\langle c, t\rangle \hookrightarrow$ *Stat29* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

-- We will now derive a variant recursive function definition method which is simpler (but more limited) than that which our prior finite_recursive_fcn THEORY affords. This applies to cases in which the value $f(s)$ can be defined in terms of $f(sl(s))$ alone, for some strict subset $sl(s)$ (most typically $s \setminus \{\mathbf{arb}(s)\}$). For this, we simply specialize the THEORY finite_recursive_fcn.

THEORY finite_tailrecursive_fcn $(f(t), g(a,x,t), sl(x))$

$\langle\forall x \mid sl(x) \subseteq x \;\&\; (x \neq \emptyset \rightarrow sl(x) \neq x)\rangle$

END finite_tailrecursive_fcn

ENTER_THEORY finite_tailrecursive_fcn

APPLY $\langle h_\Theta : h_\Theta\rangle$ finite_recursive_fcn $\left(f(b,x,t) \mapsto \textbf{if } b = \emptyset \textbf{ then } f(t) \textbf{ else } \mathbf{arb}(b) \textbf{ fi}, g(a,y,x,t) \mapsto g(a,x,t), P(a,y,x,t) \mapsto y = sl(x)\right) \Rightarrow$

Theorem 446 (finite_tailrecursive_fcn · 0)
$\langle\forall s, t \mid \mathsf{Finite}(s) \rightarrow h_\Theta(s,t) = \textbf{if } \{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \;\&\; y = sl(s)\} = \emptyset \textbf{ then } f(t) \textbf{ else } \mathbf{arb}(\{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \;\&\; y = sl(s)\}) \textbf{ fi}\rangle$.

-- The clumsy recursive relationship appearing in this last Theorem can be simplified by noting that the condition $\{g(h_\Theta(y,T),S,T) : y \subseteq S \mid y \neq S \;\&\; y = sel(S)\} = \emptyset$ is equivalent to $S = \emptyset$, and that when $S \neq \emptyset$ $\{g(h_\Theta(y,T),S,T) : y \subseteq S \mid y \neq S \;\&\; y = sel(S)\}$ is just $\{g\}\left(h_\Theta(sel(S),T),S,T\right)$. The proof which follows does this.

Theorem 447 (finite_tailrecursive_fcn · 1) $\mathsf{Finite}(S) \rightarrow h_\Theta(S,T) = \textbf{if } S = \emptyset \textbf{ then } f(T) \textbf{ else } g\left(h_\Theta(sl(S),T),S,T\right) \textbf{ fi}$. PROOF:

Suppose_not $(s,t) \Rightarrow$ $\mathsf{Finite}(s) \;\&\; h_\Theta(s,t) \neq \textbf{if } s = \emptyset \textbf{ then } f(t) \textbf{ else } g\left(h_\Theta(sl(s),t),s,t\right) \textbf{ fi}$

Tfinite_tailrecursive_fcn · 0 $\Rightarrow$ *Stat0* :

$\langle \forall s,t \mid \mathsf{Finite}(s) \rightarrow h_\Theta(s,t) = \textbf{if } \{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\} = \emptyset \textbf{ then } f(t) \textbf{ else arb}(\{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\}) \textbf{ fi}\rangle$

$\langle s,t\rangle \hookrightarrow Stat0 \Rightarrow$ $h_\Theta(s,t) =$
$\textbf{if } \{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\} = \emptyset \textbf{ then } f(t) \textbf{ else arb}(\{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\}) \textbf{ fi}$

Suppose $\Rightarrow$ $s = \emptyset$

ELEM $\Rightarrow$ $Stat1 : \{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\} \neq \emptyset$

$\langle c\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $s \neq \emptyset$

Assump $\Rightarrow$ $Stat2 : \langle \forall x \mid sl(x) \subseteq x \,\&\, (x \neq \emptyset \rightarrow sl(x) \neq x)\rangle$

$\langle s\rangle \hookrightarrow Stat2 \Rightarrow$ $sl(s) \subseteq s \,\&\, sl(s) \neq s$

Suppose $\Rightarrow$ $Stat3 : \{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\} = \emptyset$

$\langle sl(s)\rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $h_\Theta(s,t) \neq g\Big(h_\Theta(sl(s),t),s,t\Big) \,\&\, Stat4 : h_\Theta(s,t) \in \{g(h_\Theta(y,t),s,t) : y \subseteq s \mid y \neq s \,\&\, y = sl(s)\}$

$\langle y\rangle \hookrightarrow Stat4 \Rightarrow$ $y = sl(s) \,\&\, h_\Theta(s,t) = g(h_\Theta(y,t),s,t)$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY finite_tailrecursive_fcn

THEORY finite_tailrecursive_fcn$(f(t), g(a,x,t), sl(x))$
$\langle \forall x \mid sl(x) \subseteq x \,\&\, (x \neq \emptyset \rightarrow sl(x) \neq x)\rangle$
$\Rightarrow (h_\Theta)$
$\langle \forall s,t \mid \mathsf{Finite}(s) \rightarrow h_\Theta(s,t) = \textbf{if } s = \emptyset \textbf{ then } f(t) \textbf{ else } g\Big(h_\Theta(sl(s),t),s,t\Big) \textbf{ fi}\rangle$
END finite_tailrecursive_fcn

-- We will now derive a variant recursive function definition method which is simpler (but more limited) than that which our prior finite_recursive_fcn THEORY affords. This applies to cases in which the value $f(s)$ can be defined in terms of $f(s\setminus\{\mathbf{arb}(s)\})$ alone. For this, we simply specialize the THEORY finite_recursive_fcn.

THEORY finite_tailrecursive_fcn$_1(f(t), g(a,x,t))$
END finite_recursive_fcn$_1$

ENTER_THEORY finite_tailrecursive_fcn$_1$

Theorem 448 (finite_tailrecursive_fcn$_1 \cdot 0$) $X\setminus\{\mathbf{arb}(X)\} \subseteq X \,\&\, (X \neq \emptyset \rightarrow X\setminus\{\mathbf{arb}(X)\} \neq X)$. PROOF:

Suppose_not$(x) \Rightarrow$ $x\setminus\{\mathbf{arb}(x)\} \nsubseteq x \,\&\, (x \neq \emptyset \rightarrow x\setminus\{\mathbf{arb}(x)\} \neq x)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle \mathsf{h}_\Theta : \mathsf{h}_\Theta \rangle$ finite_tailrecursive_fcn$\big(\mathsf{f}(\mathsf{t}) \mapsto \mathsf{f}(\mathsf{t}), \mathsf{g}(\mathsf{a},\mathsf{x},\mathsf{t}) \mapsto \mathsf{g}(\mathsf{a},\mathsf{x},\mathsf{t}), \mathsf{sl}(\mathsf{x}) \mapsto \mathsf{x} \setminus \{\mathbf{arb}(\mathsf{x})\}\big) \Rightarrow$

Theorem 449 (finite_tailrecursive_fcn$_1 \cdot 1$) $\langle \forall \mathsf{s},\mathsf{t} \mid \mathsf{Finite}(\mathsf{s}) \rightarrow \mathsf{h}_\Theta(\mathsf{s},\mathsf{t}) = \textbf{if } \mathsf{s} = \emptyset \textbf{ then } \mathsf{f}(\mathsf{t}) \textbf{ else } \mathsf{g}\big(\mathsf{h}_\Theta(\mathsf{s} \setminus \{\mathbf{arb}(\mathsf{s})\}, \mathsf{t}), \mathsf{s}, \mathsf{t}\big) \textbf{ fi} \rangle$.

ENTER_THEORY Set_theory

DISPLAY finite_tailrecursive_fcn$_1$

THEORY finite_tailrecursive_fcn$_1\big(\mathsf{f}(\mathsf{t}), \mathsf{g}(\mathsf{a},\mathsf{x},\mathsf{t})\big)$
$\Rightarrow (\mathsf{h}_\Theta)$
$\langle \forall \mathsf{s},\mathsf{t} \mid \mathsf{Finite}(\mathsf{s}) \rightarrow \mathsf{h}_\Theta(\mathsf{s},\mathsf{t}) = \textbf{if } \mathsf{s} = \emptyset \textbf{ then } \mathsf{f}(\mathsf{t}) \textbf{ else } \mathsf{g}\big(\mathsf{h}_\Theta(\mathsf{s} \setminus \{\mathbf{arb}(\mathsf{s})\}, \mathsf{t}), \mathsf{s}, \mathsf{t}\big) \textbf{ fi} \rangle$
END finite_tailrecursive_fcn$_1$

-- Our next variant recursive function definition scheme is even simpler, but applies only when the function being defined is monadic, ather than dyadic as above. To derive it we simply specialize the preceding theory.

THEORY finite_tailrecursive_fcn$_2\big(\mathsf{f}_0, \mathsf{g}_2(\mathsf{a},\mathsf{x})\big)$
END finite_tailrecursive_fcn$_2$

ENTER_THEORY finite_tailrecursive_fcn$_2$

APPLY $\langle \mathsf{h}_\Theta : \mathsf{h} \rangle$ finite_tailrecursive_fcn$_1\big(\mathsf{f}(\mathsf{t}) \mapsto \mathsf{f}_0, \mathsf{g}(\mathsf{a},\mathsf{x},\mathsf{t}) \mapsto \mathsf{g}_2(\mathsf{a},\mathsf{x})\big) \Rightarrow$

Theorem 450 (finite_tailrecursive_fcn$_2 \cdot 0$) $\langle \forall \mathsf{s},\mathsf{t} \mid \mathsf{Finite}(\mathsf{s}) \rightarrow \mathsf{h}(\mathsf{s},\mathsf{t}) = \textbf{if } \mathsf{s} = \emptyset \textbf{ then } \mathsf{f}_0 \textbf{ else } \mathsf{g}_2\big(\mathsf{h}(\mathsf{s} \setminus \{\mathbf{arb}(\mathsf{s})\}, \mathsf{t}), \mathsf{s}\big) \textbf{ fi} \rangle$.

DEF $00f$. $\quad \mathsf{h}_\Theta(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathsf{h}(\mathsf{X}, \emptyset)$

Theorem 451 (finite_tailrecursive_fcn$_2 \cdot 1$) $\mathsf{Finite}(\mathsf{S}) \rightarrow \mathsf{h}_\Theta(\mathsf{S}) = \textbf{if } \mathsf{S} = \emptyset \textbf{ then } \mathsf{f}_0 \textbf{ else } \mathsf{g}_2\big(\mathsf{h}_\Theta(\mathsf{S} \setminus \{\mathbf{arb}(\mathsf{S})\}), \mathsf{S}\big) \textbf{ fi}$. PROOF:

Suppose_not(s) $\Rightarrow$ $\quad \mathsf{Finite}(\mathsf{s})\ \&\ \mathsf{h}_\Theta(\mathsf{s}) \neq \textbf{if } \mathsf{s} = \emptyset \textbf{ then } \mathsf{f}_0 \textbf{ else } \mathsf{g}_2\big(\mathsf{h}_\Theta(\mathsf{s} \setminus \{\mathbf{arb}(\mathsf{s})\}), \mathsf{s}\big) \textbf{ fi}$
Tfinite_tailrecursive_fcn2 $\cdot 0 \Rightarrow$ $\quad$ *Stat1* : $\langle \forall \mathsf{s},\mathsf{t} \mid \mathsf{Finite}(\mathsf{s}) \rightarrow \mathsf{h}(\mathsf{s},\mathsf{t}) = \textbf{if } \mathsf{s} = \emptyset \textbf{ then } \mathsf{f}_0 \textbf{ else } \mathsf{g}_2\big(\mathsf{h}(\mathsf{s} \setminus \{\mathbf{arb}(\mathsf{s})\}, \mathsf{t}), \mathsf{s}\big) \textbf{ fi} \rangle$
$\langle \mathsf{s}, \emptyset \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad \mathsf{h}(\mathsf{s},\emptyset) = \textbf{if } \mathsf{s} = \emptyset \textbf{ then } \mathsf{f}_0 \textbf{ else } \mathsf{g}_2\big(\mathsf{h}(\mathsf{s} \setminus \{\mathbf{arb}(\mathsf{s})\}, \emptyset), \mathsf{s}\big) \textbf{ fi}$
Use_def$(\mathsf{h}_\Theta) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

ENTER_THEORY Set_theory

DISPLAY finite_tailrecursive_fcn$_2$

THEORY finite_tailrecursive_fcn$_2(f_0, g_2(a, x))$
$\Rightarrow (h_\Theta)$
$\langle\forall s \mid \mathrm{Finite}(s) \rightarrow h_\Theta(s) = \textbf{if } s = \emptyset \textbf{ then } f_0 \textbf{ else } g_2(h_\Theta(s \setminus \{\mathbf{arb}(s)\}), s) \textbf{ fi}\rangle$
END finite_tailrecursive_fcn$_2$

-- Our next aim is to introduce the very important and often used 'theory of sigma', which shows that for any commutative operation $\oplus$ defined on a set s and having the algebraic properties of addition, there exists a summation operator sigma, defined for all finite, single-valued mappings g having values in s, which represents the repeated sum customarily written using an informal 'three dots' notation as $g(x_1) \oplus g(x_2) \oplus \ldots \oplus g(xn)$. This summation operator sends the empty mapping into the additive zero element of the underlying domain s and maps any singleton $\{[x, y]\}$ with $y \in s$ into y. The sigma operator is additive over pairs of mappings having disjoint domains, and, more generally, if g is decomposed in any way into a collection of disjoint parts gj, then sigma(g) is the sum of all the values sigma(gj), where gj runs over all the parts into which g has been decomposed. This theory will be used to define sums of finite sequences of elements of many kinds, e. g. integers, signed integers, rational, real and complex numbers, functions whose values are integers, signed integers, rational, real or complex numbers, etc. It can also be used to define extended product operators PI(g) of the kind which would ordinarily be written less formally as $g(x_1) * g(x_2) * \ldots * g(xn)$ etc.

THEORY sigma_theory$(s, x \oplus y, e)$
$e \in s$
$\langle\forall x \in s \mid x \oplus e = x\rangle$
$\langle\forall x \in s, y \in s \mid x \oplus y = y \oplus x\rangle$
$\langle\forall x \in s, y \in s \mid x \oplus y \in s\rangle$
$\langle\forall x \in s, y \in s, z \in s \mid (x \oplus y) \oplus z = x \oplus (y \oplus z)\rangle$
END sigma_theory

ENTER_THEORY sigma_theory

-- Our first step is to define the sigma operation y applying our recursive definition schema to the conditional expression appearing in the following formula.

APPLY $\langle h_\Theta : \Sigma_\Theta\rangle$ finite_tailrecursive_fcn$_2(f_0 \mapsto e, g_2(y, x) \mapsto y \oplus \mathbf{arb}(x)^{[2]}) \Rightarrow$

Theorem 452 (sigma_theory$_0$) $\langle\forall x \mid \mathsf{Finite}(x) \rightarrow \Sigma_\Theta(x) = \textbf{if } x = \emptyset \textbf{ then } e \textbf{ else } \Sigma_\Theta(x \setminus \{\mathbf{arb}(x)\}) \oplus \mathbf{arb}(x)^{[2]} \textbf{ fi}\rangle$.

-- Next we begin the sequence of proofs belonging to the present theory by showing that, when applied to the null set, the sigma-operation associated with the generic addition operator $\oplus$ yields the additive zero element of the underlying domain s.

Theorem 453 (sigma_theory$_1$) $\Sigma_\Theta(\emptyset) = e$. PROOF:

Suppose_not $\Rightarrow$ $\Sigma_\Theta(\emptyset) \neq e$
T161 $\Rightarrow$ $\mathsf{Finite}(\emptyset)$
Tsigma_theory0 $\Rightarrow$ *Stat1* : $\langle\forall x \mid \mathsf{Finite}(x) \rightarrow \Sigma_\Theta(x) = \textbf{if } x = \emptyset \textbf{ then } e \textbf{ else } \Sigma_\Theta(x \setminus \{\mathbf{arb}(x)\}) \oplus \mathbf{arb}(x)^{[2]} \textbf{ fi}\rangle$
$\langle\emptyset\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that when applied to any singleton set $\{[v, y]\}$ with y in the domain s underlying $\oplus$ the sigma-operation simply yields y:

Theorem 454 (sigma_theory$_2$) $X^{[2]} \in s \rightarrow \Sigma_\Theta(\{X\}) = X^{[2]}$. PROOF:

Suppose_not(x) $\Rightarrow$ $x^{[2]} \in s \;\&\; \Sigma_\Theta(\{x\}) \neq x^{[2]}$

-- This follows immediately by defintion of the function Σ_Θ and from the fact that the costant e appearing in the preceding heorm is the additive zero element of the underlying domain s.

$\langle x\rangle \hookrightarrow$ *T174* $\Rightarrow$ $\mathsf{Finite}(\{x\})$
Tsigma_theory0 $\Rightarrow$ *Stat0* : $\langle\forall x \mid \mathsf{Finite}(x) \rightarrow \Sigma_\Theta(x) = \textbf{if } x = \emptyset \textbf{ then } e \textbf{ else } \Sigma_\Theta(x \setminus \{\mathbf{arb}(x)\}) \oplus \mathbf{arb}(x)^{[2]} \textbf{ fi}\rangle$
$\langle\{x\}\rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\Sigma_\Theta(\{x\}) = \Sigma_\Theta(\{x\} \setminus \{\mathbf{arb}(\{x\})\}) \oplus \mathbf{arb}(\{x\})^{[2]}$
TELEM $\Rightarrow$ $\{x\} \setminus \{\mathbf{arb}(\{x\})\} = \emptyset \;\&\; \mathbf{arb}(\{x\}) = x$
Tsigma_theory1 $\Rightarrow$ $\Sigma_\Theta(\emptyset) = e$
EQUAL $\Rightarrow$ $\Sigma_\Theta(\{x\}) = e \oplus x^{[2]}$
Assump $\Rightarrow$ $e \in s \;\&\;$ *Stat1* : $\langle\forall x \in s \mid x \oplus e = x\rangle \;\&\;$ *Stat2* : $\langle\forall x \in s, y \in s \mid x \oplus y = y \oplus x\rangle$
$\langle e, x^{[2]}\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\Sigma_\Theta(\{x\}) = x^{[2]} \oplus e$
$\langle x^{[2]}\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemm tells us that when applied to a finite map F whose range is included in the domain s underlying $\oplus$, the sigma-operation yields an element of s:

Theorem 455 (sigma_theory$_3$) $\text{Finite}(F) \,\&\, \mathbf{range}(F) \subseteq s \rightarrow \Sigma_\Theta(F) \in s$. PROOF:

Suppose_not$(f_1) \Rightarrow$ $\text{Finite}(f_1) \,\&\, \mathbf{range}(f_1) \subseteq s \,\&\, \Sigma_\Theta(f_1) \notin s$

-- For, assuming the contrary, there would be a map f contradicting the statement which we want to prove which was also inclusion-minimal. By an earlier theorem, such an f would be non-null.

APPLY $\langle m_\Theta : f\rangle$ finite_induction$(n \mapsto f_1, P(x) \mapsto \mathbf{range}(x) \subseteq s \,\&\, \Sigma_\Theta(x) \notin s) \Rightarrow$

$f \subseteq f_1 \,\&\, \mathbf{range}(f) \subseteq s \,\&\, \Sigma_\Theta(f) \notin s \,\&\, \mathit{Stat1}: \langle \forall g \subseteq f \,|\, g \neq f \rightarrow \neg(\mathbf{range}(g) \subseteq s \,\&\, \Sigma_\Theta(g) \notin s)\rangle$

$\langle f_1, f\rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\text{Finite}(f)$

Suppose $\Rightarrow$ $f = \emptyset$

$\mathit{Tsigma\_theory1} \Rightarrow$ $\Sigma_\Theta(\emptyset) = e$

Assump $\Rightarrow$ $e \in s$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $f \supseteq f \setminus \{\mathbf{arb}(f)\} \,\&\, f \setminus \{\mathbf{arb}(f)\} \neq f \,\&\, \mathbf{arb}(f) \in f$

-- But then removal of $\mathbf{arb}(f)$ from f would produce a map strictly included in f whose sigma-image would not belong to s, a contradiction which proves the desired statement.

$\langle f \setminus \{\mathbf{arb}(f)\}\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\neg(\mathbf{range}(f \setminus \{\mathbf{arb}(f)\}) \subseteq s \,\&\, \Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\}) \notin s)$

Use_def(range) $\Rightarrow$ $\mathbf{range}(f \setminus \{\mathbf{arb}(f)\}) = \{x^{[2]} : x \in f \setminus \{\mathbf{arb}(f)\}\} \,\&\, \mathbf{range}(f) = \{x^{[2]} : x \in f\}$

Set_monot $\Rightarrow$ $\{x^{[2]} : x \in f \setminus \{\mathbf{arb}(f)\}\} \subseteq \{x^{[2]} : x \in f\}$

ELEM $\Rightarrow$ $\Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\}) \in s$

$\mathit{Tsigma\_theory0} \Rightarrow$ $\mathit{Stat0}: \langle \forall x \,|\, \text{Finite}(x) \rightarrow \Sigma_\Theta(x) = \textbf{if } x = \emptyset \textbf{ then } e \textbf{ else } \Sigma_\Theta(x \setminus \{\mathbf{arb}(x)\}) \oplus \mathbf{arb}(x)^{[2]} \textbf{ fi}\rangle$

$\langle f\rangle \hookrightarrow \mathit{Stat0} \Rightarrow$ $\Sigma_\Theta(f) = \Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\}) \oplus \mathbf{arb}(f)^{[2]}$

Suppose $\Rightarrow$ $\mathit{Stat2}: \mathbf{arb}(f)^{[2]} \notin \{x^{[2]} : x \in f\}$

$\langle \mathbf{arb}(f)\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{arb}(f)^{[2]} \in s$

Assump $\Rightarrow$ $\mathit{Stat3}: \langle \forall x \in s, y \in s \,|\, x \oplus y \in s\rangle$

$\langle \Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\}), \mathbf{arb}(f)^{[2]}\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem shows that in the recursive definition of $\Sigma_\Theta(f)$ (where f is a finite non-null map whose range is included in the domain underlying $\oplus$), any element c of f can take the place of $\mathbf{arb}(f)$.

Theorem 456 (sigma_theory$_4$) $C \in F \,\&\, \text{Finite}(F) \,\&\, \mathbf{range}(F) \subseteq s \rightarrow \Sigma_\Theta(F) = \Sigma_\Theta(F \setminus \{C\}) \oplus C^{[2]}$. PROOF:

Suppose_not$(f_1, s, c_1) \Rightarrow$ $\mathit{Stat0a}: \text{Finite}(f_1) \,\&\, \mathbf{range}(f_1) \subseteq s \,\&\, c_1 \in f_1 \,\&\, \Sigma_\Theta(f_1) \neq \Sigma_\Theta(f_1 \setminus \{c_1\}) \oplus c_1^{[2]}$

-- For, assuming the contrary, there would exist an inclusion-minimal map f contradicting our assertion, so that f would have an element c such that $\Sigma_\Theta(f) \neq \Sigma_\Theta(f \setminus \{c\}) \oplus c^{[2]}$, whereas $\Sigma_\Theta(f) = \Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\}) \oplus \mathbf{arb}(f)^{[2]}$ by definition.

Suppose $\Rightarrow$ $\quad$ *Stat1* : $\neg\langle \exists c \in f_1 \mid \Sigma_\Theta(f_1) \neq \Sigma_\Theta(f_1 \backslash \{c\}) \oplus c^{[2]}\rangle$
$\langle c_1 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad \mathsf{Finite}(f_1)\ \&\ \mathbf{range}(f_1) \subseteq s\ \&\ \langle \exists c \in f_1 \mid \Sigma_\Theta(f_1) \neq \Sigma_\Theta(f_1 \backslash \{c\}) \oplus c^{[2]}\rangle$
APPLY $\langle m_\Theta : f\rangle$ finite_induction $(n \mapsto f_1, P(x) \mapsto \mathbf{range}(x) \subseteq s\ \&\ \langle \exists c \in x \mid \Sigma_\Theta(x) \neq \Sigma_\Theta(x \backslash \{c\}) \oplus c^{[2]}\rangle) \Rightarrow$
$\quad f \subseteq f_1\ \&\ \mathbf{range}(f) \subseteq s\ \&$ *Stat2* : $\langle \exists c \in f \mid \Sigma_\Theta(f) \neq \Sigma_\Theta(f \backslash \{c\}) \oplus c^{[2]}\rangle\ \&$ *Stat3* : $\langle \forall g \subseteq f \mid g \neq f \rightarrow \mathbf{range}(g) \not\subseteq s\ \&\ \langle \exists d \in g \mid \Sigma_\Theta(g) \neq \Sigma_\Theta(g \backslash \{d\}) \oplus d^{[2]}\rangle\rangle$
$\langle f_1, f\rangle \hookrightarrow$ *T162* $\Rightarrow$ $\quad \mathsf{Finite}(f)$
$\langle c \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\quad c \in f\ \&\ \Sigma_\Theta(f) \neq \Sigma_\Theta(f \backslash \{c\}) \oplus c^{[2]}$
Tsigma_theory0 $\Rightarrow$ $\quad$ *Stat0* : $\langle \forall x \mid \mathsf{Finite}(x) \rightarrow \Sigma_\Theta(x) = \mathbf{if}\ x = \emptyset\ \mathbf{then}\ e\ \mathbf{else}\ \Sigma_\Theta(x \backslash \{\mathbf{arb}(x)\}) \oplus \mathbf{arb}(x)^{[2]}\ \mathbf{fi}\rangle$
$\langle f \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\quad \Sigma_\Theta(f) = \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\}) \oplus \mathbf{arb}(f)^{[2]}$

-- Given the counterexample that we have assumed, it is clear that f includes $\{c\}$ strictly. Moreover, the assumed minimality of f implies that any element of $f \backslash \{\mathbf{arb}(f)\}$, in particular c, can be used to calculate $\Sigma_\Theta(f \backslash \{c\})$.

Suppose $\Rightarrow$ $\quad \mathbf{arb}(f) = c$
EQUAL $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad \mathbf{arb}(f) \neq c\ \&\ f \backslash \{c\} \neq \emptyset\ \&\ f \backslash \{\mathbf{arb}(f)\} \subseteq f\ \&\ f \backslash \{\mathbf{arb}(f)\} \neq f\ \&\ c \in f \backslash \{\mathbf{arb}(f)\}$
$\langle f \backslash \{\mathbf{arb}(f)\}, f\rangle \hookrightarrow$ *T60* $\Rightarrow$ $\quad \mathbf{range}(f \backslash \{\mathbf{arb}(f)\}) \subseteq \mathbf{range}(f)$
ELEM $\Rightarrow$ $\quad \mathbf{range}(f \backslash \{\mathbf{arb}(f)\}) \subseteq s$
$\langle f \backslash \{\mathbf{arb}(f)\}\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad$ *Stat4* : $\neg\langle \exists c \in f \backslash \{\mathbf{arb}(f)\} \mid \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\}) \neq \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \oplus c^{[2]}\rangle$
$\langle c \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\quad \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\}) = \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \oplus c^{[2]}$

-- Using the minimality of f once more we can derive the following expression for $\Sigma_\Theta(f \backslash \{c\})$.

ELEM $\Rightarrow$ $\quad f \backslash \{c\} \subseteq f\ \&\ f \backslash \{c\} \neq f\ \&\ \mathbf{arb}(f) \in f \backslash \{c\}$
$\langle f \backslash \{c\}, f\rangle \hookrightarrow$ *T60* $\Rightarrow$ $\quad \mathbf{range}(f \backslash \{c\}) \subseteq \mathbf{range}(f)\ \&\ \mathbf{range}(f \backslash \{c\}) \subseteq s$
$\langle f \backslash \{c\}\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad$ *Stat5* : $\neg\langle \exists d \in f \backslash \{c\} \mid \Sigma_\Theta(f \backslash \{c\}) \neq \Sigma_\Theta(f \backslash \{c\} \backslash \{d\}) \oplus d^{[2]}\rangle$
$\langle \mathbf{arb}(f)\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\quad \Sigma_\Theta(f \backslash \{c\}) = \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \oplus \mathbf{arb}(f)^{[2]}$

-- Using the commutativity and associativity of $\oplus$ we can now derive an easy contradiction which completes our proof.

$\langle c, f\rangle \hookrightarrow$ *T56* $\Rightarrow$ $\quad c^{[2]} \in s$
$\langle \mathbf{arb}(f), f\rangle \hookrightarrow$ *T56* $\Rightarrow$ $\quad \mathbf{arb}(f)^{[2]} \in s$
$\langle f, f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}\rangle \hookrightarrow$ *T162* $\Rightarrow$ $\quad \mathsf{Finite}(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\})$
$\langle f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}, f\rangle \hookrightarrow$ *T60([Stat0a, ∩])* $\Rightarrow$ $\quad \mathbf{range}(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \subseteq s$
$\langle f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}\rangle \hookrightarrow$ *Tsigma_theory3* $\Rightarrow$ $\quad \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \in s$
Assump $\Rightarrow$ $\quad$ *Stat6* : $\langle \forall x \in s, y \in s, z \in s \mid (x \oplus y) \oplus z = x \oplus (y \oplus z)\rangle$
$\langle \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}), \mathbf{arb}(f)^{[2]}, c^{[2]}\rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $\quad \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \oplus \mathbf{arb}(f)^{[2]} \oplus c^{[2]} =$
$\quad \Sigma_\Theta(f \backslash \{\mathbf{arb}(f)\} \backslash \{c\}) \oplus (\mathbf{arb}(f)^{[2]} \oplus c^{[2]})$
Assump $\Rightarrow$ $\quad$ *Stat7* : $\langle \forall x \in s, y \in s \mid x \oplus y = y \oplus x\rangle$

$\langle \mathbf{arb}(f)^{[2]}, c^{[2]} \rangle \hookrightarrow \mathit{Stat7} \Rightarrow \quad \mathbf{arb}(f)^{[2]} \oplus c^{[2]} = c^{[2]} \oplus \mathbf{arb}(f)^{[2]}$

$\langle \Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\} \setminus \{c\}), c^{[2]}, \mathbf{arb}(f)^{[2]} \rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad \Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\} \setminus \{c\}) \oplus (c^{[2]} \oplus \mathbf{arb}(f)^{[2]}) =$
$\Sigma_\Theta(f \setminus \{\mathbf{arb}(f)\} \setminus \{c\}) \oplus c^{[2]} \oplus \mathbf{arb}(f)^{[2]}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that if map f like that considered in the previous theorems is divided into two disjoint parts f_1 and f_2, we have $\Sigma_\Theta(f) = \Sigma_\Theta(f_1) \oplus \Sigma_\Theta(f_2)$.

Theorem 457 (sigma_theory$_5$) $\mathsf{Finite}(F)\ \&\ \mathsf{Is\_map}(F)\ \&\ \mathbf{range}(F) \subseteq s \rightarrow \Sigma_\Theta(F) = \Sigma_\Theta(F_{|\mathbf{domain}(F) \cap T}) \oplus \Sigma_\Theta(F_{|\mathbf{domain}(F) \setminus T})$. PROOF:

Suppose_not$(f_1, s, t_1) \Rightarrow \quad \mathsf{Finite}(f_1)\ \&\ \mathsf{Is_map}(f_1)\ \&\ \mathbf{range}(f_1) \subseteq s\ \&\ \Sigma_\Theta(f_1) \neq \Sigma_\Theta(f_{1|\mathbf{domain}(f_1) \cap t_1}) \oplus \Sigma_\Theta(f_{1|\mathbf{domain}(f_1) \setminus t_1})$

-- Assuming that there exists a counterexample to our thorem, we can choose an inclusion-minimal map f that contradicts it, along with a set t for which $\Sigma_\Theta(f) \neq \Sigma_\Theta(f_{|\mathbf{domain}(f) \cap t}) \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t})$.

Suppose $\Rightarrow \quad \mathit{Stat0}: \langle \forall t \mid \Sigma_\Theta(f_1) = \Sigma_\Theta(f_{1|\mathbf{domain}(f_1) \cap t}) \oplus \Sigma_\Theta(f_{1|\mathbf{domain}(f_1) \setminus t}) \rangle$

$\langle t_1 \rangle \hookrightarrow \mathit{Stat0} \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \neg \langle \forall t \mid \Sigma_\Theta(f_1) = \Sigma_\Theta(f_{1|\mathbf{domain}(f_1) \cap t}) \oplus \Sigma_\Theta(f_{1|\mathbf{domain}(f_1) \setminus t}) \rangle$

APPLY $\langle m_\Theta : f \rangle$ finite_induction $(n \mapsto f_1, P(x) \mapsto \mathbf{range}(x) \subseteq s\ \&\ \mathsf{Is\_map}(x)\ \&\ \neg \langle \forall t \mid \Sigma_\Theta(x) = \Sigma_\Theta(x_{|\mathbf{domain}(x) \cap t}) \oplus \Sigma_\Theta(x_{|\mathbf{domain}(x) \setminus t}) \rangle) \Rightarrow$
$\mathit{Stat1}: f \subseteq f_1\ \&\ \mathbf{range}(f) \subseteq s\ \&\ \mathsf{Is\_map}(f)\ \&\ \mathit{Stat2}: \neg \langle \forall t \mid \Sigma_\Theta(f) = \Sigma_\Theta(f_{|\mathbf{domain}(f) \cap t}) \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}) \rangle\ \&\ \mathit{Stat3}: \langle \forall g \subseteq f \mid g \neq f \rightarrow \mathbf{range}(g) \not\subseteq s\ \&\ \mathsf{Is\_map}(g)\ \&\ \neg \langle \forall$

$\langle f_1, f \rangle \hookrightarrow \mathit{T162} \Rightarrow \quad \mathsf{Finite}(f)$

$\langle t \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \Sigma_\Theta(f) \neq \Sigma_\Theta(f_{|\mathbf{domain}(f) \cap t}) \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t})$

-- Next we can decompose f as the disjoint union of $f_{|\mathbf{domain}(f) \cap t}$ with $f_{|\mathbf{domain}(f) \setminus t}$.

$\mathit{TELEM} \Rightarrow \quad \mathbf{domain}(f) = \mathbf{domain}(f) \cap t \cup (\mathbf{domain}(f) \setminus t)$

EQUAL $\Rightarrow \quad f_{|\mathbf{domain}(f)} = f_{|\mathbf{domain}(f) \cap t \cup (\mathbf{domain}(f) \setminus t)}$

$\langle f, \mathbf{domain}(f) \cap t, \mathbf{domain}(f) \setminus t \rangle \hookrightarrow \mathit{T58} \Rightarrow \quad f_{|\mathbf{domain}(f)} = f_{|\mathbf{domain}(f) \cap t} \cup f_{|\mathbf{domain}(f) \setminus t}$

$\langle f, t \rangle \hookrightarrow \mathit{T63} \Rightarrow \quad f_{|\mathbf{domain}(f) \cap t} = f_{|t}$

$\langle f \setminus \{c\}, t \rangle \hookrightarrow \mathit{T63} \Rightarrow \quad (f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t} = (f \setminus \{c\})_{|t}$

$\langle f \rangle \hookrightarrow \mathit{T62} \Rightarrow \quad f = f_{|t} \cup f_{|\mathbf{domain}(f) \setminus t}$

-- It follows from the following assumptions and instances of theorems of the present theory ...

Assump $\Rightarrow \quad \mathit{Stat4}: e \in s$

Assump $\Rightarrow \quad \mathit{Stat5}: \langle \forall x \in s \mid x \oplus e = x \rangle$

$\text{Assump} \Rightarrow \quad Stat6: \langle \forall x \in s, y \in s \mid x \oplus y = y \oplus x \rangle$
$Tsigma\_theory1 \Rightarrow \quad \Sigma_\Theta(\emptyset) = e$
$\langle f, \mathbf{domain}(f) \backslash t \rangle \hookrightarrow T47 \Rightarrow \quad f_{|\mathbf{domain}(f) \backslash t} \subseteq f$
$\langle f, f_{|\mathbf{domain}(f) \backslash t} \rangle \hookrightarrow T162 \Rightarrow \quad \text{Finite}(f_{|\mathbf{domain}(f) \backslash t})$
$\langle f, \mathbf{domain}(f) \backslash t \rangle \hookrightarrow T72 \Rightarrow \quad \mathbf{range}(f_{|\mathbf{domain}(f) \backslash t}) \subseteq s$
$\langle f_{|\mathbf{domain}(f) \backslash t} \rangle \hookrightarrow Tsigma\_theory3 \Rightarrow \quad \Sigma_\Theta(f_{|\mathbf{domain}(f) \backslash t}) \in s$

-- ...that t must intersect the domain of f; more specifically, $f_{|t}$ cannot be null.

$\text{Suppose} \Rightarrow \quad f_{|t} = \emptyset$
$\langle Stat1 \rangle \text{ ELEM} \Rightarrow \quad f = f_{|\mathbf{domain}(f) \backslash t}$
$\text{EQUAL } \langle Stat1 \rangle \Rightarrow \quad \Sigma_\Theta(f) = \Sigma_\Theta(f_{|\mathbf{domain}(f) \backslash t})$
$\langle \Sigma_\Theta(f_{|\mathbf{domain}(f) \backslash t}) \rangle \hookrightarrow Stat5 \Rightarrow \quad \Sigma_\Theta(f) = \Sigma_\Theta(f_{|\mathbf{domain}(f) \backslash t}) \oplus e$
$\text{EQUAL } \langle Stat1 \rangle \Rightarrow \quad \Sigma_\Theta(f) \neq e \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \backslash t})$
$\langle e, \Sigma_\Theta(f_{|\mathbf{domain}(f) \backslash t}) \rangle \hookrightarrow Stat6 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat7: f_{|t} \neq \emptyset$

-- Hence we can pick an element c from the restricted map $f_{|\mathbf{domain}(f) \cap t}$, which must satisfy $\mathbf{domain}(f \backslash \{c\}) \backslash t = \mathbf{domain}(f) \backslash t$.

$\langle c \rangle \hookrightarrow Stat7(\langle Stat1 \rangle) \Rightarrow \quad c \in f_{|\mathbf{domain}(f) \cap t}$
$\text{Use\_def}(|) \Rightarrow \quad Stat8: c \in \{x \in f \mid x^{[1]} \in \mathbf{domain}(f) \cap t\}$
$\langle \, \rangle \hookrightarrow Stat8(\langle Stat8 \rangle) \Rightarrow \quad c^{[1]} \in \mathbf{domain}(f) \cap t$
$\text{Suppose} \Rightarrow \quad \mathbf{domain}(f \backslash \{c\}) \backslash t \neq \mathbf{domain}(f) \backslash t$
$\langle f \backslash \{c\}, f \rangle \hookrightarrow T60 \Rightarrow \quad Stat9: \mathbf{domain}(f \backslash \{c\}) \backslash t \not\supseteq \mathbf{domain}(f) \backslash t$
$\langle b \rangle \hookrightarrow Stat9(\langle Stat9 \rangle) \Rightarrow \quad b \in \mathbf{domain}(f) \backslash t \;\&\; b \notin \mathbf{domain}(f \backslash \{c\})$
$\text{Use\_def}(\mathbf{domain}) \Rightarrow \quad b \in \{x^{[1]} : x \in f\} \backslash t \;\&\; b \notin \mathbf{domain}(f \backslash \{c\})$
$\langle Stat1 \rangle \text{ ELEM} \Rightarrow \quad Stat10: b \in \{x^{[1]} : x \in f\}$
$\langle b_1 \rangle \hookrightarrow Stat10(\langle Stat10 \rangle) \Rightarrow \quad b = b_1^{[1]} \;\&\; b_1 \in f$
$\text{Use\_def}(\mathbf{domain}) \Rightarrow \quad Stat11: b \notin \{x^{[1]} : x \in f \backslash \{c\}\}$
$\langle b_1 \rangle \hookrightarrow Stat11(\langle Stat9 \rangle) \Rightarrow \quad b_1 \notin f \backslash \{c\}$
$\langle Stat1 \rangle \text{ ELEM} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat11a: \mathbf{domain}(f \backslash \{c\}) \backslash t = \mathbf{domain}(f) \backslash t$

-- It easily follows that $f_{|\mathbf{domain}(f) \backslash t} = f \backslash \{c\}_{|\mathbf{domain}(f) \backslash t}$; for, ...

$\langle f \backslash \{c\}, \{c\}, \mathbf{domain}(f) \backslash t \rangle \hookrightarrow T59 \Rightarrow \quad (f \backslash \{c\} \cup \{c\})_{|\mathbf{domain}(f) \backslash t} =$
$\quad (f \backslash \{c\})_{|\mathbf{domain}(f) \backslash t} \cup \{c\}_{|\mathbf{domain}(f) \backslash t}$

$\langle \mathit{Stat1} \rangle$ ELEM $\Rightarrow$ $\mathsf{f} = \mathsf{f} \setminus \{\mathsf{c}\} \cup \{\mathsf{c}\}$
EQUAL $\langle \mathit{Stat11a} \rangle \Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} = (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f} \setminus \{\mathsf{c}\}) \setminus \mathsf{t}} \cup \{\mathsf{c}\}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$

-- …assuming that $\{\mathsf{c}\}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} \neq \emptyset$ would contradict the fact, already established, that $\mathsf{c}^{[1]} \in \mathsf{t}$.

Suppose $\Rightarrow$ $\mathit{Stat12}: \{\mathsf{c}\}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} \neq \emptyset$
$\langle \mathsf{a} \rangle \hookrightarrow \mathit{Stat12} \Rightarrow$ $\mathsf{a} \in \{\mathsf{c}\}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$
Use_def $(|) \Rightarrow$ $\mathit{Stat13}: \mathsf{a} \in \{\mathsf{p} \in \{\mathsf{c}\} \mid \mathsf{p}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}\}$
$\langle \, \rangle \hookrightarrow \mathit{Stat13} \Rightarrow$ $\mathsf{a} \in \{\mathsf{c}\} \;\&\; \mathsf{a}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}$
$\langle \mathit{Stat8} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat12a}: \mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} = (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f} \setminus \{\mathsf{c}\}) \setminus \mathsf{t}}$

-- It is also easy to sharpen the preceding observation into the equality

$$\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} = \mathsf{f} \setminus \{\mathsf{c}\}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} .$$

Suppose $\Rightarrow$ $\mathit{Stat14}: \mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} \neq (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$
Use_def $(|) \Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} = \{\mathsf{x} \in \mathsf{f} \mid \mathsf{x}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}\} \;\&\; (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} = \{\mathsf{x} \in \mathsf{f} \setminus \{\mathsf{c}\} \mid \mathsf{x}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}\}$
$\langle \mathit{Stat14} \rangle$ ELEM $\Rightarrow$ $\mathsf{f} = \mathsf{f} \setminus \{\mathsf{c}\} \cup (\mathsf{f} \setminus (\mathsf{f} \setminus \{\mathsf{c}\}))$
$\langle \mathsf{f} \setminus \{\mathsf{c}\}, \mathsf{f} \setminus (\mathsf{f} \setminus \{\mathsf{c}\}), \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t} \rangle \hookrightarrow \mathit{T59} \Rightarrow$
$\mathsf{f} \setminus \{\mathsf{c}\} \cup (\mathsf{f} \setminus (\mathsf{f} \setminus \{\mathsf{c}\}))_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} \supseteq (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$
EQUAL $\langle \mathit{Stat14} \rangle \Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} \supseteq (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$
$\langle \mathit{Stat14} \rangle$ ELEM $\Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} \not\subseteq (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$
$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat14} \Rightarrow$ $\mathit{Stat15}: \mathsf{d} \in \{\mathsf{x} \in \mathsf{f} \mid \mathsf{x}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}\} \;\&\; \mathit{Stat16}: \mathsf{d} \notin \{\mathsf{x} \in \mathsf{f} \setminus \{\mathsf{c}\} \mid \mathsf{x}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}\}$
$\langle \mathsf{d}_1 \rangle \hookrightarrow \mathit{Stat15} \Rightarrow$ $\mathsf{d} \in \mathsf{f} \;\&\; \mathsf{d}^{[1]} \in \mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}$
$\langle \mathsf{d} \rangle \hookrightarrow \mathit{Stat16} \Rightarrow$ $\mathsf{d} \notin \mathsf{f} \setminus \{\mathsf{c}\}$
$\langle \mathit{Stat1} \rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat14a}: \mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}} = (\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}$

-- The assumed minimality of f now leads to the equality

$$\Sigma_\Theta(\mathsf{f} \setminus \{\mathsf{c}\}) = \Sigma_\Theta(\mathsf{f} \setminus \{\mathsf{c}\}_{|\mathbf{domain}(\mathsf{f} \setminus \{\mathsf{c}\}) \cap \mathsf{t}}) \oplus \Sigma_\Theta(\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}) ,$$

Suppose $\Rightarrow$ $\Sigma_\Theta(\mathsf{f} \setminus \{\mathsf{c}\}) \neq \Sigma_\Theta((\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f} \setminus \{\mathsf{c}\}) \cap \mathsf{t}}) \oplus \Sigma_\Theta((\mathsf{f} \setminus \{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f} \setminus \{\mathsf{c}\}) \setminus \mathsf{t}})$
$\langle \mathit{Stat11a} \rangle$ ELEM $\Rightarrow$ $\mathsf{f} \setminus \{\mathsf{c}\} \subseteq \mathsf{f} \;\&\; \mathsf{f} \setminus \{\mathsf{c}\} \neq \mathsf{f}$
$\langle \mathsf{f} \setminus \{\mathsf{c}\}, \mathsf{f} \rangle \hookrightarrow \mathit{T47}(\langle \mathit{Stat1} \rangle) \Rightarrow$ $\mathsf{Is\_map}(\mathsf{f} \setminus \{\mathsf{c}\})$
$\langle \mathsf{f} \setminus \{\mathsf{c}\}, \mathsf{f} \rangle \hookrightarrow \mathit{T60}(\langle \mathit{Stat1} \rangle) \Rightarrow$ $\mathbf{range}(\mathsf{f} \setminus \{\mathsf{c}\}) \subseteq \mathsf{s}$

$\langle f, f \setminus \{c\} \rangle \hookrightarrow \mathit{T162}(\langle \mathit{Stat1} \rangle) \Rightarrow \quad \mathsf{Finite}(f \setminus \{c\})$
$\langle f \setminus \{c\} \rangle \hookrightarrow \mathit{Stat3}(\langle \mathit{Stat14a} \rangle) \Rightarrow \quad \mathit{Stat17}:$
$\quad \langle \forall t \mid \Sigma_\Theta(f \setminus \{c\}) = \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \oplus \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \setminus t}) \rangle$
$\langle t \rangle \hookrightarrow \mathit{Stat17}(\langle \mathit{Stat14a} \rangle) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat17a}: \ \Sigma_\Theta(f \setminus \{c\}) = \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \oplus \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \setminus t})$

-- whence, thanks to the assumptions and earlier theorems of the present theory, ...

$\langle c, f \rangle \hookrightarrow \mathit{Tsigma\_theory4} \Rightarrow \quad \Sigma_\Theta(f) = \Sigma_\Theta(f \setminus \{c\}) \oplus c^{[2]}$
$\mathsf{EQUAL} \ \langle \mathit{Stat11a} \rangle \Rightarrow \quad \Sigma_\Theta(f) = \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}) \oplus c^{[2]}$
$\mathsf{Assump} \Rightarrow \quad \mathit{Stat18}: \ \langle \forall x \in s, y \in s \mid x \oplus y \in s \rangle$
$\mathsf{Assump} \Rightarrow \quad \mathit{Stat19}: \ \langle \forall x \in s, y \in s, z \in s \mid (x \oplus y) \oplus z = x \oplus (y \oplus z) \rangle$

$\langle f, \mathbf{domain}(f) \setminus t \rangle \hookrightarrow \mathit{T43}(\langle \mathit{Stat17a} \rangle) \Rightarrow \quad f_{|\mathbf{domain}(f) \setminus t} \subseteq f$
$\langle f, f_{|\mathbf{domain}(f) \setminus t} \rangle \hookrightarrow \mathit{T162} \Rightarrow \quad \mathsf{Finite}(f_{|\mathbf{domain}(f) \setminus t})$
$\langle f_{|\mathbf{domain}(f) \setminus t}, f \rangle \hookrightarrow \mathit{T60} \Rightarrow \quad \mathbf{range}(f_{|\mathbf{domain}(f) \setminus t}) \subseteq s$

$\langle f \setminus \{c\}, \mathbf{domain}(f \setminus \{c\}) \cap t \rangle \hookrightarrow \mathit{T43} \Rightarrow \quad (f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t} \subseteq f$
$\langle f, (f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t} \rangle \hookrightarrow \mathit{T162} \Rightarrow \quad \mathsf{Finite}((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t})$
$\langle (f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}, f \rangle \hookrightarrow \mathit{T60} \Rightarrow \quad \mathbf{range}((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \subseteq s$
$\langle (f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t} \rangle \hookrightarrow \mathit{Tsigma\_theory3} \Rightarrow \quad \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \in s$

-- ... we find that

$$\Sigma_\Theta(f) = \Sigma_\Theta(f \setminus \{c\}_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \oplus c^{[2]} \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t})$$

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat33}: \ c^{[2]} \notin s$
$\langle \mathit{Stat1}, \mathit{Stat33}, * \rangle \ \mathsf{ELEM} \Rightarrow \quad c^{[2]} \notin \mathbf{range}(f)$
$\mathsf{Use\_def}(\mathbf{range}) \Rightarrow \quad \mathit{Stat20}: \ c^{[2]} \notin \{x^{[2]} : x \in f\}$
$\langle c \rangle \hookrightarrow \mathit{Stat20}(\langle \mathit{Stat20} \rangle) \Rightarrow \quad c \notin f$
$\langle f, \mathsf{dom}(f) \cap t \rangle \hookrightarrow \mathit{T43} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad c^{[2]} \in s$
$\langle \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}), \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}), c^{[2]} \rangle \hookrightarrow \mathit{Stat19} \Rightarrow \quad \Sigma_\Theta(f) =$
$\quad \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\})} \cap t) \oplus (\Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}) \oplus c^{[2]})$
$\langle \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}), c^{[2]}, \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}) \rangle \hookrightarrow \mathit{Stat19} \Rightarrow \quad \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \oplus (c^{[2]} \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t})) =$
$\quad \Sigma_\Theta((f \setminus \{c\})_{|\mathbf{domain}(f \setminus \{c\}) \cap t}) \oplus c^{[2]} \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t})$
$\langle \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}), c^{[2]} \rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t}) \oplus c^{[2]} = c^{[2]} \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus t})$

EQUAL $\langle \mathit{Stat1} \rangle \Rightarrow$ $\Sigma_\Theta(\mathsf{f}) = \Sigma_\Theta((\mathsf{f}\backslash\{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}\backslash\{\mathsf{c}\})\,\cap\,\mathsf{t}}) \oplus \mathsf{c}^{[2]} \oplus \Sigma_\Theta(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\backslash\mathsf{t}})$

-- Addition of the first two terms on the right-hand side of this equality is easily seen to yield $\Sigma_\Theta(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}})$; thus, by using Theorem $\mathsf{sigma\_theory}_4$ we get a contradiction which proves the present theorem.

$\langle \{\mathsf{c}\}, \mathsf{f}\backslash\{\mathsf{c}\}, \mathsf{t} \rangle \hookrightarrow \mathit{T59}([]) \Rightarrow$ $(\mathsf{f}\backslash\{\mathsf{c}\} \cup \{\mathsf{c}\})_{|\mathsf{t}} = (\mathsf{f}\backslash\{\mathsf{c}\})_{|\mathsf{t}} \cup \{\mathsf{c}\}_{|\mathsf{t}}$
EQUAL $\langle \mathit{Stat1} \rangle \Rightarrow$ $\mathsf{f}_{|\mathsf{t}} = (\mathsf{f}\backslash\{\mathsf{c}\})_{|\mathsf{t}} \cup \{\mathsf{c}\}_{|\mathsf{t}}$
Suppose $\Rightarrow$ $\{\mathsf{c}\}_{|\mathsf{t}} \neq \{\mathsf{c}\}$
$\langle \{\mathsf{c}\}, \mathsf{t} \rangle \hookrightarrow \mathit{T43} \Rightarrow$ $\mathit{Stat21}: \{\mathsf{c}\}_{|\mathsf{t}} \not\supseteq \{\mathsf{c}\}$
$\langle \mathsf{dd} \rangle \hookrightarrow \mathit{Stat21} \Rightarrow$ $\mathsf{c} \notin \{\mathsf{c}\}_{|\mathsf{t}}$
Use_def(|) $\Rightarrow$ $\mathit{Stat22}: \mathsf{c} \notin \{\mathsf{x} \in \{\mathsf{c}\} \mid \mathsf{x}^{[1]} \in \mathsf{t}\}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat22} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} = (\mathsf{f}\backslash\{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}\backslash\{\mathsf{c}\})\,\cap\,\mathsf{t}} \cup \{\mathsf{c}\}$
$\langle \mathsf{f}\backslash\{\mathsf{c}\}, \mathbf{domain}(\mathsf{f}\backslash\{\mathsf{c}\}) \cap \mathsf{t} \rangle \hookrightarrow \mathit{T43} \Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} \backslash \{\mathsf{c}\} = (\mathsf{f}\backslash\{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}\backslash\{\mathsf{c}\})\,\cap\,\mathsf{t}}$
EQUAL $\langle \mathit{Stat19} \rangle \Rightarrow$ $\Sigma_\Theta(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} \backslash \{\mathsf{c}\}) = \Sigma_\Theta((\mathsf{f}\backslash\{\mathsf{c}\})_{|\mathbf{domain}(\mathsf{f}\backslash\{\mathsf{c}\})\,\cap\,\mathsf{t}})$

$\langle \mathsf{f}, \mathbf{domain}(\mathsf{f}) \cap \mathsf{t} \rangle \hookrightarrow \mathit{T43} \Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} \subseteq \mathsf{f}$
$\langle \mathsf{f}, \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} \rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\mathsf{Finite}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}})$
$\langle \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}}, \mathsf{f} \rangle \hookrightarrow \mathit{T60} \Rightarrow$ $\mathbf{range}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}}) \subseteq \mathsf{s}$
$\langle \mathsf{c}, \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} \rangle \hookrightarrow \mathit{Tsigma\_theory4} \Rightarrow$ $\Sigma_\Theta(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}}) = \Sigma_\Theta(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\,\cap\,\mathsf{t}} \backslash \{\mathsf{c}\}) \oplus \mathsf{c}^{[2]}$
EQUAL $\langle \mathit{Stat1} \rangle \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem shows that if a map f with values in s is decomposed in any way into a collection of disjoint parts fj, then sigma(f) is the sum of all the values sigma(fj), where fj runs over all the parts into which f has been decomposed. To state this result formally, we make use of an auxiliary single-valued mapping g having the same domain as f, and decompose the domain of f into the collection of sets $\mathsf{g}^{\,\urcorner}\{\mathsf{y}\}$, where y varies over the range of g.

-- Rearrangement - of - sums Theorem

Theorem 458 ($\mathsf{sigma\_theory}_6$) $\mathsf{Finite}(\mathsf{F}) \,\&\, \mathsf{Svm}(\mathsf{F}) \,\&\, \mathsf{Svm}(\mathsf{G}) \,\&\, \mathbf{domain}(\mathsf{F}) = \mathbf{domain}(\mathsf{G}) \,\&\, \mathbf{range}(\mathsf{F}) \subseteq \mathsf{s} \rightarrow \Sigma_\Theta(\mathsf{F}) = \Sigma_\Theta(\{[\mathsf{y}, \Sigma_\Theta(\mathsf{F}_{|\mathsf{G}^{\urcorner}\{\mathsf{y}\}})] : \mathsf{y} \in \mathbf{range}(\mathsf{G})\})$. PROOF:

-- Assuming by contradiction the statement to be false, we could take a counterexample f, g with f incusion-minimal.

Suppose_not$(\mathsf{f}_1, \mathsf{g}_1) \Rightarrow$ $\mathit{Stat0a}: \mathsf{Finite}(\mathsf{f}_1) \,\&\, \mathsf{Svm}(\mathsf{f}_1) \,\&\, \mathsf{Svm}(\mathsf{g}_1) \,\&\, \mathbf{domain}(\mathsf{f}_1) = \mathbf{domain}(\mathsf{g}_1) \,\&\, \mathbf{range}(\mathsf{f}_1) \subseteq \mathsf{s} \,\&\, \Sigma_\Theta(\mathsf{f}_1) \neq \Sigma_\Theta(\{[\mathsf{y}, \Sigma_\Theta(\mathsf{f}_{1|\mathsf{g}_1^{\urcorner}\{\mathsf{y}\}})] : \mathsf{y} \in \mathbf{range}(\mathsf{g}_1)\})$
Suppose $\Rightarrow$ $\mathit{Stat0}: \neg\langle \exists \mathsf{g} \mid \mathsf{Svm}(\mathsf{g}) \,\&\, \mathbf{domain}(\mathsf{f}_1) = \mathbf{domain}(\mathsf{g}) \,\&\, \Sigma_\Theta(\mathsf{f}_1) \neq \Sigma_\Theta(\{[\mathsf{y}, \Sigma_\Theta(\mathsf{f}_{1|\mathsf{g}^{\urcorner}\{\mathsf{y}\}})] : \mathsf{y} \in \mathbf{range}(\mathsf{g})\})\rangle$

$\langle g_1 \rangle \hookrightarrow Stat0 \Rightarrow$ false; Discharge $\Rightarrow$ $\langle \exists g \mid Svm(g) \,\&\, \mathbf{domain}(f_1) = \mathbf{domain}(g) \,\&\, \Sigma_\Theta(f_1) \neq \Sigma_\Theta(\{[y, \Sigma_\Theta(f_{1|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\})\rangle$

APPLY $\langle m_\Theta : f \rangle$ finite_induction$\big(n \mapsto f_1, P(x) \mapsto \big(Svm(x) \,\&\, \mathbf{range}(x) \subseteq s \,\&\, \langle \exists g \mid Svm(g) \,\&\, \mathbf{domain}(x) = \mathbf{domain}(g) \,\&\, \Sigma_\Theta(x) \neq \Sigma_\Theta(\{[y, \Sigma_\Theta(x_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\})\rangle\big)\big) \Rightarrow$

$Stat1 : f \subseteq f_1 \,\&\, Svm(f) \,\&\, \mathbf{range}(f) \subseteq s \,\&\, Stat2 : \langle \exists g \mid Svm(g) \,\&\, \mathbf{domain}(f) = \mathbf{domain}(g) \,\&\, \Sigma_\Theta(f) \neq \Sigma_\Theta(\{[y, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\})\rangle \,\&\, Stat4 : \langle \forall h \subseteq f \mid h \neq f$

$\langle f_1, f \rangle \hookrightarrow T162 \Rightarrow$ Finite(f)

$\langle g \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat3 : Svm(g) \,\&\, \mathbf{domain}(f) = \mathbf{domain}(g) \,\&\, \Sigma_\Theta(f) \neq \Sigma_\Theta(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g)\})$

-- Note that **range**(g) cannot be null, else $\mathbf{domain}(f) = \mathbf{domain}(g)$ would be null too, which would quickly lead us to a contradiction.

Suppose $\Rightarrow$ $\mathbf{range}(g) = \emptyset$

$\langle g \rangle \hookrightarrow T78 \Rightarrow$ $Stat5 : \mathbf{domain}(f) = \emptyset$

APPLY $\langle\,\rangle$ setformer$_0$(e(x) $\mapsto x^{[1]}$, s $\mapsto$ f, P(x) $\mapsto$ false) $\Rightarrow$ $f \neq \emptyset \rightarrow \{x^{[1]} : x \in f\} \neq \emptyset$

Use_def(**domain**) $\Rightarrow$ $Stat6 : f \neq \emptyset \rightarrow \mathbf{domain}(f) \neq \emptyset$

$\langle Stat5, Stat6 \rangle$ ELEM $\Rightarrow$ $f = \emptyset$

Suppose $\Rightarrow$ $Stat7 : \{[y, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\} \neq \emptyset$

$\langle d \rangle \hookrightarrow Stat7 \Rightarrow$ false; Discharge $\Rightarrow$ $\{[y, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\} = \emptyset$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat8 : \mathbf{range}(g) \neq \emptyset$

-- Hence we can pick a y from **range**(g), and decompose the domain of f as disjoint union

$$\mathbf{domain}(f) = g^{\urcorner}\{y\} \cup g^{\urcorner}\mathbf{range}(g) \setminus \{y\},$$

where $g^{\urcorner}\{8y\} \neq \emptyset$. By Theorem sigma_theory5, we will have

$$\Sigma_\Theta(f) = \Sigma_\Theta(f_{|g^{\urcorner}\{y\}}) \oplus \Sigma_\Theta(g^{\urcorner}\mathbf{range}(g) \setminus \{y\}).$$

$\langle y \rangle \hookrightarrow Stat8 \Rightarrow$ $y \in \mathbf{range}(g)$

$\langle y, g \rangle \hookrightarrow T152 \Rightarrow$ $Stat9 : g^{\urcorner}\{y\} \neq \emptyset$

$\langle g, \{y\} \rangle \hookrightarrow T150 \Rightarrow$ $\mathbf{domain}(f) = g^{\urcorner}\{y\} \cup g^{\urcorner}\mathbf{range}(g) \setminus \{y\}$

$\langle \{y\}, \mathbf{range}(g) \setminus \{y\}, g \rangle \hookrightarrow T151 \Rightarrow$ $g^{\urcorner}\{y\} \cap g^{\urcorner}\mathbf{range}(g) \setminus \{y\} = \emptyset$

Use_def(Svm) $\Rightarrow$ $Stat10 : Is\_map(f) \,\&\, \mathbf{range}(f) \subseteq s \,\&\, Finite(f)$

$\langle f, s, g^{\urcorner}\{y\} \rangle \hookrightarrow Tsigma\_theory5(\langle Stat10 \rangle) \Rightarrow$ $\Sigma_\Theta(f) = \Sigma_\Theta(f_{|\mathbf{domain}(f) \cap g^{\urcorner}\{y\}}) \oplus \Sigma_\Theta(f_{|\mathbf{domain}(f) \setminus g^{\urcorner}\{y\}})$

$\langle Stat9 \rangle$ ELEM $\Rightarrow$ $\mathbf{domain}(f) \cap g^{\urcorner}\{y\} = g^{\urcorner}\{y\} \,\&\, \mathbf{domain}(f) \setminus g^{\urcorner}\{y\} = g^{\urcorner}\mathbf{range}(g) \setminus \{y\}$

EQUAL $\Rightarrow$ $Stat11 : \Sigma_\Theta(f) = \Sigma_\Theta(f_{|g^{\urcorner}\{y\}}) \oplus \Sigma_\Theta(f_{|g^{\urcorner}\mathbf{range}(g) \setminus \{y\}})$

-- Before we can exploit the assumed minimality of f, we must show that the restriction of f to $g^{\urcorner}\mathbf{range}(g) \setminus \{y\}$ is strictly included in f.

Suppose $\Rightarrow$ $\neg(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}} \subseteq \mathsf{f} \;\&\; \mathsf{f} \neq \mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}})$

$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat9} \Rightarrow$ $\mathsf{x} \in \mathsf{g} \Lsh \{\mathsf{y}\}$

Suppose $\Rightarrow$ $[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] \notin \mathsf{f}$

$\langle \mathsf{f} \rangle \hookrightarrow \mathit{T65} \Rightarrow$ $\mathit{Stat12}:\ [\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] \notin \{[\mathsf{xx}, \mathsf{f}\upharpoonright\mathsf{xx}] : \mathsf{xx} \in \mathbf{domain}(\mathsf{f})\}$

$\langle \mathsf{g}, \{\mathsf{x}\} \rangle \hookrightarrow \mathit{T150} \Rightarrow$ $\mathsf{x} \in \mathbf{domain}(\mathsf{f})$

$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat12} \Rightarrow$ false; Discharge $\Rightarrow$ $[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] \in \mathsf{f}$

$\langle \mathsf{f}, \mathbf{domain}(\mathsf{f})\setminus \mathsf{g} \Lsh \{\mathsf{y}\} \rangle \hookrightarrow \mathit{T43} \Rightarrow$ $\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}} \subseteq \mathsf{f}$

$\langle \mathsf{f}, \mathbf{domain}(\mathsf{f})\setminus \mathsf{g} \Lsh \{\mathsf{y}\} \rangle \hookrightarrow \mathit{T52} \Rightarrow$ $\mathsf{Svm}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}})$

EQUAL $\Rightarrow$ $\mathsf{Svm}(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}})$

$\langle \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}} \rangle \hookrightarrow \mathit{T65} \Rightarrow$ $\mathit{Stat13}:\ \left\{\left[\mathsf{xx}, \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}}\upharpoonright\mathsf{xx}\right] : \mathsf{xx} \in \mathbf{domain}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}})\right\} = \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}}$

$\langle \mathsf{f}, \mathbf{domain}(\mathsf{f})\setminus \mathsf{g} \Lsh \{\mathsf{y}\} \rangle \hookrightarrow \mathit{T84} \Rightarrow$ $\mathsf{x} \notin \mathbf{domain}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}})$

Suppose $\Rightarrow$ $[\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] \in \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}}$

EQUAL $\Rightarrow$ $\mathit{Stat93}:\ [\mathsf{x}, \mathsf{f}\upharpoonright\mathsf{x}] \in \left\{\left[\mathsf{xx}, \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}}\upharpoonright\mathsf{xx}\right] : \mathsf{xx} \in \mathbf{domain}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}})\right\}$

-- ?? EQUAL $\Rightarrow$ Stat93: [x, f [x]] in {[x, (f ON (domain (f)-(g INV_IM {y}))) [x]]: x in domain (f ON (domain (f)-(g INV_IM {y})))}

$\langle \mathsf{x}' \rangle \hookrightarrow \mathit{Stat93} \Rightarrow$ $\mathit{Stat13a}:\ \mathsf{x} = \mathsf{x}' \;\&\; \mathsf{x}' \in \mathbf{domain}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}})$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{f} \neq \mathsf{f}_{|\mathbf{domain}(\mathsf{f})\setminus\mathsf{g}\Lsh\{\mathsf{y}\}}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat14}:\ \mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}} \subseteq \mathsf{f} \;\&\; \mathsf{f} \neq \mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}$

-- Simplification of the expression for $\Sigma_\Theta(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}})$ which results from this remark leads us to the equality

$$\Sigma_\Theta(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}) = \Sigma_\Theta\left(\left\{\left[\mathsf{v}, \Sigma_\Theta(\mathsf{f}_{|\mathsf{g}\Lsh\{\mathsf{v}\}})\right] : \mathsf{v} \in \mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}\right\}\right).$$

Suppose $\Rightarrow$ $\mathit{Stat15}:\ \Sigma_\Theta(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}) \neq \Sigma_\Theta\left(\left\{\left[\mathsf{v}, \Sigma_\Theta(\mathsf{f}_{|\mathsf{g}\Lsh\{\mathsf{v}\}})\right] : \mathsf{v} \in \mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}\right\}\right)$

Suppose $\Rightarrow$ $\mathbf{range}(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}) \not\subseteq \mathsf{s}$

$\langle \mathsf{f}, \mathsf{g} \Lsh \mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\} \rangle \hookrightarrow \mathit{T43}([]) \Rightarrow$ $\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}} \subseteq \mathsf{f}$

$\langle \mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}, \mathsf{f} \rangle \hookrightarrow \mathit{T60}(\langle \mathit{Stat1} \rangle) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{range}(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}) \subseteq \mathsf{s}$

$\langle \mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}} \rangle \hookrightarrow \mathit{Stat4}(\langle \mathit{Stat14} \rangle) \Rightarrow$
$\neg\mathsf{Svm}(\mathsf{f}_{|\mathsf{g}\Lsh\mathbf{range}(\mathsf{g})\setminus\{\mathsf{y}\}}) \vee$

$$\neg\left\langle \exists k \mid \mathsf{Svm}(k) \,\&\, \mathbf{domain}(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) = \mathbf{domain}(k) \,\&\, \Sigma_\Theta(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) \neq \Sigma_\Theta\left(\left\{\left[v, \Sigma_\Theta((f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|k\Lsh\{v\}})\right] : v\in\mathbf{range}(k)\right\}\right)\right\rangle$$

$\langle f, g \Lsh \mathbf{range}(g)\setminus\{y\}\rangle \hookrightarrow \mathit{T43} \Rightarrow$ $f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}} \subseteq f$

$\langle f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}\rangle \hookrightarrow \mathit{T48} \Rightarrow$ $\mathit{Stat16}:\ \neg$

$$\left\langle \exists k \mid \mathsf{Svm}(k) \,\&\, \mathbf{domain}(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) = \mathbf{domain}(k) \,\&\, \Sigma_\Theta(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) \neq \Sigma_\Theta\left(\left\{\left[v, \Sigma_\Theta((f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|k\Lsh\{v\}})\right] : v\in\mathbf{range}(k)\right\}\right)\right\rangle$$

$\langle g, g \Lsh \mathbf{range}(g)\setminus\{y\}\rangle \hookrightarrow \mathit{T43} \Rightarrow$ $g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}} \subseteq g$

$\langle g, g \Lsh \mathbf{range}(g)\setminus\{y\}\rangle \hookrightarrow \mathit{T52} \Rightarrow$ $\mathsf{Svm}(g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})$

$\langle f, g \Lsh \mathbf{range}(g)\setminus\{y\}\rangle \hookrightarrow \mathit{T84} \Rightarrow$ $\mathbf{domain}(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) = \mathbf{domain}(f) \cap g \Lsh \mathbf{range}(g)\setminus\{y\}$

$\langle g, g \Lsh \mathbf{range}(g)\setminus\{y\}\rangle \hookrightarrow \mathit{T84} \Rightarrow$ $\mathbf{domain}(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) = \mathbf{domain}(g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})$

$\langle g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}\rangle \hookrightarrow \mathit{Stat16} \Rightarrow$ $\Sigma_\Theta(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) =$

$$\Sigma_\Theta\left(\left\{\left[v, \Sigma_\Theta((f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}\Lsh\{v\}})\right] : v\in\mathbf{range}(g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})\right\}\right)$$

$\langle g, \mathbf{range}(g)\setminus\{y\}\rangle \hookrightarrow \mathit{T157} \Rightarrow$ $\mathbf{range}(g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) = \mathbf{range}(g)\setminus\{y\}$

EQUAL $\Rightarrow$ $\mathit{Stat17}:\ \Sigma_\Theta(f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}) = \Sigma_\Theta\left(\left\{\left[v, \Sigma_\Theta((f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}\Lsh\{v\}})\right] : v\in\mathbf{range}(g)\setminus\{y\}\right\}\right)$

Suppose $\Rightarrow$ $\mathit{Stat18}:\ \neg\langle \forall v\in\mathbf{range}(g)\setminus\{y\} \mid (f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|g_{|g\Lsh\mathbf{range}(g)\setminus\{y\}}\Lsh\{v\}} = f_{|g\Lsh\{v\}}\rangle$

Loc_def $\Rightarrow$ $ry = \mathbf{range}(g)\setminus\{y\}$
Loc_def $\Rightarrow$ $gry = g \Lsh ry$
Loc_def $\Rightarrow$ $gory = g_{|gry}$
EQUAL $\Rightarrow$ $\mathit{Stat19}:\ \neg\langle \forall v\in ry \mid (f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|gory\Lsh\{v\}} = f_{|g\Lsh\{v\}}\rangle$

$\langle v'\rangle \hookrightarrow \mathit{Stat19} \Rightarrow$ $\{v'\} \subseteq ry \,\&\, (f_{|g\Lsh\mathbf{range}(g)\setminus\{y\}})_{|gory\Lsh\{v'\}} \neq f_{|g\Lsh\{v'\}}$

Loc_def $\Rightarrow$ $w' = \{v'\}$
EQUAL $\Rightarrow$ $(f_{|g\Lsh ry})_{|gory\Lsh w'} \neq f_{|g\Lsh w'}$
$\langle w', ry, g\rangle \hookrightarrow \mathit{T159}(\langle \mathit{Stat19}\rangle) \Rightarrow$ $g\Lsh w' = g \Lsh ry \cap g \Lsh w'$
Suppose $\Rightarrow$ $(f_{|g\Lsh ry})_{|g\Lsh w'} \neq f_{|g\Lsh w'}$
$\langle f, g\Lsh ry, g\Lsh w'\rangle \hookrightarrow \mathit{T160} \Rightarrow$ $(f_{|g\Lsh ry})_{|g\Lsh w'} = f_{|g\Lsh ry\,\cap\, g\Lsh w'}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $(f_{|g\Lsh ry})_{|g\Lsh w'} = f_{|g\Lsh w'}$
Suppose $\Rightarrow$ $gory \Lsh w' = g \Lsh w'$
EQUAL $\Rightarrow$ $(f_{|g\Lsh ry})_{|gory\Lsh w'} = (f_{|g\Lsh ry})_{|g\Lsh w'}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $gory \Lsh w' \neq g \Lsh w'$

$\langle g, ry, w'\rangle \hookrightarrow T158 \Rightarrow \quad g_{|g \upharpoonleft ry} \upharpoonleft w' = g \upharpoonleft w'$
EQUAL $\Rightarrow \quad gory \upharpoonleft w' = g \upharpoonleft w'$
ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ *Stat20* : $\left\langle \forall v \in \mathbf{range}(g)\setminus\{y\} \mid (f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}})_{|g_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}} \upharpoonleft \{v\}} = f_{|g \upharpoonleft \{v\}}\right\rangle$

Suppose $\Rightarrow \quad$ *Stat21* : $\left\{\left[v, \Sigma_{\Theta}((f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}})_{|g_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}} \upharpoonleft \{v\}})\right] : v \in \mathbf{range}(g)\setminus\{y\}\right\} \neq$
$\quad \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\}$
$\langle vq\rangle \hookrightarrow Stat21 \Rightarrow \quad vq \in \mathbf{range}(g)\setminus\{y\} \ \& \ \Sigma_{\Theta}((f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}})_{|g_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}} \upharpoonleft \{vq\}}) \neq \Sigma_{\Theta}(f_{|g \upharpoonleft \{vq\}})$

$\langle vq\rangle \hookrightarrow Stat20 \Rightarrow \quad (f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}})_{|g_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}} \upharpoonleft \{vq\}} = f_{|g \upharpoonleft \{vq\}}$

EQUAL $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ *Stat22* : $\left\{\left[v, \Sigma_{\Theta}((f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}})_{|g_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}} \upharpoonleft \{v\}})\right] : v \in \mathbf{range}(g)\setminus\{y\}\right\} = \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\}$
EQUAL $\langle Stat22, Stat17\rangle \Rightarrow \quad$ *Stat23* : $\Sigma_{\Theta}(f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}}) = \Sigma_{\Theta}(\{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\})$
$\langle Stat23, Stat15\rangle$ ELEM $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ *Stat24* : $\Sigma_{\Theta}(f_{|g \upharpoonleft \mathbf{range}(g)\setminus\{y\}}) = \Sigma_{\Theta}(\{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\})$

-- Next we ascertain all conditions needed for instantiation of the variables of Theorem sigma_theory4 to $[y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})]$ and to $\{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\}$ respectively.

Suppose $\Rightarrow \quad \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\} \cup \{[y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})]\} \neq \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\}$
Set_monot $\Rightarrow \quad \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\} \subseteq \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\}$
Suppose $\Rightarrow \quad$ *Stat25* : $[y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})] \notin \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\}$
$\langle y\rangle \hookrightarrow Stat25 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ *Stat26* : $\{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\} \cup \{[y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})]\} \not\supseteq \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\}$
$\langle a\rangle \hookrightarrow Stat26 \Rightarrow \quad$ *Stat27* :
$\quad a \in \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\} \ \& \ a \notin \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\} \ \& \ a \neq [y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})]$
$\langle u, u\rangle \hookrightarrow Stat27 \Rightarrow \quad$ *Stat28* : $a = [u, \Sigma_{\Theta}(f_{|g \upharpoonleft \{u\}})] \ \& \ u \in \mathbf{range}(g) \ \& \ u \notin \mathbf{range}(g)\setminus\{y\} \ \& \ a \neq [y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})]$
$\langle Stat28\rangle$ ELEM $\Rightarrow \quad u = y \ \& \ \Sigma_{\Theta}(f_{|g \upharpoonleft \{u\}}) \neq \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})$
EQUAL $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ *Stat29* : $\{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\setminus\{y\}\} \cup \{[y, \Sigma_{\Theta}(f_{|g \upharpoonleft \{y\}})]\} = \{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\}$

-- One of the conditions needed for the application of Theorem sigma_theory4 is proved as follows, exploiting the known fact that a single-valued map is finite if and only if its domain is finite.

Suppose $\Rightarrow \quad \neg\mathsf{Finite}(\{[v, \Sigma_{\Theta}(f_{|g \upharpoonleft \{v\}})] : v \in \mathbf{range}(g)\})$
$\langle g\rangle \hookrightarrow T148 \Rightarrow \quad \#g = \#\mathbf{domain}(g)$
$\langle f\rangle \hookrightarrow T148 \Rightarrow \quad \#f = \#\mathbf{domain}(f)$
$\langle f\rangle \hookrightarrow T166 \Rightarrow \quad \mathsf{Finite}(\#f)$

EQUAL $\Rightarrow$ $\quad \mathsf{Finite}(\#\mathbf{domain}(g))$

ELEM $\Rightarrow$ $\quad \mathsf{Svm}\big(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big) \,\&\, \mathbf{domain}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}) = \mathbf{range}(g)$

$\langle \mathbf{domain}(g)\rangle \hookrightarrow T166 \Rightarrow$ $\quad \mathsf{Finite}(\mathbf{domain}(g))$

$\langle g\rangle \hookrightarrow T165 \Rightarrow$ $\quad \mathsf{Finite}(\mathbf{range}(g))$

EQUAL $\Rightarrow$ $\quad \mathsf{Finite}\big(\mathbf{domain}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\})\big)$

$\langle \mathbf{domain}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\})\rangle \hookrightarrow T166 \Rightarrow$

$\quad \mathsf{Finite}\big(\#\mathbf{domain}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\})\big)$

$\big\langle \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big\rangle \hookrightarrow T148 \Rightarrow$ $\quad \#\mathbf{domain}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}) =$

$\quad \#\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}$

EQUAL $\Rightarrow$ $\quad \mathsf{Finite}\big(\#\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big)$

$\big\langle \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big\rangle \hookrightarrow T166 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad Stat30: \ \mathsf{Finite}\big(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big)$

Suppose $\Rightarrow$ $\quad Stat30a: \ \mathbf{range}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}) \not\subseteq s$

$\langle Stat30a\rangle$ ELEM $\Rightarrow$ $\quad Stat31: \ \{\Sigma_\Theta(f_{|g^\urcorner\{v\}}) : v \in \mathbf{range}(g)\} \not\subseteq s$

$\langle b\rangle \hookrightarrow Stat31([]) \Rightarrow$ $\quad Stat32: \ b \in \{\Sigma_\Theta(f_{|g^\urcorner\{v\}}) : v \in \mathbf{range}(g)\} \,\&\, b \notin s$

$\langle w\rangle \hookrightarrow Stat32([]) \Rightarrow$ $\quad b = \Sigma_\Theta(f_{|g^\urcorner\{w\}}) \,\&\, w \in \mathbf{range}(g)$

$\langle f, g^\urcorner\{w\}\rangle \hookrightarrow T43 \Rightarrow$ $\quad f_{|g^\urcorner\{w\}} \subseteq f$

$\langle f_{|g^\urcorner\{w\}}, f\rangle \hookrightarrow T48 \Rightarrow$ $\quad \mathsf{Svm}(f_{|g^\urcorner\{w\}})$

$\langle f, f_{|g^\urcorner\{w\}}\rangle \hookrightarrow T162 \Rightarrow$ $\quad \mathsf{Finite}(f_{|g^\urcorner\{w\}})$

$\langle f_{|g^\urcorner\{w\}}, f\rangle \hookrightarrow T60 \Rightarrow$ $\quad \mathbf{range}(f_{|g^\urcorner\{w\}}) \subseteq \mathbf{range}(f)$

$\langle f_{|g^\urcorner\{w\}}\rangle \hookrightarrow Tsigma\_theory3([Stat0a, \cap]) \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad Stat33: \ \mathbf{range}(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}) \subseteq s$

$\big\langle [y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})], \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big\rangle \hookrightarrow Tsigma\_theory4(\langle Stat29, Stat30, Stat33\rangle) \Rightarrow$ $\quad \Sigma_\Theta\big(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big) =$

$\quad \Sigma_\Theta\big(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\} \setminus \{[y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})]\}\big) \oplus \Sigma_\Theta(f_{|g^\urcorner\{y\}})$

Suppose $\Rightarrow$ $\quad Stat34: \ [y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})] \in \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}$

$\langle v\rangle \hookrightarrow Stat34 \Rightarrow$ $\quad Stat35: \ v \neq y \,\&\, [y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})] = [v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})]$

$\langle Stat35\rangle$ ELEM $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad Stat36: \ [y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})] \notin \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}$

$\langle Stat29, Stat36\rangle$ ELEM $\Rightarrow$

$\quad [y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})] \in \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\} \,\&$

$\qquad \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\} \setminus \{[y, \Sigma_\Theta(f_{|g^\urcorner\{y\}})]\} = \{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}$

EQUAL $\Rightarrow$ $\quad Stat37: \ \Sigma_\Theta\big(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g)\}\big) = \Sigma_\Theta\big(\{[v, \Sigma_\Theta(f_{|g^\urcorner\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}\big) \oplus \Sigma_\Theta(f_{|g^\urcorner\{y\}})$

-- Before we can exploit commutativity, we must check that the two addends on the left-hand side of the equality just obtained belong to s. Thanks to the Theorem sigma_theory_3, it suffices to show that both of them are finite and have range contained in s.

$\langle f, g^\urcorner\{y\}\rangle \hookrightarrow T43 \Rightarrow$ $\quad f_{|g^\urcorner\{y\}} \subseteq f$

$\langle f, f_{|g^{\urcorner}\{y\}} \rangle \hookrightarrow T162 \Rightarrow \quad \mathsf{Finite}(f_{|g^{\urcorner}\{y\}})$
$\langle f_{|g^{\urcorner}\{y\}}, f \rangle \hookrightarrow T60 \Rightarrow \quad \mathbf{range}(f_{|g^{\urcorner}\{y\}}) \subseteq \mathbf{range}(f)$
$\langle f_{|g^{\urcorner}\{y\}} \rangle \hookrightarrow Tsigma\_theory3 \Rightarrow \quad Stat38: \ \Sigma_\Theta(f_{|g^{\urcorner}\{y\}}) \in s$
$\mathsf{Set\_monot} \Rightarrow \quad \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\} \subseteq \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g)\}$
$\langle \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g)\}, \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\} \rangle \hookrightarrow T162 \Rightarrow$
$\quad \mathsf{Finite}(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\})$
$\langle \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}, \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g)\} \rangle \hookrightarrow T60 \Rightarrow$
$\quad \mathbf{range}(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}) \subseteq s$
$\langle \{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\} \rangle \hookrightarrow Tsigma\_theory3 \Rightarrow \quad Stat39:$
$\quad \Sigma_\Theta(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}) \in s$
$\mathsf{Assump} \Rightarrow \quad Stat40: \ \langle \forall t \in s, z \in s \mid t \oplus z = z \oplus t \rangle$
$\langle \Sigma_\Theta(f_{|g^{\urcorner}\{y\}}), \Sigma_\Theta(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}) \rangle \hookrightarrow Stat40 \Rightarrow \quad Stat41:$
$\quad \Sigma_\Theta(f_{|g^{\urcorner}\{y\}}) \oplus \Sigma_\Theta(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}) =$
$\quad\quad \Sigma_\Theta(\{[v, \Sigma_\Theta(f_{|g^{\urcorner}\{v\}})] : v \in \mathbf{range}(g) \setminus \{y\}\}) \oplus \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})$
$\mathsf{EQUAL} \ \langle Stat41, Stat37, Stat11, Stat24, Stat3 \rangle \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following theorem is a specialized variant of the rearrangement-of-sums theorem, providing essentially the same conclusion as that given by the rearrangement-of-sums theorem, but from hypotheses that are sometimes more convenient.

-- Sum Permutation Theorem

Theorem 459 (sigma_theory$_7$) $\mathsf{Finite}(F) \,\&\, \mathsf{Svm}(F) \,\&\, 1\text{–}1(G) \,\&\, \mathbf{domain}(F) = \mathbf{domain}(G) \,\&\, \mathbf{range}(F) \subseteq s \rightarrow \Sigma_\Theta(F) = \Sigma_\Theta(\{[y, F \upharpoonright (G^{\leftarrow} \upharpoonright y)] : y \in \mathbf{range}(G)\})$. PROOF:

$\mathsf{Suppose\_not}(f, g) \Rightarrow \quad \mathsf{Finite}(f) \,\&\, \mathsf{Finite}(g) \,\&\, \mathsf{Svm}(f) \,\&\, 1\text{–}1(g) \,\&\, \mathbf{domain}(f) = \mathbf{domain}(g) \,\&\, \mathbf{range}(f) \subseteq s \,\&\, \Sigma_\Theta(f) \neq \Sigma_\Theta(\{[y, f \upharpoonright (g^{\leftarrow} \upharpoonright y)] : y \in \mathbf{range}(g)\})$

-- For suppose that f and g furnish a counterexample to our assertion.

$\mathsf{Use\_def}(1\text{–}1) \Rightarrow \quad \mathsf{Svm}(g)$
$\mathsf{Suppose} \Rightarrow \quad \{[y, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\} = \{[y, f \upharpoonright (g^{\leftarrow} \upharpoonright y)] : y \in \mathbf{range}(g)\}$
$\mathsf{EQUAL} \Rightarrow \quad \Sigma_\Theta(\{[y, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\}) = \Sigma_\Theta(\{[y, f \upharpoonright (g^{\leftarrow} \upharpoonright y)] : y \in \mathbf{range}(g)\})$
$\langle f, g \rangle \hookrightarrow Tsigma\_theory6 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat1: \ \{[y, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}})] : y \in \mathbf{range}(g)\} \neq \{[y, f \upharpoonright (g^{\leftarrow} \upharpoonright y)] : y \in \mathbf{range}(g)\}$
$\langle y \rangle \hookrightarrow Stat1 \Rightarrow \quad y \in \mathbf{range}(g) \,\&\, \Sigma_\Theta(f_{|g^{\urcorner}\{y\}}) \neq f \upharpoonright (g^{\leftarrow} \upharpoonright y)$
$\langle g, y \rangle \hookrightarrow T155 \Rightarrow \quad g^{\urcorner}\{y\} = \{g^{\leftarrow} \upharpoonright y\}$
$\langle g \rangle \hookrightarrow T89 \Rightarrow \quad \mathbf{range}(g) = \mathbf{domain}(g^{\leftarrow}) \,\&\, \mathbf{domain}(g) = \mathbf{range}(g^{\leftarrow})$
$\langle y, g^{\leftarrow} \rangle \hookrightarrow T64 \Rightarrow \quad g^{\leftarrow} \upharpoonright y \in \mathbf{domain}(f)$
$\mathsf{Use\_def}(\mathsf{Svm}) \Rightarrow \quad \mathsf{Is\_map}(f)$
$\langle f, g^{\leftarrow} \upharpoonright y \rangle \hookrightarrow T69 \Rightarrow \quad [g^{\leftarrow} \upharpoonright y, f \upharpoonright (g^{\leftarrow} \upharpoonright y)] \in f$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{f}_{|\{\mathsf{g}^{\leftarrow}\restriction \mathsf{y}\}} \neq \{[\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]\}$

$\mathsf{Use\_def}\,(|) \Rightarrow \quad \{\mathsf{p} : \mathsf{p} \in \mathsf{f} \mid \mathsf{p}^{[1]} \in \{\mathsf{g}^{\leftarrow}\restriction \mathsf{y}\}\} \neq \{[\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]\}$

$\mathsf{Suppose} \Rightarrow \quad Stat2 : [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})] \notin \{\mathsf{p} : \mathsf{p} \in \mathsf{f} \mid \mathsf{p}^{[1]} \in \{\mathsf{g}^{\leftarrow}\restriction \mathsf{y}\}\}$

$TELEM \Rightarrow \quad [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]^{[1]} = \mathsf{g}^{\leftarrow}\restriction \mathsf{y}$

$\langle [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})] \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat3 : \{\mathsf{p} : \mathsf{p} \in \mathsf{f} \mid \mathsf{p}^{[1]} \in \{\mathsf{g}^{\leftarrow}\restriction \mathsf{y}\}\} \not\subseteq \{[\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]\}$

$\langle \mathsf{q} \rangle \hookrightarrow Stat3 \Rightarrow \quad Stat4 : \mathsf{q} \in \{\mathsf{p} : \mathsf{p} \in \mathsf{f} \mid \mathsf{p}^{[1]} \in \{\mathsf{g}^{\leftarrow}\restriction \mathsf{y}\}\} \;\&\; \mathsf{q} \neq [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]$

$\langle \mathsf{p} \rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{p} \in \mathsf{f} \;\&\; \mathsf{p}^{[1]} = \mathsf{g}^{\leftarrow}\restriction \mathsf{y} \;\&\; \mathsf{p} \neq [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]$

$\mathsf{Use\_def}\,(\mathsf{Svm}) \Rightarrow \quad Stat5 : \langle \forall \mathsf{x} \in \mathsf{f}, \mathsf{y} \in \mathsf{f} \mid \mathsf{x}^{[1]} = \mathsf{y}^{[1]} \rightarrow \mathsf{x} = \mathsf{y} \rangle$

$\langle \mathsf{p}, [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})] \rangle \hookrightarrow Stat5 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{f}_{|\{\mathsf{g}^{\leftarrow}\restriction \mathsf{y}\}} = \{[\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]\}$

$\mathsf{Suppose} \Rightarrow \quad [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]^{[2]} \notin \mathsf{s}$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y}) \notin \mathsf{s}$

$\langle \mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f} \rangle \hookrightarrow T64 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]^{[2]} \in \mathsf{s}$

$\langle [\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})] \rangle \hookrightarrow Tsigma\_theory2 \Rightarrow \quad \Sigma_{\Theta}\big(\{[\mathsf{g}^{\leftarrow}\restriction \mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})]\}\big) =$
$\quad \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

ENTER_THEORY Set_theory

DISPLAY sigma_theory

THEORY sigma_theory$(\mathsf{s}, \mathsf{x} \oplus \mathsf{y}, \mathsf{e})$

-- Contains some elementary lemmas about single - valued functions

$\mathsf{e} \in \mathsf{s}$

$\langle \forall \mathsf{x} \in \mathsf{s} \mid \mathsf{x} \oplus \mathsf{e} = \mathsf{x} \rangle$

$\langle \forall \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{s} \mid \mathsf{x} \oplus \mathsf{y} = \mathsf{y} \oplus \mathsf{x} \rangle$

$\langle \forall \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{s} \mid \mathsf{x} \oplus \mathsf{y} \in \mathsf{s} \rangle$

$\langle \forall \mathsf{x} \in \mathsf{s}, \mathsf{y} \in \mathsf{s}, \mathsf{z} \in \mathsf{s} \mid (\mathsf{x} \oplus \mathsf{y}) \oplus \mathsf{z} = \mathsf{x} \oplus (\mathsf{y} \oplus \mathsf{z}) \rangle$

$\Rightarrow (\Sigma_{\Theta})$

$\Sigma_{\Theta}(\emptyset) = \mathsf{e}$

$\langle \forall \mathsf{x} \mid \mathsf{x}^{[2]} \in \mathsf{s} \rightarrow \Sigma_{\Theta}(\{\mathsf{x}\}) = \mathsf{x}^{[2]} \rangle$

$\langle \forall \mathsf{f} \mid \mathsf{Finite}(\mathsf{f}) \;\&\; \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \rightarrow \Sigma_{\Theta}(\mathsf{f}) \in \mathsf{s} \rangle$

$\langle \forall \mathsf{f}, \mathsf{c} \in \mathsf{f} \mid \mathsf{Finite}(\mathsf{f}) \;\&\; \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \rightarrow \Sigma_{\Theta}(\mathsf{f}) = \Sigma_{\Theta}(\mathsf{f} \setminus \{\mathsf{c}\}) \oplus \mathsf{c}^{[2]} \rangle$

$\langle \forall \mathsf{f} \mid \mathsf{Finite}(\mathsf{f}) \;\&\; \mathsf{Is\_map}(\mathsf{f}) \;\&\; \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \rightarrow \langle \forall \mathsf{t} \mid \Sigma_{\Theta}(\mathsf{f}) = \Sigma_{\Theta}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \cap \mathsf{t}}) \oplus \Sigma_{\Theta}(\mathsf{f}_{|\mathbf{domain}(\mathsf{f}) \setminus \mathsf{t}}) \rangle \rangle$

$\langle \forall \mathsf{f}, \mathsf{g} \mid \mathsf{Finite}(\mathsf{f}) \;\&\; \mathsf{Svm}(\mathsf{f}) \;\&\; \mathsf{Svm}(\mathsf{g}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathbf{domain}(\mathsf{g}) \;\&\; \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \rightarrow \Sigma_{\Theta}(\mathsf{f}) = \Sigma_{\Theta}\big(\{[\mathsf{y}, \Sigma_{\Theta}(\mathsf{f}_{|\mathsf{g}^{\urcorner}\{\mathsf{y}\}})] : \mathsf{y} \in \mathbf{range}(\mathsf{g})\}\big) \rangle$

$\langle \forall \mathsf{f}, \mathsf{g} \mid \mathsf{Finite}(\mathsf{f}) \;\&\; \mathsf{Svm}(\mathsf{f}) \;\&\; \mathsf{1\text{–}1}(\mathsf{g}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathbf{domain}(\mathsf{g}) \;\&\; \mathbf{range}(\mathsf{f}) \subseteq \mathsf{s} \rightarrow \Sigma_{\Theta}(\mathsf{f}) = \Sigma_{\Theta}\big(\{[\mathsf{y}, \mathsf{f}\restriction(\mathsf{g}^{\leftarrow}\restriction \mathsf{y})] : \mathsf{y} \in \mathbf{range}(\mathsf{g})\}\big) \rangle$

END sigma_theory

12 Equivalence relationships and Classes; Linear orderings

-- Next we introduce another tool used constantly, the theory of equivalence classes, which tells us that a two-variable predicate $\mathsf{P}(\mathsf{x},\mathsf{y})$ on a set s can be represented in the form $\mathsf{f}(\mathsf{x}) = \mathsf{f}(\mathsf{y})$ using a one-variable auxiliary function f if and only if P is transitive and reflexive. Since it is obvious that any relationship of the form $\mathsf{f}(\mathsf{x}) = \mathsf{f}(\mathsf{y})$ must be transitive and reflexive, we need only consider the converse. To construct the mapping f, we decompose the domain of P into 'equivalence classes', namely the collection of all sets each containing all elements x such that $\mathsf{P}(\mathsf{x},\mathsf{y})$ holds for a given y, and then simply map each x into the equivalence class to which it belongs.

THEORY equivalence_classes$\big(\mathsf{P}(\mathsf{x},\mathsf{y}),\mathsf{s}\big)$
-- Theory of equivalence classes
$\big\langle\forall \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{s}\,|\,\big(\mathsf{P}(\mathsf{x},\mathsf{y})\leftrightarrow\mathsf{P}(\mathsf{y},\mathsf{x})\big)\ \&\ \mathsf{P}(\mathsf{x},\mathsf{x})\big\rangle$
$\big\langle\forall \mathsf{x}\in\mathsf{s},\mathsf{y}\in\mathsf{s},\mathsf{z}\in\mathsf{s}\,|\,\mathsf{P}(\mathsf{x},\mathsf{y})\ \&\ \mathsf{P}(\mathsf{y},\mathsf{z})\rightarrow\mathsf{P}(\mathsf{x},\mathsf{z})\big\rangle$
END equivalence_classes

ENTER_THEORY equivalence_classes

-- The formal definitions of the notions described just above are as follows. The first definition is that of the equivalence class to which a given element of s belongs, and the second is that of the collection of all equivalence classes.

DEF equivalence_classes $\cdot\, 0a$. $\quad \mathsf{f}_\Theta(\mathsf{X}) \ =_{\text{Def}}\ \{\mathsf{z}\in\mathsf{s}\,|\,\mathsf{P}(\mathsf{X},\mathsf{z})\}$
DEF equivalence_classes $\cdot\, 0b$. $\quad \mathsf{Eqc}_\Theta \ =_{\text{Def}}\ \{\mathsf{f}_\Theta(\mathsf{x}) : \mathsf{x}\in\mathsf{s}\}$

-- We now show that, as promised, the transitive relationship $\mathsf{P}(\mathsf{x},\mathsf{y})$ is equivalent to $\mathsf{f}_\Theta(\mathsf{x}) = \mathsf{f}_\Theta(\mathsf{y})$.

Theorem 460 (equivalence_classes$_1$) $\mathsf{X},\mathsf{Y}\in\mathsf{s}\rightarrow\big(\mathsf{P}(\mathsf{X},\mathsf{Y})\leftrightarrow\mathsf{f}_\Theta(\mathsf{X})=\mathsf{f}_\Theta(\mathsf{Y})\big)$. PROOF:

Suppose_not$(\mathsf{x},\mathsf{s},\mathsf{y})$ $\Rightarrow$ $\quad \mathsf{x},\mathsf{y}\in\mathsf{s}\ \&\ \neg\big(\mathsf{P}(\mathsf{x},\mathsf{y})\leftrightarrow\mathsf{f}_\Theta(\mathsf{x})=\mathsf{f}_\Theta(\mathsf{y})\big)$

-- For if $\mathsf{P}(\mathsf{x},\mathsf{y})$ is true, then since P is a transitive relation any w satisfying $\mathsf{P}(\mathsf{x},\mathsf{w})$ must also satisfy $\mathsf{P}(\mathsf{x},\mathsf{w})$ and so by definition we must have $\mathsf{f}_\Theta(\mathsf{x}) = \mathsf{f}_\Theta(\mathsf{y})$.

Suppose $\Rightarrow$ $\quad \mathsf{P}(\mathsf{x},\mathsf{y})\ \&\ \mathsf{f}_\Theta(\mathsf{x})\neq\mathsf{f}_\Theta(\mathsf{y})$
Use_def(f_Θ) $\Rightarrow$ $\quad$ *Stat1*: $\{\mathsf{z}\in\mathsf{s}\,|\,\mathsf{P}(\mathsf{x},\mathsf{z})\}\neq\{\mathsf{z}\in\mathsf{s}\,|\,\mathsf{P}(\mathsf{y},\mathsf{z})\}$
$\langle\mathsf{c}\rangle\hookrightarrow$*Stat1* $\Rightarrow$ $\quad \mathsf{c}\in\mathsf{s}\ \&\ \big(\mathsf{P}(\mathsf{x},\mathsf{c})\ \&\ \neg\mathsf{P}(\mathsf{y},\mathsf{c})\big)\vee\big(\neg\mathsf{P}(\mathsf{x},\mathsf{c})\ \&\ \mathsf{P}(\mathsf{y},\mathsf{c})\big)$
Assump $\Rightarrow$ $\quad$ *Stat2*: $\big\langle\forall\mathsf{u}\in\mathsf{s},\mathsf{v}\in\mathsf{s}\,|\,\big(\mathsf{P}(\mathsf{u},\mathsf{v})\leftrightarrow\mathsf{P}(\mathsf{v},\mathsf{u})\big)\ \&\ \mathsf{P}(\mathsf{u},\mathsf{u})\big\rangle$

$\text{Assump} \Rightarrow \quad \mathit{Stat3}:\ \langle\forall u \in s, v \in s, w \in s \mid P(u,v) \,\&\, P(v,w) \rightarrow P(u,w)\rangle$
$\text{Suppose} \Rightarrow \quad P(x,c) \,\&\, \neg P(y,c)$
$\langle x,y\rangle\hookrightarrow \mathit{Stat2} \Rightarrow \quad P(y,x)$
$\langle y,x,c\rangle\hookrightarrow \mathit{Stat3} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \neg P(x,c) \,\&\, P(y,c)$
$\langle x,y,c\rangle\hookrightarrow \mathit{Stat3} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \neg P(x,y) \,\&\, f_\Theta(x) = f_\Theta(y)$

-- Conversely if $f_\Theta(x) = f_\Theta(y)$ is true, then since x plainly belongs to $f_\Theta(x)$, it must also belong to $f_\Theta(y)$, and so by definition we must have $P(x,y)$, thereby completing the proof of our theorem.

$\text{Use\_def}(f_\Theta) \Rightarrow \quad \{z \in s \mid P(x,z)\} = \{z \in s \mid P(y,z)\}$
$\text{Suppose} \Rightarrow \quad \mathit{Stat4}:\ y \notin \{z \in s \mid P(y,z)\}$
$\langle\ \rangle\hookrightarrow \mathit{Stat4} \Rightarrow \quad \neg P(y,y)$
$\langle y,y\rangle\hookrightarrow \mathit{Stat2} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \mathit{Stat5}:\ y \in \{z \in s \mid P(x,z)\}$
$\langle\ \rangle\hookrightarrow \mathit{Stat5} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$ QED

-- Next we prove the elementary facts that f_Θ maps elements of s into elements of the equivalence class set Eqc_Θ, and that for each equivalence class y, $\mathbf{arb}(y)$ is a member of s whose equivalence class is y.

Theorem 461 (equivalence_classes$_2$) $X \in s \rightarrow f_\Theta(X) \in Eqc_\Theta$. PROOF:

$\text{Suppose\_not}(x, s, Eqc_\Theta, y) \Rightarrow \quad x \in s \,\&\, f_\Theta(x) \notin Eqc_\Theta$

$\text{Use\_def}(Eqc_\Theta) \Rightarrow \quad x \in s \,\&\, \mathit{Stat1}:\ f_\Theta(x) \notin \{f_\Theta(x) : x \in s\}$
$\langle x\rangle\hookrightarrow \mathit{Stat1} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$ QED

Theorem 462 (equivalence_classes2b) $X \in Eqc_\Theta \rightarrow \mathbf{arb}(X) \in s \,\&\, f_\Theta(\mathbf{arb}(X)) = X$. PROOF:

$\text{Suppose\_not}(y, s, Eqc_\Theta) \Rightarrow \quad y \in Eqc_\Theta \,\&\, \mathbf{arb}(y) \notin s \vee f_\Theta(\mathbf{arb}(y)) \neq y$
$\text{Use\_def}(Eqc_\Theta) \Rightarrow \quad \mathit{Stat1}:\ y \in \{f_\Theta(x) : x \in s\}$
$\langle c\rangle\hookrightarrow \mathit{Stat1} \Rightarrow \quad y = f_\Theta(c) \,\&\, c \in s$
$\text{Use\_def}(f_\Theta) \Rightarrow \quad f_\Theta(c) = \{zz \in s \mid P(c,zz)\}$
$\text{Suppose} \Rightarrow \quad c \notin f_\Theta(c)$
$\text{ELEM} \Rightarrow \quad \mathit{Stat5}:\ c \notin \{zz \in s \mid P(c,zz)\}$
$\text{Assump} \Rightarrow \quad \mathit{Stat2}:\ \langle\forall x \in s, y \in s \mid \big(P(x,y) \leftrightarrow P(y,x)\big) \,\&\, P(x,x)\rangle \,\&\, \mathit{Stat3}:\ \langle\forall x \in s, y \in s, z \in s \mid P(x,y) \,\&\, P(y,z) \rightarrow P(x,z)\rangle$
$\langle c\rangle\hookrightarrow \mathit{Stat5} \Rightarrow \quad \neg P(c,c)$
$\langle c,c\rangle\hookrightarrow \mathit{Stat2} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad c \in f_\Theta(c)$
$\text{ELEM} \Rightarrow \quad f_\Theta(c) \neq \emptyset$

$\langle f_\Theta(c)\rangle \hookrightarrow T0 \Rightarrow \quad \mathbf{arb}(f_\Theta)(c) \in f_\Theta(c)$
Use_def $(f_\Theta) \Rightarrow \quad \mathbf{arb}(f_\Theta)(c) \in \{zz \in s \mid P(c, zz)\}$
EQUAL $\Rightarrow \quad$ *Stat6* : $\mathbf{arb}(y) \in \{zz \in s \mid P(c, zz)\}$
$\langle\ \rangle \hookrightarrow Stat6 \Rightarrow \quad \mathbf{arb}(y) \in s \mathbin{\&} P(c, \mathbf{arb}(y))$
ELEM $\Rightarrow \quad f_\Theta(\mathbf{arb}(y)) \neq f_\Theta(c)$
Use_def $(f_\Theta) \Rightarrow \quad$ *Stat7* : $\{zz \in s \mid P(\mathbf{arb}(y), zz)\} \neq \{zz \in s \mid P(c, zz)\}$
$\langle d\rangle \hookrightarrow Stat7 \Rightarrow \quad d \in s \mathbin{\&} \big(P(\mathbf{arb}(y), d) \mathbin{\&} \neg P(c, d)\big) \vee \big(\neg P(\mathbf{arb}(y), d) \mathbin{\&} P(c, d)\big)$
Suppose $\Rightarrow \quad P(\mathbf{arb}(y), d) \mathbin{\&} \neg P(c, d)$
$\langle c, \mathbf{arb}(y), d\rangle \hookrightarrow Stat3 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \neg P(\mathbf{arb}(y), d) \mathbin{\&} P(c, d)$
$\langle \mathbf{arb}(y), c\rangle \hookrightarrow Stat2 \Rightarrow \quad P(\mathbf{arb}(y), c)$
$\langle \mathbf{arb}(y), c, d\rangle \hookrightarrow Stat3 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Our next, quite trivial result rounds out the previous theorem by proving that $\mathbf{arb}(f_\Theta)(x)$ is equivalent to x for any $x \in s$.

Theorem 463 (equivalence_classes$_3$) $X \in s \to P\big(x, \mathbf{arb}(f_\Theta)(x)\big)$. PROOF:

Suppose_not $(u, s) \Rightarrow \quad u \in s \mathbin{\&} \neg P\big(u, \mathbf{arb}(f_\Theta)(u)\big)$
$\langle Eqc_\Theta, u, s\rangle \hookrightarrow Tequivalence\_classes2 \Rightarrow \quad f_\Theta(u) \in Eqc_\Theta$
$\langle Eqc_\Theta, f_\Theta(u), s\rangle \hookrightarrow Tequivalence\_classes2b \Rightarrow \quad \mathbf{arb}(f_\Theta)(u) \in s \mathbin{\&} f_\Theta\big(\mathbf{arb}(f_\Theta)(u)\big) = f_\Theta(u)$
$\langle u, s, \mathbf{arb}(f_\Theta)(u)\rangle \hookrightarrow Tequivalence\_classes1 \Rightarrow \quad P\big(u, \mathbf{arb}(f_\Theta)(u)\big)$
ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY equivalence_classes

THEORY equivalence_classes(P, s)
-- Theory of equivalence classes
$\langle \forall x \in s, y \in s \mid \big(P(x, y) \leftrightarrow P(y, x)\big) \mathbin{\&} P(x, x)\rangle$
$\langle \forall x \in s, y \in s, z \in s \mid P(x, y) \mathbin{\&} P(y, z) \to P(x, z)\rangle$
$\Rightarrow (f_\Theta, Eqc_\Theta)$
$\langle \forall x \in s \mid f_\Theta(x) \in Eqc_\Theta\rangle \mathbin{\&} \langle \forall y \in Eqc_\Theta \mid \mathbf{arb}(y) \in s \mathbin{\&} f_\Theta(\mathbf{arb}(y)) = y\rangle$
$\langle \forall x \in s, y \in s \mid P(x, y) \leftrightarrow f_\Theta(x) = f_\Theta(y)\rangle$
$\langle \forall x \in s \mid P\big(x, \mathbf{arb}(f_\Theta)(x)\big)\rangle$
END equivalence_classes

-- Next we introduce another tool used constantly, the theory of strict linear orderings, which tells us that a two-variable predicate $x \lhd y$ which enjoys transitivity, irreflexivity and trichotomy on a set s induces various other useful predicates and operations, among which maximum and least-upper-bound operations, both associating a value in s to every finite subset of s.

THEORY $\mathsf{linear\_order}(s, X \lhd Y)$
$\langle \forall x \in s, y \in s, z \in s \mid x \lhd y \,\&\, y \lhd z \rightarrow x \lhd z \rangle$
$\langle \forall x \in s \mid \neg x \lhd x \rangle$
$\langle \forall x \in s, y \in s \mid x \lhd y \vee x = y \vee y \lhd x \rangle$
END $\mathsf{linear\_order}$

ENTER_THEORY $\mathsf{linear\_order}$

-- Less - than - or - equal comparison

DEF 10030. $\mathsf{le}_\Theta(X, Y) \quad \leftrightarrow_{\mathrm{Def}} \quad X \lhd Y \vee X = Y$

-- The ordering relation $\lhd$ can be defined on a larger domain than the one, s, underlying the present theory; on the other hand, the following operation $\mathsf{smaller}_\Theta$ relativizes the comparisons to s, taking s itself as the conventional smallest element.

-- Choice of the smaller

DEF 10031. $\mathsf{smaller}_\Theta(X, Y) \quad =_{\mathrm{Def}} \quad \textbf{if } X \notin s \vee Y \notin s \textbf{ then } s \textbf{ else if } X \lhd Y \textbf{ then } X \textbf{ else } Y \textbf{ fi fi}$

-- The following easy theorem shows that, much like the strict order $\lhd$, the associated 'less-than-or-equal' relation is transitive.

Theorem 464 ($\mathsf{linear\_order}_1$) $\{U, V, W\} \subseteq s \,\&\, \mathsf{le}_\Theta(U, V) \,\&\, \mathsf{le}_\Theta(V, W) \rightarrow \mathsf{le}_\Theta(U, W)$. PROOF:

Suppose_not$(x, y, z) \Rightarrow \quad \{x, y, z\} \subseteq s \,\&\, \mathsf{le}_\Theta(x, y) \,\&\, \mathsf{le}_\Theta(y, z) \,\&\, \neg \mathsf{le}_\Theta(x, z)$
Use_def$(\mathsf{le}_\Theta) \Rightarrow \quad x \lhd y \vee x = y \,\&\, y \lhd z \vee y = z \,\&\, \neg(x \lhd z \vee x = z)$
Suppose $\Rightarrow \quad x = y \,\&\, y = z$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow \quad (x \lhd y \,\&\, y \lhd z) \vee (x \lhd y \,\&\, y = z) \vee (x = y \,\&\, y \lhd z)$
Suppose $\Rightarrow \quad x \lhd y \,\&\, y \lhd z$
Assump $\Rightarrow \quad$ *Stat1* : $\langle \forall x \in s, y \in s, z \in s \mid x \lhd y \,\&\, y \lhd z \rightarrow x \lhd z \rangle$
$\langle x, y, z \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow \quad (x \lhd y \,\&\, y = z) \vee (x = y \,\&\, y \lhd z)$
Suppose $\Rightarrow \quad x \lhd y \,\&\, y = z$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow \quad x = y \,\&\, y \lhd z$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The irreflexivity property of the strict order $\lhd$ induces the following property of the associated 'less-than-or-equal' relation.

Theorem 465 (linear_order$_2$) $\{U, V\} \subseteq s \ \& \ le_\Theta(U, V) \ \& \ le_\Theta(V, U) \rightarrow U = V$. PROOF:

Suppose_not$(x, y) \Rightarrow \quad \{x, y\} \subseteq s \ \& \ le_\Theta(x, y) \ \& \ le_\Theta(y, x) \ \& \ x \neq y$
Use_def$(le_\Theta) \Rightarrow \quad x \lhd y \ \& \ y \lhd x$
Assump $\Rightarrow \quad$ *Stat1* : $\langle \forall x \in s, y \in s, z \in s \mid x \lhd y \ \& \ y \lhd z \longrightarrow x \lhd z \rangle$
$\langle x, y, x \rangle \hookrightarrow$ *Stat1* $\Rightarrow \quad x \lhd x$
Assump $\Rightarrow \quad$ *Stat2* : $\langle \forall x \in s \mid \neg x \lhd x \rangle$
$\langle x \rangle \hookrightarrow$ *Stat2* $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The trichotomic property of the strict order $\lhd$ induces the following dichotomic property of the associated 'less-than-or-equal' relation: of any two elements of the domain s, one does not exceed the other.

Theorem 466 (linear_order$_3$) $\{U, V\} \subseteq s \rightarrow le_\Theta(U, V) \lor le_\Theta(V, U)$. PROOF:

Suppose_not$(x, y) \Rightarrow \quad \{x, y\} \subseteq s \ \& \ \neg le_\Theta(x, y) \ \& \ \neg le_\Theta(y, x)$
Use_def$(le_\Theta) \Rightarrow \quad x \neq y \ \& \ \neg x \lhd y \ \& \ \neg y \lhd x$
Assump $\Rightarrow \quad$ *Stat1* : $\langle \forall x \in s, y \in s \mid x \lhd y \lor x = y \lor y \lhd x \rangle$
$\langle x, y \rangle \hookrightarrow$ *Stat1* $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The following theorem extends to three elements the dichotomic property seen above, stating that of any three elements in the domain s one is smallest.

Theorem 467 (linear_order$_4$) $\{U, V, W\} \subseteq s \rightarrow \big(le_\Theta(U, V) \ \& \ le_\Theta(U, W)\big) \lor \big(le_\Theta(V, U) \ \& \ le_\Theta(V, W)\big) \lor \big(le_\Theta(W, U) \ \& \ le_\Theta(W, V)\big)$. PROOF:

Suppose_not$(x, y, z) \Rightarrow \quad$ *Stat0* : $\{x, y, z\} \subseteq s \ \& \ \neg\big(le_\Theta(x, y) \ \& \ le_\Theta(x, z)\big) \ \& \ \neg\big(le_\Theta(y, x) \ \& \ le_\Theta(y, z)\big) \ \& \ \neg\big(le_\Theta(z, x) \ \& \ le_\Theta(z, y)\big)$
$\langle x, y \rangle \hookrightarrow$ *Tlinear_order_3* $\Rightarrow \quad le_\Theta(x, y) \lor le_\Theta(y, x)$
$\langle y, z \rangle \hookrightarrow$ *Tlinear_order_3* $\Rightarrow \quad le_\Theta(y, z) \lor le_\Theta(z, y)$
$\langle x, z \rangle \hookrightarrow$ *Tlinear_order_3* $\Rightarrow \quad le_\Theta(x, z) \lor le_\Theta(z, x)$
Suppose $\Rightarrow \quad$ *Stat1* : $le_\Theta(x, y) \ \& \ le_\Theta(y, z)$
$\langle x, y, z \rangle \hookrightarrow$ *Tlinear_order_1*([*Stat0*, *Stat1*]) $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad le_\Theta(y, x) \ \& \ le_\Theta(z, y)$
$\langle z, y, x \rangle \hookrightarrow$ *Tlinear_order_1* $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The operation which chooses the smaller of its two arguments is commutative:

Theorem 468 (linear_order$_5$) $smaller_\Theta(X, Y) = smaller_\Theta(Y, X)$. PROOF:

$\mathsf{Suppose\_not}(x, y) \Rightarrow$ $\mathsf{smaller}_\Theta(x, y) \neq \mathsf{smaller}_\Theta(y, x)$
$\mathsf{Suppose} \Rightarrow$ $x \notin s \vee y \notin s$
$\mathsf{Use\_def}(\mathsf{smaller}_\Theta) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $x, y \in s$
$\mathsf{Assump} \Rightarrow$ *Stat1* : $\langle \forall x \in s, y \in s \mid x \lhd y \vee x = y \vee y \lhd x \rangle$
$\langle x, y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x \lhd y \vee x = y \vee y \lhd x$
$\mathsf{Suppose} \Rightarrow$ $x = y$
$\mathsf{EQUAL} \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $x \lhd y \vee y \lhd x$
$\mathsf{Suppose} \Rightarrow$ $x \lhd y \;\&\; y \lhd x$
$\mathsf{Assump} \Rightarrow$ *Stat2* : $\langle \forall x \in s, y \in s, z \in s \mid x \lhd y \;\&\; y \lhd z \longrightarrow x \lhd z \rangle$
$\langle x, y, x \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $x \lhd x$
$\mathsf{Assump} \Rightarrow$ *Stat3* : $\langle \forall x \in s \mid \neg x \lhd x \rangle$
$\langle x \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $\neg(x \lhd y \;\&\; y \lhd x)$
$\mathsf{Suppose} \Rightarrow$ $x \lhd y$
$\mathsf{Use\_def}(\mathsf{smaller}_\Theta) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $y \lhd x$
$\mathsf{Use\_def}(\mathsf{smaller}_\Theta) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- The set s, which we have taken as the conventional minimum, acts as the unit element of the operation which chooses the smaller of its two arguments:

Theorem 469 ($\mathsf{linear\_order}_6$) $\mathsf{smaller}_\Theta(X, s) = s \;\&\; \mathsf{smaller}_\Theta(s, X) = s$. PROOF:

$\mathsf{Suppose\_not}(x) \Rightarrow$ $\mathsf{smaller}_\Theta(x, s) \neq s \vee \mathsf{smaller}_\Theta(s, x) \neq s$
TELEM $\Rightarrow$ $s \notin s$
$\mathsf{Use\_def}(\mathsf{smaller}_\Theta) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- Every doubleton subset of s is closed with respect to the operation which chooses the smaller of its two arguments:

Theorem 470 ($\mathsf{linear\_order}_7$) $\{X, Y\} \subseteq s \rightarrow \mathsf{smaller}_\Theta(X, Y) \in \{X, Y\}$. PROOF:

$\mathsf{Suppose\_not}(x, y) \Rightarrow$ $\{x, y\} \subseteq s \;\&\; \mathsf{smaller}_\Theta(x, y) \notin \{x, y\}$
$\mathsf{Use\_def}(\mathsf{smaller}_\Theta) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

-- When x, y are elements of s such that x is less than or equal to y, is x, the smaller is x:

Theorem 471 ($\mathsf{linear\_order}_8$) $\{X, Y\} \subseteq s \;\&\; X \lhd Y \vee X = Y \rightarrow \mathsf{smaller}_\Theta(X, Y) = X \;\&\; \mathsf{smaller}_\Theta(Y, X) = X$. PROOF:

$\text{Suppose\_not}(x,y) \Rightarrow \quad \{x,y\} \subseteq s \;\&\; x \lhd y \vee x = y \;\&\; \text{smaller}_\Theta(x,y) \neq x \vee \text{smaller}_\Theta(y,x) \neq x$
$\langle x,y\rangle \hookrightarrow \textit{Tlinear\_order\_5} \Rightarrow \quad \textit{Stat1}: \; x,y \in s \;\&\; \text{smaller}_\Theta(x,y) \neq x$
$\text{Suppose} \Rightarrow \quad x \lhd y$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad x = y$
$\langle x,y\rangle \hookrightarrow \textit{Tlinear\_order\_7}(\langle \textit{Stat1}\rangle) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

Theorem 472 (linear_order$_9$) $\{X,Y\} \subseteq s \rightarrow \big(\text{smaller}_\Theta(X,Y) = X \leftrightarrow X \lhd Y \vee X = Y\big)$. PROOF:

$\text{Suppose\_not}(x,y) \Rightarrow \quad \{x,y\} \subseteq s \;\&\; \text{le}_\Theta(x,y) \;\&\; \big(\text{smaller}_\Theta(x,y) \neq x \leftrightarrow x \lhd y \vee x = y\big)$
$\text{Suppose} \Rightarrow \quad x \lhd y \vee x = y$
$\langle x,y\rangle \hookrightarrow \textit{Tlinear\_order\_8} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{smaller}_\Theta(x,y) = x$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad$ QED

-- The operation which chooses the smaller of its two arguments enjoys associativity:

Theorem 473 (linear_order$_{10}$) $\text{smaller}_\Theta\big(X, (\text{smaller}_\Theta)(Y,ZZ)\big) = \text{smaller}_\Theta\big((\text{smaller}_\Theta)(X,Y), ZZ\big)$. PROOF:

$\text{Suppose\_not}(x,y,w) \Rightarrow \quad \text{smaller}_\Theta\big(x, \text{smaller}_\Theta(y,w)\big) \neq \text{smaller}_\Theta\big(\text{smaller}_\Theta(x,y), w\big)$

-- For, assuming by contradiction that x, y, w make a counter-example, it turns out that all three must belong to s; ...

$\textit{TELEM} \Rightarrow \quad s \notin s$
$\text{Suppose} \Rightarrow \quad y \notin s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(y,w) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(x,y) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(x,s) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(s,w) = s$
$\text{EQUAL} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad y \in s$
$\text{Suppose} \Rightarrow \quad x \notin s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(x,y) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(s,w) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta\big(x, \text{smaller}_\Theta(y,w)\big) = s$
$\text{EQUAL} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad x \in s$
$\text{Suppose} \Rightarrow \quad w \notin s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(y,w) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta(x,s) = s$
$\text{Use\_def}(\text{smaller}_\Theta) \Rightarrow \quad \text{smaller}_\Theta\big(\text{smaller}_\Theta(x,y), w\big) = s$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $w \in s$

-- ...but then, whichever of the three is smallest, we readily reach a contradiction. This leads us to the desired conclusion.

$\langle x, y, w \rangle \hookrightarrow$ *Tlinear_order_4* $\Rightarrow$ $(\text{le}_\Theta(x, y) \,\&\, \text{le}_\Theta(x, w)) \vee (\text{le}_\Theta(y, x) \,\&\, \text{le}_\Theta(y, w)) \vee (\text{le}_\Theta(w, x) \,\&\, \text{le}_\Theta(w, y))$
Suppose $\Rightarrow$ $\text{le}_\Theta(x, y) \,\&\, \text{le}_\Theta(x, w)$
Use_def (le_Θ) $\Rightarrow$ $x \lhd y \vee x = y$
Use_def (le_Θ) $\Rightarrow$ $x \lhd w \vee x = w$
$\langle x, y \rangle \hookrightarrow$ *Tlinear_order_8* $\Rightarrow$ $\text{smaller}_\Theta(x, y) = x$
$\langle x, w \rangle \hookrightarrow$ *Tlinear_order_8* $\Rightarrow$ $\text{smaller}_\Theta(x, w) = x$
$\langle y, w \rangle \hookrightarrow$ *Tlinear_order_7* $\Rightarrow$ $\text{smaller}_\Theta(y, w) \in \{y, w\}$
Suppose $\Rightarrow$ $\text{smaller}_\Theta(y, w) = y$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\text{smaller}_\Theta(y, w) = w$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $(\text{le}_\Theta(y, x) \,\&\, \text{le}_\Theta(y, w)) \vee (\text{le}_\Theta(w, x) \,\&\, \text{le}_\Theta(w, y))$
Suppose $\Rightarrow$ $\text{le}_\Theta(y, x) \,\&\, \text{le}_\Theta(y, w)$
Use_def (le_Θ) $\Rightarrow$ $y \lhd x \vee y = x$
Use_def (le_Θ) $\Rightarrow$ $y \lhd w \vee y = w$
$\langle y, x \rangle \hookrightarrow$ *Tlinear_order_8* $\Rightarrow$ $\text{smaller}_\Theta(x, y) = y$
$\langle y, w \rangle \hookrightarrow$ *Tlinear_order_8* $\Rightarrow$ $\text{smaller}_\Theta(y, w) = y$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\text{le}_\Theta(w, x) \,\&\, \text{le}_\Theta(w, y)$
Use_def (le_Θ) $\Rightarrow$ $w \lhd x \vee w = x$
Use_def (le_Θ) $\Rightarrow$ $w \lhd y \vee w = y$
$\langle w, y \rangle \hookrightarrow$ *Tlinear_order_8* $\Rightarrow$ $\text{smaller}_\Theta(y, w) = w$
$\langle w, x \rangle \hookrightarrow$ *Tlinear_order_8* $\Rightarrow$ $\text{smaller}_\Theta(x, w) = w$
$\langle x, y \rangle \hookrightarrow$ *Tlinear_order_7* $\Rightarrow$ $\text{smaller}_\Theta(x, y) \in \{x, y\}$
Suppose $\Rightarrow$ $\text{smaller}_\Theta(x, y) = x$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\text{smaller}_\Theta(x, y) = y$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The set $s \cup \{s\}$ is closed with respect to the operation which chooses the smaller of its two arguments:

Theorem 474 (linear_order_{11}) $\langle \forall x \in s \cup \{s\}, y \in s \cup \{s\} \mid \text{smaller}_\Theta(x, y) \in s \cup \{s\} \rangle$. PROOF:

Suppose_not (x, y) $\Rightarrow$ $x, y \in s \cup \{s\} \,\&\, \text{smaller}_\Theta(x, y) \notin s \cup \{s\}$
Suppose $\Rightarrow$ $x = s \vee y = s$
Use_def (smaller_Θ) $\Rightarrow$ false; Discharge $\Rightarrow$ $x, y \in s$
$\langle x, y \rangle \hookrightarrow$ *Tlinear_order_7* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following definition of the upper bounds of a set t takes into account only the part of t which consists of elements of the domain s underlying the present theory.

-- Upper bounds

DEF 10032. $\mathsf{ubs}_\Theta(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \{\mathsf{x} \in \mathsf{s} \,|\, \langle \forall \mathsf{y} \in \mathsf{X} \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{x}) = \mathsf{y}\rangle\}$

-- The maximum of a set t is the upper bound of t — if any — which belongs to t. If there is no such element, we conventionally take the domain s underlying the present theory as the maximum.

-- Maximum of a set

DEF 10033. $\mathsf{max}_\Theta(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{\mathsf{s}\} \cup \mathsf{X} \cap \mathsf{ubs}_\Theta(\mathsf{X}))$

-- In a way similar to the maximum, we conventionally take the least upper bound of a set t, when none proper exists, to be s.

-- Least upper bound of a set

DEF 10035. $\mathsf{lub}_\Theta(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{\mathsf{s}\} \cup \{\mathsf{x} \in \mathsf{ubs}_\Theta(\mathsf{X}) \,|\, \mathsf{ubs}_\Theta(\mathsf{X}) \subseteq \mathsf{ubs}_\Theta(\{\mathsf{x}\})\})$

-- It readily follows from the definitions that the upper bounds of $\emptyset$ form the entire s (which also equals the conventional maximum of $\emptyset$).

Theorem 475 (linear_order$_{12}$) $\mathsf{ubs}_\Theta(\emptyset) = \mathsf{s} \;\&\; \mathsf{max}_\Theta(\emptyset) = \mathsf{s}$. PROOF:

Suppose_not $\Rightarrow$ $\mathsf{ubs}_\Theta(\emptyset) \neq \mathsf{s} \vee \mathsf{max}_\Theta(\emptyset) \neq \mathsf{s}$

Use_def $(\mathsf{max}_\Theta) \Rightarrow$ $\mathit{Stat1}: \; \mathsf{ubs}_\Theta(\emptyset) \neq \mathsf{s}$

$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathsf{c} \in \mathsf{ubs}_\Theta(\emptyset) \leftrightarrow \mathsf{c} \notin \mathsf{s}$

Use_def $(\mathsf{ubs}_\Theta) \Rightarrow$ $\mathsf{c} \in \{\mathsf{x} \in \mathsf{s} \,|\, \langle \forall \mathsf{y} \in \emptyset \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{x}) = \mathsf{y}\rangle\} \leftrightarrow \mathsf{c} \notin \mathsf{s}$

Suppose $\Rightarrow$ $\mathit{Stat2}: \; \mathsf{c} \in \{\mathsf{x} \in \mathsf{s} \,|\, \langle \forall \mathsf{y} \in \emptyset \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{x}) = \mathsf{y}\rangle\}$

$\langle\;\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat3}: \; \mathsf{c} \notin \{\mathsf{x} \in \mathsf{s} \,|\, \langle \forall \mathsf{y} \in \emptyset \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{x}) = \mathsf{y}\rangle\}$

$\langle\;\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $\mathit{Stat5}: \; \neg\langle \forall \mathsf{y} \in \emptyset \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{c}) = \mathsf{y}\rangle$

$\langle \mathsf{y} \rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- As defined above, the set of upper bounds and the maximum of any set t do not depend on those elements of t which lie outside s.

Theorem 476 (linear_order$_{13}$) $\mathsf{ubs}_\Theta(T) \subseteq \mathsf{s} \;\&\; \mathsf{ubs}_\Theta(T) = \mathsf{ubs}_\Theta(T \cap \mathsf{s}) \;\&\; \mathsf{max}_\Theta(T) = \mathsf{max}_\Theta(T \cap \mathsf{s})$. PROOF:

Suppose_not$(\mathsf{t}) \Rightarrow$ $\mathsf{ubs}_\Theta(\mathsf{t}) \not\subseteq \mathsf{s} \vee \mathsf{ubs}_\Theta(\mathsf{t}) \neq \mathsf{ubs}_\Theta(\mathsf{t} \cap \mathsf{s}) \vee \mathsf{max}_\Theta(\mathsf{t}) \neq \mathsf{max}_\Theta(\mathsf{t} \cap \mathsf{s})$

Use_def $(\mathsf{ubs}_\Theta) \Rightarrow$ $\mathsf{ubs}_\Theta(\mathsf{t}) = \{\mathsf{x} \in \mathsf{s} \,|\, \langle \forall \mathsf{y} \in \mathsf{t} \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{x}) = \mathsf{y}\rangle\}$

Suppose $\Rightarrow$ $\mathit{Stat1}: \; \{\mathsf{x} \in \mathsf{s} \,|\, \langle \forall \mathsf{y} \in \mathsf{t} \cap \mathsf{s} \,|\, \mathsf{smaller}_\Theta(\mathsf{y}, \mathsf{x}) = \mathsf{y}\rangle\} \not\subseteq \mathsf{s}$

$\langle c\rangle \hookrightarrow Stat1 \Rightarrow$ $c \notin s \;\&\; Stat2:\; c \in \{x \in s \,|\, \langle \forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y,x) = y\rangle\}$
$\langle\,\rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat3:\; \mathsf{ubs}_\Theta(t) \subseteq s \;\&\; \mathsf{ubs}_\Theta(t) \neq \mathsf{ubs}_\Theta(t \cap s) \vee \mathsf{max}_\Theta(t) \neq \mathsf{max}_\Theta(t \cap s)$
Suppose $\Rightarrow$ $Stat4:\; \mathsf{ubs}_\Theta(t) \neq \mathsf{ubs}_\Theta(t \cap s)$
Use_def $(\mathsf{ubs}_\Theta) \Rightarrow$ $\{x \in s \,|\, \langle \forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y,x) = y\rangle\} \neq \{x \in s \,|\, \langle \forall y \in t \cap s \cap s \,|\, \mathsf{smaller}_\Theta(y,x) = y\rangle\}$
$TELEM \Rightarrow$ $t \cap s = t \cap s \cap s$
EQUAL $\langle Stat4\rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat5:\; \mathsf{ubs}_\Theta(t) = \mathsf{ubs}_\Theta(t \cap s) \;\&\; \mathsf{max}_\Theta(t) \neq \mathsf{max}_\Theta(t \cap s)$
Use_def $(\mathsf{max}_\Theta) \Rightarrow$ $\mathbf{arb}(\{s\} \cup t \cap \mathsf{ubs}_\Theta(t)) \neq \mathbf{arb}(\{s\} \cup t \cap s \cap \mathsf{ubs}_\Theta(t \cap s))$
$\langle Stat3, Stat5\rangle$ ELEM $\Rightarrow$ $t \cap \mathsf{ubs}_\Theta(t) = t \cap s \cap \mathsf{ubs}_\Theta(t \cap s)$
EQUAL $\langle Stat5\rangle \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Every non-null finite subset t of s is endowed with maximum. Such maximum belongs to t and exceeds every other element of t.

Theorem 477 (linear_order$_{14}$) $\mathsf{Finite}(T) \;\&\; X \in T \;\&\; T \subseteq s \rightarrow \mathsf{max}_\Theta(T) \in T \;\&\; X = \mathsf{max}_\Theta(T) \vee X \lhd \mathsf{max}_\Theta(T)$. PROOF:

Suppose_not$(t_0, x_0) \Rightarrow$ $Stat0:\; \mathsf{Finite}(t_0) \;\&\; x_0 \in t_0 \;\&\; t_0 \subseteq s \;\&\; \neg(\mathsf{max}_\Theta(t_0) \in t_0 \;\&\; x_0 = \mathsf{max}_\Theta(t_0) \vee x_0 \lhd \mathsf{max}_\Theta(t_0))$

-- For, if not, then we could take an inclusion-minimal t for which an x_1 exists violating the statement.

Suppose $\Rightarrow$ $Stat1:\; \langle \forall x \in t_0 \,|\, \mathsf{max}_\Theta(t_0) \in t_0 \;\&\; x = \mathsf{max}_\Theta(t_0) \vee x \lhd \mathsf{max}_\Theta(t_0)\rangle$
$\langle x_0\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $\neg\langle \forall x \in t_0 \,|\, \mathsf{max}_\Theta(t_0) \in t_0 \;\&\; x = \mathsf{max}_\Theta(t_0) \vee x \lhd \mathsf{max}_\Theta(t_0)\rangle$
APPLY $\langle m_\Theta : t\rangle$ finite_induction $(n \mapsto t_0, P(k) \mapsto \neg\langle \forall x \in k \,|\, \mathsf{max}_\Theta(k) \in k \;\&\; x = \mathsf{max}_\Theta(k) \vee x \lhd \mathsf{max}_\Theta(k)\rangle) \Rightarrow$
$t \subseteq t_0 \;\&\; Stat2:\; \neg\langle \forall x \in t \,|\, \mathsf{max}_\Theta(t) \in t \;\&\; x = \mathsf{max}_\Theta(t) \vee x \lhd \mathsf{max}_\Theta(t)\rangle \;\&\; Stat3:\; \langle \forall k \subseteq t \,|\, k \neq t \rightarrow \langle \forall x \in k \,|\, \mathsf{max}_\Theta(k) \in k \;\&\; x = \mathsf{max}_\Theta(k) \vee x \lhd \mathsf{max}_\Theta(k)\rangle\rangle$
$\langle x_1\rangle \hookrightarrow Stat2 \Rightarrow$ $x_1 \in t \;\&\; \mathsf{max}_\Theta(t) \notin t \vee (x_1 \neq \mathsf{max}_\Theta(t) \;\&\; \neg x_1 \lhd \mathsf{max}_\Theta(t))$
ELEM $\Rightarrow$ $Stat4:\; t \subseteq s \;\&\; x_1 \in t \;\&\; x_1 \neq \mathsf{max}_\Theta(t) \;\&\; \mathsf{max}_\Theta(t) \notin t \vee \neg x_1 \lhd \mathsf{max}_\Theta(t)$
$\langle t \setminus \{x_1\}\rangle \hookrightarrow Stat3(\langle Stat4\rangle) \Rightarrow$ $Stat5:$
$\langle \forall x \in t \setminus \{x_1\} \,|\, \mathsf{max}_\Theta(t \setminus \{x_1\}) \in t \setminus \{x_1\} \;\&\; x = \mathsf{max}_\Theta(t \setminus \{x_1\}) \vee x \lhd \mathsf{max}_\Theta(t \setminus \{x_1\})\rangle$

-- Observe that x_1 cannot be the sole member of t. This entails that the maximum of $t \setminus \{x_1\}$ belongs to $t \setminus \{x_1\}$ and exceeds any element y of $t \setminus \{x_1\}$.

Suppose $\Rightarrow$ $t = \{x_1\}$
Suppose $\Rightarrow$ $x_1 \notin \mathsf{ubs}_\Theta(t)$
Use_def $(\mathsf{ubs}_\Theta) \Rightarrow$ $Stat6:\; x_1 \notin \{x \in s \,|\, \langle \forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y,x) = y\rangle\}$
$\langle\,\rangle \hookrightarrow Stat6(\langle Stat4\rangle) \Rightarrow$ $Stat7:\; \neg\langle \forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y,x_1) = y\rangle$
$\langle y_1\rangle \hookrightarrow Stat7(\langle Stat4\rangle) \Rightarrow$ $Stat8:\; y_1 = x_1 \;\&\; \mathsf{smaller}_\Theta(y_1,x_1) \neq y_1$
EQUAL $\langle Stat7\rangle \Rightarrow$ $Stat9:\; \mathsf{smaller}_\Theta(x_1,x_1) \neq x_1$
$\langle x_1, x_1\rangle \hookrightarrow Tlinear\_order\_7([Stat4, Stat9]) \Rightarrow$ false; Discharge $\Rightarrow$ $\{x_1\} = t \cap \mathsf{ubs}_\Theta(t)$
Use_def $(\mathsf{max}_\Theta) \Rightarrow$ $Stat10:\; \mathsf{max}_\Theta(t) = \mathbf{arb}(\{s, x_1\})$

$\langle Stat10, Stat4\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat11:\ t\setminus\{x_1\}\neq\emptyset$
$\langle c\rangle\hookrightarrow Stat11(\langle Stat11\rangle)\Rightarrow$ $Stat12:\ c\in t\setminus\{x_1\}$
$\langle c\rangle\hookrightarrow Stat5(\langle Stat12\rangle)\Rightarrow$ $Stat13:\ \mathsf{max}_\Theta(t\setminus\{x_1\})\in t\setminus\{x_1\}$
Use_def (max$_\Theta$) $\Rightarrow$ $Stat14:\ \mathsf{max}_\Theta(t\setminus\{x_1\})=\mathbf{arb}(\{s\}\cup(t\setminus\{x_1\})\cap\mathsf{ubs}_\Theta(t\setminus\{x_1\}))$
$\langle Stat4, Stat13, Stat14\rangle$ ELEM $\Rightarrow$ $\mathsf{max}_\Theta(t\setminus\{x_1\})\in\mathsf{ubs}_\Theta(t\setminus\{x_1\})$
Use_def (ubs$_\Theta$) $\Rightarrow$ $Stat15:\ \mathsf{max}_\Theta(t\setminus\{x_1\})\in\{x\in s\mid\langle\forall y\in(t\setminus\{x_1\})\cap s\mid\mathsf{smaller}_\Theta(y,x)=y\rangle\}$
$\langle\ \rangle\hookrightarrow Stat15(\langle Stat15\rangle)\Rightarrow$ $Stat16:\ \langle\forall y\in(t\setminus\{x_1\})\cap s\mid\mathsf{smaller}_\Theta(y,\mathsf{max}_\Theta(t\setminus\{x_1\}))=y\rangle$

-- In view of thrichotomy, we need to consider only two cases: either $x_1\lhd\mathsf{max}_\Theta(t)$ or $\mathsf{max}_\Theta(t)\lhd x_1$.

Assump $\Rightarrow$ $Stat17:\ \langle\forall x\in s, y\in s\mid x\lhd y\vee x=y\vee y\lhd x\rangle$
$\langle x_1,\mathsf{max}_\Theta(t\setminus\{x_1\})\rangle\hookrightarrow Stat17([Stat4, Stat13])\Rightarrow$ $x_1\lhd\mathsf{max}_\Theta(t\setminus\{x_1\})\vee\mathsf{max}_\Theta(t\setminus\{x_1\})\lhd x_1$
Use_def (max$_\Theta$) $\Rightarrow$ $Stat27:\ \mathsf{max}_\Theta(t)=\mathbf{arb}(\{s\}\cup t\cap\mathsf{ubs}_\Theta(t))$

-- Assuming that x_1 is smaller than the maximum of $t\setminus\{x_1\}$, we derive that this maximum is also the maximum of t, which leads to a contradiction.

Suppose $\Rightarrow$ $Stat18:\ x_1\lhd\mathsf{max}_\Theta(t\setminus\{x_1\})$
Suppose $\Rightarrow$ $Stat19:\ \neg\langle\forall y\in t\cap s\mid\mathsf{smaller}_\Theta(y,\mathsf{max}_\Theta(t\setminus\{x_1\}))=y\rangle$
$\langle y_2\rangle\hookrightarrow Stat19(\langle Stat19\rangle)\Rightarrow$ $Stat20:\ y_2\in t\cap s\ \&\ \mathsf{smaller}_\Theta(y_2,\mathsf{max}_\Theta(t\setminus\{x_1\}))\neq y_2$
Suppose $\Rightarrow$ $y_2\in t\setminus\{x_1\}$
$\langle y_2\rangle\hookrightarrow Stat16\Rightarrow$ false; Discharge $\Rightarrow$ $y_2=x_1$
EQUAL $\langle Stat20\rangle\Rightarrow$ $Stat21:\ \mathsf{smaller}_\Theta(x_1,\mathsf{max}_\Theta(t\setminus\{x_1\}))\neq x_1$
Use_def (smaller$_\Theta$) $\Rightarrow$ $Stat22:\ \mathsf{smaller}_\Theta(x_1,\mathsf{max}_\Theta(t\setminus\{x_1\}))=\textbf{if }x_1\notin s\vee\mathsf{max}_\Theta(t\setminus\{x_1\})\notin s\textbf{ then }s\textbf{ else if }x_1\lhd\mathsf{max}_\Theta(t\setminus\{x_1\})\textbf{ then }x_1\textbf{ else }\mathsf{max}_\Theta(t\setminus\{x_1\})\textbf{ fi fi}$
$\langle Stat22, Stat4, Stat13, Stat18, Stat21\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat23:\ \langle\forall y\in t\cap s\mid\mathsf{smaller}_\Theta(y,\mathsf{max}_\Theta(t\setminus\{x_1\}))=y\rangle$
Suppose $\Rightarrow$ $Stat23a:\ \mathsf{max}_\Theta(t)\neq\mathsf{max}_\Theta(t\setminus\{x_1\})$
Suppose $\Rightarrow$ $Stat24:\ \mathsf{max}_\Theta(t\setminus\{x_1\})\notin t\cap\mathsf{ubs}_\Theta(t)$
$\langle Stat24, Stat13\rangle$ ELEM $\Rightarrow$ $\mathsf{max}_\Theta(t\setminus\{x_1\})\notin\mathsf{ubs}_\Theta(t)$
Use_def (ubs$_\Theta$) $\Rightarrow$ $Stat25:\ \mathsf{max}_\Theta(t\setminus\{x_1\})\notin\{x\in s\mid\langle\forall y\in t\cap s\mid\mathsf{smaller}_\Theta(y,x)=y\rangle\}$
$\langle\ \rangle\hookrightarrow Stat25(\langle Stat13, Stat4, Stat23\rangle)\Rightarrow$ false; Discharge $\Rightarrow$ $Stat26:\ \mathsf{max}_\Theta(t\setminus\{x_1\})\in t\cap\mathsf{ubs}_\Theta(t)$
Suppose $\Rightarrow$ $Stat28:\ \mathsf{max}_\Theta(t)=s$
$\langle Stat4, Stat27, Stat28\rangle$ ELEM $\Rightarrow$ $\mathsf{max}_\Theta(t\setminus\{x_1\})\notin t\cap\mathsf{ubs}_\Theta(t)$
$\langle Stat26\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat29:\ \mathsf{max}_\Theta(t)\in t\cap\mathsf{ubs}_\Theta(t)$
Use_def (ubs$_\Theta$) $\Rightarrow$ $Stat30:\ \mathsf{max}_\Theta(t)\in\{x\in s\mid\langle\forall y\in t\cap s\mid\mathsf{smaller}_\Theta(y,x)=y\rangle\}$
$\langle\ \rangle\hookrightarrow Stat30(\langle Stat30\rangle)\Rightarrow$ $Stat31:\ \langle\forall y\in t\cap s\mid\mathsf{smaller}_\Theta(y,\mathsf{max}_\Theta(t))=y\rangle$
$\langle\mathsf{max}_\Theta(t\setminus\{x_1\})\rangle\hookrightarrow Stat31(\langle Stat31, Stat13, Stat4\rangle)\Rightarrow$ $\mathsf{smaller}_\Theta(\mathsf{max}_\Theta(t\setminus\{x_1\}),\mathsf{max}_\Theta(t))=\mathsf{max}_\Theta(t\setminus\{x_1\})$
$\langle\mathsf{max}_\Theta(t)\rangle\hookrightarrow Stat23([Stat29, Stat4])\Rightarrow$ $\mathsf{smaller}_\Theta(\mathsf{max}_\Theta(t),\mathsf{max}_\Theta(t\setminus\{x_1\}))=\mathsf{max}_\Theta(t)$
$\langle\mathsf{max}_\Theta(t\setminus\{x_1\}),\mathsf{max}_\Theta(t)\rangle\hookrightarrow Tlinear\_order\_5(\langle Stat23a\rangle)\Rightarrow$ false; Discharge $\Rightarrow$ $Stat32:\ \mathsf{max}_\Theta(t)=\mathsf{max}_\Theta(t\setminus\{x_1\})$
EQUAL $\Rightarrow$ $Stat33:\ x_1\lhd\mathsf{max}_\Theta(t)$

$\langle$*Stat32, Stat33, Stat13, Stat4*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat34* : $\mathsf{max}_\Theta(\mathsf{t}\setminus\{\mathsf{x}_1\}) \lhd \mathsf{x}_1$

-- Hence the maximum of $t\setminus\{x_1\}$ must be smaller than x_1, and therefore x_1 turns out to be the maximum of t. But this leads to a contradiction too; hence we conclude that the desired statement holds.

Suppose $\Rightarrow$ *Stat35* : $\neg\langle\forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y, x_1) = y\rangle$
$\langle y_3\rangle \hookrightarrow$ *Stat35*($\langle$*Stat35*$\rangle$) $\Rightarrow$ *Stat36* : $y_3 \in t \;\&\; \mathsf{smaller}_\Theta(y_3, x_1) \neq y_3$
$\langle y_3, x_1\rangle \hookrightarrow$ *Tlinear_order_7*([*Stat36, Stat4*]) $\Rightarrow$ $\mathsf{smaller}_\Theta(y_3, x_1) = x_1$
EQUAL $\langle$*Stat36*$\rangle$ $\Rightarrow$ $x_1 \neq y_3$
Suppose $\Rightarrow$ *Stat37* : $y_3 \in t\setminus\{x_1\}$
$\langle y_3\rangle \hookrightarrow$ *Stat16* $\Rightarrow$ *Stat37a* : $\mathsf{smaller}_\Theta(y_3, \mathsf{max}_\Theta(t\setminus\{x_1\})) = y_3$
Use_def (le_Θ) $\Rightarrow$ *Stat38* : $\mathsf{le}_\Theta(\mathsf{max}_\Theta(t\setminus\{x_1\}), x_1)$
Use_def ($\mathsf{smaller}_\Theta$) $\Rightarrow$ *Stat39* : $\mathsf{smaller}_\Theta(y_3, \mathsf{max}_\Theta(t\setminus\{x_1\})) = \textbf{if } y_3 \notin s \vee \mathsf{max}_\Theta(t\setminus\{x_1\}) \notin s \textbf{ then } s \textbf{ else if } y_3 \lhd \mathsf{max}_\Theta(t\setminus\{x_1\}) \textbf{ then } y_3 \textbf{ else } \mathsf{max}_\Theta(t\setminus\{x_1\}) \textbf{ fi fi}$
$\langle$*Stat39, Stat37, Stat4, Stat13, Stat37a*$\rangle$ ELEM $\Rightarrow$ $y_3 \lhd \mathsf{max}_\Theta(t\setminus\{x_1\}) \vee y_3 = \mathsf{max}_\Theta(t\setminus\{x_1\})$
Use_def (le_Θ) $\Rightarrow$ *Stat40* : $\mathsf{le}_\Theta(y_3, \mathsf{max}_\Theta(t\setminus\{x_1\}))$
$\langle y_3, \mathsf{max}_\Theta(t\setminus\{x_1\}), x_1\rangle \hookrightarrow$ *Tlinear_order_1*($\langle$*Stat38, Stat40, Stat37, Stat4, Stat13*$\rangle$) $\Rightarrow$ $\mathsf{le}_\Theta(y_3, x_1)$
Use_def (le_Θ) $\Rightarrow$ *Stat41* : $y_3 \lhd x_1$
Use_def ($\mathsf{smaller}_\Theta$) $\Rightarrow$ *Stat42* : $\mathsf{smaller}_\Theta(y_3, x_1) = \textbf{if } y_3 \notin s \vee x_1 \notin s \textbf{ then } s \textbf{ else if } y_3 \lhd x_1 \textbf{ then } y_3 \textbf{ else } x_1 \textbf{ fi fi}$
$\langle$*Stat42, Stat37, Stat4, Stat41, Stat36*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $y_3 = x_1$
$\langle$*Stat36*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat43* : $\langle\forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y, x_1) = y\rangle$
Suppose $\Rightarrow$ *Stat43a* : $\mathsf{max}_\Theta(t) \neq x_1$
Suppose $\Rightarrow$ *Stat44* : $x_1 \notin t \cap \mathsf{ubs}_\Theta(t)$
$\langle$*Stat44, Stat4*$\rangle$ ELEM $\Rightarrow$ $x_1 \notin \mathsf{ubs}_\Theta(t)$
Use_def (ubs_Θ) $\Rightarrow$ *Stat45* : $x_1 \notin \{x \in s \,|\, \langle\forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y, x) = y\rangle\}$
$\langle\,\rangle \hookrightarrow$ *Stat45*([*Stat4, Stat43*]) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat46* : $x_1 \in t \cap \mathsf{ubs}_\Theta(t)$
Suppose $\Rightarrow$ *Stat48* : $\mathsf{max}_\Theta(t) = s$
$\langle$*Stat4, Stat27, Stat48*$\rangle$ ELEM $\Rightarrow$ $x_1 \notin t \cap \mathsf{ubs}_\Theta(t)$
$\langle$*Stat46*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat49* : $\mathsf{max}_\Theta(t) \in t \cap \mathsf{ubs}_\Theta(t)$
Use_def (ubs_Θ) $\Rightarrow$ *Stat50* : $\mathsf{max}_\Theta(t) \in \{x \in s \,|\, \langle\forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y, x) = y\rangle\}$
$\langle\,\rangle \hookrightarrow$ *Stat30*($\langle$*Stat50*$\rangle$) $\Rightarrow$ *Stat51* : $\langle\forall y \in t \cap s \,|\, \mathsf{smaller}_\Theta(y, \mathsf{max}_\Theta(t)) = y\rangle$
$\langle x_1\rangle \hookrightarrow$ *Stat51*([*Stat51, Stat4*]) $\Rightarrow$ $\mathsf{smaller}_\Theta(x_1, \mathsf{max}_\Theta(t)) = x_1$
$\langle \mathsf{max}_\Theta(t)\rangle \hookrightarrow$ *Stat43*([*Stat49, Stat4*]) $\Rightarrow$ $\mathsf{smaller}_\Theta(\mathsf{max}_\Theta(t), x_1) = \mathsf{max}_\Theta(t)$
$\langle x_1, \mathsf{max}_\Theta(t)\rangle \hookrightarrow$ *Tlinear_order_5*($\langle$*Stat43a*$\rangle$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat52* : $\mathsf{max}_\Theta(t) = x_1$
EQUAL $\Rightarrow$ *Stat53* : $x_1 \lhd \mathsf{max}_\Theta(t)$
$\langle$*Stat52, Stat53, Stat4*$\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY linear_order

THEORY linear_order$(s, X \lhd Y)$
$\langle \forall x \in s, y \in s, z \in s \mid x \lhd y \;\&\; y \lhd z \rightarrow x \lhd z \rangle$
$\langle \forall x \in s \mid \neg x \lhd x \rangle$
$\langle \forall x \in s, y \in s \mid x \lhd y \vee x = y \vee y \lhd x \rangle$
$\Rightarrow (le_\Theta, smaller_\Theta, ubs_\Theta, max_\Theta, lub_\Theta)$
$\langle \forall x, y \mid le_\Theta(x, y) = x \lhd y \vee x = y \rangle$
$\langle \forall x, y \mid smaller_\Theta(x, y) = \textbf{if } x \notin s \vee y \notin s \textbf{ then } s \textbf{ else if } x \lhd y \textbf{ then } x \textbf{ else } y \textbf{ fi fi} \rangle$
$\langle \forall u, v, w \mid \{u, v, w\} \subseteq s \;\&\; le_\Theta(u, v) \;\&\; le_\Theta(v, w) \rightarrow le_\Theta(u, w) \rangle$
$\langle \forall u, v \mid \{u, v\} \subseteq s \;\&\; le_\Theta(u, v) \;\&\; le_\Theta(v, u) \rightarrow u = v \rangle$
$\langle \forall u, v \mid \{u, v\} \subseteq s \rightarrow le_\Theta(u, v) \vee le_\Theta(v, u) \rangle$
$\langle \forall u, v, w \mid \{u, v, w\} \subseteq s \rightarrow \big(le_\Theta(u, v) \;\&\; le_\Theta(u, w)\big) \vee \big(le_\Theta(v, u) \;\&\; le_\Theta(v, w)\big) \vee \big(le_\Theta(w, u) \;\&\; le_\Theta(w, v)\big) \rangle$
$\langle \forall x, y \mid smaller_\Theta(x, y) = smaller_\Theta(y, x) \rangle$
$\langle \forall x \mid smaller_\Theta(x, s) = s \;\&\; smaller_\Theta(s, x) = s \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq s \rightarrow smaller_\Theta(x, y) \in \{x, y\} \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq s \;\&\; x \lhd y \vee x = y \rightarrow smaller_\Theta(x, y) = x \;\&\; smaller_\Theta(y, x) = x \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq s \rightarrow \big(smaller_\Theta(x, y) = x \leftrightarrow x \lhd y \vee x = y\big) \rangle$
$\langle \forall x, y, z \mid smaller_\Theta\big(x, (smaller_\Theta)(y, z)\big) = smaller_\Theta\big((smaller_\Theta)(x, y), z\big) \rangle$
$\langle \forall x \in s \cup \{s\}, y \in s \cup \{s\} \mid smaller_\Theta(x, y) \in s \cup \{s\} \rangle$
$\langle \forall t \mid ubs_\Theta(t) = \{x \in s \mid \langle \forall y \in t \cap s \mid smaller_\Theta(y, x) = y \rangle \} \rangle$
$\langle \forall t \mid max_\Theta(t) = \mathbf{arb}(\{s\} \cup t \cap ubs_\Theta(t)) \rangle$
$\langle \forall t \mid lub_\Theta(t) = \mathbf{arb}(\{s\} \cup \{x \in ubs_\Theta(t) \mid ubs_\Theta(t) \subseteq ubs_\Theta(\{x\})\}) \rangle$
$ubs_\Theta(\emptyset) = s \;\&\; max_\Theta(\emptyset) = s$
$\langle \forall t \mid ubs_\Theta(t) \subseteq s \;\&\; ubs_\Theta(t) = ubs_\Theta(t \cap s) \;\&\; max_\Theta(t) = max_\Theta(t \cap s) \rangle$
$\langle \forall t, x \mid Finite(t) \;\&\; x \in t \;\&\; t \subseteq s \rightarrow max_\Theta(t) \in t \;\&\; x = max_\Theta(t) \vee x \lhd max_\Theta(t) \rangle$
END linear_order

13 Various other inductive principles

-- For subsequent use, we reformulate a few special cases of the principle of transfinite definition as THEORYs that can be applied internally within the proofs of theorems.

THEORY transfinite_definition_0_params$\big(g(x), h(x)\big)$
END transfinite_definition_0_params

ENTER_THEORY transfinite_definition_0_params

DEF transfinite_definition_0_params $\cdot\, 0a$. $\quad f_\Theta(X) \quad =_{\text{Def}} \quad g\Big(\{h(f_\Theta(t)) : t \in X\}\Big)$

Theorem 478 (transfinite_definition_0_params$_1$) $f_\Theta(X) = g\Big(\{h(f_\Theta(t)) : t \in X\}\Big)$. PROOF:

Suppose_not$(x) \Rightarrow \quad f_\Theta(x) \neq g\Big(\{h(f_\Theta(t)) : t \in x\}\Big)$

Use_def$(f_\Theta) \Rightarrow \quad f_\Theta(x) = g\Big(\{h(f_\Theta(t)) : t \in x\}\Big)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY transfinite_definition_0_params$_1$

THEORY transfinite_definition_0_params$(g(x), h(x))$
$\Rightarrow (f_\Theta)$
$\langle \forall x \mid f_\Theta(x) = g\Big(\{h(f_\Theta(t)) : t \in x\}\Big)\rangle$
END transfinite_definition_0_params

THEORY transfinite_definition_1_params$(g(x, a), h(x, a))$
END transfinite_definition_1_params

ENTER_THEORY transfinite_definition_1_params

DEF transfinite_definition_1_params $\cdot\, 0a$. $\quad f_\Theta(X, Y) \quad =_{\text{Def}} \quad g\Big(\{h(f_\Theta(t), Y) : t \in X\}, Y\Big)$

Theorem 479 (transfinite_definition_1_params$_1$) $f_\Theta(X, A) = g\Big(\{h(f_\Theta(t, A)) : t \in X\}, A\Big)$. PROOF:

Suppose_not$(x, a) \Rightarrow \quad f_\Theta(x, a) \neq g\Big(\{h(f_\Theta(t), a) : t \in x\}, a\Big)$

Use_def$(f_\Theta) \Rightarrow \quad f_\Theta(x, a) = g\Big(\{h(f_\Theta(t), a) : t \in x\}, a\Big)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY transfinite_definition_0_params$_1$

THEORY $\mathsf{transfinite\_definition\_0\_params}(\mathsf{g}(\mathsf{x},\mathsf{a}),\mathsf{h}(\mathsf{x},\mathsf{a}))$
$\Rightarrow (\mathsf{f}_\Theta)$
$\langle\forall \mathsf{x},\mathsf{a} \,|\, \mathsf{f}_\Theta(\mathsf{x},\mathsf{a}) = \mathsf{g}\Big(\{\mathsf{h}(\mathsf{f}_\Theta(\mathsf{t}),\mathsf{a}) : \mathsf{t}\in\mathsf{x}\},\mathsf{a}\Big)\rangle$
END $\mathsf{transfinite\_definition\_0\_params}$

-- Our next proof establishes a first, purely set-theoretic form of the well-known Zorn's Lemma. We prove that if t is any collection of sets such that every subfamily of t linearly ordered by inclusion admits an upper bound in t, then t has an element maximal for inclusion, i. e. not properly included in any other element of t.

Theorem 480 (335) $\langle\forall x\subseteq T \,|\, \langle\forall u\in x, v\in x \,|\, u\supseteq v \vee v\supseteq u\rangle \rightarrow \langle\exists w\in T,\forall y\in x \,|\, w\supseteq y\rangle\rangle \rightarrow \langle\exists y\in T,\forall x\in T \,|\, \neg(x\supseteq y \,\&\, x\neq y)\rangle$. PROOF:

Suppose_not(t) $\Rightarrow$ *Stat1* :
$\langle\forall x\subseteq t \,|\, \langle\forall u\in x, v\in x \,|\, u\supseteq v \vee v\supseteq u\rangle \longrightarrow \langle\exists w\in t,\forall y\in x \,|\, w\supseteq y\rangle\rangle \,\&\, \mathit{Stat2}: \neg\langle\exists y\in t,\forall x\in t \,|\, \neg(x\supseteq y \,\&\, x\neq y)\rangle$

-- For supposing the contrary, we can define a mapping of t into t which sends each element of t into a strictly larger element, and also a mapping of every subset of t linearly ordered by inclusion into a n upper bound for it in t.

Loc_def $\Rightarrow$ $\mathsf{larger} = \{[x,\mathbf{arb}(\{y\in t \,|\, y\supseteq x \,\&\, y\neq x\})] : x\in t\}$
APPLY $\langle\,\rangle$ fcn_symbol$(\mathsf{f}(x)\mapsto \mathbf{arb}(\{y\in t \,|\, y\supseteq x \,\&\, y\neq x\}), \mathsf{g}\mapsto \mathsf{larger}, \mathsf{s}\mapsto t)$ $\Rightarrow$
$\mathsf{Svm}(\mathsf{larger}) \,\&\, \mathit{Stat3}: \langle\forall x \,|\, \mathsf{larger}\upharpoonright x = \textbf{if } x\in t \textbf{ then } \mathbf{arb}(\{y\in t \,|\, y\supseteq x \,\&\, y\neq x\}) \textbf{ else } \emptyset \textbf{ fi}\rangle$
Loc_def $\Rightarrow$ $\mathsf{upper\_bound} = \{[x,\mathbf{arb}(\{y\in t \,|\, \langle\forall u\in x \,|\, y\supseteq u\rangle\})] : x\in\mathcal{P}t\}$
APPLY $\langle\,\rangle$ fcn_symbol$\Big(\mathsf{f}(x)\mapsto \mathbf{arb}\Big(\{y\in t \,|\, \langle\forall u\in x \,|\, y\supseteq u\rangle\}\Big), \mathsf{g}\mapsto \mathsf{upper_bound}, \mathsf{s}\mapsto\mathcal{P}t\Big)$ $\Rightarrow$
$\mathsf{Svm}(\mathsf{upper\_bound}) \,\&\, \mathit{Stat4}: \langle\forall x \,|\, \mathsf{upper\_bound}\upharpoonright x = \textbf{if } x\in\mathcal{P}t \textbf{ then } \mathbf{arb}(\{y\in t \,|\, \langle\forall u\in x \,|\, y\supseteq u\rangle\}) \textbf{ else } \emptyset \textbf{ fi}\rangle$
Loc_def $\Rightarrow$ $s = \bigcup t$

-- Now we use the functions 'upper_bound' and 'larger' to introduce the following (recursively defined) function, which we will then show maps each ordinal into t, and is strictly monotone increasing.

APPLY $\langle \mathsf{f}_\Theta : \mathsf{Zo}\rangle$ transfinite_definition_0_params$(\mathsf{g}(x)\mapsto \mathsf{larger}\upharpoonright(\mathsf{upper_bound}\upharpoonright x), \mathsf{h}(x)\mapsto x)$ $\Rightarrow$
$\mathit{Stat5}: \langle\forall x \,|\, \mathsf{Zo}(x) = \mathsf{larger}\upharpoonright\big(\mathsf{upper\_bound}\upharpoonright\{\mathsf{Zo}(y) : y\in x\}\big)\rangle$
Suppose $\Rightarrow$ $\mathit{Stat6}: \langle\exists x \,|\, \mathcal{O}(x) \,\&\, \mathsf{Zo}(x)\notin t \vee \langle\exists u\in x \,|\, \neg\big(\mathsf{Zo}(x)\supseteq\mathsf{Zo}(u) \,\&\, \mathsf{Zo}(x)\neq\mathsf{Zo}(u)\big)\rangle\rangle$

-- For if there exists some counterexample to this last assertion, then by transfinite induction there exists a smallest such counterexample c.

$\langle d \rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad \mathcal{O}(d) \,\&\, Zo(d) \notin t \vee \langle \exists u \in d \mid \neg(Zo(d) \supseteq Zo(u) \,\&\, Zo(d) \neq Zo(u)) \rangle$

APPLY $\langle mt_\Theta : c \rangle$ transfinite_induction$\Big(n \mapsto d, P(x) \mapsto \Big(\mathcal{O}(x) \,\&\, Zo(x) \notin t \vee \langle \exists u \in x \mid \neg(Zo(x) \supseteq Zo(u) \,\&\, Zo(x) \neq Zo(u)) \rangle\Big)\Big) \Rightarrow$

$\mathit{Stat6a}: \langle \forall x \mid \big(\mathcal{O}(c) \,\&\, Zo(c) \notin t \vee \langle \exists u \in c \mid \neg(Zo(c) \supseteq Zo(u) \,\&\, Zo(c) \neq Zo(u)) \rangle\big) \,\&\, \Big(x \in c \to \neg\big(\mathcal{O}(x) \,\&\, Zo(x) \notin t \vee \langle \exists u \in x \mid \neg(Zo(x) \supseteq Zo(u) \,\&\, Zo(x) \neq Zo(u)) \rangle\big)\Big) \rangle$

$\langle \emptyset \rangle \hookrightarrow \mathit{Stat6a} \Rightarrow \quad \mathcal{O}(c) \,\&\, Zo(c) \notin t \vee \langle \exists u \in c \mid \neg(Zo(c) \supseteq Zo(u) \,\&\, Zo(c) \neq Zo(u)) \rangle$

Suppose $\Rightarrow \quad \mathit{Stat7a}: \neg\langle \forall x \in c \mid \neg\big(\mathcal{O}(x) \,\&\, Zo(x) \notin t \vee \langle \exists u \in x \mid \neg(Zo(x) \supseteq Zo(u) \,\&\, Zo(x) \neq Zo(u)) \rangle\big) \rangle$

$\langle x_0 \rangle \hookrightarrow \mathit{Stat7a} \Rightarrow \quad x_0 \in c \,\&\, \mathcal{O}(x_0) \,\&\, Zo(x_0) \notin t \vee \langle \exists u \in x_0 \mid \neg(Zo(x_0) \supseteq Zo(u) \,\&\, Zo(x_0) \neq Zo(u)) \rangle$

$\langle x_0 \rangle \hookrightarrow \mathit{Stat6a} \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \mathit{Stat7}: \langle \forall x \in c \mid \neg\big(\mathcal{O}(x) \,\&\, Zo(x) \notin t \vee \langle \exists u \in x \mid \neg(Zo(x) \supseteq Zo(u) \,\&\, Zo(x) \neq Zo(u)) \rangle\big) \rangle$

-- For this minimal counterexample c, the set $\{Zo(y) : y \in c\}$ must be a collection of subsets of t and must be linearly ordered by inclusion.

Suppose $\Rightarrow \quad \mathit{Stat8}: t \not\supseteq \{Zo(y) : y \in c\}$

$\langle x_1 \rangle \hookrightarrow \mathit{Stat8} \Rightarrow \quad \mathit{Stat9}: x_1 \in \{Zo(y) : y \in c\} \,\&\, x_1 \notin t$

$\langle y_1 \rangle \hookrightarrow \mathit{Stat9} \Rightarrow \quad y_1 \in c \,\&\, x_1 = Zo(y_1)$

$\langle y_1 \rangle \hookrightarrow \mathit{Stat7} \Rightarrow \quad \neg(\mathcal{O}(y_1) \,\&\, Zo(y_1) \notin t)$

$\langle c, y_1 \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(y_1)$

$\langle \mathit{Stat9} \rangle$ ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad t \supseteq \{Zo(y) : y \in c\}$

Suppose $\Rightarrow \quad \mathit{Stat10}: \neg\langle \forall u \in \{Zo(y) : y \in c\}, v \in \{Zo(y) : y \in c\} \mid u \supseteq v \vee v \supseteq u \rangle$

$\langle a, b \rangle \hookrightarrow \mathit{Stat10} \Rightarrow \quad \mathit{Stat11}: a, b \in \{Zo(y) : y \in c\} \,\&\, \neg(a \supseteq b \vee b \supseteq a)$

$\langle o_1, o_2 \rangle \hookrightarrow \mathit{Stat11} \Rightarrow \quad \mathit{Stat11a}: o_1, o_2 \in c \,\&\, \neg(Zo(o_1) \supseteq Zo(o_2) \vee Zo(o_2) \supseteq Zo(o_1))$

$\langle c, o_1 \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(o_1)$

$\langle c, o_2 \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(o_2)$

$\langle o_1 \rangle \hookrightarrow \mathit{Stat7} \Rightarrow \quad \mathit{Stat12}: \neg\langle \exists u \in o_1 \mid \neg(Zo(o_1) \supseteq Zo(u) \,\&\, Zo(o_1) \neq Zo(u)) \rangle$

$\langle o_2 \rangle \hookrightarrow \mathit{Stat7} \Rightarrow \quad \mathit{Stat13}: \neg\langle \exists u \in o_2 \mid \neg(Zo(o_2) \supseteq Zo(u) \,\&\, Zo(o_2) \neq Zo(u)) \rangle$

$\langle o_1, o_2 \rangle \hookrightarrow \mathit{T28} \Rightarrow \quad o_1 \in o_2 \vee o_2 \in o_1 \vee o_1 = o_2$

Suppose $\Rightarrow \quad o_1 = o_2$

EQUAL $\Rightarrow \quad Zo(o_1) = Zo(o_2)$

$\langle \mathit{Stat11a} \rangle$ ELEM $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad o_1 \in o_2 \vee o_2 \in o_1$

Suppose $\Rightarrow \quad o_2 \in o_1$

$\langle o_2 \rangle \hookrightarrow \mathit{Stat12} \Rightarrow \quad$ false; Discharge $\Rightarrow \quad o_1 \in o_2$

$\langle o_1 \rangle \hookrightarrow \mathit{Stat13} \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \langle \forall u \in \{Zo(y) : y \in c\}, v \in \{Zo(y) : y \in c\} \mid u \supseteq v \vee v \supseteq u \rangle$

-- Thus, by definition, $\{Zo(y) : y \in c\}$ must have an upper bound cb which is a subset of t, and therefore by the axiom of choice upper_bound$\restriction \{Zo(z_1) : z_1 \in c\}$ must belong to t and include every element of $\{Zo(y) : y \in c\}$.

$\langle \{Zo(z_1) : z_1 \in c\} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathit{Stat14}: \langle \exists w \in t, \forall y \in \{Zo(z_1) : z_1 \in c\} \mid w \supseteq y \rangle$

$\langle \mathsf{cb} \rangle \hookrightarrow Stat14 \Rightarrow$ $\mathsf{cb} \in \mathsf{t} \;\&\; \langle \forall \mathsf{y} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{cb} \supseteq \mathsf{y} \rangle$
$\langle \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \rangle \hookrightarrow Stat4 \Rightarrow$ $\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} =$
$\quad$ **if** $\{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \in \mathcal{P}\mathsf{t}$ **then** $\mathbf{arb}(\{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\})$ **else** $\emptyset$ **fi**
Suppose $\Rightarrow$ $\{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \notin \mathcal{P}\mathsf{t}$
Use_def$(\mathcal{P}) \Rightarrow$ $Stat15 :\ \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \notin \{\mathsf{x} :\ \mathsf{x} \subseteq \mathsf{t}\}$
$\langle \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \rangle \hookrightarrow Stat15 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} = \mathbf{arb}(\{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\})$
Suppose $\Rightarrow$ $Stat16 :\ \{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\} = \emptyset$
$\langle \mathsf{cb} \rangle \hookrightarrow Stat16 \Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\} \neq \emptyset$
$\langle \{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\} \rangle \hookrightarrow T0 \Rightarrow$ $\mathbf{arb}(\{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\}) \in$
$\quad \{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\}$
$\langle Stat14 \rangle$ ELEM $\Rightarrow$ $Stat17 :\ \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \in \{\mathsf{y} \in \mathsf{t} \mid \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{y} \supseteq \mathsf{u} \rangle\}$
$\langle\, \rangle \hookrightarrow Stat17 \Rightarrow$ $\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \in \mathsf{t} \;\&\; Stat18 :\ \langle \forall \mathsf{u} \in \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \mid \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \supseteq \mathsf{u} \rangle$

-- It follows by a second use of the axiom of choice that $\mathsf{larger} \restriction (\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}) = \mathsf{Zo}(\mathsf{c})$ is an element of t properly including every element of $\{\mathsf{Zo}(\mathsf{y}) :\ \mathsf{y} \in \mathsf{c}\}$. This refutes our earlier supposition, and so lets us conclude that Zo sends ordinals into t and is strictly monotone increasing.

$\langle \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \rangle \hookrightarrow Stat3 \Rightarrow$ $\mathsf{larger} \restriction (\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}) =$
$\quad \mathbf{arb}(\{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\})$
$\langle \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat19 :\ \neg$
$\quad \langle \forall \mathsf{x} \in \mathsf{t} \mid \neg(\mathsf{x} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{x} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}) \rangle$
$\langle \mathsf{cu} \rangle \hookrightarrow Stat19 \Rightarrow$ $\mathsf{cu} \in \mathsf{t} \;\&\; \mathsf{cu} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{cu} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}$
Suppose $\Rightarrow$ $Stat20 :\ \{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\} = \emptyset$
$\langle \mathsf{cu} \rangle \hookrightarrow Stat20 \Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\} \neq \emptyset$
$\langle \{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\} \rangle \hookrightarrow T0 \Rightarrow$
$\quad \mathbf{arb}(\{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\}) \in$
$\qquad \{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\}$
$\langle Stat17 \rangle$ ELEM $\Rightarrow$ $\mathsf{larger} \restriction (\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}) \in \{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\}$
$\langle \mathsf{c} \rangle \hookrightarrow Stat5 \Rightarrow$ $Stat21 :\ \mathsf{Zo}(\mathsf{c}) \in \{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{y} \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}\}$
$\langle\, \rangle \hookrightarrow Stat21 \Rightarrow$ $\mathsf{Zo}(\mathsf{c}) \in \mathsf{t} \;\&\; \mathsf{Zo}(\mathsf{c}) \supseteq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \;\&\; \mathsf{Zo}(\mathsf{c}) \neq \mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}$
ELEM $\Rightarrow$ $Stat22 :\ \langle \exists \mathsf{u} \in \mathsf{c} \mid \neg(\mathsf{Zo}(\mathsf{c}) \supseteq \mathsf{Zo}(\mathsf{u}) \;\&\; \mathsf{Zo}(\mathsf{c}) \neq \mathsf{Zo}(\mathsf{u})) \rangle$
$\langle \mathsf{cv} \rangle \hookrightarrow Stat22 \Rightarrow$ $\mathsf{cv} \in \mathsf{c} \;\&\; \neg(\mathsf{Zo}(\mathsf{c}) \supseteq \mathsf{Zo}(\mathsf{cv}) \;\&\; \mathsf{Zo}(\mathsf{c}) \neq \mathsf{Zo}(\mathsf{cv}))$
$\langle Stat19 \rangle$ ELEM $\Rightarrow$ $\mathsf{upper\_bound} \restriction \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\} \not\supseteq \mathsf{Zo}(\mathsf{cv})$
$\langle \mathsf{Zo}(\mathsf{cv}) \rangle \hookrightarrow Stat18 \Rightarrow$ $Stat23 :\ \mathsf{Zo}(\mathsf{cv}) \notin \{\mathsf{Zo}(\mathsf{z}_1) :\ \mathsf{z}_1 \in \mathsf{c}\}$
$\langle \mathsf{cv} \rangle \hookrightarrow Stat23 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat24 :\ \neg \langle \exists \mathsf{x} \mid \mathcal{O}(\mathsf{x}) \;\&\; \mathsf{Zo}(\mathsf{x}) \notin \mathsf{t} \vee \langle \exists \mathsf{u} \in \mathsf{x} \mid \neg(\mathsf{Zo}(\mathsf{x}) \supseteq \mathsf{Zo}(\mathsf{u}) \;\&\; \mathsf{Zo}(\mathsf{x}) \neq \mathsf{Zo}(\mathsf{u})) \rangle \rangle$

-- Thus Zo is a 1-1 map of all ordinals into the set t, a thing impossible. The easiest way of seeing this is to consider the restriction of Zo to an ordinal greater than the cardinality of t, for example to $\#\mathcal{P}t$; this can certainly have no 1-1 map into t, giving a contradiction which proves our assertion.

$\langle t\rangle \hookrightarrow T228 \Rightarrow \quad \#t \in \#\mathcal{P}t$
$\langle t\rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#t)$
$\langle \mathcal{P}t\rangle \hookrightarrow T121 \Rightarrow \quad \mathcal{O}(\#\mathcal{P}t)$
APPLY $\langle x_\Theta : o_3, y_\Theta : o_4\rangle$ fcn_symbol$(g \mapsto \{[x, Zo(x)] : x \in \#\mathcal{P}t\}, f(x) \mapsto Zo(x), s \mapsto \#\mathcal{P}t) \Rightarrow$
$\quad \mathbf{domain}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\}) = \#\mathcal{P}t \;\&\; \mathbf{range}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\}) = \{Zo(x) : x \in \#\mathcal{P}t\} \;\&\; (o_3, o_4 \in \#\mathcal{P}t \;\&\; Zo(o_3) = Zo(o_4) \;\&\; o_3 \neq o_4) \vee 1\text{–}1(\{[x, Zo(x)] : x \in \#$
Suppose $\Rightarrow \quad \neg 1\text{–}1(\{[x, Zo(x)] : x \in \#\mathcal{P}t\})$
ELEM $\Rightarrow \quad o_3, o_4 \in \#\mathcal{P}t \;\&\; Zo(o_3) = Zo(o_4) \;\&\; o_3 \neq o_4$
$\langle \#\mathcal{P}t, o_3\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(o_3)$
$\langle \#\mathcal{P}t, o_4\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(o_4)$
$\langle o_3, o_4\rangle \hookrightarrow T28 \Rightarrow \quad o_3 \in o_4 \vee o_4 \in o_3$
Suppose $\Rightarrow \quad o_3 \in o_4$
$\langle o_4\rangle \hookrightarrow Stat24 \Rightarrow \quad Stat25: \neg\langle \exists u \in o_4 \mid \neg(Zo(o_4) \supseteq Zo(u) \;\&\; Zo(o_4) \neq Zo(u))\rangle$
$\langle o_3\rangle \hookrightarrow Stat25 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad o_4 \in o_3$
$\langle o_3\rangle \hookrightarrow Stat24 \Rightarrow \quad Stat26: \neg\langle \exists u \in o_3 \mid \neg(Zo(o_3) \supseteq Zo(u) \;\&\; Zo(o_3) \neq Zo(u))\rangle$
$\langle o_4\rangle \hookrightarrow Stat26 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad 1\text{–}1(\{[x, Zo(x)] : x \in \#\mathcal{P}t\})$
Suppose $\Rightarrow \quad t \not\supseteq \mathbf{range}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\})$
ELEM $\Rightarrow \quad Stat27: t \not\supseteq \{Zo(x) : x \in \#\mathcal{P}t\}$
$\langle e\rangle \hookrightarrow Stat27 \Rightarrow \quad Stat28: e \in \{Zo(x) : x \in \#\mathcal{P}t\} \;\&\; e \notin t$
$\langle e_2\rangle \hookrightarrow Stat28 \Rightarrow \quad e_2 \in \#\mathcal{P}t \;\&\; e = Zo(e_2)$
$\langle \#\mathcal{P}t, e_2\rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(e_2)$
$\langle e_2\rangle \hookrightarrow Stat24 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad t \supseteq \mathbf{range}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\})$
$\langle \mathbf{range}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\}), t\rangle \hookrightarrow T144 \Rightarrow$
$\quad \#t \supseteq \#\mathbf{range}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\})$
$\langle \{[x, Zo(x)] : x \in \#\mathcal{P}t\}\rangle \hookrightarrow T131 \Rightarrow \quad \#t \supseteq \#\mathbf{domain}(\{[x, Zo(x)] : x \in \#\mathcal{P}t\})$
EQUAL $\Rightarrow \quad \#t \supseteq \#\#\mathcal{P}t$
$\langle \mathcal{P}t\rangle \hookrightarrow T140 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- The following corollary of the preceding theorem shows that if s is any member of a family t of sets sets satisfying the hypotheses of that theorem, then s is contained in an element of t maximal in t.

Theorem 481 (336) $\langle \forall x \subseteq T \mid \langle \forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u\rangle \rightarrow \langle \exists w \in T, \forall y \in x \mid w \supseteq y\rangle\rangle \rightarrow$
$\langle \forall u \in T, \exists y \in T \mid y \supseteq u \;\&\; \langle \forall x \in T \mid \neg(x \supseteq y \;\&\; x \neq y)\rangle\rangle$. PROOF:

Suppose_not$(t) \Rightarrow \quad Stat1:$
$\quad \langle \forall x \subseteq T \mid \langle \forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u\rangle \rightarrow \langle \exists w \in T, \forall y \in x \mid w \supseteq y\rangle\rangle \;\&\; Stat2: \neg$

$\langle\forall u \in t, \exists y \in t \mid y \supseteq u \;\&\; \langle\forall x \in t \mid \neg(x \supseteq y \;\&\; x \neq y)\rangle\rangle$

-- For suppose that $u \in t$ contradicts the conclusion of our theorem, and consider the subset tt of all elements of t which contain u. It is clear that every collection of subsets of tt linearly ordered by inclusion has an upper bound in tt, and so by the preceding theorem tt contains an element ma maximal for inclusions among all the sets in tt.

$\langle u\rangle \hookrightarrow Stat2 \Rightarrow\quad u \in t \;\&\; Stat3:\; \neg\langle\exists y \in t \mid y \supseteq u \;\&\; \langle\forall x \in t \mid \neg(x \supseteq y \;\&\; x \neq y)\rangle\rangle$
Loc_def $\Rightarrow\quad tt = \{x \in t \mid x \supseteq u\}$
Suppose $\Rightarrow\quad Stat4:\; t \not\supseteq tt$
$\langle c\rangle \hookrightarrow Stat4 \Rightarrow\quad c \notin t \;\&\; Stat5:\; c \in \{x \in t \mid x \supseteq u\}$
$\langle\;\rangle \hookrightarrow Stat5 \Rightarrow\quad$ false; Discharge $\Rightarrow\quad t \supseteq tt$
Suppose $\Rightarrow\quad Stat6:\; \neg\langle\forall x \subseteq tt \mid \langle\forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u\rangle \rightarrow \langle\exists w \in tt, \forall y \in x \mid w \supseteq y\rangle\rangle$
$\langle d\rangle \hookrightarrow Stat6 \Rightarrow\quad d \subseteq tt \;\&\; \langle\forall u \in d, v \in d \mid u \supseteq v \vee v \supseteq u\rangle \;\&\; Stat7:\; \neg\langle\exists w \in tt, \forall y \in d \mid w \supseteq y\rangle$
$\langle d\rangle \hookrightarrow Stat1 \Rightarrow\quad Stat8:\; \langle\exists w \in t, \forall y \in d \mid w \supseteq y\rangle$
$\langle wd\rangle \hookrightarrow Stat8 \Rightarrow\quad wd \in t \;\&\; Stat9:\; \langle\forall y \in d \mid wd \supseteq y\rangle$

-- Since $u \in tt$, d cannot be null, from which it is easily seen that wd must contain u, and so $wd \in tt$, Thu it follows by The0rem 332a that tt has an element wa maximal (for inclusion) in

Suppose $\Rightarrow\quad u \notin tt$
ELEM $\Rightarrow\quad Stat10:\; u \notin \{x \in t \mid x \supseteq u\}$
$\langle\;\rangle \hookrightarrow Stat10 \Rightarrow\quad$ false; Discharge $\Rightarrow\quad u \in tt$
Suppose $\Rightarrow\quad d = \emptyset$
$\langle u\rangle \hookrightarrow Stat7 \Rightarrow\quad Stat11:\; \neg\langle\forall y \in d \mid u \supseteq y\rangle$
$\langle a\rangle \hookrightarrow Stat11 \Rightarrow\quad$ false; Discharge $\Rightarrow\quad Stat12:\; d \neq \emptyset$
$\langle b\rangle \hookrightarrow Stat12 \Rightarrow\quad b \in d$
$\langle b\rangle \hookrightarrow Stat9 \Rightarrow\quad wd \supseteq b$
ELEM $\Rightarrow\quad Stat13:\; b \in \{x \in t \mid x \supseteq u\}$
$\langle\;\rangle \hookrightarrow Stat13 \Rightarrow\quad wd \supseteq u$
Suppose $\Rightarrow\quad wd \notin tt$
ELEM $\Rightarrow\quad Stat14:\; wd \notin \{x \in t \mid x \supseteq u\}$
$\langle\;\rangle \hookrightarrow Stat14 \Rightarrow\quad$ false; Discharge $\Rightarrow\quad wd \in tt$
$\langle wd\rangle \hookrightarrow Stat7 \Rightarrow\quad$ false; Discharge $\Rightarrow\quad \langle\forall x \subseteq tt \mid \langle\forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u\rangle \rightarrow \langle\exists w \in tt, \forall y \in x \mid w \supseteq y\rangle\rangle$
$\langle tt\rangle \hookrightarrow T335 \Rightarrow\quad Stat15:\; \langle\exists y \in tt, \forall x \in tt \mid \neg(x \supseteq y \;\&\; x \neq y)\rangle$
$\langle ma\rangle \hookrightarrow Stat15 \Rightarrow\quad ma \in tt \;\&\; Stat16:\; \langle\forall x \in tt \mid \neg(x \supseteq ma \;\&\; x \neq ma)\rangle$

-- But it is easily seen that ma is maximal in the whole collection t, and so our theorem is proved.

$\langle \mathsf{ma} \rangle \hookrightarrow Stat3 \Rightarrow \quad \neg\big(\mathsf{ma} \supseteq \mathsf{u} \ \& \ \langle \forall \mathsf{x} \in \mathsf{t} \,|\, \neg(\mathsf{x} \supseteq \mathsf{ma} \ \& \ \mathsf{x} \neq \mathsf{ma}) \rangle\big)$
ELEM $\Rightarrow$ $\quad Stat17: \ \mathsf{ma} \in \{\mathsf{x} \in \mathsf{t} \,|\, \mathsf{x} \supseteq \mathsf{u}\}$
$\langle\ \rangle \hookrightarrow Stat17 \Rightarrow \quad \mathsf{ma} \supseteq \mathsf{u}$
ELEM $\Rightarrow$ $\quad Stat18: \ \neg\langle \forall \mathsf{x} \in \mathsf{t} \,|\, \neg(\mathsf{x} \supseteq \mathsf{ma} \ \& \ \mathsf{x} \neq \mathsf{ma}) \rangle$
$\langle \mathsf{e} \rangle \hookrightarrow Stat18 \Rightarrow \quad \mathsf{e} \in \mathsf{t} \ \& \ \mathsf{e} \supseteq \mathsf{ma} \ \& \ \mathsf{e} \neq \mathsf{ma}$
ELEM $\Rightarrow$ $\quad \mathsf{e} \supseteq \mathsf{u}$
Suppose $\Rightarrow$ $\quad \mathsf{e} \notin \mathsf{tt}$
ELEM $\Rightarrow$ $\quad Stat19: \ \mathsf{e} \notin \{\mathsf{x} \in \mathsf{t} \,|\, \mathsf{x} \supseteq \mathsf{u}\}$
$\langle\ \rangle \hookrightarrow Stat19 \Rightarrow \quad$ false; Discharge $\Rightarrow$ $\quad \mathsf{e} \in \mathsf{tt}$
$\langle \mathsf{e} \rangle \hookrightarrow Stat16 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- Next we note a special case common in applications of Theorem 336, namely that in which the union of any linearly ordered collection of elements of t is a subset of t.

Theorem 482 (337) $\langle \forall \mathsf{x} \subseteq T \,|\, \langle \forall \mathsf{u} \in \mathsf{x}, \mathsf{v} \in \mathsf{x} \,|\, \mathsf{u} \supseteq \mathsf{v} \vee \mathsf{v} \supseteq \mathsf{u} \rangle \rightarrow \bigcup \mathsf{x} \in T \rangle \rightarrow \langle \forall \mathsf{u} \in T, \exists \mathsf{y} \in T \,|\, \mathsf{y} \supseteq \mathsf{u} \ \& \ \langle \forall \mathsf{x} \in T \,|\, \neg(\mathsf{x} \supseteq \mathsf{y} \ \& \ \mathsf{x} \neq \mathsf{y}) \rangle \rangle$. PROOF:

Suppose_not(t) $\Rightarrow$ $\quad Stat1:$
$\langle \forall \mathsf{x} \subseteq \mathsf{t} \,|\, \langle \forall \mathsf{u} \in \mathsf{x}, \mathsf{v} \in \mathsf{x} \,|\, \mathsf{u} \supseteq \mathsf{v} \vee \mathsf{v} \supseteq \mathsf{u} \rangle \rightarrow \bigcup \mathsf{x} \in \mathsf{t} \rangle \ \& \ \neg\langle \forall \mathsf{u} \in \mathsf{t}, \exists \mathsf{y} \in \mathsf{t} \,|\, \mathsf{y} \supseteq \mathsf{u} \ \& \ \langle \forall \mathsf{x} \in \mathsf{t} \,|\, \neg(\mathsf{x} \supseteq \mathsf{y} \ \& \ \mathsf{x} \neq \mathsf{y}) \rangle \rangle$

-- For given any subcollection of t linearly ordered by inclusion, $\bigcup$t plainly includes all the sets in t, and so our present assertion follows immediately from the preceding theorem.

Suppose $\Rightarrow$ $\quad Stat2: \ \neg\langle \forall \mathsf{x} \subseteq \mathsf{t} \,|\, \langle \forall \mathsf{u} \in \mathsf{x}, \mathsf{v} \in \mathsf{x} \,|\, \mathsf{u} \supseteq \mathsf{v} \vee \mathsf{v} \supseteq \mathsf{u} \rangle \rightarrow \langle \exists \mathsf{w} \in \mathsf{t}, \forall \mathsf{y} \in \mathsf{x} \,|\, \mathsf{w} \supseteq \mathsf{y} \rangle \rangle$
$\langle \mathsf{a} \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{a} \subseteq \mathsf{t} \ \& \ \langle \forall \mathsf{u} \in \mathsf{a}, \mathsf{v} \in \mathsf{a} \,|\, \mathsf{u} \supseteq \mathsf{v} \vee \mathsf{v} \supseteq \mathsf{u} \rangle \ \& \ Stat3: \ \neg\langle \exists \mathsf{w} \in \mathsf{t}, \forall \mathsf{y} \in \mathsf{a} \,|\, \mathsf{w} \supseteq \mathsf{y} \rangle$
$\langle \mathsf{a} \rangle \hookrightarrow Stat1 \Rightarrow \quad \bigcup \mathsf{a} \in \mathsf{t}$
$\langle \bigcup \mathsf{a} \rangle \hookrightarrow Stat3 \Rightarrow \quad Stat4: \ \neg\langle \forall \mathsf{y} \in \mathsf{a} \,|\, \bigcup \mathsf{a} \supseteq \mathsf{y} \rangle$
$\langle \mathsf{b} \rangle \hookrightarrow Stat4 \Rightarrow \quad \mathsf{b} \in \mathsf{a} \ \& \ Stat5: \ \bigcup \mathsf{a} \not\supseteq \mathsf{b}$
$\langle \mathsf{c} \rangle \hookrightarrow Stat5 \Rightarrow \quad \mathsf{c} \in \mathsf{b} \ \& \ \mathsf{c} \notin \bigcup \mathsf{a}$
Use_def($\bigcup$) $\Rightarrow$ $\quad Stat6: \ \mathsf{c} \notin \{\mathsf{y} : \mathsf{x} \in \mathsf{a}, \mathsf{y} \in \mathsf{x}\}$
$\langle \mathsf{b}, \mathsf{c} \rangle \hookrightarrow Stat6 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- We shall now show how Theorem 337 can be used to construct a so-called 'ultrafilter' containing any given set theoretic 'filter'. The definitions involved are as follows. A collection t of subsets of a set s is called a 'filter' in s if (a) $\emptyset \notin$ t; (b) t any superset of any member of t belongs to t; (c) any intersection of two elements of t belongs to t. A filter in s is called an 'ultrafilter' in s if for any subset u of s, either u or s\u belongs to t. The formal definitions are as follows:

DEF 332a. Filter(X, Y) $\leftrightarrow_{\text{Def}}$ $\mathsf{X} \subseteq \mathcal{P}\mathsf{Y} \ \& \ \emptyset \notin \mathsf{X} \ \& \ \langle \forall \mathsf{x} \in \mathsf{X}, \mathsf{y} \in \mathsf{X} \,|\, \mathsf{x} \cap \mathsf{y} \in \mathsf{X} \rangle \ \& \ \langle \forall \mathsf{x} \in \mathsf{X}, \mathsf{y} \subseteq \mathsf{Y} \,|\, \mathsf{y} \supseteq \mathsf{x} \rightarrow \mathsf{y} \in \mathsf{X} \rangle$
DEF 332b. Ultrafilter(X, Y) $\leftrightarrow_{\text{Def}}$ $\mathsf{Filter}(\mathsf{X}, \mathsf{Y}) \ \& \ \langle \forall \mathsf{y} \subseteq \mathsf{Y} \,|\, \mathsf{y} \in \mathsf{X} \vee \mathsf{Y} \backslash \mathsf{y} \in \mathsf{X} \rangle$

-- It is easily seen that the union $\bigcup C$ of any linearly ordered collection C of filters in a set S is also a filter in S. Indeed, (a) since $\emptyset$ is not in any element of C it is not in $\bigcup C$; (b) If x is a superset of an element y of $\bigcup C$, then since y must belong to some member t of C, x must also belong to t and hence to $\bigcup C$; (c) given any two elements A and B of $\bigcup C$, A (resp. B) must belong to some element (i. e. filter) FA (resp. FB) of C, But then, since the elements of C are linearly ordered by inclusion, one of the two filters FA and FB, say FA, but included the other. Hence A and B are both members of FA It follows using Theorem 337 that every filter is contained in a maximal filter. But it is easily seen that a filter t in s is maximal (among all filters in s) if and only if it is an ultrafilter in s. For if t is an ultrafilter it cannot be enlarged by adding any subset x of s not already in t, since $s\backslash x$ must already belong to t, and thus addition of x would force $x \cap (s\backslash x) = \emptyset$ to belong to the resulting filter, which is impossible by definition of a filter. Conversely, if neither x nor $s\backslash x$ belong to t we can extend t to the larger filter t' consisting of all subsets of s which include a set of the form $f \cap x$, where f belongs to t. Indeed, it is clear that the family t' of sets defined in this way is closed under intersection, and that it includes any superset o any of its sets. Hence to show that t' is a filter we have only to show that no set of the form $f \cap x$ can be null. But if $f \cap x$ were null, we would have $s\backslash x \supseteq f$, so $s\backslash x$ would be a member of t, contrary to assumption. The formal versions of the preceding informal arguments are as follows.

Theorem 483 (338) $\langle \forall t \in TP \,|\, \text{Filter}(t, S) \rangle \,\&\, \langle \forall u \in TP, v \in TP \,|\, u \supseteq v \vee v \supseteq u \rangle \rightarrow \text{Filter}(\bigcup TP, S)$. PROOF:

Suppose_not$(t', s) \Rightarrow$ $Stat1: \langle \forall t \in t' \,|\, \text{Filter}(t, s) \rangle \,\&\, Stat2: \langle \forall u \in t', v \in t' \,|\, u \supseteq v \vee v \supseteq u \rangle \,\&\, \neg\text{Filter}(\bigcup t', s)$

Use_def(Filter) $\Rightarrow$ $\neg(\bigcup t' \subseteq \mathcal{P}s \,\&\, \emptyset \notin \bigcup t' \,\&\, \langle \forall x \in \bigcup t', y \in \bigcup t' \,|\, x \cap y \in \bigcup t' \rangle \,\&\, \langle \forall x \in \bigcup t', y \subseteq s \,|\, y \supseteq x \rightarrow y \in \bigcup t' \rangle)$

Suppose $\Rightarrow$ $Stat3: \bigcup t' \not\subseteq \mathcal{P}s$

$\langle a \rangle \hookrightarrow Stat3 \Rightarrow$ $a \in \bigcup t' \,\&\, a \notin \mathcal{P}s$

Use_def$(\bigcup) \Rightarrow$ $Stat4: a \in \{x : y \in t', x \in y\}$

Use_def$(\mathcal{P}) \Rightarrow$ $a \notin \{x : x \subseteq s\}$

$\langle b, c \rangle \hookrightarrow Stat4 \Rightarrow$ $b \in t' \,\&\, c \in b \,\&\, a = c$

$\langle b \rangle \hookrightarrow Stat1 \Rightarrow$ $\text{Filter}(b, s)$

Use_def(Filter) $\Rightarrow$ $b \subseteq \mathcal{P}s$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\bigcup t' \subseteq \mathcal{P}s$

Suppose $\Rightarrow$ $\emptyset \in \bigcup t'$

Use_def$(\bigcup) \Rightarrow$ $Stat5: \emptyset \in \{x : y \in t', x \in y\}$

$\langle b_2, c_2 \rangle \hookrightarrow Stat5 \Rightarrow$ $b_2 \in t' \,\&\, \emptyset \in b_2$

$\langle b_2 \rangle \hookrightarrow Stat1 \Rightarrow$ $\text{Filter}(b_2, s)$

Use_def(Filter) $\Rightarrow$ false; Discharge $\Rightarrow$ $\emptyset \notin \bigcup t'$

Suppose $\Rightarrow$ $Stat6: \neg\langle \forall x \in \bigcup t', y \subseteq s \,|\, y \supseteq x \rightarrow y \in \bigcup t' \rangle$

$\langle b_3, c_3 \rangle \hookrightarrow Stat6 \Rightarrow$ $b_3 \in \bigcup t' \,\&\, c_3 \subseteq s \,\&\, c_3 \supseteq b_3 \,\&\, c_3 \notin \bigcup t'$

Use_def$(\bigcup) \Rightarrow$ $Stat7: b_3 \in \{x : y \in t', x \in y\}$

$\langle b_4, c_4 \rangle \hookrightarrow Stat7 \Rightarrow \quad b_4 \in t' \,\&\, b_3 \in b_4$
$\langle b_4 \rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{Filter}(b_4, s)$
Use_def(Filter) $\Rightarrow \quad Stat8: \ \langle \forall x \in b_4, y \subseteq s \mid y \supseteq x \rightarrow y \in b_4 \rangle$
$\langle b_3, c_3 \rangle \hookrightarrow Stat8 \Rightarrow \quad c_3 \in b_4$
Use_def($\bigcup$) $\Rightarrow \quad Stat9: \ c_3 \notin \{x : \ y \in t', x \in y\}$
$\langle b_4, c_3 \rangle \hookrightarrow Stat9 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad Stat10: \ \neg\langle \forall x \in \bigcup t', y \in \bigcup t' \mid x \cap y \in \bigcup t' \rangle$

-- Thus only the third clause in the definition of 'Filter' could be false for t'. But since the elements of t' are linearly ordered by inclusion it is easily seen that this clause must also be true, so t' must be a filter, as asserted.

$\langle a_2, a_3 \rangle \hookrightarrow Stat10 \Rightarrow \quad a_2, a_3 \in \bigcup t' \,\&\, a_2 \cap a_3 \notin \bigcup t'$
Use_def($\bigcup$) $\Rightarrow \quad Stat11: \ a_2, a_3 \in \{x : \ y \in t', x \in y\}$
$\langle b_5, c_5, b_6, c_6 \rangle \hookrightarrow Stat11 \Rightarrow \quad b_5 \in t' \,\&\, a_2 \in b_5 \,\&\, b_6 \in t' \,\&\, a_3 \in b_6$
$\langle b_5, b_6 \rangle \hookrightarrow Stat2 \Rightarrow \quad b_5 \supseteq b_6 \vee b_6 \supseteq b_5$
Suppose $\Rightarrow \quad b_5 \supseteq b_6$
ELEM $\Rightarrow \quad a_3 \in b_5$
$\langle b_5 \rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{Filter}(b_5, s)$
Use_def(Filter) $\Rightarrow \quad Stat12: \ \langle \forall x \in b_5, y \in b_5 \mid x \cap y \in b_5 \rangle$
$\langle a_2, a_3 \rangle \hookrightarrow Stat12 \Rightarrow \quad a_2 \cap a_3 \in b_5$
Use_def($\bigcup$) $\Rightarrow \quad Stat13: \ a_2 \cap a_3 \notin \{x : \ y \in t', x \in y\}$
$\langle b_5, a_2 \cap a_3 \rangle \hookrightarrow Stat13 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad b_6 \supseteq b_5$
ELEM $\Rightarrow \quad a_2 \in b_6$
$\langle b_6 \rangle \hookrightarrow Stat1 \Rightarrow \quad \mathsf{Filter}(b_6, s)$
Use_def(Filter) $\Rightarrow \quad Stat14: \ \langle \forall x \in b_6, y \in b_6 \mid x \cap y \in b_6 \rangle$
$\langle a_2, a_3 \rangle \hookrightarrow Stat14 \Rightarrow \quad a_2 \cap a_3 \in b_6$
Use_def($\bigcup$) $\Rightarrow \quad Stat15: \ a_2 \cap a_3 \notin \{x : \ y \in t', x \in y\}$
$\langle b_6, a_2 \cap a_3 \rangle \hookrightarrow Stat15 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad$ QED

-- Next we prove the lemma, anticipated above, that a filter is maximal if and only if it is an ultrafilter.

Theorem 484 (339) $\mathsf{Filter}(T, S) \,\&\, \langle \forall x \subseteq \mathcal{P}S \mid x \supseteq T \,\&\, \mathsf{Filter}(x, S) \rightarrow x = T \rangle \leftrightarrow \mathsf{Ultrafilter}(T, S)$. PROOF:

Suppose_not(t, s) $\Rightarrow \quad \neg\big(\mathsf{Filter}(t, s) \,\&\, \langle \forall x \subseteq \mathcal{P}s \mid x \supseteq t \,\&\, \mathsf{Filter}(x, s) \rightarrow x = t \rangle \leftrightarrow \mathsf{Ultrafilter}(t, s)\big)$

-- For it is easily seen that an ultrafilter must be a maximal filter.

Suppose $\Rightarrow \quad \mathsf{Ultrafilter}(t, s) \,\&\, \neg\big(\mathsf{Filter}(t, s) \,\&\, \langle \forall x \subseteq \mathcal{P}s \mid x \supseteq t \,\&\, \mathsf{Filter}(x, s) \rightarrow x = t \rangle\big)$
Use_def(Ultrafilter) $\Rightarrow \quad Stat1: \ \neg\langle \forall x \subseteq \mathcal{P}s \mid x \supseteq t \,\&\, \mathsf{Filter}(x, s) \rightarrow x = t \rangle$

$\langle t_2 \rangle \hookrightarrow Stat1 \Rightarrow \quad t_2 \subseteq \mathcal{P}s \;\&\; t_2 \supseteq t \;\&\; \text{Filter}(t_2, s) \;\&\; t_2 \neq t$

ELEM $\Rightarrow \quad Stat2: \; t \not\supseteq t_2$

$\langle a \rangle \hookrightarrow Stat2 \Rightarrow \quad a \in t_2 \;\&\; a \notin t$

Use_def (Filter) $\Rightarrow \quad t_2 \subseteq \mathcal{P}s$

ELEM $\Rightarrow \quad a \in \mathcal{P}s$

Use_def ($\mathcal{P}$) $\Rightarrow \quad Stat3: \; a \in \{x : x \subseteq s\}$

$\langle c' \rangle \hookrightarrow Stat3 \Rightarrow \quad a \subseteq s$

Use_def (Ultrafilter) $\Rightarrow \quad Stat4: \; \langle \forall y \subseteq s \,|\, y \in t \vee s \backslash y \in t \rangle$

$\langle a \rangle \hookrightarrow Stat4 \Rightarrow \quad s \backslash a \in t_2$

Use_def (Filter) $\Rightarrow \quad \emptyset \notin t_2 \;\&\; Stat5: \; \langle \forall x \in t_2, y \in t_2 \,|\, x \cap y \in t_2 \rangle$

$\langle a, s \backslash a \rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \neg\text{Ultrafilter}(t, s) \;\&\; \text{Filter}(t, s) \;\&\; \langle \forall x \subseteq \mathcal{P}s \,|\, x \supseteq t \;\&\; \text{Filter}(x, s) \rightarrow x = t \rangle$

-- Thus if our theorem is false t must be a maximal filter but not an ultrafilter.

Use_def (Ultrafilter) $\Rightarrow \quad Stat6: \; \neg\langle \forall y \subseteq s \,|\, y \in t \vee s \backslash y \in t \rangle$

$\langle c \rangle \hookrightarrow Stat6 \Rightarrow \quad c \subseteq s \;\&\; c \notin t \;\&\; s \backslash c \notin t$

Loc_def $\Rightarrow \quad t_3 = \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\}$

Suppose $\Rightarrow \quad \neg\text{Filter}(t_3)$

Use_def (Filter) $\Rightarrow \quad \neg(t_3 \subseteq \mathcal{P}s \;\&\; \emptyset \notin t_3 \;\&\; \langle \forall x \in t_3, y \in t_3 \,|\, x \cap y \in t_3 \rangle \;\&\; \langle \forall x \in t_3, y \subseteq s \,|\, y \supseteq x \rightarrow y \in t_3 \rangle)$

Suppose $\Rightarrow \quad Stat7: \; t_3 \not\subseteq \mathcal{P}s$

$\langle b \rangle \hookrightarrow Stat7 \Rightarrow \quad Stat8: \; b \in \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\} \;\&\; b \notin \mathcal{P}s$

$\langle b_2 \rangle \hookrightarrow Stat8 \Rightarrow \quad b \subseteq s$

Use_def ($\mathcal{P}$) $\Rightarrow \quad Stat9: \; b \notin \{x : x \subseteq s\}$

$\langle b \rangle \hookrightarrow Stat9 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad t_3 \subseteq \mathcal{P}s$

Suppose $\Rightarrow \quad Stat10: \; \neg\langle \forall x \in t_3, y \subseteq s \,|\, y \supseteq x \rightarrow y \in t_3 \rangle$

$\langle b_3, c_3 \rangle \hookrightarrow Stat10 \Rightarrow \quad Stat11:$

$\quad b_3 \in \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\} \;\&\; c_3 \subseteq s \;\&\; c_3 \supseteq b_3 \;\&\; Stat12: \; c_3 \notin \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\}$

$\langle b_4 \rangle \hookrightarrow Stat11 \Rightarrow \quad b_4 \subseteq s \;\&\; \langle \exists y \in t \,|\, b_4 \supseteq y \cap c \rangle \;\&\; b_3 = b_4$

EQUAL $\Rightarrow \quad b_3 \subseteq s \;\&\; Stat13: \; \langle \exists y \in t \,|\, b_3 \supseteq y \cap c \rangle$

$\langle c_4 \rangle \hookrightarrow Stat13 \Rightarrow \quad c_4 \in t \;\&\; b_3 \supseteq c_4 \cap c$

$\langle c_3 \rangle \hookrightarrow Stat12 \Rightarrow \quad Stat14: \; \neg\langle \exists y \in t \,|\, c_3 \supseteq y \cap c \rangle$

$\langle c_4 \rangle \hookrightarrow Stat14 \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \langle \forall x \in t_3, y \subseteq s \,|\, y \supseteq x \rightarrow y \in t_3 \rangle$

Suppose $\Rightarrow \quad Stat15: \; \neg\langle \forall x \in t_3, y \in t_3 \,|\, x \cap y \in t_3 \rangle$

$\langle b_5, c_5 \rangle \hookrightarrow Stat15 \Rightarrow \quad Stat16:$

$\quad b_5 \in \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\} \;\&\; Stat17:$

$\qquad c_5 \in \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\} \;\&\; Stat18: \; b_5 \cap c_5 \notin \{x : x \subseteq s \,|\, \langle \exists y \in t \,|\, x \supseteq y \cap c \rangle\}$

$\langle b_6 \rangle \hookrightarrow Stat16 \Rightarrow \quad b_6 \subseteq s \;\&\; Stat19: \; \langle \exists y \in t \,|\, b_6 \supseteq y \cap c \rangle \;\&\; b_5 = b_6$

EQUAL $\Rightarrow \quad b_5 \subseteq s \;\&\; \langle \exists y \in t \,|\, b_5 \supseteq y \cap c \rangle$

$\langle c_6 \rangle \hookrightarrow Stat17 \Rightarrow \quad c_6 \subseteq s \;\&\; Stat20: \; \langle \exists y \in t \,|\, c_6 \supseteq y \cap c \rangle \;\&\; c_5 = c_6$

EQUAL $\Rightarrow \quad c_5 \subseteq s \;\&\; \langle \exists y \in t \,|\, c_5 \supseteq y \cap c \rangle$

$\langle d_6 \rangle \hookrightarrow Stat19 \Rightarrow \quad d_6 \in t \;\&\; b_5 \supseteq d_6 \cap c$
$\langle e_6 \rangle \hookrightarrow Stat20 \Rightarrow \quad e_6 \in t \;\&\; c_5 \supseteq e_6 \cap c$
ELEM $\Rightarrow \quad b_5 \cap c_5 \supseteq d_6 \cap e_6 \cap c$
Use_def(Filter) $\Rightarrow \quad Stat21: \langle \forall x \in t, y \in t \mid x \cap y \in t \rangle$
$\langle d_6, e_6 \rangle \hookrightarrow Stat21 \Rightarrow \quad d_6 \cap e_6 \in t$
$\langle b_5 \cap c_5 \rangle \hookrightarrow Stat18 \Rightarrow \quad Stat22: \neg\langle \exists y \in t \mid b_5 \cap c_5 \supseteq y \cap c \rangle$
$\langle d_6 \cap e_6 \rangle \hookrightarrow Stat22 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad Stat23: \emptyset \in \{x : x \subseteq s \mid \langle \exists y \in t \mid x \supseteq y \cap c \rangle\}$
$\langle x \rangle \hookrightarrow Stat23 \Rightarrow \quad Stat24: \langle \exists y \in t \mid x \supseteq y \cap c \rangle \;\&\; x = \emptyset$
$\langle y \rangle \hookrightarrow Stat24 \Rightarrow \quad y \in t \;\&\; y \cap c = \emptyset$
Use_def(Filter) $\Rightarrow \quad t \subseteq \mathcal{P}s \;\&\; Stat25: \langle \forall x \in t, y \subseteq s \mid y \supseteq x \rightarrow y \in t \rangle$
ELEM $\Rightarrow \quad y \in \mathcal{P}s$
Use_def($\mathcal{P}$) $\Rightarrow \quad Stat26: y \in \{x : x \subseteq s\}$
$\langle yy \rangle \hookrightarrow Stat26 \Rightarrow \quad y \subseteq s$
ELEM $\Rightarrow \quad y \subseteq s \backslash c$
$\langle y, s\backslash c \rangle \hookrightarrow Stat25 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ QED

-- Since Theorem 338 tells us that the hypothesis of Theorem 337 is valid for the set of filters in a set s, the following conclusion results immediately.

Theorem 485 (340) $\;$ $\mathrm{Filter}(T, S) \rightarrow \langle \exists u \mid u \supseteq T \;\&\; \mathrm{Ultrafilter}(T, S) \rangle$. PROOF:

Suppose_not(t, s) $\Rightarrow \quad \mathrm{Filter}(t, s) \;\&\; \neg\langle \exists u \mid u \supseteq t \;\&\; \mathrm{Ultrafilter}(t, s) \rangle$

-- For let t, s be a counterexample to our assetion, and consider the collection filters_in_s of all filters in s. Since it is easily seen that filters_in_s atisfies the hypothesis of theorem 337, it follows that t is contained in some maximal filter.

Loc_def $\Rightarrow \quad \mathrm{filters\_in\_s} = \{f \subseteq \mathcal{P}s \mid \mathrm{Filter}(f, s)\}$
Suppose $\Rightarrow \quad Stat1: \neg\langle \forall x \subseteq \mathrm{filters\_in\_s} \mid \langle \forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u \rangle \rightarrow \bigcup x \in \mathrm{filters\_in\_s} \rangle$
$\langle x_1 \rangle \hookrightarrow Stat1 \Rightarrow \quad x_1 \subseteq \mathrm{filters\_in\_s} \;\&\; \langle \forall u \in x_1, v \in x_1 \mid u \supseteq v \vee v \supseteq u \rangle \;\&\; \bigcup x_1 \notin \mathrm{filters\_in\_s}$
ELEM $\Rightarrow \quad Stat2: x_1 \subseteq \{f \subseteq \mathcal{P}s \mid \mathrm{Filter}(f, s)\} \;\&\; Stat3: \bigcup x_1 \notin \{f \subseteq \mathcal{P}s \mid \mathrm{Filter}(f, s)\}$
Suppose $\Rightarrow \quad Stat4: \neg\langle \forall y \in x_1 \mid \mathrm{Filter}(y, s) \rangle$
$\langle y_1 \rangle \hookrightarrow Stat4 \Rightarrow \quad y_1 \in x_1 \;\&\; \neg\mathrm{Filter}(y_1, s)$
ELEM $\Rightarrow \quad Stat5: y_1 \in \{f \subseteq \mathcal{P}s \mid \mathrm{Filter}(f, s)\}$
$\langle \; \rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \langle \forall y \in x_1 \mid \mathrm{Filter}(y, s) \rangle$
$\langle x_1 \rangle \hookrightarrow T338 \Rightarrow \quad \mathrm{Filter}(\bigcup x_1, s)$
$\langle \; \rangle \hookrightarrow Stat3 \Rightarrow \quad Stat6: \bigcup x_1 \not\subseteq \mathcal{P}s$
$\langle c \rangle \hookrightarrow Stat6 \Rightarrow \quad c \in \bigcup x_1 \;\&\; c \notin \mathcal{P}s$
Use_def($\bigcup$) $\Rightarrow \quad Stat7: c \in \{x : y \in x_1, x \in y\}$
$\langle y_2, c_2 \rangle \hookrightarrow Stat7 \Rightarrow \quad y_2 \in x_1 \;\&\; c_2 \in y_2 \;\&\; c_2 = c$

ELEM $\Rightarrow$ $\quad$ *Stat8* : $y_2 \in \{f \subseteq \mathcal{P}s \mid \text{Filter}(f, s)\} \;\&\; c \in y_2$
$\langle\,\rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $\quad$ $\text{Filter}(y_2, s)$
Use_def(Filter) $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ $\langle \forall x \subseteq \text{filters\_in\_s} \mid \langle \forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u\rangle \rightarrow \bigcup x \in \text{filters\_in\_s}\rangle$
$\langle \text{filters\_in\_s}\rangle \hookrightarrow$ *T337* $\Rightarrow$ $\quad$ *Stat9* : $\langle \forall u \in \text{filters\_in\_s}, \exists y \in \text{filters\_in\_s} \mid y \supseteq u \;\&\; \langle \forall x \in \text{filters\_in\_s} \mid \neg(x \supseteq y \;\&\; x \neq y)\rangle\rangle$
Suppose $\Rightarrow$ $\quad$ $t \notin \text{filters\_in\_s}$
ELEM $\Rightarrow$ $\quad$ *Stat10* : $t \notin \{f \subseteq \mathcal{P}s \mid \text{Filter}(f, s)\}$
$\langle\,\rangle \hookrightarrow$ *Stat10* $\Rightarrow$ $\quad$ *Stat11* : $t \not\subseteq \mathcal{P}s$
Use_def(Filter) $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ $t \in \text{filters\_in\_s}$
$\langle t\rangle \hookrightarrow$ *Stat9* $\Rightarrow$ $\quad$ *Stat12* : $\langle \exists y \in \text{filters\_in\_s} \mid y \supseteq t \;\&\; \langle \forall x \in \text{filters\_in\_s} \mid \neg(x \supseteq y \;\&\; x \neq y)\rangle\rangle$
$\langle \text{tm}\rangle \hookrightarrow$ *Stat12* $\Rightarrow$ $\quad$ $\text{tm} \in \text{filters\_in\_s} \;\&\; \text{tm} \supseteq t \;\&\;$ *Stat13* : $\langle \forall x \in \text{filters\_in\_s} \mid \neg(x \supseteq \text{tm} \;\&\; x \neq \text{tm})\rangle$

-- But it follows using Theorem 339 that tm must be an ultrafilter, and so our theorem is proved.

Suppose $\Rightarrow$ $\quad$ *Stat14* : $\neg\langle \forall x \subseteq \mathcal{P}s \mid x \supseteq \text{tm} \;\&\; \text{Filter}(x, s) \rightarrow x = \text{tm}\rangle$
$\langle t_2\rangle \hookrightarrow$ *Stat14* $\Rightarrow$ $\quad$ $t_2 \subseteq \mathcal{P}s \;\&\; t_2 \supseteq \text{tm} \;\&\; \text{Filter}(t_2, s) \;\&\; t_2 \neq \text{tm}$
$\langle t_2\rangle \hookrightarrow$ *Stat13* $\Rightarrow$ $\quad$ *Stat15* : $t_2 \notin \{f \subseteq \mathcal{P}s \mid \text{Filter}(f, s)\}$
$\langle t_2\rangle \hookrightarrow$ *Stat15* $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

14 Formal fractions and rational numbers

-- We have seen above that the signed integers is a collection of quantities into which the unsigned integers can be embedded in a manner preserving all the basic algebraic operations on unsigned integers. In this sense, the unsigned integers are an 'extension' of the unsigned integers. This is the first of several extensions, each of which serves to simplify some aspect of the collection of numbers being extended. As previously noted, extension of the unsigned integers to the signed integers serves to simplify the properties of subtraction. Three extensions subsequent to this respectively introduce (i) the rational numbers, thereby simplifying division; (ii) the real numbers, thereby ensuring that every polynomial which takes on both positive and negative values also takes on the zero value; (iii) the complex numbers, thereby ensuring that every polynomial other than a simple constant has at least one zero. We shall see that these extended families of numbers have many deep properties other than the basic properties noted. Of these extensions it is the introduction of real numbers which will involve the deepest construction. In the present section we begin to walk the path outlined above by introducing the rational numbers. This is done in two steps. First we introduce the formal fractions and the elementary algebraic operations on them. Formal fractions are simply ordered pairs of signed integers $[\mathsf{m},\mathsf{n}]$, m being the fraction's 'numerator' and n its denominator. Then an eqivalence relation beween fractions is introduced. This amounts to the fractions becoming identical when each is reduced to 'lowest terms' by division of its nuerator and denoinator by thier greatest common factor, but is more conveniently expressed by the condition that

DEF 35. $\mathsf{Fr} =_{\mathrm{Def}} \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{Z}, \mathsf{y} \in \mathbb{Z} \mid \mathsf{y} \neq [\emptyset,\emptyset]\}$

DEF 36. $\mathsf{X} \approx_{\mathsf{Fr}} \mathsf{Y} \leftrightarrow_{\mathrm{Def}} \mathsf{X}^{[1]} *_{\mathbb{Z}} \mathsf{Y}^{[2]} = \mathsf{X}^{[2]} *_{\mathbb{Z}} \mathsf{Y}^{[1]}$

-- Nonnegative fraction

DEF 36*a*. $\mathsf{Fr\_is\_nonneg}(\mathsf{X}) =_{\mathrm{Def}} \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{X}^{[1]} *_{\mathbb{Z}} \mathsf{X}^{[2]})$

-- Our next few results show that the binary predicate $\approx_{\mathsf{Fr}}$ is an equivalence relation. We begin by showing that $\approx_{\mathsf{Fr}}$ is symmetric and transitive. The proof simply uses the definition to our expand our assertion into simple algebraic relationships for signed integers, which then follow by elementary algebraic manipulation.

Theorem 486 (341) $\langle \forall \mathsf{x} \in \mathsf{Fr}, \mathsf{y} \in \mathsf{Fr} \mid (\mathsf{x} \approx_{\mathsf{Fr}} \mathsf{y} \leftrightarrow \mathsf{y} \approx_{\mathsf{Fr}} \mathsf{x}) \;\&\; \mathsf{x} \approx_{\mathsf{Fr}} \mathsf{x}\rangle$. PROOF:

Suppose_not $\Rightarrow$ *Stat0*: $\neg\langle \forall \mathsf{x} \in \mathsf{Fr}, \mathsf{Y} \in \mathsf{Fr} \mid (\mathsf{x} \approx_{\mathsf{Fr}} \mathsf{y} \leftrightarrow \mathsf{y} \approx_{\mathsf{Fr}} \mathsf{x}) \;\&\; \mathsf{x} \approx_{\mathsf{Fr}} \mathsf{x}\rangle$

$\langle \mathsf{x},\mathsf{y}\rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\mathsf{x},\mathsf{y} \in \mathsf{Fr} \;\&\; \neg(\mathsf{x} \approx_{\mathsf{Fr}} \mathsf{y} \leftrightarrow \mathsf{y} \approx_{\mathsf{Fr}} \mathsf{x}) \vee \neg \mathsf{x} \approx_{\mathsf{Fr}} \mathsf{x}$

Use_def(Fr) $\Rightarrow$ *Stat1*: $\mathsf{x},\mathsf{y} \in \{[\mathsf{u},\mathsf{v}] : \mathsf{u} \in \mathbb{Z}, \mathsf{v} \in \mathbb{Z} \mid \mathsf{v} \neq [\emptyset,\emptyset]\}$

$\langle \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2*: $\mathsf{x} = [\mathsf{a},\mathsf{b}] \;\&\; \mathsf{a},\mathsf{b} \in \mathbb{Z} \;\&\; \mathsf{y} = [\mathsf{c},\mathsf{d}] \;\&\; \mathsf{c},\mathsf{d} \in \mathbb{Z}$

$\langle$*Stat2*$\rangle$ ELEM $\Rightarrow$ $\mathsf{x}^{[1]} = \mathsf{a} \;\&\; \mathsf{x}^{[2]} = \mathsf{b}$

$\langle Stat2, Stat2\rangle$ ELEM $\Rightarrow$ $y^{[1]} = c \mathbin{\&} y^{[2]} = d$

Suppose $\Rightarrow$ $\neg x \approx_{Fr} x$

Use_def$(\approx_{Fr}) \Rightarrow$ $x^{[1]} *_{\mathbb{Z}} x^{[2]} \neq x^{[2]} *_{\mathbb{Z}} x^{[1]}$

$\langle x^{[1]}, x^{[2]}\rangle \hookrightarrow T307([Stat0, \cap]) \Rightarrow$ false; Discharge $\Rightarrow$ $\neg(x \approx_{Fr} y \leftrightarrow y \approx_{Fr} x)$

Suppose $\Rightarrow$ $x \approx_{Fr} y \mathbin{\&} \neg y \approx_{Fr} x$

Use_def$(\approx_{Fr}) \Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} y^{[1]} \mathbin{\&} y^{[1]} *_{\mathbb{Z}} x^{[2]} \neq y^{[2]} *_{\mathbb{Z}} x^{[1]}$

EQUAL $\Rightarrow$ $a *_{\mathbb{Z}} d = b *_{\mathbb{Z}} c \mathbin{\&} c *_{\mathbb{Z}} b \neq d *_{\mathbb{Z}} a$

ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg x \approx_{Fr} y \mathbin{\&} y \approx_{Fr} x$

Use_def$(\approx_{Fr}) \Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} \neq x^{[2]} *_{\mathbb{Z}} y^{[1]} \mathbin{\&} y^{[1]} *_{\mathbb{Z}} x^{[2]} = y^{[2]} *_{\mathbb{Z}} x^{[1]}$

EQUAL $\Rightarrow$ $a *_{\mathbb{Z}} d \neq b *_{\mathbb{Z}} c \mathbin{\&} c *_{\mathbb{Z}} b = d *_{\mathbb{Z}} a$

ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 487 (342) $\langle \forall x \in Fr, y \in Fr, zz \in Fr \mid x \approx_{Fr} y \mathbin{\&} y \approx_{Fr} zz \rightarrow x \approx_{Fr} zz\rangle$. PROOF:

Suppose_not $\Rightarrow$ $Stat0 : \neg\langle \forall x \in Fr, y \in Fr, zz \in Fr \mid x \approx_{Fr} y \mathbin{\&} y \approx_{Fr} zz \rightarrow x \approx_{Fr} zz\rangle$

$\langle x, y, zz\rangle \hookrightarrow Stat0 \Rightarrow$ $x, y, zz \in Fr \mathbin{\&} x \approx_{Fr} y \mathbin{\&} y \approx_{Fr} zz \mathbin{\&} \neg x \approx_{Fr} zz$

Use_def(Fr) $\Rightarrow$ $Stat1 :$

$x \in \{[u, v] : u \in \mathbb{Z}, v \in \mathbb{Z} \mid v \neq [\emptyset, \emptyset]\} \mathbin{\&} Stat2 :$

$y \in \{[u, v] : u \in \mathbb{Z}, v \in \mathbb{Z} \mid v \neq [\emptyset, \emptyset]\} \mathbin{\&} Stat3 : zz \in \{[u, v] : u \in \mathbb{Z}, v \in \mathbb{Z} \mid v \neq [\emptyset, \emptyset]\}$

$\langle a, b\rangle \hookrightarrow Stat1 \Rightarrow$ $Stat4 : x = [a, b] \mathbin{\&} a, b \in \mathbb{Z}$

$\langle Stat4\rangle$ ELEM $\Rightarrow$ $x^{[1]} = a \mathbin{\&} x^{[2]} = b$

$\langle c, d\rangle \hookrightarrow Stat2 \Rightarrow$ $Stat5 : y = [c, d] \mathbin{\&} c, d \in \mathbb{Z} \mathbin{\&} d \neq [\emptyset, \emptyset]$

$\langle Stat5\rangle$ ELEM $\Rightarrow$ $y^{[1]} = c \mathbin{\&} y^{[2]} = d$

$\langle e, f\rangle \hookrightarrow Stat3 \Rightarrow$ $Stat6 : zz = [e, f] \mathbin{\&} e, f \in \mathbb{Z}$

$\langle Stat6\rangle$ ELEM $\Rightarrow$ $zz^{[1]} = e \mathbin{\&} zz^{[2]} = f$

Use_def$(\approx_{Fr}) \Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} y^{[1]} \mathbin{\&} y^{[1]} *_{\mathbb{Z}} zz^{[2]} = y^{[2]} *_{\mathbb{Z}} zz^{[1]} \mathbin{\&} x^{[1]} *_{\mathbb{Z}} zz^{[2]} \neq x^{[2]} *_{\mathbb{Z}} zz^{[1]}$

EQUAL $\Rightarrow$ $a *_{\mathbb{Z}} d = b *_{\mathbb{Z}} c \mathbin{\&} c *_{\mathbb{Z}} f = d *_{\mathbb{Z}} e \mathbin{\&} a *_{\mathbb{Z}} f \neq b *_{\mathbb{Z}} e$

EQUAL $\Rightarrow$ $a *_{\mathbb{Z}} d *_{\mathbb{Z}} f = b *_{\mathbb{Z}} c *_{\mathbb{Z}} f$

ALGEBRA $\langle Stat4, Stat5, Stat6\rangle \Rightarrow$ $a *_{\mathbb{Z}} d *_{\mathbb{Z}} f = a *_{\mathbb{Z}} f *_{\mathbb{Z}} d \mathbin{\&} (b *_{\mathbb{Z}} c) *_{\mathbb{Z}} f = b *_{\mathbb{Z}} (c *_{\mathbb{Z}} f)$

EQUAL $\Rightarrow$ $a *_{\mathbb{Z}} d *_{\mathbb{Z}} f = b *_{\mathbb{Z}} (d *_{\mathbb{Z}} e)$

ALGEBRA $\Rightarrow$ $a *_{\mathbb{Z}} f *_{\mathbb{Z}} d = b *_{\mathbb{Z}} (d *_{\mathbb{Z}} e)$

ALGEBRA $\Rightarrow$ $b *_{\mathbb{Z}} (d *_{\mathbb{Z}} e) = b *_{\mathbb{Z}} e *_{\mathbb{Z}} d$

ALGEBRA $\Rightarrow$ $a *_{\mathbb{Z}} f, b *_{\mathbb{Z}} e \in \mathbb{Z}$

ELEM $\Rightarrow$ $a *_{\mathbb{Z}} f *_{\mathbb{Z}} d = b *_{\mathbb{Z}} e *_{\mathbb{Z}} d$

$\langle d, a *_{\mathbb{Z}} f, b *_{\mathbb{Z}} e\rangle \hookrightarrow T332 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now that we know that $\approx_{Fr}$ is an equivalence relationship, we can apply the equivalence_classes theory to it, to derive

APPLY $\langle \mathsf{Eqc}_\Theta : \mathbb{Q}, \mathsf{f}_\Theta : \mathsf{Fr\_to\_}\mathbb{Q} \rangle$ equivalence_classes$\big(\mathsf{P}(x,y) \mapsto x \approx_{Fr} y, s \mapsto \mathsf{Fr}\big) \Rightarrow$

Theorem 488 (343) $\langle \forall x, y \mid x, y \in \mathsf{Fr} \rightarrow (x \approx_{Fr} y \leftrightarrow \mathsf{Fr\_to\_}\mathbb{Q}(x) = \mathsf{Fr\_to\_}\mathbb{Q}(y)) \rangle \;\&\; \langle \forall x \mid x \in \mathbb{Q} \rightarrow \mathbf{arb}(x) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(x)) = x \rangle \;\&\; \langle \forall x \mid x \in \mathsf{Fr} \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}(x) \in \mathbb{Q} \rangle \;\&\; \langle \forall x \mid x \in \mathsf{Fr} \rightarrow x \approx_{Fr} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})\,(x) \rangle.$

-- [Note: $\mathbb{Q}$ is the set of rational numbers.]

Theorem 489 (344) $X \in \mathsf{Fr} \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}(X) \in \mathbb{Q} \;\&\; X \approx_{Fr} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})\,(X)$. PROOF:

Suppose_not$(x) \Rightarrow$ $\quad x \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(x) \notin \mathbb{Q} \lor \neg x \approx_{Fr} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})\,(x)$
T343 $\Rightarrow$ $\quad$ *Stat1* : $\langle \forall x \mid x \in \mathsf{Fr} \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}(x) \in \mathbb{Q} \rangle \;\&\; \langle \forall x \mid x \in \mathsf{Fr} \rightarrow x \approx_{Fr} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})\,(x) \rangle$
$\langle x, x \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

Theorem 490 (345) $X, Y \in \mathsf{Fr} \rightarrow \big(X \approx_{Fr} Y \leftrightarrow \mathsf{Fr\_to\_}\mathbb{Q}(X) = \mathsf{Fr\_to\_}\mathbb{Q}(Y)\big)$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $\quad x, y \in \mathsf{Fr} \;\&\; \neg\big(x \approx_{Fr} y \leftrightarrow \mathsf{Fr\_to\_}\mathbb{Q}(x) = \mathsf{Fr\_to\_}\mathbb{Q}(y)\big)$
T343 $\Rightarrow$ $\quad$ *Stat1* : $\langle \forall x, y \mid x, y \in \mathsf{Fr} \rightarrow (x \approx_{Fr} y \leftrightarrow \mathsf{Fr\_to\_}\mathbb{Q}(x) = \mathsf{Fr\_to\_}\mathbb{Q}(y)) \rangle$
$\langle x, y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

Theorem 491 (346) $Y \in \mathbb{Q} \rightarrow \mathbf{arb}(Y) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(Y)) = Y$. PROOF:

Suppose_not$(y) \Rightarrow$ $\quad y \in \mathbb{Q} \;\&\; \mathbf{arb}(y) \notin \mathsf{Fr} \lor \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(y)) \neq y$
T343 $\Rightarrow$ $\quad$ *Stat1* : $\langle \forall y \mid y \in \mathbb{Q} \rightarrow \mathbf{arb}(y) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(y)) = y \rangle$
$\langle y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

DEF 37. $\quad \mathbf{0}_{\mathbb{Q}} =_{\mathrm{Def}} \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], [1, \emptyset]])$
DEF 37a. $\quad \mathbf{1}_{\mathbb{Q}} =_{\mathrm{Def}} \mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [1, \emptyset]])$

-- Rational Sum

DEF 38. $\quad X +_{\mathbb{Q}} Y =_{\mathrm{Def}} \mathsf{Fr\_to\_}\mathbb{Q}(\big[\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(Y)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(Y)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(X)^{[2]}, \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(Y)^{[2]}\big])$

-- Rational product

DEF 39. $\quad X *_{\mathbb{Q}} Y =_{\mathrm{Def}} \mathsf{Fr\_to\_}\mathbb{Q}(\big[\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(Y)^{[1]}, \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(Y)^{[2]}\big])$

-- Reciprocal

DEF 40. $\quad \mathsf{Recip}_{\mathbb{Q}}(X) =_{\mathrm{Def}} \mathsf{Fr\_to\_}\mathbb{Q}(\big[\mathbf{arb}(X)^{[2]}, \mathbf{arb}(X)^{[1]}\big])$

-- Rational quotient

DEF 41. $\quad X /_{\mathbb{Q}} Y \quad =_{\text{Def}} \quad X *_{\mathbb{Q}} \text{Recip}_{\mathbb{Q}}(Y)$

-- Rational negative

DEF 42. $\quad \text{Rev}_{\mathbb{Q}}(X) \quad =_{\text{Def}} \quad \text{Fr\_to\_}\mathbb{Q}\left(\left[\text{Rev}_{\mathbb{Z}}(\mathbf{arb}(X)^{[1]}), \mathbf{arb}(X)^{[2]}\right] \right)$

-- Nonnegative Rational

DEF 43. $\quad \text{is\_nonneg}_{\mathbb{Q}}(X) \quad =_{\text{Def}} \quad \text{is\_nonneg}_{\mathbb{N}}(\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(X)^{[2]})$

-- Rational Subtraction

DEF 44. $\quad X -_{\mathbb{Q}} Y \quad =_{\text{Def}} \quad X +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(Y)$

-- Rational Comparison

DEF 45. $\quad X >_{\mathbb{Q}} Y \quad \leftrightarrow_{\text{Def}} \quad \text{is\_nonneg}_{\mathbb{Q}}(X -_{\mathbb{Q}} Y) \ \& \ X \neq Y$

THEORY $\text{Ordered\_add}(g, e, x \oplus y, x \text{ minz } y, \text{rvz}(x), \text{nneg}(x))$

$e \in g \ \& \ \langle \forall x \in g \mid x \oplus e = x \ \& \ x \oplus \text{rvz}(x) = e \ \& \ \text{rvz}(x) \in g \rangle$

$\langle \forall x \in g, y \in g \mid x \oplus y \in g \ \& \ x \oplus y = y \oplus x \ \& \ x \oplus \text{rvz}(y) = x \text{ minz } y \rangle$

$\langle \forall x \in g, y \in g, z \in g \mid (x \oplus y) \oplus z = x \oplus (y \oplus z) \rangle$

$\langle \forall x \in g, y \in g \mid \text{nneg}(x) \ \& \ \text{nneg}(y) \rightarrow \text{nneg}(x \oplus y) \rangle$

$\langle \forall x \in g \mid \text{nneg}(x) \vee \text{nneg}(\text{rvz}(x)) \ \& \ \left(\text{nneg}(x) \ \& \ \text{nneg}(\text{rvz}(x)) \rightarrow x = e\right) \rangle$

END Ordered_add

ENTER_THEORY Ordered_add

-- Note that no theorems need to be proved since a decision algorithm is available

DEF $00j$. $\quad X \succcurlyeq_{\ominus} Y \quad =_{\text{Def}} \quad \text{nneg}(X \oplus \text{rvz}(Y))$

DEF $00k$. $\quad X \preccurlyeq_{\ominus} Y \quad =_{\text{Def}} \quad Y \succcurlyeq_{\ominus} X$

DEF $00m$. $\quad X \succ_{\ominus} Y \quad \leftrightarrow_{\text{Def}} \quad X \succcurlyeq_{\ominus} Y \ \& \ X \neq Y$

DEF $00n$. $\quad X \prec_{\ominus} Y \quad =_{\text{Def}} \quad Y \succ_{\ominus} X$

-- needed to interface Otter - based THEORY orderedGroups

Theorem 492 (Ordered_add · 0) $\left(X \preccurlyeq_{\ominus} Y \leftrightarrow \text{nneg}(Y \oplus \text{rvz}(X))\right) \ \& \ \left(X, Y \in g \rightarrow \left(X \succ_{\ominus} Y \leftrightarrow \text{nneg}(X \oplus \text{rvz}(Y)) \ \& \ X \neq Y\right)\right) \ \& \ \left(X, Y \in g \rightarrow \left(X \succ_{\ominus} Y \leftrightarrow \text{nneg}(X \text{ minz } Y) \ \& \ X \neq\right.\right.$

Suppose_not$(x, y) \Rightarrow \quad \left(x \preccurlyeq_{\ominus} y \leftrightarrow \neg\text{nneg}(y \oplus \text{rvz}(x))\right) \vee$

$\left(x, y \in g \ \& \ \left(x \succ_{\ominus} y \leftrightarrow \neg\text{nneg}(x \oplus \text{rvz}(y)) \vee x = y\right)\right) \vee \left(x, y \in g \ \& \ \left(x \succ_{\ominus} y \leftrightarrow \neg\text{nneg}(x \text{ minz } y) \vee x = y\right)\right)$

Suppose $\Rightarrow \quad x \preccurlyeq_{\ominus} y \leftrightarrow \neg\text{nneg}(y \oplus \text{rvz}(x))$

Use_def$(\preccurlyeq_{\ominus}) \Rightarrow \quad x \preccurlyeq_{\ominus} y \leftrightarrow y \succcurlyeq_{\ominus} x$

Use_def$(\succcurlyeq_{\ominus}) \Rightarrow \quad y \succcurlyeq_{\ominus} x \leftrightarrow \text{nneg}(y \oplus \text{rvz}(x))$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\Big(x, y \in g \;\&\; \big(x \succ_\Theta y \leftrightarrow \neg \mathrm{nneg}(x \oplus \mathrm{rvz}(y)) \vee x = y\big)\Big) \vee \big(x, y \in g \;\&\; (x \succ_\Theta y \leftrightarrow \neg \mathrm{nneg}(x \;\mathrm{minz}\; y) \vee x = y)\big)$

Assump $\Rightarrow$ *Stat1*: $\langle \forall x \in g, y \in g \mid x \oplus y \in g \;\&\; x \oplus y = y \oplus x \;\&\; x \oplus \mathrm{rvz}(y) = x \;\mathrm{minz}\; y\rangle$
$\langle x, y\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x \oplus \mathrm{rvz}(y) = x \;\mathrm{minz}\; y$
EQUAL $\Rightarrow$ $x, y \in g \;\&\; \big(x \succ_\Theta y \leftrightarrow \neg \mathrm{nneg}(x \oplus \mathrm{rvz}(y)) \vee x = y\big)$
Use_def ($\succ_\Theta$) $\Rightarrow$ $x, y \in g \;\&\; \big(x \succcurlyeq_\Theta y \;\&\; x \neq y \leftrightarrow \neg \mathrm{nneg}(x \oplus \mathrm{rvz}(y)) \vee x = y\big)$
Use_def ($\succcurlyeq_\Theta$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 493 (Ordered_add · 1) $X, Y \in g \;\&\; X = Y \vee \neg X \succcurlyeq_\Theta Y \rightarrow Y \succcurlyeq_\Theta X$. PROOF:

Suppose_not(c, c') $\Rightarrow$ *Stat0*: $c, c' \in g \;\&\; \neg c \succcurlyeq_\Theta c' \;\&\; c = c' \vee \neg c' \succcurlyeq_\Theta c$
Use_def ($\succcurlyeq_\Theta$) $\Rightarrow$ $\neg \mathrm{nneg}(c \oplus \mathrm{rvz}(c')) \;\&\; c = c' \vee \neg \mathrm{nneg}(c' \oplus \mathrm{rvz}(c))$
Assump $\Rightarrow$ $e \in g \;\&$ *Stat1*: $\langle \forall x \in g \mid x \oplus e = x \;\&\; x \oplus \mathrm{rvz}(x) = e \;\&\; \mathrm{rvz}(x) \in g\rangle$
Assump $\Rightarrow$ *Stat3*: $\langle \forall x \in g, y \in g \mid x \oplus y \in g \;\&\; x \oplus y = y \oplus x \;\&\; x \oplus \mathrm{rvz}(y) = x \;\mathrm{minz}\; y\rangle$
Suppose $\Rightarrow$ *Stat2*: $\neg\langle \forall x \in g \mid e \oplus x = x\rangle$
$\langle x\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $x \in g \;\&\; e \oplus x \neq x$
$\langle x, e\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $x \oplus e \neq x$
$\langle x\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat4*: $\langle \forall x \in g \mid e \oplus x = x\rangle$
Assump $\Rightarrow$ *Stat5*: $\langle \forall x \in g \mid \mathrm{nneg}(x) \vee \mathrm{nneg}(\mathrm{rvz}(x)) \;\&\; \big(\mathrm{nneg}(x) \;\&\; \mathrm{nneg}(\mathrm{rvz}(x)) \rightarrow x = e\big)\rangle$
Suppose $\Rightarrow$ $c = c'$
EQUAL $\Rightarrow$ $\neg \mathrm{nneg}(c \oplus \mathrm{rvz}(c))$
$\langle c\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $c \oplus \mathrm{rvz}(c) = e$
EQUAL $\Rightarrow$ $\neg \mathrm{nneg}(e)$
$\langle e\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathrm{rvz}(e) \in g$
$\langle \mathrm{rvz}(e)\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $e \oplus \mathrm{rvz}(e) = \mathrm{rvz}(e)$
$\langle e\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $e \oplus \mathrm{rvz}(e) = e$
EQUAL $\Rightarrow$ $\mathrm{rvz}(e) = e$
$\langle e\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\mathrm{nneg}(e) \vee \mathrm{nneg}(\mathrm{rvz}(e))$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \mathrm{nneg}(c' \oplus \mathrm{rvz}(c))$
$\langle c\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat22*: $\mathrm{rvz}(c) \in g$
$\langle c'\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat33*: $\mathrm{rvz}(c') \in g$
$\langle c, \mathrm{rvz}(c')\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $c \oplus \mathrm{rvz}(c') \in g$
$\langle c', \mathrm{rvz}(c)\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $c' \oplus \mathrm{rvz}(c) \in g$
$\langle c \oplus \mathrm{rvz}(c')\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ *Stat6*: $\mathrm{nneg}\big(\mathrm{rvz}(c \oplus \mathrm{rvz}(c'))\big)$
$\langle c' \oplus \mathrm{rvz}(c)\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $\mathrm{nneg}\big(\mathrm{rvz}(c' \oplus \mathrm{rvz}(c))\big)$
$\langle c \oplus \mathrm{rvz}(c')\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathrm{rvz}(c \oplus \mathrm{rvz}(c')) \in g$

$\langle c' \oplus rvz(c) \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz\big(c' \oplus rvz(c)\big) \in g$

$\text{Assump} \Rightarrow \quad Stat7: \ \langle \forall x \in g, y \in g \,|\, nneg(x) \,\&\, nneg(y) \rightarrow nneg(x \oplus y)\rangle$

$\langle rvz(c \oplus rvz(c')), rvz(c' \oplus rvz(c)) \rangle \hookrightarrow Stat7(\langle Stat6 \rangle) \Rightarrow \quad nneg\Big(rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big)\Big)$

$\langle rvz(c \oplus rvz(c')), rvz(c' \oplus rvz(c)) \rangle \hookrightarrow Stat3(\langle Stat6 \rangle) \Rightarrow \quad rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big) \in g$

$\langle c', rvz(c) \rangle \hookrightarrow Stat3([Stat0, Stat22]) \Rightarrow \quad c' \oplus rvz(c) = rvz(c) \oplus c'$

$\langle c, rvz(c') \rangle \hookrightarrow Stat3([Stat0, Stat33]) \Rightarrow \quad c \oplus rvz(c') = rvz(c') \oplus c$

$\text{EQUAL} \Rightarrow \quad nneg\Big(rvz\big(rvz(c') \oplus c\big)\Big) \,\&\, \neg nneg\big(rvz(c) \oplus c'\big) \,\&\, rvz\big(rvz(c') \oplus c\big) \in g$

$\langle rvz(rvz(c') \oplus c), rvz(c \oplus rvz(c')) \oplus rvz(c' \oplus rvz(c)) \rangle \hookrightarrow Stat7 \Rightarrow$

$$nneg\Bigg(rvz\big(rvz(c') \oplus c\big) \oplus \Big(rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big)\Big)\Bigg)$$

$\text{Assump} \Rightarrow \quad Stat9: \ \langle \forall x \in g, y \in g, z \in g \,|\, (x \oplus y) \oplus z = x \oplus (y \oplus z)\rangle$

$\text{Suppose} \Rightarrow \quad rvz\big(rvz(c') \oplus c\big) \oplus \Big(rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big)\Big) \neq rvz(c) \oplus c'$

$\text{Suppose} \Rightarrow \quad Stat8: \ \neg\langle \forall x \in g, y \in g \,|\, rvz(x) \oplus (x \oplus y) = y\rangle$

$\langle x_1, y_1 \rangle \hookrightarrow Stat8 \Rightarrow \quad x_1, y_1 \in g \,\&\, rvz(x_1) \oplus (x_1 \oplus y_1) \neq y_1$

$\langle x_1 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(x_1) \in g$

$\langle rvz(x_1), x_1, y_1 \rangle \hookrightarrow Stat9(\langle Stat8 \rangle) \Rightarrow \quad rvz(x_1) \oplus (x_1 \oplus y_1) = (rvz(x_1) \oplus x_1) \oplus y_1$

$\langle rvz(x_1), x_1 \rangle \hookrightarrow Stat3 \Rightarrow \quad rvz(x_1) \oplus x_1 = x_1 \oplus rvz(x_1)$

$\langle x_1 \rangle \hookrightarrow Stat1 \Rightarrow \quad x_1 \oplus rvz(x_1) = e$

$\langle y_1 \rangle \hookrightarrow Stat4 \Rightarrow \quad e \oplus y_1 = y_1$

$\text{EQUAL } \langle Stat8 \rangle \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat10: \ \langle \forall x \in g, y \in g \,|\, rvz(x) \oplus (x \oplus y) = y\rangle$

$\text{Suppose} \Rightarrow \quad Stat11: \ \neg\langle \forall x \in g, y \in g \,|\, rvz\big(x \oplus rvz(y)\big) = y \oplus rvz(x)\rangle$

$\langle x_2, y_2 \rangle \hookrightarrow Stat11 \Rightarrow \quad x_2, y_2 \in g \,\&\, rvz\big(x_2 \oplus rvz(y_2)\big) \neq y_2 \oplus rvz(x_2)$

$\text{Suppose} \Rightarrow \quad Stat12: \ \neg\langle \forall x \in g \,|\, rvz\big(rvz(x)\big) = x\rangle$

$\langle x_3 \rangle \hookrightarrow Stat12 \Rightarrow \quad x_3 \in g \,\&\, rvz\big(rvz(x_3)\big) \neq x_3$

$\langle x_3 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(x_3) \in g \,\&\, x_3 \oplus rvz(x_3) = e$

$\langle rvz(x_3), x_3 \rangle \hookrightarrow Stat3 \Rightarrow \quad rvz(x_3) \oplus x_3 = x_3 \oplus rvz(x_3)$

$\langle rvz(x_3), x_3 \rangle \hookrightarrow Stat10 \Rightarrow \quad rvz\big(rvz(x_3)\big) \oplus \big(rvz(x_3) \oplus x_3\big) = x_3$

$\langle rvz(x_3) \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz\big(rvz(x_3)\big) \in g$

$\langle rvz(rvz(x_3)) \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz\big(rvz(x_3)\big) \oplus e = rvz\big(rvz(x_3)\big)$

$\text{EQUAL } \langle Stat12 \rangle \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat20: \ \langle \forall x \in g \,|\, rvz\big(rvz(x)\big) = x\rangle$

$\text{Suppose} \Rightarrow \quad Stat13: \ \neg\langle \forall x \in g, y \in g \,|\, rvz(x) \oplus (y \oplus x) = y\rangle$

$\langle x_4, y_4 \rangle \hookrightarrow Stat13 \Rightarrow \quad x_4, y_4 \in g \,\&\, rvz(x_4) \oplus (y_4 \oplus x_4) \neq y_4$

$\langle x_4, y_4 \rangle \hookrightarrow Stat3 \Rightarrow \quad y_4 \oplus x_4 = x_4 \oplus y_4$

$\langle x_4, y_4 \rangle \hookrightarrow Stat10 \Rightarrow \quad rvz(x_4) \oplus (x_4 \oplus y_4) = y_4$

$\text{EQUAL } \langle Stat13 \rangle \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat14: \ \langle \forall x \in g, y \in g \,|\, rvz(x) \oplus (y \oplus x) = y\rangle$

$\text{Suppose} \Rightarrow \quad Stat15: \ \neg\langle \forall x \in g, y \in g \,|\, rvz(x \oplus y) \oplus x = rvz(y)\rangle$

$\langle x_5, y_5 \rangle \hookrightarrow Stat15 \Rightarrow \quad x_5, y_5 \in g \;\&\; rvz(x_5 \oplus y_5) \oplus x_5 \neq rvz(y_5)$
$\langle x_5, y_5 \rangle \hookrightarrow Stat3 \Rightarrow \quad x_5 \oplus y_5 \in g$
$\langle x_5 \oplus y_5 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(x_5 \oplus y_5) \in g$
$\langle y_5 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(y_5) \in g$
$\langle x_5 \oplus y_5, rvz(y_5) \rangle \hookrightarrow Stat14 \Rightarrow \quad rvz(x_5 \oplus y_5) \oplus \big(rvz(y_5) \oplus (x_5 \oplus y_5)\big) = rvz(y_5)$
$\langle y_5, x_5 \rangle \hookrightarrow Stat14 \Rightarrow \quad rvz(y_5) \oplus (x_5 \oplus y_5) = x_5$
EQUAL $\langle Stat15 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat16: \langle \forall x \in g, y \in g \mid rvz(x \oplus y) \oplus x = rvz(y) \rangle$
Suppose $\Rightarrow$ $Stat17: \neg\langle \forall x \in g, y \in g \mid rvz(x \oplus y) = rvz(x) \oplus rvz(y) \rangle$
$\langle x_6, y_6 \rangle \hookrightarrow Stat17 \Rightarrow \quad x_6, y_6 \in g \;\&\; rvz(x_6 \oplus y_6) \neq rvz(x_6) \oplus rvz(y_6)$
$\langle x_6, y_6 \rangle \hookrightarrow Stat3 \Rightarrow \quad x_6 \oplus y_6 \in g$
$\langle y_6 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(y_6) \in g$
$\langle rvz(y_6), x_6 \oplus y_6 \rangle \hookrightarrow Stat16 \Rightarrow \quad rvz\big(rvz(y_6) \oplus (x_6 \oplus y_6)\big) \oplus rvz(y_6) = rvz(x_6 \oplus y_6)$
$\langle y_6, x_6 \rangle \hookrightarrow Stat14 \Rightarrow \quad rvz(y_6) \oplus (x_6 \oplus y_6) = x_6$
EQUAL $\langle Stat17 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat18: \langle \forall x \in g, y \in g \mid rvz(x \oplus y) = rvz(x) \oplus rvz(y) \rangle$
$\langle y_2 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(y_2) \in g$
$\langle x_2, rvz(y_2) \rangle \hookrightarrow Stat18 \Rightarrow \quad rvz\big(x_2 \oplus rvz(y_2)\big) = rvz(x_2) \oplus rvz\big(rvz(y_2)\big)$
$\langle y_2 \rangle \hookrightarrow Stat20 \Rightarrow \quad rvz\big(rvz(y_2)\big) = y_2$
$\langle x_2 \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(x_2) \in g$
$\langle rvz(x_2), y_2 \rangle \hookrightarrow Stat3 \Rightarrow \quad rvz(x_2) \oplus y_2 = y_2 \oplus rvz(x_2)$
EQUAL $\langle Stat11 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat30: \langle \forall x \in g, y \in g \mid rvz\big(x \oplus rvz(y)\big) = y \oplus rvz(x) \rangle$
$\langle c, c' \rangle \hookrightarrow Stat30 \Rightarrow \quad Stat31a: \; rvz\big(c \oplus rvz(c')\big) = c' \oplus rvz(c)$
Suppose $\Rightarrow$ $Stat31: \; rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big) \neq e$
$\langle c', c \rangle \hookrightarrow Stat30 \Rightarrow \quad rvz\big(c' \oplus rvz(c)\big) = c \oplus rvz(c')$
$\langle c', rvz(c), c \oplus rvz(c') \rangle \hookrightarrow Stat9 \Rightarrow \quad \big(c' \oplus rvz(c)\big) \oplus \big(c \oplus rvz(c')\big) = c' \oplus \big(rvz(c) \oplus \big(c \oplus rvz(c')\big)\big)$
$\langle c, rvz(c') \rangle \hookrightarrow Stat10 \Rightarrow \quad rvz(c) \oplus \big(c \oplus rvz(c')\big) = rvz(c')$
$\langle c' \rangle \hookrightarrow Stat1 \Rightarrow \quad c' \oplus rvz(c') = e$
EQUAL $\langle Stat31a \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat32: \; rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big) = e$
$\langle c, rvz(c') \rangle \hookrightarrow Stat3 \Rightarrow \quad c \oplus rvz(c') = rvz(c') \oplus c$
$\langle c', rvz(c) \rangle \hookrightarrow Stat3 \Rightarrow \quad c' \oplus rvz(c) = rvz(c) \oplus c'$
$\langle c, rvz(c') \rangle \hookrightarrow Stat30 \Rightarrow \quad rvz\big(c \oplus rvz(c')\big) = c' \oplus rvz(c)$
$\langle rvz(c) \oplus c' \rangle \hookrightarrow Stat1 \Rightarrow \quad rvz(c) \oplus c' \oplus e = rvz(c) \oplus c'$
EQUAL $\langle Stat32 \rangle \Rightarrow$ $rvz\big(rvz(c') \oplus c\big) \oplus \Big(rvz\big(c \oplus rvz(c')\big) \oplus rvz\big(c' \oplus rvz(c)\big)\Big) = rvz(c) \oplus c'$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY Ordered_add

THEORY $\mathsf{Ordered\_add}(g, e, \oplus, \mathsf{minz}, \mathsf{rvz}, \mathsf{nneg})$

$e \in g \;\&\; \langle \forall x \in g \mid x \oplus e = x \;\&\; x \oplus \mathsf{rvz}(x) = e \;\&\; \mathsf{rvz}(x) \in g \rangle$

$\langle \forall x \in g, y \in g \mid x \oplus y \in g \;\&\; x \oplus y = y \oplus x \;\&\; x \oplus \mathsf{rvz}(y) = x \;\mathsf{minz}\; y \rangle$

$\langle \forall x \in g, y \in g, z \in g \mid (x \oplus y) \oplus z = x \oplus (y \oplus z) \rangle$

$\langle \forall x \in g, y \in g \mid \mathsf{nneg}(x) \;\&\; \mathsf{nneg}(y) \rightarrow \mathsf{nneg}(x \oplus y) \rangle$

$\langle \forall x \in g \mid \mathsf{nneg}(x) \vee \mathsf{nneg}(\mathsf{rvz}(x)) \;\&\; \mathsf{nneg}(x) \;\&\; \mathsf{nneg}(\mathsf{rvz}(x)) \rightarrow x = e \rangle$

$\Rightarrow (\succcurlyeq_{\ominus}, \preccurlyeq_{\ominus}, \succ_{\ominus}, \prec_{\ominus})$

$\langle \forall x, y \mid x \succcurlyeq_{\ominus} y \leftrightarrow \mathsf{nneg}(x \oplus \mathsf{rvz}(y)) \rangle$

$\langle \forall x, y \mid x \preccurlyeq_{\ominus} y \leftrightarrow y \succcurlyeq_{\ominus} x \rangle$

$\langle \forall x, y \mid x \succ_{\ominus} y \leftrightarrow x \succcurlyeq_{\ominus} y \;\&\; x \neq y \rangle$

$\langle \forall x, y \mid x \prec_{\ominus} y \leftrightarrow y \succ_{\ominus} x \rangle$

$\langle \forall x, y \mid x \preccurlyeq_{\ominus} y \leftrightarrow \mathsf{nneg}(y \oplus \mathsf{rvz}(x)) \rangle$

$\langle \forall x, y \mid x, y \in g \rightarrow \big(x \succ_{\ominus} y \leftrightarrow \mathsf{nneg}(x \oplus \mathsf{rvz}(y)) \;\&\; x \neq y\big) \rangle$

$\langle \forall x, y \mid x, y \in g \rightarrow (x \succ_{\ominus} y \leftrightarrow \mathsf{nneg}(x \;\mathsf{minz}\; y) \;\&\; x \neq y) \rangle$

$\langle \forall x, y \mid x, y \in g \;\&\; x = y \vee \neg x \succcurlyeq_{\ominus} y \rightarrow y \succcurlyeq_{\ominus} x \rangle$

END $\mathsf{Ordered\_add}$

Theorem 494 (347) $X \in \mathbb{Z} \rightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(X) \vee \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(X)) \;\&\; \big(\mathsf{is\_nonneg}_{\mathbb{N}}(X) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(X)) \rightarrow X = [\emptyset, \emptyset]\big)$. PROOF:

Suppose_not$(x) \Rightarrow$ *Stat1* : $x \in \mathbb{Z} \;\&\; \neg\big(\mathsf{is\_nonneg}_{\mathbb{N}}(x) \vee \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x))\big) \vee \Big(\big(\mathsf{is\_nonneg}_{\mathbb{N}}(x) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x))\big) \;\&\; x \neq [\emptyset, \emptyset]\Big)$

$\langle x \rangle \hookrightarrow$ *T292* $\Rightarrow$ *Stat2* : $x = \left[x^{[1]}, x^{[2]}\right] \;\&\; x^{[1]} = \emptyset \vee x^{[2]} = \emptyset \;\&\; x^{[1]}, x^{[2]} \in \mathbb{N}$

Use_def$(\mathsf{is_nonneg}_{\mathbb{N}}) \Rightarrow$ *Stat3* : $(\mathsf{is\_nonneg}_{\mathbb{N}}(x) \leftrightarrow x^{[1]} \supseteq x^{[2]}) \;\&\; \big(\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x)) \leftrightarrow \mathsf{Rev}_{\mathbb{Z}}^{[1]}(x) \supseteq \mathsf{Rev}_{\mathbb{Z}}^{[2]}(x)\big)$

Use_def$(\mathsf{Rev}_{\mathbb{Z}}) \Rightarrow$ *Stat4* : $\mathsf{Rev}_{\mathbb{Z}}(x) = \left[x^{[2]}, x^{[1]}\right]$

$\langle$ *Stat4* $\rangle$ ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}^{[1]}(x) = x^{[2]} \;\&\; \mathsf{Rev}_{\mathbb{Z}}^{[2]}(x) = x^{[1]}$

$\langle$ *Stat2*, $* \rangle$ ELEM $\Rightarrow$ *Stat5* : $\mathsf{is\_nonneg}_{\mathbb{N}}(x) \vee \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x))$

$\langle$ *Stat1*, *Stat5*, $* \rangle$ ELEM $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{N}}(x) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x)) \;\&\; x \neq [\emptyset, \emptyset]$

ELEM $\Rightarrow$ *Stat6* : $x^{[1]} = x^{[2]}$

$\langle$ *Stat2*, *Stat6* $\rangle$ ELEM $\Rightarrow$ $x^{[1]} = \emptyset \;\&\; x^{[2]} = \emptyset$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 495 (348) $X, Y \in \mathbb{Z} \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(X) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(Y) \rightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(X +_{\mathbb{Z}} Y) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(X *_{\mathbb{Z}} Y)$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $x, y \in \mathbb{Z} \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(x) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(y) \;\&\; \neg(\mathsf{is\_nonneg}_{\mathbb{N}}(x +_{\mathbb{Z}} y) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(x *_{\mathbb{Z}} y))$

$\langle x \rangle \hookrightarrow$ *T292* $\Rightarrow$ *Stat1* : $x = \left[x^{[1]}, x^{[2]}\right] \;\&\; x^{[1]} = \emptyset \vee x^{[2]} = \emptyset \;\&\; x^{[1]}, x^{[2]} \in \mathbb{N}$

$\langle y\rangle \hookrightarrow T292 \Rightarrow \quad Stat2: \ y = [y^{[1]}, y^{[2]}] \ \& \ y^{[1]} = \emptyset \lor y^{[2]} = \emptyset \ \& \ y^{[1]}, y^{[2]} \in \mathbb{N}$
$\mathsf{Use\_def}(\mathsf{is\_nonneg}_{\mathbb{N}}) \Rightarrow \quad Stat3: \ x^{[1]} \supseteq x^{[2]} \ \& \ y^{[1]} \supseteq y^{[2]}$
$\langle Stat1, Stat2, Stat3, *\rangle \ \mathsf{ELEM} \Rightarrow \quad Stat4: \ x^{[2]} = \emptyset \ \& \ y^{[2]} = \emptyset$
$\mathsf{EQUAL} \Rightarrow \quad Stat5: \ x = [x^{[1]}, \emptyset] \ \& \ y = [y^{[1]}, \emptyset]$
$\mathsf{Use\_def}(+_{\mathbb{Z}}) \Rightarrow \quad x +_{\mathbb{Z}} y = \mathsf{Red}([x^{[1]} + y^{[1]}, x^{[2]} + y^{[2]}])$
$\mathsf{EQUAL} \Rightarrow \quad x +_{\mathbb{Z}} y = \mathsf{Red}([x^{[1]} + y^{[1]}, \emptyset + \emptyset])$
$\mathsf{ALGEBRA} \Rightarrow \quad \emptyset + \emptyset = \emptyset \ \& \ x^{[1]} + y^{[1]} \in \mathbb{N}$
$\mathsf{EQUAL} \Rightarrow \quad x +_{\mathbb{Z}} y = \mathsf{Red}([x^{[1]} + y^{[1]}, \emptyset])$
$\langle x^{[1]} + y^{[1]}\rangle \hookrightarrow T310 \Rightarrow \quad Stat6: \ x +_{\mathbb{Z}} y = [x^{[1]} + y^{[1]}, \emptyset]$
$\langle Stat6\rangle \ \mathsf{ELEM} \Rightarrow \quad x +_{\mathbb{Z}} y^{[1]} \supseteq x +_{\mathbb{Z}} y^{[2]}$
$\mathsf{Use\_def}(\mathsf{is\_nonneg}_{\mathbb{N}}) \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}(x +_{\mathbb{Z}} y)$
$\mathsf{Use\_def}(*_{\mathbb{Z}}) \Rightarrow \quad x *_{\mathbb{Z}} y = \mathsf{Red}([x^{[1]} * y^{[1]} + x^{[2]} * y^{[2]}, x^{[1]} * y^{[2]} + y^{[1]} * x^{[2]}])$
$\mathsf{EQUAL} \Rightarrow \quad x *_{\mathbb{Z}} y = \mathsf{Red}([x^{[1]} * y^{[1]} + \emptyset * \emptyset, x^{[1]} * \emptyset + y^{[1]} * \emptyset])$
$\mathsf{ALGEBRA} \Rightarrow \quad x^{[1]} * y^{[1]} + \emptyset * \emptyset = x^{[1]} * y^{[1]} \ \& \ x^{[1]} * \emptyset + y^{[1]} * \emptyset = \emptyset \ \& \ x^{[1]} * y^{[1]} \in \mathbb{N}$
$\mathsf{EQUAL} \Rightarrow \quad x *_{\mathbb{Z}} y = \mathsf{Red}([x^{[1]} * y^{[1]}, \emptyset])$
$\langle x^{[1]} * y^{[1]}\rangle \hookrightarrow T310 \Rightarrow \quad Stat7: \ x *_{\mathbb{Z}} y = [x^{[1]} * y^{[1]}, \emptyset]$
$\langle Stat7\rangle \ \mathsf{ELEM} \Rightarrow \quad x *_{\mathbb{Z}} y^{[1]} \supseteq x *_{\mathbb{Z}} y^{[2]}$
$\mathsf{Use\_def}(\mathsf{is\_nonneg}_{\mathbb{N}}) \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}(x *_{\mathbb{Z}} y)$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

APPLY $\langle \succcurlyeq_\Theta : \geqslant_{\mathbb{Z}}, \preccurlyeq_\Theta : \leqslant_{\mathbb{Z}}, \succ_\Theta : >_{\mathbb{Z}}, \prec_\Theta : <_{\mathbb{Z}} \rangle$ $\mathsf{Ordered\_add}(g \mapsto \mathbb{Z}, e \mapsto [\emptyset, \emptyset], \oplus \mapsto +_{\mathbb{Z}}, \mathsf{minz} \mapsto -_{\mathbb{Z}}, \mathsf{rvz} \mapsto \mathsf{Rev}_{\mathbb{Z}}, \mathsf{nneg} \mapsto \mathsf{is\_nonneg}_{\mathbb{N}}) \Rightarrow$

Theorem 496 (349) $\big(X \geqslant_{\mathbb{Z}} Y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(X +_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(Y))\big) \ \& \ (X \leqslant_{\mathbb{Z}} Y \leftrightarrow Y \geqslant_{\mathbb{Z}} X) \ \& \ (X >_{\mathbb{Z}} Y \leftrightarrow X \geqslant_{\mathbb{Z}} Y \ \& \ X \neq Y) \ \& \ (X <_{\mathbb{Z}} Y \leftrightarrow Y >_{\mathbb{Z}} X)$.

Theorem 497 (350) $X \in \mathbb{Z} \to \mathsf{is\_nonneg}_{\mathbb{N}}(X *_{\mathbb{Z}} X)$. PROOF:

$\mathsf{Suppose\_not}(x) \Rightarrow \quad x \in \mathbb{Z} \ \& \ \neg\mathsf{is\_nonneg}_{\mathbb{N}}(x *_{\mathbb{Z}} x)$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}(x)$
$\langle x, x\rangle \hookrightarrow T348 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{N}}(x)$
$\langle x\rangle \hookrightarrow T347 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x))$
$\langle x\rangle \hookrightarrow T314 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(x) \in \mathbb{Z}$
$\langle \mathsf{Rev}_{\mathbb{Z}}(x), \mathsf{Rev}_{\mathbb{Z}}(x)\rangle \hookrightarrow T348 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x) *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(x))$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(x) *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(x) = x *_{\mathbb{Z}} x$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 498 (351) $X, Y \in \mathbb{Z} \ \& \ X \neq [\emptyset, \emptyset] \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}(X) \to \big(\mathsf{is\_nonneg}_{\mathbb{N}}(X *_{\mathbb{Z}} Y) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(Y)\big)$. PROOF:

$\mathsf{Suppose\_not}(x,y) \Rightarrow\quad x,y \in \mathbb{Z} \,\&\, x \neq [\emptyset,\emptyset] \,\&\, \mathsf{is\_nonneg}_{\mathbb{N}}(x) \,\&\, \neg\big(\mathsf{is\_nonneg}_{\mathbb{N}}(x *_{\mathbb{Z}} y) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(y)\big)$

$\mathsf{Suppose} \Rightarrow\quad \mathsf{is\_nonneg}_{\mathbb{N}}(y)$

$\langle x,y\rangle \hookrightarrow T348 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad \neg\mathsf{is\_nonneg}_{\mathbb{N}}(y) \,\&\, \mathsf{is\_nonneg}_{\mathbb{N}}(x *_{\mathbb{Z}} y)$

$\langle y\rangle \hookrightarrow T347 \Rightarrow\quad \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{S\_rev}(y))$

$\langle y\rangle \hookrightarrow T314 \Rightarrow\quad \mathsf{S\_rev}(y) \in \mathbb{Z}$

$\langle x, \mathsf{Rev}_{\mathbb{Z}}(y)\rangle \hookrightarrow T348 \Rightarrow\quad \mathsf{is\_nonneg}_{\mathbb{N}}(x *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(y))$

$\mathsf{ALGEBRA} \Rightarrow\quad x *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(y) = \mathsf{Rev}_{\mathbb{Z}}(x *_{\mathbb{Z}} y) \,\&\, x *_{\mathbb{Z}} y \in \mathbb{Z}$

$\mathsf{EQUAL} \Rightarrow\quad \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(x *_{\mathbb{Z}} y))$

$\langle x *_{\mathbb{Z}} y\rangle \hookrightarrow T347 \Rightarrow\quad x *_{\mathbb{Z}} y = [\emptyset,\emptyset]$

$\langle y,x\rangle \hookrightarrow T330 \Rightarrow\quad Stat1:\ y = [\emptyset,\emptyset]$

$\mathsf{Use\_def}(\mathsf{is\_nonneg}_{\mathbb{N}}) \Rightarrow\quad y^{[1]} \not\supseteq y^{[2]}$

$\langle Stat1\rangle\ \mathsf{ELEM} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

Theorem 499 (352) $X \in \mathsf{Fr} \leftrightarrow X = \left[X^{[1]}, X^{[2]}\right] \,\&\, X^{[1]}, X^{[2]} \in \mathbb{Z} \,\&\, X^{[2]} \neq [\emptyset,\emptyset]$. PROOF:

$\mathsf{Suppose\_not}(x) \Rightarrow\quad \neg(x \in \mathsf{Fr} \leftrightarrow x = \left[x^{[1]}, x^{[2]}\right] \,\&\, x^{[1]}, x^{[2]} \in \mathbb{Z} \,\&\, x^{[2]} \neq [\emptyset,\emptyset])$

$\mathsf{Suppose} \Rightarrow\quad Stat1:\ x \in \mathsf{Fr} \,\&\, \neg(x = \left[x^{[1]}, x^{[2]}\right] \,\&\, x^{[1]}, x^{[2]} \in \mathbb{Z} \,\&\, x^{[2]} \neq [\emptyset,\emptyset])$

$\mathsf{Use\_def}(\mathsf{Fr}) \Rightarrow\quad Stat2:\ x \in \{[u,y] :\ u \in \mathbb{Z}, y \in \mathbb{Z} \,|\, y \neq [\emptyset,\emptyset]\}$

$\langle u,y\rangle \hookrightarrow Stat2 \Rightarrow\quad Stat3:\ x = [u,y] \,\&\, u,y \in \mathbb{Z} \,\&\, y \neq [\emptyset,\emptyset]$

$\langle Stat3, *\rangle\ \mathsf{ELEM} \Rightarrow\quad Stat3a:\ x = [u,y]$

$\langle Stat3a\rangle\ \mathsf{ELEM} \Rightarrow\quad Stat4:\ x = \left[x^{[1]}, x^{[2]}\right] \,\&\, x^{[1]} = u \,\&\, x^{[2]} = y$

$\langle Stat3, Stat4, *\rangle\ \mathsf{ELEM} \Rightarrow\quad Stat5:\ x^{[1]}, x^{[2]} \in \mathbb{Z} \,\&\, x^{[2]} \neq [\emptyset,\emptyset]$

$\langle Stat1, Stat4, Stat5, *\rangle\ \mathsf{ELEM} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad x = \left[x^{[1]}, x^{[2]}\right] \,\&\, x^{[1]}, x^{[2]} \in \mathbb{Z} \,\&\, x^{[2]} \neq [\emptyset,\emptyset] \,\&\, x \notin \mathsf{Fr}$

$\mathsf{Use\_def}(\mathsf{Fr}) \Rightarrow\quad Stat7:\ x \notin \{[u,y] :\ u \in \mathbb{Z}, y \in \mathbb{Z} \,|\, y \neq [\emptyset,\emptyset]\}$

$\langle x^{[1]}, x^{[2]}\rangle \hookrightarrow Stat7 \Rightarrow\quad \neg(x = \left[x^{[1]}, x^{[2]}\right] \,\&\, x^{[1]}, x^{[2]} \in \mathbb{Z} \,\&\, x^{[2]} \neq [\emptyset,\emptyset])$

$\mathsf{ELEM} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

Theorem 500 (353) $N \in \mathbb{Q} \rightarrow \mathbf{arb}(N) \in \mathsf{Fr} \,\&\, \mathbf{arb}(N) = \left[\mathbf{arb}(N)^{[1]}, \mathbf{arb}(N)^{[2]}\right] \,\&\, \mathbf{arb}(N)^{[1]}, \mathbf{arb}(N)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(N)^{[2]} \neq [\emptyset,\emptyset]$. PROOF:

$\mathsf{Suppose\_not}(n) \Rightarrow\quad n \in \mathbb{Q} \,\&\, \neg(\mathbf{arb}(n) \in \mathsf{Fr} \,\&\, \mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \,\&\, \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(n)^{[2]} \neq [\emptyset,\emptyset])$

$\langle n\rangle \hookrightarrow T346 \Rightarrow\quad \mathbf{arb}(n) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$

$\langle \mathbf{arb}(n)\rangle \hookrightarrow T352 \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad$ QED

-- Next, as a preliminary toward consideration of the properties of products of rational numbers, we prove that if two pairs x, y and w, zz of formal fractions are given, with $x \approx_{Fr} y$ and similarly for x, y, then the formal product of x by w represents the same rational number as the formal product of y by zz.

Theorem 501 (354) $X, Y \in Fr \; \& \; X \approx_{Fr} Y \; \& \; W, ZZ \in Fr \; \& \; W \approx_{Fr} ZZ \rightarrow$
$\left[X^{[1]} *_{\mathbb{Z}} W^{[2]} +_{\mathbb{Z}} W^{[1]} *_{\mathbb{Z}} X^{[2]}, X^{[2]} *_{\mathbb{Z}} W^{[2]}\right] \approx_{Fr} \left[Y^{[1]} *_{\mathbb{Z}} ZZ^{[2]} +_{\mathbb{Z}} ZZ^{[1]} *_{\mathbb{Z}} Y^{[2]}, Y^{[2]} *_{\mathbb{Z}} ZZ^{[2]}\right]$. PROOF:

Suppose_not$(x, y, w, zz) \Rightarrow$
$x, y \in Fr \; \& \; x \approx_{Fr} y \; \& \; w, zz \in Fr \; \& \; w \approx_{Fr} zz \; \&$
$\neg \left[x^{[1]} *_{\mathbb{Z}} w^{[2]} +_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} x^{[2]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right] \approx_{Fr} \left[y^{[1]} *_{\mathbb{Z}} zz^{[2]} +_{\mathbb{Z}} zz^{[1]} *_{\mathbb{Z}} y^{[2]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right]$
$\langle x \rangle \hookrightarrow T352 \Rightarrow \quad x = \left[x^{[1]}, x^{[2]}\right] \; \& \; x^{[1]}, x^{[2]} \in \mathbb{Z}$
$\langle y \rangle \hookrightarrow T352 \Rightarrow \quad y = \left[y^{[1]}, y^{[2]}\right] \; \& \; y^{[1]}, y^{[2]} \in \mathbb{Z}$
$\langle w \rangle \hookrightarrow T352 \Rightarrow \quad w = \left[w^{[1]}, w^{[2]}\right] \; \& \; w^{[1]}, w^{[2]} \in \mathbb{Z}$
$\langle zz \rangle \hookrightarrow T352 \Rightarrow \quad zz = \left[zz^{[1]}, zz^{[2]}\right] \; \& \; zz^{[1]}, zz^{[2]} \in \mathbb{Z}$
Use_def$(\approx_{Fr}) \Rightarrow \quad x^{[1]} *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} y^{[1]} \; \& \; w^{[1]} *_{\mathbb{Z}} zz^{[2]} = w^{[2]} *_{\mathbb{Z}} zz^{[1]}$
Loc_def $\Rightarrow \quad$ *Stat1* : $prod_1 = \left[x^{[1]} *_{\mathbb{Z}} w^{[2]} +_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} x^{[2]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right]$
$\langle Stat1 \rangle$ ELEM $\Rightarrow \quad$ *Stat2* : ${prod_1}^{[1]} = x^{[1]} *_{\mathbb{Z}} w^{[2]} +_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} x^{[2]}$
$\langle Stat1 \rangle$ ELEM $\Rightarrow \quad$ *Stat3* : ${prod_1}^{[2]} = x^{[2]} *_{\mathbb{Z}} w^{[2]}$
Loc_def $\Rightarrow \quad$ *Stat4* : $prod_2 = \left[y^{[1]} *_{\mathbb{Z}} zz^{[2]} +_{\mathbb{Z}} zz^{[1]} *_{\mathbb{Z}} y^{[2]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right]$
$\langle Stat4 \rangle$ ELEM $\Rightarrow \quad$ *Stat5* : ${prod_2}^{[1]} = y^{[1]} *_{\mathbb{Z}} zz^{[2]} +_{\mathbb{Z}} zz^{[1]} *_{\mathbb{Z}} y^{[2]}$
$\langle Stat4 \rangle$ ELEM $\Rightarrow \quad$ *Stat6* : ${prod_2}^{[2]} = y^{[2]} *_{\mathbb{Z}} zz^{[2]}$
EQUAL $\Rightarrow \quad \neg prod_1 \approx_{Fr} prod_2$
Use_def$(\approx_{Fr}) \Rightarrow \quad$ *Stat7* : ${prod_1}^{[1]} *_{\mathbb{Z}} {prod_2}^{[2]} \neq {prod_1}^{[2]} *_{\mathbb{Z}} {prod_2}^{[1]}$
EQUAL $\langle$*Stat2, Stat3, Stat5, Stat6, Stat7*$\rangle \Rightarrow \quad (x^{[1]} *_{\mathbb{Z}} w^{[2]} +_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} x^{[2]}) *_{\mathbb{Z}} (y^{[2]} *_{\mathbb{Z}} zz^{[2]}) \neq$
$x^{[2]} *_{\mathbb{Z}} w^{[2]} *_{\mathbb{Z}} (y^{[1]} *_{\mathbb{Z}} zz^{[2]} +_{\mathbb{Z}} zz^{[1]} *_{\mathbb{Z}} y^{[2]})$

-- To conclude our proof we simply apply the distributive law for signed integers to the left and right hand sides of the last inequality seen above,

ALGEBRA $\Rightarrow \quad x^{[2]} *_{\mathbb{Z}} w^{[2]} *_{\mathbb{Z}} (y^{[1]} *_{\mathbb{Z}} zz^{[2]} +_{\mathbb{Z}} zz^{[1]} *_{\mathbb{Z}} y^{[2]}) = x^{[2]} *_{\mathbb{Z}} y^{[1]} *_{\mathbb{Z}} (w^{[2]} *_{\mathbb{Z}} zz^{[2]}) +_{\mathbb{Z}} w^{[2]} *_{\mathbb{Z}} zz^{[1]} *_{\mathbb{Z}} (y^{[2]} *_{\mathbb{Z}} x^{[2]})$
ALGEBRA $\Rightarrow \quad (x^{[1]} *_{\mathbb{Z}} w^{[2]} +_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} x^{[2]}) *_{\mathbb{Z}} (y^{[2]} *_{\mathbb{Z}} zz^{[2]}) = x^{[1]} *_{\mathbb{Z}} y^{[2]} *_{\mathbb{Z}} (w^{[2]} *_{\mathbb{Z}} zz^{[2]}) +_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} zz^{[2]} *_{\mathbb{Z}} (y^{[2]} *_{\mathbb{Z}} x^{[2]})$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- The following corollary restates the preceding lemma in an obvious way.

Theorem 502 (355) $X, Y \in Fr \; \& \; X \approx_{Fr} Y \; \& \; W, ZZ \in Fr \; \& \; W \approx_{Fr} ZZ \rightarrow$
$Fr\_to\_\mathbb{Q}(\left[X^{[1]} *_{\mathbb{Z}} W^{[2]} +_{\mathbb{Z}} W^{[1]} *_{\mathbb{Z}} X^{[2]}, X^{[2]} *_{\mathbb{Z}} W^{[2]}\right]) = Fr\_to\_\mathbb{Q}(\left[Y^{[1]} *_{\mathbb{Z}} ZZ^{[2]} +_{\mathbb{Z}} ZZ^{[1]} *_{\mathbb{Z}} Y^{[2]}, Y^{[2]} *_{\mathbb{Z}} ZZ^{[2]}\right])$. PROOF:

Suppose_not $(\mathsf{x},\mathsf{y},\mathsf{w},\mathsf{zz}) \Rightarrow$

$\mathsf{x},\mathsf{y} \in \mathsf{Fr} \;\&\; \mathsf{x} \approx_{\mathsf{Fr}} \mathsf{y} \;\&\; \mathsf{w},\mathsf{zz} \in \mathsf{Fr} \;\&\; \mathsf{w} \approx_{\mathsf{Fr}} \mathsf{zz} \;\&$

$\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[2]} +_{\mathbb{Z}} \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{x}^{[2]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}]) \neq \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} +_{\mathbb{Z}} \mathsf{zz}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}])$

$\langle \mathsf{x} \rangle \hookrightarrow T352 \Rightarrow$ $\quad Stat1: \mathsf{x} = [\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] \;\&\; \mathsf{x}^{[1]}, \mathsf{x}^{[2]} \in \mathbb{Z} \;\&\; \mathsf{x}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{w} \rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathsf{w} = [\mathsf{w}^{[1]}, \mathsf{w}^{[2]}] \;\&\; \mathsf{w}^{[1]}, \mathsf{w}^{[2]} \in \mathbb{Z} \;\&\; \mathsf{w}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{w}^{[2]}, \mathsf{x}^{[2]} \rangle \hookrightarrow T330([Stat1, \cap]) \Rightarrow$ $\quad \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{y} \rangle \hookrightarrow T352 \Rightarrow$ $\quad Stat2: \mathsf{y} = [\mathsf{y}^{[1]}, \mathsf{y}^{[2]}] \;\&\; \mathsf{y}^{[1]}, \mathsf{y}^{[2]} \in \mathbb{Z} \;\&\; \mathsf{y}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{zz} \rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathsf{zz} = [\mathsf{zz}^{[1]}, \mathsf{zz}^{[2]}] \;\&\; \mathsf{zz}^{[1]}, \mathsf{zz}^{[2]} \in \mathbb{Z} \;\&\; \mathsf{zz}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{zz}^{[2]}, \mathsf{y}^{[2]} \rangle \hookrightarrow T330([Stat2, \cap]) \Rightarrow$ $\quad \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} \neq [\emptyset, \emptyset]$

ALGEBRA $\Rightarrow$ $\quad \mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[2]} +_{\mathbb{Z}} \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{x}^{[2]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $\quad \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $\quad \mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} +_{\mathbb{Z}} \mathsf{zz}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $\quad \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} \in \mathbb{Z}$

$\langle [\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[2]} +_{\mathbb{Z}} \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{x}^{[2]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}] \rangle \hookrightarrow T352 \Rightarrow$

$[\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[2]} +_{\mathbb{Z}} \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{x}^{[2]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}] \in \mathsf{Fr}$

$\langle [\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} +_{\mathbb{Z}} \mathsf{zz}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}] \rangle \hookrightarrow T352 \Rightarrow$

$[\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} +_{\mathbb{Z}} \mathsf{zz}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}] \in \mathsf{Fr}$

$\langle \mathsf{x},\mathsf{y},\mathsf{w},\mathsf{zz} \rangle \hookrightarrow T354 \Rightarrow$

$[\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[2]} +_{\mathbb{Z}} \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{x}^{[2]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}] \approx_{\mathsf{Fr}} [\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} +_{\mathbb{Z}} \mathsf{zz}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}]$

$\langle [\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[2]} +_{\mathbb{Z}} \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{x}^{[2]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}], [\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} +_{\mathbb{Z}} \mathsf{zz}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}] \rangle \hookrightarrow T345 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

Theorem 503 (356) $\quad \mathsf{X},\mathsf{Y} \in \mathsf{Fr} \;\&\; \mathsf{X} \approx_{\mathsf{Fr}} \mathsf{Y} \;\&\; \mathsf{W},\mathsf{ZZ} \in \mathsf{Fr} \;\&\; \mathsf{W} \approx_{\mathsf{Fr}} \mathsf{ZZ} \to [\mathsf{X}^{[1]} *_{\mathbb{Z}} \mathsf{W}^{[1]}, \mathsf{X}^{[2]} *_{\mathbb{Z}} \mathsf{W}^{[2]}] \approx_{\mathsf{Fr}} [\mathsf{Y}^{[1]} *_{\mathbb{Z}} \mathsf{ZZ}^{[1]}, \mathsf{Y}^{[2]} *_{\mathbb{Z}} \mathsf{ZZ}^{[2]}]$. PROOF:

Suppose_not $(\mathsf{x},\mathsf{y},\mathsf{w},\mathsf{zz}) \Rightarrow$ $\quad \mathsf{x},\mathsf{y} \in \mathsf{Fr} \;\&\; \mathsf{x} \approx_{\mathsf{Fr}} \mathsf{y} \;\&\; \mathsf{w},\mathsf{zz} \in \mathsf{Fr} \;\&\; \mathsf{w} \approx_{\mathsf{Fr}} \mathsf{zz} \;\&$

$\neg [\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[1]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}] \approx_{\mathsf{Fr}} [\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[1]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}]$

$\langle \mathsf{x} \rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathsf{x} = [\mathsf{x}^{[1]}, \mathsf{x}^{[2]}] \;\&\; \mathsf{x}^{[1]}, \mathsf{x}^{[2]} \in \mathbb{Z}$

$\langle \mathsf{y} \rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathsf{y} = [\mathsf{y}^{[1]}, \mathsf{y}^{[2]}] \;\&\; \mathsf{y}^{[1]}, \mathsf{y}^{[2]} \in \mathbb{Z}$

$\langle \mathsf{w} \rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathsf{w} = [\mathsf{w}^{[1]}, \mathsf{w}^{[2]}] \;\&\; \mathsf{w}^{[1]}, \mathsf{w}^{[2]} \in \mathbb{Z}$

$\langle \mathsf{zz} \rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathsf{zz} = [\mathsf{zz}^{[1]}, \mathsf{zz}^{[2]}] \;\&\; \mathsf{zz}^{[1]}, \mathsf{zz}^{[2]} \in \mathbb{Z}$

Use_def $(\approx_{\mathsf{Fr}}) \Rightarrow$ $\quad \mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{y}^{[2]} = \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{y}^{[1]} \;\&\; \mathsf{w}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[2]} = \mathsf{w}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[1]}$

Use_def $(\approx_{\mathsf{Fr}}) \Rightarrow$ $\quad Stat1:$

$[\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[1]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}]^{[1]} *_{\mathbb{Z}} [\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[1]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}]^{[2]} \neq$

$[\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[1]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}]^{[2]} *_{\mathbb{Z}} [\mathsf{y}^{[1]} *_{\mathbb{Z}} \mathsf{zz}^{[1]}, \mathsf{y}^{[2]} *_{\mathbb{Z}} \mathsf{zz}^{[2]}]^{[1]}$

ELEM $\Rightarrow$ $\quad [\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[1]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}]^{[1]} = \mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[1]}$

ELEM $\Rightarrow$ $\quad [\mathsf{x}^{[1]} *_{\mathbb{Z}} \mathsf{w}^{[1]}, \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}]^{[2]} = \mathsf{x}^{[2]} *_{\mathbb{Z}} \mathsf{w}^{[2]}$

ELEM $\Rightarrow$ $\left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right]^{[1]} = y^{[1]} *_{\mathbb{Z}} zz^{[1]}$

ELEM $\Rightarrow$ $\left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right]^{[2]} = y^{[2]} *_{\mathbb{Z}} zz^{[2]}$

EQUAL $\langle$*Stat1*$\rangle$ $\Rightarrow$ $x^{[1]} *_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} (y^{[2]} *_{\mathbb{Z}} zz^{[2]}) \neq x^{[2]} *_{\mathbb{Z}} w^{[2]} *_{\mathbb{Z}} (y^{[1]} *_{\mathbb{Z}} zz^{[1]})$

ALGEBRA $\Rightarrow$ $x^{[1]} *_{\mathbb{Z}} w^{[1]} *_{\mathbb{Z}} (y^{[2]} *_{\mathbb{Z}} zz^{[2]}) = x^{[1]} *_{\mathbb{Z}} y^{[2]} *_{\mathbb{Z}} (w^{[1]} *_{\mathbb{Z}} zz^{[2]})$

ALGEBRA $\Rightarrow$ $x^{[2]} *_{\mathbb{Z}} w^{[2]} *_{\mathbb{Z}} (y^{[1]} *_{\mathbb{Z}} zz^{[1]}) = x^{[2]} *_{\mathbb{Z}} y^{[1]} *_{\mathbb{Z}} (w^{[2]} *_{\mathbb{Z}} zz^{[1]})$

ELEM $\Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} *_{\mathbb{Z}} (w^{[1]} *_{\mathbb{Z}} zz^{[2]}) \neq x^{[2]} *_{\mathbb{Z}} y^{[1]} *_{\mathbb{Z}} (w^{[2]} *_{\mathbb{Z}} zz^{[1]})$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following corollary restates the preceding lemma in an obvious way.

Theorem 504 (357) $X, Y \in Fr \;\&\; X \approx_{Fr} Y \;\&\; W, ZZ \in Fr \;\&\; W \approx_{Fr} ZZ \rightarrow Fr\_to\_\mathbb{Q}(\left[X^{[1]} *_{\mathbb{Z}} W^{[1]}, X^{[2]} *_{\mathbb{Z}} W^{[2]}\right]) = Fr\_to\_\mathbb{Q}(\left[Y^{[1]} *_{\mathbb{Z}} ZZ^{[1]}, Y^{[2]} *_{\mathbb{Z}} ZZ^{[2]}\right])$. PROOF:

Suppose_not(x, y, w, zz) $\Rightarrow$ $x, y \in Fr \;\&\; x \approx_{Fr} y \;\&\; w, zz \in Fr \;\&\; w \approx_{Fr} zz \;\&$
$Fr\_to\_\mathbb{Q}(\left[x^{[1]} *_{\mathbb{Z}} w^{[1]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right]) \neq Fr\_to\_\mathbb{Q}(\left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right])$

$\langle x \rangle \hookrightarrow$ *T352* $\Rightarrow$ *Stat1*: $x = \left[x^{[1]}, x^{[2]}\right] \;\&\; x^{[1]}, x^{[2]} \in \mathbb{Z} \;\&\; x^{[2]} \neq [\emptyset, \emptyset]$

$\langle w \rangle \hookrightarrow$ *T352* $\Rightarrow$ $w = \left[w^{[1]}, w^{[2]}\right] \;\&\; w^{[1]}, w^{[2]} \in \mathbb{Z} \;\&\; w^{[2]} \neq [\emptyset, \emptyset]$

$\langle w^{[2]}, x^{[2]} \rangle \hookrightarrow$ *T330*([*Stat1*, $\cap$]) $\Rightarrow$ $x^{[2]} *_{\mathbb{Z}} w^{[2]} \neq [\emptyset, \emptyset]$

$\langle y \rangle \hookrightarrow$ *T352* $\Rightarrow$ *Stat2*: $y = \left[y^{[1]}, y^{[2]}\right] \;\&\; y^{[1]}, y^{[2]} \in \mathbb{Z} \;\&\; y^{[2]} \neq [\emptyset, \emptyset]$

$\langle zz \rangle \hookrightarrow$ *T352* $\Rightarrow$ $zz = \left[zz^{[1]}, zz^{[2]}\right] \;\&\; zz^{[1]}, zz^{[2]} \in \mathbb{Z} \;\&\; zz^{[2]} \neq [\emptyset, \emptyset]$

$\langle zz^{[2]}, y^{[2]} \rangle \hookrightarrow$ *T330*([*Stat2*, $\cap$]) $\Rightarrow$ $y^{[2]} *_{\mathbb{Z}} zz^{[2]} \neq [\emptyset, \emptyset]$

ALGEBRA $\Rightarrow$ $x^{[1]} *_{\mathbb{Z}} w^{[1]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $x^{[2]} *_{\mathbb{Z}} w^{[2]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $y^{[1]} *_{\mathbb{Z}} zz^{[1]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $y^{[2]} *_{\mathbb{Z}} zz^{[2]} \in \mathbb{Z}$

$\left\langle \left[x^{[1]} *_{\mathbb{Z}} w^{[1]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right] \right\rangle \hookrightarrow$ *T352* $\Rightarrow$ $\left[x^{[1]} *_{\mathbb{Z}} w^{[1]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right] \in Fr$

$\left\langle \left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right] \right\rangle \hookrightarrow$ *T352* $\Rightarrow$ $\left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right] \in Fr$

$\langle x, y, w, zz \rangle \hookrightarrow$ *T356* $\Rightarrow$ $\left[x^{[1]} *_{\mathbb{Z}} w^{[1]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right] \approx_{Fr} \left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right]$

$\left\langle \left[x^{[1]} *_{\mathbb{Z}} w^{[1]}, x^{[2]} *_{\mathbb{Z}} w^{[2]}\right], \left[y^{[1]} *_{\mathbb{Z}} zz^{[1]}, y^{[2]} *_{\mathbb{Z}} zz^{[2]}\right] \right\rangle \hookrightarrow$ *T345* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemma gives a variant of the formula for the sum of two rationals; this will be useful later.

Theorem 505 (358) $X \in \mathbb{Q} \;\&\; Y, ZZ \in \mathbb{Z} \;\&\; ZZ \neq [\emptyset, \emptyset] \rightarrow X +_{\mathbb{Q}} Fr\_to\_\mathbb{Q}([Y, ZZ]) = Fr\_to\_\mathbb{Q}(\left[\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} ZZ +_{\mathbb{Z}} \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} Y, \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} ZZ\right])$. PROOF:

Suppose_not(x, y, zz) $\Rightarrow$ $x \in \mathbb{Q} \;\&\; y, zz \in \mathbb{Z} \;\&\; zz \neq [\emptyset, \emptyset] \;\&$
$x +_{\mathbb{Q}} Fr\_to\_\mathbb{Q}([y, zz]) \neq Fr\_to\_\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} zz +_{\mathbb{Z}} \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} y, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

-- For let x, y, zz be a counterexample to our assertion. Since x is rational, both arb (x) and [y, zz] are fractions. By Theorem 352, [y, zz] is a fraction equivalent to the standard representative, arb (Fr_to_Ra ([y, zz])) of its equivalence class.

$\langle x \rangle \hookrightarrow T346 \Rightarrow$ $Stat0: \mathbf{arb}(x) \in \mathsf{Fr}$

$\langle \mathbf{arb}(x) \rangle \hookrightarrow T352 \Rightarrow$ $\mathbf{arb}(x) = \left[\mathbf{arb}(x)^{[1]}, \mathbf{arb}(x)^{[2]}\right] \ \& \ \mathbf{arb}(x)^{[1]}, \mathbf{arb}(x)^{[2]} \in \mathbb{Z} \ \& \ \mathbf{arb}(x)^{[2]} \neq [\emptyset, \emptyset]$

$\langle [y, zz] \rangle \hookrightarrow T352 \Rightarrow$ $[y, zz] \in \mathsf{Fr}$

$\langle [y, zz] \rangle \hookrightarrow T344 \Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) \in \mathbb{Q}$

$\langle \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) \rangle \hookrightarrow T353 \Rightarrow$ $\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([y, zz]) \in \mathsf{Fr}$

$\langle [y, zz] \rangle \hookrightarrow T344 \Rightarrow$ $[y, zz] \approx_{\mathsf{Fr}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([y, zz])$

$T341 \Rightarrow$ $Stat1: \langle \forall v \in \mathsf{Fr}, w \in \mathsf{Fr} \mid (v \approx_{\mathsf{Fr}} w \leftrightarrow w \approx_{\mathsf{Fr}} v) \ \& \ v \approx_{\mathsf{Fr}} v \rangle$

$\langle \mathbf{arb}(x), \mathbf{arb}(x) \rangle \hookrightarrow Stat1([Stat0, Stat0]) \Rightarrow$ $\mathbf{arb}(x) \approx_{\mathsf{Fr}} \mathbf{arb}(x)$

-- Using theorem 355, we can therefore replace arb (Fr_to_Ra ([y, zz]) by [y, zz] in the standard expression for the rational product seen below.

$\mathsf{Use\_def}(+_{\mathbb{Q}}) \Rightarrow$ $x +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([y, zz]) +_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([y, zz]) *_{\mathbb{Z}} \mathbf{arb}(x)^{[2]}, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([y, zz])\right])$

$\langle \mathbf{arb}(x), \mathbf{arb}(x), [y, zz], \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([y, zz]) \rangle \hookrightarrow T355 \Rightarrow$

$$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([y, zz]) +_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([y, zz]) *_{\mathbb{Z}} \mathbf{arb}(x)^{[2]}, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([y, zz])\right]) =$$
$$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} [y, zz]^{[2]} +_{\mathbb{Z}} [y, zz]^{[1]} *_{\mathbb{Z}} \mathbf{arb}(x)^{[2]}, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} [y, zz]^{[2]}\right])$$

$\mathsf{ELEM} \Rightarrow$ $[y, zz]^{[2]} = zz \ \& \ [y, zz]^{[1]} = y$

$\mathsf{EQUAL} \Rightarrow$ $x +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} zz +_{\mathbb{Z}} y *_{\mathbb{Z}} \mathbf{arb}(x)^{[2]}, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

$\mathsf{ALGEBRA} \Rightarrow$ $y *_{\mathbb{Z}} \mathbf{arb}(x)^{[2]} = \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} y$

$\mathsf{EQUAL} \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ QED

Theorem 506 (359) $X \in \mathbb{Q} \ \& \ Y, ZZ \in \mathbb{Z} \ \& \ ZZ \neq [\emptyset, \emptyset] \rightarrow X *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([Y, ZZ]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} Y, \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} ZZ\right])$. PROOF:

$\mathsf{Suppose\_not}(x, y, zz) \Rightarrow$ $x \in \mathbb{Q} \ \& \ y, zz \in \mathbb{Z} \ \& \ zz \neq [\emptyset, \emptyset] \ \& \ x *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) \neq \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} y, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

$\langle x \rangle \hookrightarrow T353 \Rightarrow$ $\mathbf{arb}(x) = \left[\mathbf{arb}(x)^{[1]}, \mathbf{arb}(x)^{[2]}\right] \ \& \ \mathbf{arb}(x)^{[1]}, \mathbf{arb}(x)^{[2]} \in \mathbb{Z} \ \& \ \mathbf{arb}(x)^{[2]} \neq [\emptyset, \emptyset]$

$\langle [y, zz] \rangle \hookrightarrow T352 \Rightarrow$ $[y, zz] \in \mathsf{Fr}$

$\langle [y, zz] \rangle \hookrightarrow T344 \Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) \in \mathbb{Q}$

$\langle \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([y, zz]) \in \mathsf{Fr}$

$\langle [y, zz] \rangle \hookrightarrow T344 \Rightarrow$ $[y, zz] \approx_{\mathsf{Fr}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([y, zz])$

$\langle x \rangle \hookrightarrow T346 \Rightarrow$ $Stat0: \mathbf{arb}(x) \in \mathsf{Fr}$

$T341 \Rightarrow$ $Stat1: \langle \forall v \in \mathsf{Fr}, w \in \mathsf{Fr} \mid (v \approx_{\mathsf{Fr}} w \leftrightarrow w \approx_{\mathsf{Fr}} v) \ \& \ v \approx_{\mathsf{Fr}} v \rangle$

$\langle \mathbf{arb}(x), \mathbf{arb}(x) \rangle \hookrightarrow Stat1(\langle Stat0 \rangle) \Rightarrow \quad \mathbf{arb}(x) \approx_{Fr} \mathbf{arb}(x)$

$\mathsf{Use\_def}(*_{\mathbb{Q}}) \Rightarrow \quad x *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([y, zz]), \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([y, zz])\right])$

$\langle \mathbf{arb}(x), \mathbf{arb}(x), [y, zz], \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([y, zz]) \rangle \hookrightarrow T357 \Rightarrow \quad x *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) =$
$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} [y, zz]^{[1]}, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} [y, zz]^{[2]}\right])$

$\mathsf{ELEM} \Rightarrow \quad [y, zz]^{[1]} = y \;\&\; [y, zz]^{[2]} = zz$

$\mathsf{EQUAL} \Rightarrow \quad x *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} y, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false};$ $\quad \mathsf{Discharge} \Rightarrow$ QED

Theorem 507 (360) $X \in \mathsf{Fr} \rightarrow X \approx_{Fr} \left[\mathsf{Rev}_{\mathbb{Z}}(X^{[1]}), \mathsf{Rev}_{\mathbb{Z}}(X^{[2]})\right]$. PROOF:

$\mathsf{Suppose\_not}(x) \Rightarrow \quad x \in \mathsf{Fr} \;\&\; \neg x \approx_{Fr} \left[\mathsf{Rev}_{\mathbb{Z}}(x^{[1]}), \mathsf{Rev}_{\mathbb{Z}}(x^{[2]})\right]$

$\langle x \rangle \hookrightarrow T352 \Rightarrow \quad x = \left[x^{[1]}, x^{[2]}\right] \;\&\; x^{[1]}, x^{[2]} \in \mathbb{Z} \;\&\; x^{[2]} \neq [\emptyset, \emptyset]$

$\mathsf{Use\_def}(\approx_{Fr}) \Rightarrow \quad Stat1: \; x^{[1]} *_{\mathbb{Z}} \left[\mathsf{Rev}_{\mathbb{Z}}(x^{[1]}), \mathsf{Rev}_{\mathbb{Z}}(x^{[2]})\right]^{[2]} \neq x^{[2]} *_{\mathbb{Z}} \left[\mathsf{Rev}_{\mathbb{Z}}(x^{[1]}), \mathsf{Rev}_{\mathbb{Z}}(x^{[2]})\right]^{[1]}$

$\langle x^{[1]}, x^{[2]} \rangle \hookrightarrow T313 \Rightarrow \quad x^{[1]} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(x^{[2]}) = \mathsf{Rev}_{\mathbb{Z}}(x^{[1]} *_{\mathbb{Z}} x^{[2]})$

$\langle x^{[2]}, x^{[1]} \rangle \hookrightarrow T313 \Rightarrow \quad x^{[2]} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(x^{[1]}) = \mathsf{Rev}_{\mathbb{Z}}(x^{[2]} *_{\mathbb{Z}} x^{[1]})$

$\langle Stat1 \rangle \; \mathsf{ELEM} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(x^{[1]} *_{\mathbb{Z}} x^{[2]}) \neq \mathsf{Rev}_{\mathbb{Z}}(x^{[2]} *_{\mathbb{Z}} x^{[1]})$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false};$ $\quad \mathsf{Discharge} \Rightarrow$ QED

-- Next we show that if two fractions, both with positive denominators, are equivalent, then one is has a non-negative numerator if and only if the other does. This lemma prepares for the proof of the more general statement given by Theorem 364 below.

Theorem 508 (361) $X, Y \in \mathsf{Fr} \;\&\; X \approx_{Fr} Y \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(X^{[2]}) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(Y^{[2]}) \rightarrow \left(\mathsf{is\_nonneg}_{\mathbb{N}}(X^{[1]}) \vee X^{[1]} = [\emptyset, \emptyset] \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(Y^{[1]}) \vee Y^{[1]} = [\emptyset, \emptyset]\right)$. PROOF:

$\mathsf{Suppose\_not}(x, y) \Rightarrow \quad Stat1: \; x, y \in \mathsf{Fr} \;\&\; x \approx_{Fr} y \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(x^{[2]}) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(y^{[2]}) \;\&\; \neg\left(\mathsf{is\_nonneg}_{\mathbb{N}}(x^{[1]}) \vee x^{[1]} = [\emptyset, \emptyset] \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(y^{[1]}) \vee y^{[1]} = [\emptyset, \emptyset]\right)$

-- For consider a counterexample x, y. It is easily seen that if one of the fractions is zero so is the other. Hence we have only to consider the case in which one of the fractions, say x, has a positive numerator and the other has a negative numerator.

$\langle x \rangle \hookrightarrow T352 \Rightarrow \quad x = \left[x^{[1]}, x^{[2]}\right] \;\&\; x^{[1]}, x^{[2]} \in \mathbb{Z} \;\&\; x^{[2]} \neq [\emptyset, \emptyset]$

$\langle y \rangle \hookrightarrow T352 \Rightarrow \quad y = \left[y^{[1]}, y^{[2]}\right] \;\&\; y^{[1]}, y^{[2]} \in \mathbb{Z} \;\&\; y^{[2]} \neq [\emptyset, \emptyset]$

$\mathsf{Use\_def}(\approx_{Fr}) \Rightarrow \quad Stat2: \; x^{[1]} *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} y^{[1]}$

$\mathsf{Suppose} \Rightarrow \quad x^{[1]} = [\emptyset, \emptyset]$

$\mathsf{EQUAL} \Rightarrow \quad [\emptyset, \emptyset] *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} y^{[1]}$

$\langle y^{[2]} \rangle \hookrightarrow T324 \Rightarrow \quad x^{[2]} *_{\mathbb{Z}} y^{[1]} = [\emptyset, \emptyset]$

$\langle y^{[1]}, x^{[2]} \rangle \hookrightarrow T330([Stat1, \cap]) \Rightarrow \quad y^{[1]} = [\emptyset, \emptyset]$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat3: \; x^{[1]} \neq [\emptyset, \emptyset]$
Suppose $\Rightarrow$ $y^{[1]} = [\emptyset, \emptyset]$
EQUAL $\Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} [\emptyset, \emptyset]$
ALGEBRA $\Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} = [\emptyset, \emptyset] *_{\mathbb{Z}} x^{[2]}$
$\langle x^{[2]} \rangle \hookrightarrow T324 \Rightarrow \quad x^{[1]} *_{\mathbb{Z}} y^{[2]} = [\emptyset, \emptyset]$
$\langle y^{[2]}, x^{[1]} \rangle \hookrightarrow T330([Stat1, \cap]) \Rightarrow \quad x^{[1]} = [\emptyset, \emptyset]$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $Stat4: \; y^{[1]} \neq [\emptyset, \emptyset]$
$\langle Stat1, Stat3, Stat4 \rangle$ ELEM $\Rightarrow$ $\neg\left(\text{is\_nonneg}_{\mathbb{N}}(x^{[1]}) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(y^{[1]})\right)$
ALGEBRA $\Rightarrow$ $x^{[2]} *_{\mathbb{Z}} y^{[1]}, x^{[1]} *_{\mathbb{Z}} y^{[2]} \in \mathbb{Z}$

-- In this case, theorem 347 tells us that S_Rev (car (y)) must be non-negative. A bit of algebra now shows that both car (x) S_TIMES cdr (y) and S_Rev (car (x) S_TIMES cdr (y)) must be non-negative. Hence car (x) S_TIMES cdr (y) must be zero, and therefore car (x) must also be zero by theorem 330.

Suppose $\Rightarrow$ $Stat5: \; \text{is\_nonneg}_{\mathbb{N}}(x^{[1]})$
ELEM $\Rightarrow$ $\neg\text{is\_nonneg}_{\mathbb{N}}(y^{[1]})$
$\langle x^{[1]}, y^{[2]} \rangle \hookrightarrow T348 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{N}}(x^{[1]} *_{\mathbb{Z}} y^{[2]})$
$\langle y^{[1]} \rangle \hookrightarrow T347 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{N}}\left(\text{Rev}_{\mathbb{Z}}(y^{[1]})\right)$
$\langle y^{[1]} \rangle \hookrightarrow T314 \Rightarrow \quad \text{Rev}_{\mathbb{Z}}(y^{[1]}) \in \mathbb{Z}$
$\langle x^{[2]}, \text{Rev}_{\mathbb{Z}}(y^{[1]}) \rangle \hookrightarrow T348 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{N}}\left(x^{[2]} *_{\mathbb{Z}} \text{Rev}_{\mathbb{Z}}(y^{[1]})\right)$
ALGEBRA $\Rightarrow$ $Stat6: \; \text{is\_nonneg}_{\mathbb{N}}\left(\text{Rev}_{\mathbb{Z}}(x^{[2]} *_{\mathbb{Z}} y^{[1]})\right)$
EQUAL $\langle Stat2, Stat6 \rangle \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}\left(\text{Rev}_{\mathbb{Z}}(x^{[1]} *_{\mathbb{Z}} y^{[2]})\right)$
$\langle x^{[1]} *_{\mathbb{Z}} y^{[2]} \rangle \hookrightarrow T347 \Rightarrow \quad x^{[1]} *_{\mathbb{Z}} y^{[2]} = [\emptyset, \emptyset]$
$\langle y^{[2]}, x^{[1]} \rangle \hookrightarrow T330([Stat1, \cap]) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat7: \; \neg\text{is\_nonneg}_{\mathbb{N}}(x^{[1]}) \;\&\; \text{is\_nonneg}_{\mathbb{N}}(y^{[1]})$
$\langle x^{[2]}, y^{[1]} \rangle \hookrightarrow T348 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{N}}(x^{[2]} *_{\mathbb{Z}} y^{[1]})$
$\langle x^{[1]} \rangle \hookrightarrow T347 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{N}}\left(\text{Rev}_{\mathbb{Z}}(x^{[1]})\right)$
$\langle x^{[1]} \rangle \hookrightarrow T314 \Rightarrow \quad \text{Rev}_{\mathbb{Z}}(x^{[1]}) \in \mathbb{Z}$
$\langle y^{[2]}, \text{Rev}_{\mathbb{Z}}(x^{[1]}) \rangle \hookrightarrow T348 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{N}}\left(y^{[2]} *_{\mathbb{Z}} \text{Rev}_{\mathbb{Z}}(x^{[1]})\right)$
ALGEBRA $\Rightarrow$ $Stat8: \; \text{is\_nonneg}_{\mathbb{N}}\left(\text{Rev}_{\mathbb{Z}}(x^{[1]} *_{\mathbb{Z}} y^{[2]})\right)$
EQUAL $\langle Stat2, Stat8 \rangle \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}\left(\text{Rev}_{\mathbb{Z}}(x^{[2]} *_{\mathbb{Z}} y^{[1]})\right)$
$\langle x^{[2]} *_{\mathbb{Z}} y^{[1]} \rangle \hookrightarrow T347 \Rightarrow \quad x^{[2]} *_{\mathbb{Z}} y^{[1]} = [\emptyset, \emptyset]$
$\langle y^{[1]}, x^{[2]} \rangle \hookrightarrow T330([Stat1, \cap]) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem generalizes the preceding result by showing that if one of two equivalent fractions is non-negative, so is the other.

Theorem 509 (362) $X, Y \in Fr \,\&\, X \approx_{Fr} Y \to \left(\text{is\_nonneg}_{\mathbb{N}}(X^{[1]} *_{\mathbb{Z}} X^{[2]}) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(Y^{[1]} *_{\mathbb{Z}} Y^{[2]})\right)$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $\mathit{Stat0}:\ x, y \in Fr \,\&\, x \approx_{Fr} y \,\&\, \neg\left(\text{is\_nonneg}_{\mathbb{N}}(x^{[1]} *_{\mathbb{Z}} x^{[2]}) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(y^{[1]} *_{\mathbb{Z}} y^{[2]})\right)$

$\langle x \rangle \hookrightarrow T352 \Rightarrow$ $x = \left[x^{[1]}, x^{[2]}\right] \,\&\, x^{[1]}, x^{[2]} \in \mathbb{Z} \,\&\, x^{[2]} \neq [\emptyset, \emptyset]$

$\langle y \rangle \hookrightarrow T352 \Rightarrow$ $y = \left[y^{[1]}, y^{[2]}\right] \,\&\, y^{[1]}, y^{[2]} \in \mathbb{Z} \,\&\, y^{[2]} \neq [\emptyset, \emptyset]$

Loc_def $\Rightarrow$ $ax = x^{[1]}$

Loc_def $\Rightarrow$ $dx = x^{[2]}$

Loc_def $\Rightarrow$ $ay = y^{[1]}$

Loc_def $\Rightarrow$ $dy = y^{[2]}$

EQUAL $\Rightarrow$ $dx \neq [\emptyset, \emptyset] \,\&\, dy \neq [\emptyset, \emptyset]$

Use_def$(\approx_{Fr}) \Rightarrow$ $x^{[1]} *_{\mathbb{Z}} y^{[2]} = x^{[2]} *_{\mathbb{Z}} y^{[1]}$

EQUAL $\Rightarrow$ $ax *_{\mathbb{Z}} dy = dx *_{\mathbb{Z}} ay$

EQUAL $\Rightarrow$ $\mathit{Stat1}:\ ax, dx, ay, dy \in \mathbb{Z}$

EQUAL $\Rightarrow$ $ax *_{\mathbb{Z}} dy *_{\mathbb{Z}} (dx *_{\mathbb{Z}} dy) = dx *_{\mathbb{Z}} ay *_{\mathbb{Z}} (dx *_{\mathbb{Z}} dy)$

EQUAL $\Rightarrow$ $\neg\left(\text{is\_nonneg}_{\mathbb{N}}(ax *_{\mathbb{Z}} dx) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(ay *_{\mathbb{Z}} dy)\right)$

ALGEBRA $\langle \mathit{Stat1} \rangle \Rightarrow$ $dy *_{\mathbb{Z}} dy *_{\mathbb{Z}} (ax *_{\mathbb{Z}} dx) = dx *_{\mathbb{Z}} dx *_{\mathbb{Z}} (ay *_{\mathbb{Z}} dy)$

EQUAL $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}(dy *_{\mathbb{Z}} dy *_{\mathbb{Z}} (ax *_{\mathbb{Z}} dx)) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(dx *_{\mathbb{Z}} dx *_{\mathbb{Z}} (ay *_{\mathbb{Z}} dy))$

$\langle dx \rangle \hookrightarrow T350 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}(dx *_{\mathbb{Z}} dx)$

$\langle dy \rangle \hookrightarrow T350 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}(dy *_{\mathbb{Z}} dy)$

$\langle dx, dx \rangle \hookrightarrow T330([\mathit{Stat0}, \cap]) \Rightarrow$ $dx *_{\mathbb{Z}} dx \neq [\emptyset, \emptyset]$

$\langle dy, dy \rangle \hookrightarrow T330([\mathit{Stat0}, \cap]) \Rightarrow$ $dy *_{\mathbb{Z}} dy \neq [\emptyset, \emptyset]$

ALGEBRA $\Rightarrow$ $dx *_{\mathbb{Z}} dx, ay *_{\mathbb{Z}} dy \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $dy *_{\mathbb{Z}} dy, ax *_{\mathbb{Z}} dx \in \mathbb{Z}$

$\langle dy *_{\mathbb{Z}} dy, ax *_{\mathbb{Z}} dx \rangle \hookrightarrow T351 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}(dy *_{\mathbb{Z}} dy *_{\mathbb{Z}} (ax *_{\mathbb{Z}} dx)) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(ax *_{\mathbb{Z}} dx)$

$\langle dx *_{\mathbb{Z}} dx, ay *_{\mathbb{Z}} dy \rangle \hookrightarrow T351 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}(dx *_{\mathbb{Z}} dx *_{\mathbb{Z}} (ay *_{\mathbb{Z}} dy)) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(ay *_{\mathbb{Z}} dy)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 510 (363) $X \in Fr \to \left(\text{is\_nonneg}_{\mathbb{Q}}(X) \leftrightarrow \text{Fr\_is\_nonneg}\left(\left[\text{Rev}_{\mathbb{Z}}(X^{[1]}), \text{Rev}_{\mathbb{Z}}(X^{[2]})\right]\right)\right)$. PROOF:

Suppose_not$(n) \Rightarrow$ $\mathit{Stat1}:\ n \in Fr \,\&\, \neg\left(\text{Fr\_is\_nonneg}(n) \leftrightarrow \text{Fr\_is\_nonneg}\left(\left[\text{Rev}_{\mathbb{Z}}(n^{[1]}), \text{Rev}_{\mathbb{Z}}(n^{[2]})\right]\right)\right)$

$\langle n \rangle \hookrightarrow T352 \Rightarrow$ $n = \left[n^{[1]}, n^{[2]}\right] \,\&\, n^{[1]}, n^{[2]} \in \mathbb{Z} \,\&\, n^{[2]} \neq [\emptyset, \emptyset]$

Use_def(Fr_is_nonneg) $\Rightarrow$ $\mathit{Stat2}:\ \text{Fr\_is\_nonneg}(n) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(n^{[1]} *_{\mathbb{Z}} n^{[2]})$

Use_def(Fr_is_nonneg) $\Rightarrow$ $\mathit{Stat3}:\ \text{Fr\_is\_nonneg}\left(\left[\text{Rev}_{\mathbb{Z}}(n^{[1]}), \text{Rev}_{\mathbb{Z}}(n^{[2]})\right]\right) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}\left(\left[\text{Rev}_{\mathbb{Z}}(n^{[1]}), \text{Rev}_{\mathbb{Z}}(n^{[2]})\right]^{[1]} *_{\mathbb{Z}} \left[\text{Rev}_{\mathbb{Z}}(n^{[1]}), \text{Rev}_{\mathbb{Z}}(n^{[2]})\right]^{[2]}\right)$

$\langle \mathit{Stat1}, \mathit{Stat2}, \mathit{Stat3}, * \rangle$ ELEM $\Rightarrow$ $\neg\left(\text{is\_nonneg}_{\mathbb{N}}\left(\left[\text{Rev}_{\mathbb{Z}}(n^{[1]}), \text{Rev}_{\mathbb{Z}}(n^{[2]})\right]^{[1]} *_{\mathbb{Z}} \left[\text{Rev}_{\mathbb{Z}}(n^{[1]}), \text{Rev}_{\mathbb{Z}}(n^{[2]})\right]^{[2]}\right) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(n^{[1]} *_{\mathbb{Z}} n^{[2]})\right)$

$\langle n^{[1]} \rangle \hookrightarrow T314 \Rightarrow$ $\text{Rev}_{\mathbb{Z}}(n^{[1]}) \in \mathbb{Z}$

$\langle n^{[2]} \rangle \hookrightarrow T314 \Rightarrow$ $\text{Rev}_{\mathbb{Z}}(n^{[2]}) \in \mathbb{Z}$

ELEM $\Rightarrow$ $\left[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[1]}), \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[2]})\right]^{[1]} = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[1]})$

ELEM $\Rightarrow$ $\left[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[1]}), \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[2]})\right]^{[2]} = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[2]})$

EQUAL $\Rightarrow$ $\mathit{Stat4}:\ \neg\Big(\mathsf{is\_nonneg}_{\mathbb{N}}\left(\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[1]}) *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[2]})\right) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]})\Big)$

ALGEBRA $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[1]}) *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}^{[2]}) = \mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]}$

EQUAL $\langle \mathit{Stat4} \rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that if two fractions, both with positive denominators, are equivalent, then one has a non-negative numerator if and only if the other does.

Theorem 511 (364) $\mathsf{X}, \mathsf{Y} \in \mathsf{Fr}\ \&\ \mathsf{X} \approx_{\mathsf{Fr}} \mathsf{Y} \rightarrow \big(\mathsf{Fr\_is\_nonneg}(\mathsf{X}) \leftrightarrow \mathsf{Fr\_is\_nonneg}(\mathsf{Y})\big)$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathit{Stat0}:\ \mathsf{n}, \mathsf{m} \in \mathsf{Fr}\ \&\ \mathsf{n} \approx_{\mathsf{Fr}} \mathsf{m}\ \&\ \neg\big(\mathsf{Fr\_is\_nonneg}(\mathsf{n}) \leftrightarrow \mathsf{Fr\_is\_nonneg}(\mathsf{m})\big)$

$\langle \mathsf{n} \rangle \hookrightarrow \mathit{T352} \Rightarrow$ $\mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right]\ \&\ \mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{Z}\ \&\ \mathsf{n}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{m} \rangle \hookrightarrow \mathit{T352} \Rightarrow$ $\mathsf{m} = \left[\mathsf{m}^{[1]}, \mathsf{m}^{[2]}\right]\ \&\ \mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{Z}\ \&\ \mathsf{m}^{[2]} \neq [\emptyset, \emptyset]$

Use_def(Fr_is_nonneg) $\Rightarrow$ $\neg\big(\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]}) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{m}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]})\big)$

Use_def($\approx_{\mathsf{Fr}}$) $\Rightarrow$ $\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]} = \mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[1]}$

EQUAL $\Rightarrow$ $\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]} *_{\mathbb{Z}} (\mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]}) = \mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[1]} *_{\mathbb{Z}} (\mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]})$

ALGEBRA $\Rightarrow$ $\mathit{Stat1}:\ \mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]}, \mathsf{m}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $\mathit{Stat2}:\ \mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{n}^{[2]}, \mathsf{m}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $\mathsf{m}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]} *_{\mathbb{Z}} (\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]}) = \mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{n}^{[2]} *_{\mathbb{Z}} (\mathsf{m}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]})$

$\langle \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T350} \Rightarrow$ $\mathit{Stat3}:\ \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{n}^{[2]})$

$\langle \mathsf{m}^{[2]} \rangle \hookrightarrow \mathit{T350} \Rightarrow$ $\mathit{Stat4}:\ \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{m}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]})$

$\langle \mathsf{n}^{[2]}, \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T330}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathit{Stat5}:\ \mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{n}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{m}^{[2]}, \mathsf{m}^{[2]} \rangle \hookrightarrow \mathit{T330}([\mathit{Stat0}, \cap]) \Rightarrow$ $\mathit{Stat6}:\ \mathsf{m}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathsf{m}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]}, \mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]} \rangle \hookrightarrow \mathit{T351}(\langle \mathit{Stat1}, \mathit{Stat2}, \mathit{Stat4}, \mathit{Stat6} \rangle) \Rightarrow$

$\mathsf{is\_nonneg}_{\mathbb{N}}\big(\mathsf{m}^{[2]} *_{\mathbb{Z}} \mathsf{m}^{[2]} *_{\mathbb{Z}} (\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]})\big) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{n}^{[1]} *_{\mathbb{Z}} \mathsf{n}^{[2]})$

$\langle \mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{n}^{[2]}, \mathsf{m}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]} \rangle \hookrightarrow \mathit{T351}(\langle \mathit{Stat1}, \mathit{Stat2}, \mathit{Stat3}, \mathit{Stat5} \rangle) \Rightarrow$

$\mathsf{is\_nonneg}_{\mathbb{N}}\big(\mathsf{n}^{[2]} *_{\mathbb{Z}} \mathsf{n}^{[2]} *_{\mathbb{Z}} (\mathsf{m}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]})\big) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{m}^{[1]} *_{\mathbb{Z}} \mathsf{m}^{[2]})$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Now we are in position to begin consideration of the algebra of rational numbers. As a first result of this kind we prove the commutative law for rational numbers.

-- Commutativity of Addition

Theorem 512 (365) $\mathsf{N}, \mathsf{M} \in \mathbb{Q} \rightarrow \mathsf{N} +_{\mathbb{Q}} \mathsf{M} \in \mathbb{Q}\ \&\ \mathsf{N} +_{\mathbb{Q}} \mathsf{M} = \mathsf{M} +_{\mathbb{Q}} \mathsf{N}$. PROOF:

$\text{Suppose\_not}(n,m) \Rightarrow \quad n,m \in \mathbb{Q} \,\&\, n +_{\mathbb{Q}} m \notin \mathbb{Q} \vee n +_{\mathbb{Q}} m \neq m +_{\mathbb{Q}} n$

$\langle n \rangle \hookrightarrow T346 \Rightarrow \quad \mathbf{arb}(n) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$

$\langle m \rangle \hookrightarrow T346 \Rightarrow \quad \mathbf{arb}(m) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(m)) = m$

$\langle \mathbf{arb}(n) \rangle \hookrightarrow T352 \Rightarrow \quad Stat1: \ \mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \,\&\, \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathbf{arb}(m) \rangle \hookrightarrow T352 \Rightarrow \quad Stat2: \ \mathbf{arb}(m) = \left[\mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]}\right] \,\&\, \mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(m)^{[2]} \neq [\emptyset, \emptyset]$

$\text{Suppose} \Rightarrow \quad Stat3: \ n +_{\mathbb{Q}} m \notin \mathbb{Q}$

$\text{Use\_def}(+_{\mathbb{Q}}) \Rightarrow \quad \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right]) \notin \mathbb{Q}$

$T343 \Rightarrow \quad Stat4: \ \langle \forall x \,|\, x \in \text{Fr} \rightarrow \text{Fr\_to\_}\mathbb{Q}(x) \in \mathbb{Q} \rangle$

$\langle \left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right] \rangle \hookrightarrow Stat4(\langle Stat3 \rangle) \Rightarrow$

$\quad \left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right] \notin \text{Fr}$

$\text{ALGEBRA} \Rightarrow \quad \mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} \in \mathbb{Z}$

$\text{ALGEBRA} \Rightarrow \quad \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} \in \mathbb{Z}$

$\langle \mathbf{arb}(m)^{[2]}, \mathbf{arb}(n)^{[2]} \rangle \hookrightarrow T330(\langle Stat1, Stat2, * \rangle) \Rightarrow \quad \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} \neq [\emptyset, \emptyset]$

$\langle \left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right] \rangle \hookrightarrow T352 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad n +_{\mathbb{Q}} m \neq m +_{\mathbb{Q}} n$

$\text{Use\_def}(+_{\mathbb{Q}}) \Rightarrow \quad \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}\right]) \neq$

$\quad \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} +_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right])$

$\text{ALGEBRA} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$ QED

-- The following corollary of theorem 358 expresses the sum of two rationals, one of them derived explicitly from a fraction, in a form that is sometimes convenient.

Theorem 513 (366) $X \in \mathbb{Q} \,\&\, Y, ZZ \in \mathbb{Z} \,\&\, ZZ \neq [\emptyset, \emptyset] \rightarrow \text{Fr\_to\_}\mathbb{Q}([Y, ZZ]) +_{\mathbb{Q}} X = \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} ZZ +_{\mathbb{Z}} \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} Y, \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} ZZ\right])$. PROOF:

$\text{Suppose\_not}(x,y,zz) \Rightarrow \quad x \in \mathbb{Q} \,\&\, y, zz \in \mathbb{Z} \,\&\, zz \neq [\emptyset, \emptyset] \,\&$

$\quad \text{Fr\_to\_}\mathbb{Q}([y, zz]) +_{\mathbb{Q}} x \neq \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} zz +_{\mathbb{Z}} \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} y, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

$\langle x, y, zz \rangle \hookrightarrow T358 \Rightarrow \quad \text{Fr\_to\_}\mathbb{Q}([y, zz]) +_{\mathbb{Q}} x \neq x +_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([y, zz])$

$\langle [y, zz] \rangle \hookrightarrow T352 \Rightarrow \quad [y, zz] \in \text{Fr}$

$\langle [y, zz] \rangle \hookrightarrow T344 \Rightarrow \quad \text{Fr\_to\_}\mathbb{Q}([y, zz]) \in \mathbb{Q}$

$\langle x, \text{Fr\_to\_}\mathbb{Q}([y, zz]) \rangle \hookrightarrow T365 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$ QED

-- Next we express the sum of two rationals, both derived explicitly from a fraction, in a convenient form.

Theorem 514 (367) $X, Y, ZZ, W \in \mathbb{Z} \,\&\, Y \neq [\emptyset, \emptyset] \,\&\, W \neq [\emptyset, \emptyset] \rightarrow$
$\text{Fr\_to\_}\mathbb{Q}([X, Y]) +_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([ZZ, W]) = \text{Fr\_to\_}\mathbb{Q}([X *_{\mathbb{Z}} W +_{\mathbb{Z}} ZZ *_{\mathbb{Z}} Y, Y *_{\mathbb{Z}} W])$. PROOF:

Suppose_not$(x, y, zz, w) \Rightarrow$ $x, y, zz, w \in \mathbb{Z} \,\&\, y \neq [\emptyset, \emptyset] \,\&\, w \neq [\emptyset, \emptyset] \,\&$
$\text{Fr\_to\_}\mathbb{Q}([x, y]) +_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([zz, w]) \neq \text{Fr\_to\_}\mathbb{Q}([x *_{\mathbb{Z}} w +_{\mathbb{Z}} zz *_{\mathbb{Z}} y, y *_{\mathbb{Z}} w])$

$\langle [x, y] \rangle \hookrightarrow T352 \Rightarrow$ $[x, y] \in \text{Fr}$

$\langle [x, y] \rangle \hookrightarrow T344 \Rightarrow$ $\text{Fr\_to\_}\mathbb{Q}([x, y]) \in \mathbb{Q}$

$\langle \text{Fr\_to\_}\mathbb{Q}([x, y]) \rangle \hookrightarrow T353 \Rightarrow$ $\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})([x, y]) \in \text{Fr}$

$\langle \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})([x, y]) \rangle \hookrightarrow T352 \Rightarrow$ $\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) \in \mathbb{Z}$

$\langle [x, y] \rangle \hookrightarrow T344 \Rightarrow$ $[x, y] \approx_{\text{Fr}} \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})([x, y])$

$\langle \text{Fr\_to\_}\mathbb{Q}([x, y]), zz, w \rangle \hookrightarrow T358 \Rightarrow$ $\text{Fr\_to\_}\mathbb{Q}([x, y]) +_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([zz, w]) =$
$\text{Fr\_to\_}\mathbb{Q}\Big(\Big[\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([x, y]) *_{\mathbb{Z}} w +_{\mathbb{Z}} \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) *_{\mathbb{Z}} zz, \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) *_{\mathbb{Z}} w\Big]\Big)$

$\langle [zz, w] \rangle \hookrightarrow T352 \Rightarrow$ *Stat0* : $[zz, w] \in \text{Fr}$

$T341 \Rightarrow$ *Stat1* : $\langle \forall v \in \text{Fr}, u \in \text{Fr} \mid (v \approx_{\text{Fr}} u \leftrightarrow u \approx_{\text{Fr}} v) \,\&\, v \approx_{\text{Fr}} v \rangle$

$\langle [zz, w], [zz, w] \rangle \hookrightarrow Stat1 \Rightarrow$ $[zz, w] \approx_{\text{Fr}} [zz, w]$

$\langle [x, y], \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})([x, y]), [zz, w], [zz, w] \rangle \hookrightarrow T355 \Rightarrow$
$\text{Fr\_to\_}\mathbb{Q}\Big(\Big[[x, y]^{[1]} *_{\mathbb{Z}} [zz, w]^{[2]} +_{\mathbb{Z}} [zz, w]^{[1]} *_{\mathbb{Z}} [x, y]^{[2]}, [x, y]^{[2]} *_{\mathbb{Z}} [zz, w]^{[2]}\Big]\Big) =$
$\text{Fr\_to\_}\mathbb{Q}\Big(\Big[\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([x, y]) *_{\mathbb{Z}} [zz, w]^{[2]} +_{\mathbb{Z}} [zz, w]^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]), \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) *_{\mathbb{Z}} [zz, w]^{[2]}\Big]\Big)$

ELEM $\Rightarrow$ $[x, y]^{[1]} = x \,\&\, [x, y]^{[2]} = y \,\&\, [zz, w]^{[1]} = zz \,\&\, [zz, w]^{[2]} = w$

$\langle zz, \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) \rangle \hookrightarrow T307 \Rightarrow$ $zz *_{\mathbb{Z}} \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) = \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) *_{\mathbb{Z}} zz$

EQUAL $\Rightarrow$ $\text{Fr\_to\_}\mathbb{Q}([x *_{\mathbb{Z}} w +_{\mathbb{Z}} zz *_{\mathbb{Z}} y, y *_{\mathbb{Z}} w]) = \text{Fr\_to\_}\mathbb{Q}\Big(\Big[\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([x, y]) *_{\mathbb{Z}} w +_{\mathbb{Z}} \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) *_{\mathbb{Z}} zz, \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([x, y]) *_{\mathbb{Z}} w\Big]\Big)$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The proof that rational multiplication is commutative is also elementary and algebraic.

-- Commutativity of Multiplication

Theorem 515 (368) $N, M \in \mathbb{Q} \rightarrow N *_{\mathbb{Q}} M \in \mathbb{Q} \,\&\, N *_{\mathbb{Q}} M = M *_{\mathbb{Q}} N$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n, m \in \mathbb{Q} \,\&\, n *_{\mathbb{Q}} m \notin \mathbb{Q} \vee n *_{\mathbb{Q}} m \neq m *_{\mathbb{Q}} n$

$\langle n \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(n) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$

$\langle m \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(m) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(m)) = m$

$\langle \mathbf{arb}(n) \rangle \hookrightarrow T352 \Rightarrow$ *Stat1* : $\mathbf{arb}(n) = \Big[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\Big] \,\&\, \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathbf{arb}(m) \rangle \hookrightarrow T352 \Rightarrow$ *Stat2* : $\mathbf{arb}(m) = \Big[\mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]}\Big] \,\&\, \mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(m)^{[2]} \neq [\emptyset, \emptyset]$

Suppose $\Rightarrow$ *Stat3* : $n *_{\mathbb{Q}} m \notin \mathbb{Q}$

Use_def$(*_{\mathbb{Q}}) \Rightarrow$ $\text{Fr\_to\_}\mathbb{Q}\Big(\Big[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\Big]\Big) \notin \mathbb{Q}$

$T343 \Rightarrow \quad Stat4: \langle \forall x \mid x \in \mathsf{Fr} \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}(x) \in \mathbb{Q}\rangle$

$\langle \left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right] \rangle \hookrightarrow Stat4(\langle Stat3\rangle) \Rightarrow$

$\quad \left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right] \notin \mathsf{Fr}$

ALGEBRA $\Rightarrow \quad \mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]} \in \mathbb{Z}$

ALGEBRA $\Rightarrow \quad \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} \in \mathbb{Z}$

$\langle \mathbf{arb}(m)^{[2]}, \mathbf{arb}(n)^{[2]}\rangle \hookrightarrow T330(\langle Stat1, Stat2, *\rangle) \Rightarrow \quad \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]} \neq [\emptyset, \emptyset]$

$\langle \left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right] \rangle \hookrightarrow T352 \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad n *_{\mathbb{Q}} m \neq m *_{\mathbb{Q}} n$

Use_def$(*_{\mathbb{Q}}) \Rightarrow \quad \mathsf{Fr_to_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}\right]) \neq \mathsf{Fr_to_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right])$

ALGEBRA $\Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad$ QED

Theorem 516 (369) $X \in \mathbb{Q} \,\&\, Y, ZZ \in \mathbb{Z} \,\&\, ZZ \neq [\emptyset, \emptyset] \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}([Y, ZZ]) *_{\mathbb{Q}} X = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(X)^{[1]} *_{\mathbb{Z}} Y, \mathbf{arb}(X)^{[2]} *_{\mathbb{Z}} ZZ\right])$. PROOF:

Suppose_not$(x, y, zz) \Rightarrow \quad x \in \mathbb{Q} \,\&\, y, zz \in \mathbb{Z} \,\&\, zz \neq [\emptyset, \emptyset] \,\&\, \mathsf{Fr_to_}\mathbb{Q}([y, zz]) *_{\mathbb{Q}} x \neq \mathsf{Fr_to_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} y, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

$\langle x, y, zz\rangle \hookrightarrow T359 \Rightarrow \quad x *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(x)^{[1]} *_{\mathbb{Z}} y, \mathbf{arb}(x)^{[2]} *_{\mathbb{Z}} zz\right])$

$\langle [y, zz] \rangle \hookrightarrow T352 \Rightarrow \quad [y, zz] \in \mathsf{Fr}$

$\langle [y, zz] \rangle \hookrightarrow T344 \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([y, zz]) \in \mathbb{Q}$

$\langle x, \mathsf{Fr\_to\_}\mathbb{Q}([y, zz])\rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad$ QED

-- Next we prove the associative law for rational addition.

Theorem 517 (370) $K, N, M \in \mathbb{Q} \rightarrow N +_{\mathbb{Q}} (M +_{\mathbb{Q}} K) = (N +_{\mathbb{Q}} M) +_{\mathbb{Q}} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow \quad Stat1: \; k, n, m \in \mathbb{Q} \,\&\, n +_{\mathbb{Q}} (m +_{\mathbb{Q}} k) \neq n +_{\mathbb{Q}} m +_{\mathbb{Q}} k$

-- For let k, n, m be a counterexample to our assertion, so that arb (k) = [ak, dk], arb (n) = [an, dn], arb (m) = [am, dm] are fractions with signed integer numerators and denominators.

$\langle n\rangle \hookrightarrow T346 \Rightarrow \quad \mathbf{arb}(n) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$

Loc_def $\Rightarrow \quad arn = \mathbf{arb}(n)$

EQUAL $\Rightarrow \quad arn \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(arn) = n$

Loc_def $\Rightarrow \quad an = arn^{[1]}$

Loc_def $\Rightarrow \quad dn = arn^{[2]}$

$\langle arn\rangle \hookrightarrow T352 \Rightarrow \quad arn = \left[arn^{[1]}, arn^{[2]}\right] \,\&\, arn^{[1]}, arn^{[2]} \in \mathbb{Z} \,\&\, arn^{[2]} \neq [\emptyset, \emptyset]$

$\text{EQUAL} \Rightarrow \quad \mathit{Stat2}: \ \mathsf{arn} = [\mathsf{an}, \mathsf{dn}] \ \& \ \mathsf{an}, \mathsf{dn} \in \mathbb{Z} \ \& \ \mathsf{dn} \neq [\emptyset, \emptyset]$
$\langle \mathsf{m} \rangle \hookrightarrow \mathit{T346} \Rightarrow \quad \mathbf{arb}(\mathsf{m}) \in \mathsf{Fr} \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{m})) = \mathsf{m}$
$\text{Loc\_def} \Rightarrow \quad \mathsf{arm} = \mathbf{arb}(\mathsf{m})$
$\text{EQUAL} \Rightarrow \quad \mathsf{arm} \in \mathsf{Fr} \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}(\mathsf{arm}) = \mathsf{m}$
$\text{Loc\_def} \Rightarrow \quad \mathsf{am} = \mathsf{arm}^{[1]}$
$\text{Loc\_def} \Rightarrow \quad \mathsf{dm} = \mathsf{arm}^{[2]}$
$\langle \mathsf{arm} \rangle \hookrightarrow \mathit{T352} \Rightarrow \quad \mathsf{arm} = \left[\mathsf{arm}^{[1]}, \mathsf{arm}^{[2]}\right] \ \& \ \mathsf{arm}^{[1]}, \mathsf{arm}^{[2]} \in \mathbb{Z} \ \& \ \mathsf{arm}^{[2]} \neq [\emptyset, \emptyset]$
$\text{EQUAL} \Rightarrow \quad \mathit{Stat3}: \ \mathsf{arm} = [\mathsf{am}, \mathsf{dm}] \ \& \ \mathsf{am}, \mathsf{dm} \in \mathbb{Z} \ \& \ \mathsf{dm} \neq [\emptyset, \emptyset]$
$\langle \mathsf{k} \rangle \hookrightarrow \mathit{T346} \Rightarrow \quad \mathbf{arb}(\mathsf{k}) \in \mathsf{Fr} \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{k})) = \mathsf{k}$
$\text{Loc\_def} \Rightarrow \quad \mathsf{ark} = \mathbf{arb}(\mathsf{k})$
$\text{EQUAL} \Rightarrow \quad \mathsf{ark} \in \mathsf{Fr} \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}(\mathsf{ark}) = \mathsf{k}$
$\text{Loc\_def} \Rightarrow \quad \mathsf{ak} = \mathsf{ark}^{[1]}$
$\text{Loc\_def} \Rightarrow \quad \mathsf{dk} = \mathsf{ark}^{[2]}$
$\langle \mathsf{ark} \rangle \hookrightarrow \mathit{T352} \Rightarrow \quad \mathsf{ark} = \left[\mathsf{ark}^{[1]}, \mathsf{ark}^{[2]}\right] \ \& \ \mathsf{ark}^{[1]}, \mathsf{ark}^{[2]} \in \mathbb{Z} \ \& \ \mathsf{ark}^{[2]} \neq [\emptyset, \emptyset]$
$\text{EQUAL} \Rightarrow \quad \mathit{Stat4}: \ \mathsf{ark} = [\mathsf{ak}, \mathsf{dk}] \ \& \ \mathsf{ak}, \mathsf{dk} \in \mathbb{Z} \ \& \ \mathsf{dk} \neq [\emptyset, \emptyset]$
$\langle \mathsf{dk}, \mathsf{dm} \rangle \hookrightarrow \mathit{T330}(\langle \mathit{Stat4}, \mathit{Stat3}, * \rangle) \Rightarrow \quad \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk} \neq [\emptyset, \emptyset]$

-- By definition of rational addition, Fr_to_Ra ([(am S_TIMES dk) S_PLUS (ak S_TIMES dm), dm S_TIMES dk]) = m + k and Fr_to_Ra ([(an S_TIMES dm) S_PLUS (am S_TIMES dn), dn S_TIMES dm]) = n + m, so that the negative of our associative law can be written in the manner seen below.

$\text{Use\_def}(+_{\mathbb{Q}}) \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[2]} +_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}, \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[2]}\right])$
$\text{EQUAL} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}])$
$\text{Use\_def}(+_{\mathbb{Q}}) \ \langle \mathit{Stat2} \rangle \Rightarrow \quad \mathsf{n} +_{\mathbb{Q}} \mathsf{m} = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]} +_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}, \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}\right])$
$\text{EQUAL} \Rightarrow \quad \mathsf{n} +_{\mathbb{Q}} \mathsf{m} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}])$
$\text{EQUAL} \Rightarrow \quad \mathsf{n} +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}]) \neq \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]) +_{\mathbb{Q}} \mathsf{k}$
$\text{ALGEBRA} \Rightarrow \quad \mathit{Stat5}: \ \mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}, \mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} \in \mathbb{Z}$
$\langle \mathsf{dm}, \mathsf{dk} \rangle \hookrightarrow \mathit{T330} \Rightarrow \quad \mathit{Stat6}: \ \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk} \neq [\emptyset, \emptyset]$
$\langle \mathsf{n}, \mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk} \rangle \hookrightarrow \mathit{T358}(\langle \mathit{Stat1}, \mathit{Stat5}, \mathit{Stat6} \rangle) \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}) +_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}), \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})\right]) =$
$\mathsf{n} +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}])$
$\text{EQUAL} \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}) +_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})]) = \mathsf{n} +_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{k})$
$\langle \mathsf{dm}, \mathsf{dn} \rangle \hookrightarrow \mathit{T330}(\langle \mathit{Stat2}, \mathit{Stat3}, * \rangle) \Rightarrow \quad \mathit{Stat7}: \ \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} \neq [\emptyset, \emptyset]$
$\langle \mathsf{k}, \mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} \rangle \hookrightarrow \mathit{T366}(\langle \mathit{Stat1}, \mathit{Stat5}, \mathit{Stat7} \rangle) \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]) +_{\mathbb{Q}} \mathsf{k} =$
$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{k})^{[1]} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}) +_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[2]} *_{\mathbb{Z}} (\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}), \mathbf{arb}(\mathsf{k})^{[2]} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm})\right])$
$\text{EQUAL} \Rightarrow \quad \mathsf{n} +_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{ak} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}) +_{\mathbb{Z}} \mathsf{dk} *_{\mathbb{Z}} (\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}), \mathsf{dk} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm})])$
$\text{ELEM} \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}) +_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})]) \neq \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{ak} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}) +_{\mathbb{Z}} \mathsf{dk} *_{\mathbb{Z}} (\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}), \mathsf{dk} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm})])$
$\text{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \text{Discharge} \Rightarrow \quad \text{QED}$

Theorem 518 (371) $\mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q} \;\&\; (\mathsf{M} \in \mathbb{Q} \rightarrow \mathsf{M} = \mathsf{M} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$. PROOF:

Suppose_not(m) $\Rightarrow$ $\mathit{Stat9}:\ \mathbf{0}_{\mathbb{Q}} \notin \mathbb{Q} \vee \mathbf{1}_{\mathbb{Q}} \notin \mathbb{Q} \vee (\mathsf{m} \in \mathbb{Q} \;\&\; \mathsf{m} \neq \mathsf{m} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$

T291 $\Rightarrow$ $\mathit{Stat1}:\ [\emptyset,\emptyset], [1,\emptyset] \in \mathbb{Z}$

T183 $\Rightarrow$ $\mathit{Stat2}:\ 1 \neq \emptyset$

$\langle \mathit{Stat1} \rangle$ ELEM $\Rightarrow$

$$[[\emptyset,\emptyset],[1,\emptyset]] = \left[[[\emptyset,\emptyset],[1,\emptyset]]^{[1]}, [[\emptyset,\emptyset],[1,\emptyset]]^{[2]}\right] \;\&\; [[\emptyset,\emptyset],[1,\emptyset]]^{[1]} \in \mathbb{Z} \;\&\; [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} \in \mathbb{Z} \;\&\; [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} \neq [\emptyset,\emptyset]$$

$\langle [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[\emptyset,\emptyset],[1,\emptyset]] \in \mathsf{Fr}$

$\langle [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T344* $\Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]]) \in \mathbb{Q}$

Use_def($\mathbf{0}_{\mathbb{Q}}$) $\Rightarrow$ $\mathit{Stat1a}:\ \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle \mathit{Stat1} \rangle$ ELEM $\Rightarrow$

$$[[1,\emptyset],[1,\emptyset]] = \left[[[1,\emptyset],[1,\emptyset]]^{[1]}, [[1,\emptyset],[1,\emptyset]]^{[2]}\right] \;\&\; [[1,\emptyset],[1,\emptyset]]^{[1]} \in \mathbb{Z} \;\&\; [[1,\emptyset],[1,\emptyset]]^{[2]} \in \mathbb{Z} \;\&\; [[1,\emptyset],[1,\emptyset]]^{[2]} \neq [\emptyset,\emptyset]$$

$\langle [[1,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[1,\emptyset],[1,\emptyset]] \in \mathsf{Fr}$

$\langle [[1,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T344* $\Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]]) \in \mathbb{Q}$

Use_def($\mathbf{1}_{\mathbb{Q}}$) $\Rightarrow$ $\mathit{Stat2a}:\ \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle \mathit{Stat9}, \mathit{Stat1a}, \mathit{Stat2a}, * \rangle$ ELEM $\Rightarrow$ $\mathsf{m} \in \mathbb{Q} \;\&\; \mathsf{m} \neq \mathsf{m} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m} \rangle \hookrightarrow$ *T346* $\Rightarrow$ $\mathit{Stat3}:\ \mathbf{arb}(\mathsf{m}) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{m})) = \mathsf{m}$

$\langle \mathbf{arb}(\mathsf{m}) \rangle \hookrightarrow$ *T352*([*Stat1*, $\cap$]) $\Rightarrow$ $\mathit{Stat4}:$

$$\mathbf{arb}(\mathsf{m}) = \left[\mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]}\right] \;\&\; \mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(\mathsf{m})^{[2]} \neq [\emptyset,\emptyset]$$

Loc_def $\Rightarrow$ $\mathit{Stat0}:\ \mathsf{arm} = \mathbf{arb}(\mathsf{m})$

EQUAL $\Rightarrow$ $\mathit{Stat0a}:\ \mathsf{arm} \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathsf{arm}) = \mathsf{m}$

Loc_def $\Rightarrow$ $\mathsf{am} = \mathsf{arm}^{[1]}$

Loc_def $\Rightarrow$ $\mathsf{dm} = \mathsf{arm}^{[2]}$

EQUAL $\Rightarrow$ $\mathsf{arm} = [\mathsf{am}, \mathsf{dm}] \;\&\; \mathsf{am}, \mathsf{dm} \in \mathbb{Z} \;\&\; \mathsf{dm} \neq [\emptyset,\emptyset]$

Use_def($\mathbf{0}_{\mathbb{Q}}$) $\Rightarrow$ $\mathsf{m} \neq \mathsf{m} +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$

Use_def($+_{\mathbb{Q}}$) $\Rightarrow$

$$\mathsf{m} \neq \mathsf{Fr\_to\_}\mathbb{Q}\left(\left[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([[\emptyset,\emptyset],[1,\emptyset]]) +_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([[\emptyset,\emptyset],[1,\emptyset]]) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}, \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([[\emptyset,\emptyset],[1,\emptyset]])\right]\right)$$

ELEM $\Rightarrow$ $\mathit{Stat5}:\ [[\emptyset,\emptyset],[1,\emptyset]]^{[1]} = [\emptyset,\emptyset] \;\&\; [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} = [1,\emptyset]$

$\langle \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]]) \rangle \hookrightarrow$ *T346* $\Rightarrow$ $\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([[\emptyset,\emptyset],[1,\emptyset]]) \in \mathsf{Fr}$

$\langle [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T344* $\Rightarrow$

$$[[\emptyset,\emptyset],[1,\emptyset]] \approx_{\mathsf{Fr}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([[\emptyset,\emptyset],[1,\emptyset]])$$

T341 $\Rightarrow$ $\mathit{Stat77}:\ \langle \forall \mathsf{v} \in \mathsf{Fr}, \mathsf{w} \in \mathsf{Fr} \mid (\mathsf{v} \approx_{\mathsf{Fr}} \mathsf{w} \leftrightarrow \mathsf{w} \approx_{\mathsf{Fr}} \mathsf{v}) \;\&\; \mathsf{v} \approx_{\mathsf{Fr}} \mathsf{v} \rangle$

$\langle \mathbf{arb}(\mathsf{m}), \mathbf{arb}(\mathsf{m}) \rangle \hookrightarrow$ *Stat77*([*Stat0*, *Stat0a*]) $\Rightarrow$ $\mathbf{arb}(\mathsf{m}) \approx_{\mathsf{Fr}} \mathbf{arb}(\mathsf{m})$

$\langle \mathbf{arb}(\mathsf{m}), \mathbf{arb}(\mathsf{m}), [[\emptyset,\emptyset],[1,\emptyset]], \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([[\emptyset,\emptyset],[1,\emptyset]])\rangle \hookrightarrow T355 \Rightarrow$ $\quad$ *Stat6* :

$$\mathsf{m} \neq \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} +_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}, \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[2]}\right])$$

EQUAL $\langle Stat6, Stat5\rangle \Rightarrow$ $\quad \mathsf{m} \neq \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} [1,\emptyset] +_{\mathbb{Z}} [\emptyset,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}, \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} [1,\emptyset]\right])$

$\langle \mathbf{arb}(\mathsf{m})^{[1]}\rangle \hookrightarrow T324 \Rightarrow$ $\quad [1,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[1]} = \mathbf{arb}(\mathsf{m})^{[1]}$

$\langle \mathbf{arb}(\mathsf{m})^{[2]}\rangle \hookrightarrow T324 \Rightarrow$ $\quad$ *Stat7* : $[1,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]} = \mathbf{arb}(\mathsf{m})^{[2]}$

$\langle \mathbf{arb}(\mathsf{m})^{[2]}\rangle \hookrightarrow T324 \Rightarrow$ $\quad [\emptyset,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]} = [\emptyset,\emptyset]$

$\langle \mathbf{arb}(\mathsf{m})^{[1]}\rangle \hookrightarrow T328 \Rightarrow$ $\quad \mathbf{arb}(\mathsf{m})^{[1]} +_{\mathbb{Z}} [\emptyset,\emptyset] = \mathbf{arb}(\mathsf{m})^{[1]}$

ALGEBRA $\langle Stat7, Stat1, Stat4\rangle \Rightarrow$ $\quad \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} [1,\emptyset] = \mathbf{arb}(\mathsf{m})^{[2]}$

ALGEBRA $\Rightarrow$ $\quad \mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} [1,\emptyset] +_{\mathbb{Z}} [\emptyset,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]} = \mathbf{arb}(\mathsf{m})^{[1]}$

EQUAL $\Rightarrow$ $\quad$ *Stat8* : $\mathsf{m} \neq \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]}\right])$

EQUAL $\langle Stat3, Stat4, Stat8\rangle \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- Next we prove that the rational version of the algebraic law 'x + (-x) = 0'.

Theorem 519 (372) $\mathsf{M} \in \mathbb{Q} \rightarrow \mathsf{Rev}_{\mathbb{Q}}(\mathsf{M}) \in \mathbb{Q} \;\&\; \mathsf{M} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{M}) = \mathbf{0}_{\mathbb{Q}}$. PROOF:

Suppose_not$(\mathsf{m}) \Rightarrow$ $\quad \mathsf{m} \in \mathbb{Q} \;\&\; \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \notin \mathbb{Q} \vee \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \neq \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}\rangle \hookrightarrow T346 \Rightarrow$ $\quad \mathbf{arb}(\mathsf{m}) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{m})) = \mathsf{m}$

$\langle \mathbf{arb}(\mathsf{m})\rangle \hookrightarrow T352 \Rightarrow$ $\quad \mathbf{arb}(\mathsf{m}) = \left[\mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]}\right] \;\&\; \mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(\mathsf{m})^{[2]} \neq [\emptyset,\emptyset]$

Loc_def $\Rightarrow$ $\quad \mathsf{am} = \mathbf{arb}(\mathsf{m})^{[1]}$

Loc_def $\Rightarrow$ $\quad \mathsf{dm} = \mathbf{arb}(\mathsf{m})^{[2]}$

EQUAL $\Rightarrow$ $\quad$ *Stat1* : $\mathbf{arb}(\mathsf{m}) = [\mathsf{am}, \mathsf{dm}] \;\&\; \mathsf{am}, \mathsf{dm} \in \mathbb{Z} \;\&\; \mathsf{dm} \neq [\emptyset,\emptyset]$

$\langle \mathsf{dm}, \mathsf{dm}\rangle \hookrightarrow T330([Stat1, \cap]) \Rightarrow$ $\quad \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm} \neq [\emptyset,\emptyset]$

$T291 \Rightarrow$ $\quad$ *Stat2* : $[\emptyset,\emptyset], [1,\emptyset] \in \mathbb{Z}$

ALGEBRA $\Rightarrow$ $\quad \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm} \in \mathbb{Z}$

$\langle [[\emptyset,\emptyset], \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}]\rangle \hookrightarrow T352(\langle Stat1\rangle) \Rightarrow$ $\quad$ *Stat3* : $[[\emptyset,\emptyset], \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}] \in \mathsf{Fr}$

$\langle \mathsf{am}\rangle \hookrightarrow T314 \Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}) \in \mathbb{Z}$

Suppose $\Rightarrow$ $\quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \notin \mathbb{Q}$

Use_def$(\mathsf{Rev}_{\mathbb{Q}}) \Rightarrow$ $\quad$ *Stat4* : $\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{m})^{[1]}), \mathbf{arb}(\mathsf{m})^{[2]}\right]) \notin \mathbb{Q}$

$T343 \Rightarrow$ $\quad$ *Stat5* : $\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{Fr} \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}(\mathsf{x}) \in \mathbb{Q}\rangle$

$\langle \left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{m})^{[1]}), \mathbf{arb}(\mathsf{m})^{[2]}\right]\rangle \hookrightarrow Stat5(\langle Stat4\rangle) \Rightarrow$ $\quad \left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{m})^{[1]}), \mathbf{arb}(\mathsf{m})^{[2]}\right] \notin \mathsf{Fr}$

EQUAL $\Rightarrow$ $\quad [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}), \mathsf{dm}] \notin \mathsf{Fr}$

$\langle [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}), \mathsf{dm}]\rangle \hookrightarrow T352 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \neq \mathbf{0}_{\mathbb{Q}}$

Use_def $(\mathsf{Rev}_{\mathbb{Q}})$ $\Rightarrow$ $\mathsf{m} +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}\big(\big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{m})^{[1]}), \mathbf{arb}(\mathsf{m})^{[2]}\big]\big) \neq \mathbf{0}_{\mathbb{Q}}$

EQUAL $\Rightarrow$ $\mathsf{m} +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}\big([\mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}), \mathsf{dm}]\big) \neq \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}, \mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}), \mathsf{dm}\rangle \hookrightarrow T358 \Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}\big(\big[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}), \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} \mathsf{dm}\big]\big) \neq \mathbf{0}_{\mathbb{Q}}$

EQUAL $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \neq \mathsf{Fr\_to\_}\mathbb{Q}\big([\mathsf{am} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}), \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}]\big)$

ALGEBRA $\Rightarrow$ $\mathsf{am} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}) = \mathsf{am} *_{\mathbb{Z}} \mathsf{dm} -_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dm}$

ALGEBRA $\Rightarrow$ $\mathsf{am} *_{\mathbb{Z}} \mathsf{dm} \in \mathbb{Z}$

$\langle \mathsf{am} *_{\mathbb{Z}} \mathsf{dm}\rangle \hookrightarrow T327 \Rightarrow$ $\mathsf{am} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{am}) = [\emptyset, \emptyset]$

EQUAL $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \neq \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}])$

Use_def $(\mathbf{0}_{\mathbb{Q}})$ $\Rightarrow$ *Stat6* : $\mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], [1, \emptyset]]) \neq \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}])$

ELEM $\Rightarrow$ *Stat7* : $[[\emptyset, \emptyset], [1, \emptyset]]^{[1]} = [\emptyset, \emptyset]$ & $[[\emptyset, \emptyset], [1, \emptyset]]^{[2]} = [1, \emptyset]$

T183 $\Rightarrow$ *Stat8* : $1 \neq \emptyset$

$\langle$ *Stat7, Stat8, Stat2* $\rangle$ ELEM $\Rightarrow$

$[[\emptyset, \emptyset], [1, \emptyset]] = \big[[[\emptyset, \emptyset], [1, \emptyset]]^{[1]}, [[\emptyset, \emptyset], [1, \emptyset]]^{[2]}\big]$ & $[[\emptyset, \emptyset], [1, \emptyset]]^{[1]} \in \mathbb{Z}$ &

$[[\emptyset, \emptyset], [1, \emptyset]]^{[2]} \in \mathbb{Z}$ & $[[\emptyset, \emptyset], [1, \emptyset]]^{[2]} \neq [\emptyset, \emptyset]$

$\langle [[\emptyset, \emptyset], [1, \emptyset]] \rangle \hookrightarrow T352 \Rightarrow$ *Stat9* : $[[\emptyset, \emptyset], [1, \emptyset]] \in \mathsf{Fr}$

$\langle [[\emptyset, \emptyset], [1, \emptyset]], [[\emptyset, \emptyset], \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}] \rangle \hookrightarrow T345(\langle$ *Stat9, Stat3, Stat6* $\rangle) \Rightarrow$

$\neg [[\emptyset, \emptyset], [1, \emptyset]] \approx_{\mathsf{Fr}} [[\emptyset, \emptyset], \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}]$

Use_def $(\approx_{\mathsf{Fr}})$ $\Rightarrow$ *Stat10* : $[\emptyset, \emptyset] *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}) \neq [1, \emptyset] *_{\mathbb{Z}} [\emptyset, \emptyset]$

$\langle \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}\rangle \hookrightarrow T324 \Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}) = [\emptyset, \emptyset]$

$\langle$ *Stat10* $\rangle$ ELEM $\Rightarrow$ *Stat11* : $[\emptyset, \emptyset] \neq [1, \emptyset] *_{\mathbb{Z}} [\emptyset, \emptyset]$

$\langle [1, \emptyset] \rangle \hookrightarrow T324 \Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} [1, \emptyset] = [\emptyset, \emptyset]$

ALGEBRA $\langle$ *Stat11, Stat2* $\rangle$ $\Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} [1, \emptyset] = [1, \emptyset] *_{\mathbb{Z}} [\emptyset, \emptyset]$

$\langle$ *Stat11* $\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following elementary generalization of the preceding theorem gives the rational case of the inverse relationship between addition and subtraction.

Theorem 520 (373) $\mathsf{N}, \mathsf{M} \in \mathbb{Q} \rightarrow \mathsf{N} = \mathsf{M} +_{\mathbb{Q}} (\mathsf{N} -_{\mathbb{Q}} \mathsf{M})$. PROOF:

Suppose_not (n, m) $\Rightarrow$ $\mathsf{n}, \mathsf{m} \in \mathbb{Q}$ & $\mathsf{n} \neq \mathsf{m} +_{\mathbb{Q}} (\mathsf{n} -_{\mathbb{Q}} \mathsf{m})$

Use_def $(-_{\mathbb{Q}})$ $\Rightarrow$ $\mathsf{n} \neq \mathsf{m} +_{\mathbb{Q}} \big(\mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})\big)$

$\langle \mathsf{m}\rangle \hookrightarrow T372 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q}$

$\langle \mathsf{n}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow T365 \Rightarrow$ $\mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{n}$

EQUAL $\Rightarrow$ $\mathsf{n} \neq \mathsf{m} +_{\mathbb{Q}} \big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{n}\big)$

$\langle \mathsf{n}, \mathsf{m}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow T370 \Rightarrow$ $\mathsf{n} \neq \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{n}$

$\langle \mathsf{m}\rangle \hookrightarrow T372 \Rightarrow$ $\mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$

EQUAL $\Rightarrow$ $\mathsf{n} \neq \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{n}$

$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle n \rangle \hookrightarrow T371 \Rightarrow \quad n +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = n$
$\langle n, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow$ QED

-- Our next theorem states the associative law for rational multiplication.

Theorem 521 (374) $K, N, M \in \mathbb{Q} \rightarrow N *_{\mathbb{Q}} (M *_{\mathbb{Q}} K) = (N *_{\mathbb{Q}} M) *_{\mathbb{Q}} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow \quad Stat1: \; k, n, m \in \mathbb{Q} \;\&\; n *_{\mathbb{Q}} (m *_{\mathbb{Q}} k) \neq n *_{\mathbb{Q}} m *_{\mathbb{Q}} k$
$\langle n, m \rangle \hookrightarrow T368 \Rightarrow \quad n *_{\mathbb{Q}} m \in \mathbb{Q}$
$\langle n *_{\mathbb{Q}} m, k \rangle \hookrightarrow T368 \Rightarrow \quad n *_{\mathbb{Q}} (m *_{\mathbb{Q}} k) \neq k *_{\mathbb{Q}} (n *_{\mathbb{Q}} m)$
$\langle n \rangle \hookrightarrow T346 \Rightarrow \quad \mathbf{arb}(n) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$
$\langle m \rangle \hookrightarrow T346 \Rightarrow \quad \mathbf{arb}(m) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(m)) = m$
$\langle k \rangle \hookrightarrow T346 \Rightarrow \quad \mathbf{arb}(k) \in \mathsf{Fr} \;\&\; \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(k)) = k$
$\langle \mathbf{arb}(n) \rangle \hookrightarrow T352 \Rightarrow \quad \mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \;\&\; \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$
$\langle \mathbf{arb}(m) \rangle \hookrightarrow T352 \Rightarrow \quad \mathbf{arb}(m) = \left[\mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]}\right] \;\&\; \mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(m)^{[2]} \neq [\emptyset, \emptyset]$
$\langle \mathbf{arb}(k) \rangle \hookrightarrow T352 \Rightarrow \quad \mathbf{arb}(k) = \left[\mathbf{arb}(k)^{[1]}, \mathbf{arb}(k)^{[2]}\right] \;\&\; \mathbf{arb}(k)^{[1]}, \mathbf{arb}(k)^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(k)^{[2]} \neq [\emptyset, \emptyset]$
Loc_def $\Rightarrow \quad an = \mathbf{arb}(n)^{[1]}$
Loc_def $\Rightarrow \quad am = \mathbf{arb}(m)^{[1]}$
Loc_def $\Rightarrow \quad ak = \mathbf{arb}(k)^{[1]}$
Loc_def $\Rightarrow \quad dn = \mathbf{arb}(n)^{[2]}$
Loc_def $\Rightarrow \quad dm = \mathbf{arb}(m)^{[2]}$
Loc_def $\Rightarrow \quad dk = \mathbf{arb}(k)^{[2]}$
EQUAL $\Rightarrow \quad Stat2: \; \mathbf{arb}(n) = [an, dn] \;\&\; an, dn \in \mathbb{Z} \;\&\; dn \neq [\emptyset, \emptyset]$
EQUAL $\Rightarrow \quad Stat3: \; \mathbf{arb}(m) = [am, dm] \;\&\; am, dm \in \mathbb{Z} \;\&\; dm \neq [\emptyset, \emptyset]$
EQUAL $\Rightarrow \quad Stat4: \; \mathbf{arb}(k) = [ak, dk] \;\&\; ak, dk \in \mathbb{Z} \;\&\; dk \neq [\emptyset, \emptyset]$
Use_def$(*_{\mathbb{Q}}) \Rightarrow \quad n *_{\mathbb{Q}} \mathsf{Fr_to_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(k)^{[1]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(k)^{[2]}\right]) \neq k *_{\mathbb{Q}} \mathsf{Fr_to_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right])$
EQUAL $\Rightarrow \quad n *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([am *_{\mathbb{Z}} ak, dm *_{\mathbb{Z}} dk]) \neq k *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([an *_{\mathbb{Z}} am, dn *_{\mathbb{Z}} dm])$
$\langle dk, dm \rangle \hookrightarrow T330(\langle Stat3, Stat4, * \rangle) \Rightarrow \quad Stat5: \; dm *_{\mathbb{Z}} dk \neq [\emptyset, \emptyset]$
$\langle dm, dn \rangle \hookrightarrow T330(\langle Stat3, Stat2, * \rangle) \Rightarrow \quad Stat6: \; dn *_{\mathbb{Z}} dm \neq [\emptyset, \emptyset]$
ALGEBRA $\Rightarrow \quad Stat7: \; an *_{\mathbb{Z}} am, dn *_{\mathbb{Z}} dm, am *_{\mathbb{Z}} ak, dm *_{\mathbb{Z}} dk \in \mathbb{Z}$
$\langle [am *_{\mathbb{Z}} ak, dm *_{\mathbb{Z}} dk] \rangle \hookrightarrow T352([Stat7, Stat5]) \Rightarrow \quad [am *_{\mathbb{Z}} ak, dm *_{\mathbb{Z}} dk] \in \mathsf{Fr}$
$\langle [an *_{\mathbb{Z}} am, dn *_{\mathbb{Z}} dm] \rangle \hookrightarrow T352([Stat7, Stat6]) \Rightarrow \quad [an *_{\mathbb{Z}} am, dn *_{\mathbb{Z}} dm] \in \mathsf{Fr}$
$\langle n, am *_{\mathbb{Z}} ak, dm *_{\mathbb{Z}} dk \rangle \hookrightarrow T359(\langle Stat7, Stat5, Stat1, Stat2, Stat3, Stat4 \rangle) \Rightarrow \quad n *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([am *_{\mathbb{Z}} ak, dm *_{\mathbb{Z}} dk]) =$
$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} (am *_{\mathbb{Z}} ak), \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} (dm *_{\mathbb{Z}} dk)\right])$

EQUAL $\Rightarrow$ $\mathsf{n} *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{ak}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}]) = \mathsf{Fr\_to\_}\mathbb{Q}\big([\mathsf{an} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{ak}), \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})]\big)$

$\langle \mathsf{k}, \mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}\rangle \hookrightarrow T359(\langle Stat7, Stat6, Stat1, Stat2, Stat3, Stat4\rangle) \Rightarrow$ $\mathsf{k} *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]) =$
$\mathsf{Fr\_to\_}\mathbb{Q}\big(\big[\mathbf{arb}(\mathsf{k})^{[1]} *_{\mathbb{Z}} (\mathsf{an} *_{\mathbb{Z}} \mathsf{am}), \mathbf{arb}(\mathsf{k})^{[2]} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm})\big]\big)$

EQUAL $\Rightarrow$ $\mathsf{k} *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]) = \mathsf{Fr\_to\_}\mathbb{Q}\big([\mathsf{ak} *_{\mathbb{Z}} (\mathsf{an} *_{\mathbb{Z}} \mathsf{am}), \mathsf{dk} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm})]\big)$

ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we prove that a rational number remains unchanged if its numerator and denominator are multiplied by a common integer.

Theorem 522 (375) $\mathsf{K}, \mathsf{N}, \mathsf{M} \in \mathbb{Z} \,\&\, \mathsf{K} \neq [\emptyset, \emptyset] \,\&\, \mathsf{M} \neq [\emptyset, \emptyset] \rightarrow \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{N}, \mathsf{M}]) = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{K} *_{\mathbb{Z}} \mathsf{N}, \mathsf{K} *_{\mathbb{Z}} \mathsf{M}])$. PROOF:

Suppose_not$(\mathsf{k}, \mathsf{n}, \mathsf{m}) \Rightarrow$ $\mathsf{k}, \mathsf{n}, \mathsf{m} \in \mathbb{Z} \,\&\, \mathsf{k} \neq [\emptyset, \emptyset] \,\&\, \mathsf{m} \neq [\emptyset, \emptyset] \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{n}, \mathsf{m}]) \neq \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{k} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{m}])$

$\langle [\mathsf{n}, \mathsf{m}] \rangle \hookrightarrow T352 \Rightarrow$ $[\mathsf{n}, \mathsf{m}] \in \mathsf{Fr}$

ALGEBRA $\Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{m} \in \mathbb{Z}$

$\langle \mathsf{m}, \mathsf{k}\rangle \hookrightarrow T330 \Rightarrow$ $\mathsf{k} *_{\mathbb{Z}} \mathsf{m} \neq [\emptyset, \emptyset]$

$\langle [\mathsf{k} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{m}] \rangle \hookrightarrow T352 \Rightarrow$ $[\mathsf{k} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{m}] \in \mathsf{Fr}$

$\langle [\mathsf{n}, \mathsf{m}], [\mathsf{k} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{m}] \rangle \hookrightarrow T345 \Rightarrow$ $\neg [\mathsf{n}, \mathsf{m}] \approx_{\mathsf{Fr}} [\mathsf{k} *_{\mathbb{Z}} \mathsf{n}, \mathsf{k} *_{\mathbb{Z}} \mathsf{m}]$

Use_def$(\approx_{\mathsf{Fr}}) \Rightarrow$ $\mathsf{n} *_{\mathbb{Z}} (\mathsf{k} *_{\mathbb{Z}} \mathsf{m}) \neq \mathsf{m} *_{\mathbb{Z}} (\mathsf{k} *_{\mathbb{Z}} \mathsf{n})$

ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem states the distributive law for rational numbers.

Theorem 523 (376) $\mathsf{K}, \mathsf{N}, \mathsf{M} \in \mathbb{Q} \rightarrow \mathsf{N} *_{\mathbb{Q}} (\mathsf{M} +_{\mathbb{Q}} \mathsf{K}) = \mathsf{N} *_{\mathbb{Q}} \mathsf{M} +_{\mathbb{Q}} \mathsf{N} *_{\mathbb{Q}} \mathsf{K}$. PROOF:

Suppose_not$(\mathsf{k}, \mathsf{n}, \mathsf{m}) \Rightarrow$ $Stat1:\ \mathsf{k}, \mathsf{n}, \mathsf{m} \in \mathbb{Q} \,\&\, \mathsf{n} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{k}) \neq \mathsf{n} *_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} \mathsf{k}$

-- Supposing the contrary, consider the fractions defining each of our three rational numbers.

$\langle \mathsf{n}\rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\mathsf{n}) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{n})) = \mathsf{n}$

$\langle \mathsf{m}\rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\mathsf{m}) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{m})) = \mathsf{m}$

$\langle \mathsf{k}\rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\mathsf{k}) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{k})) = \mathsf{k}$

$\langle \mathbf{arb}(\mathsf{n}) \rangle \hookrightarrow T352 \Rightarrow$ $Stat2:\ \mathbf{arb}(\mathsf{n}) = \big[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\big] \,\&\, \mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(\mathsf{n})^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathbf{arb}(\mathsf{m}) \rangle \hookrightarrow T352 \Rightarrow$ $Stat3:\ \mathbf{arb}(\mathsf{m}) = \big[\mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]}\big] \,\&\, \mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(\mathsf{m})^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathbf{arb}(\mathsf{k}) \rangle \hookrightarrow T352 \Rightarrow$ $Stat4:\ \mathbf{arb}(\mathsf{k}) = \big[\mathbf{arb}(\mathsf{k})^{[1]}, \mathbf{arb}(\mathsf{k})^{[2]}\big] \,\&\, \mathbf{arb}(\mathsf{k})^{[1]}, \mathbf{arb}(\mathsf{k})^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(\mathsf{k})^{[2]} \neq [\emptyset, \emptyset]$

-- To keep our natation under control, it is convenient to introduce abbreviations for the numerators and denominators of these fractions.

Loc_def ⇒ $\mathit{Stat5}: \mathsf{an} = \mathbf{arb}(\mathsf{n})^{[1]}$

Loc_def ⇒ $\mathit{Stat6}: \mathsf{am} = \mathbf{arb}(\mathsf{m})^{[1]}$

Loc_def ⇒ $\mathit{Stat7}: \mathsf{ak} = \mathbf{arb}(\mathsf{k})^{[1]}$

Loc_def ⇒ $\mathit{Stat8}: \mathsf{dn} = \mathbf{arb}(\mathsf{n})^{[2]}$

Loc_def ⇒ $\mathit{Stat9}: \mathsf{dm} = \mathbf{arb}(\mathsf{m})^{[2]}$

Loc_def ⇒ $\mathit{Stat10}: \mathsf{dk} = \mathbf{arb}(\mathsf{k})^{[2]}$

EQUAL ⇒ $\mathit{Stat11}: \mathbf{arb}(\mathsf{n}) = [\mathsf{an}, \mathsf{dn}]\ \&\ \mathsf{an}, \mathsf{dn} \in \mathbb{Z}\ \&\ \mathsf{dn} \neq [\emptyset, \emptyset]$

EQUAL ⇒ $\mathbf{arb}(\mathsf{m}) = [\mathsf{am}, \mathsf{dm}]\ \&\ \mathsf{am}, \mathsf{dm} \in \mathbb{Z}\ \&\ \mathsf{dm} \neq [\emptyset, \emptyset]$

EQUAL ⇒ $\mathbf{arb}(\mathsf{k}) = [\mathsf{ak}, \mathsf{dk}]\ \&\ \mathsf{ak}, \mathsf{dk} \in \mathbb{Z}\ \&\ \mathsf{dk} \neq [\emptyset, \emptyset]$

-- It is obvious that the various products entering into the definition of the product and sum rationals that will concern us are all signed integers. Thus we can express all these product and sum rationals in terms of their defining fractions.

ALGEBRA ⇒ $\mathit{Stat12}: \mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{an} *_{\mathbb{Z}} \mathsf{ak}, \mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dk}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk} \in \mathbb{Z}$

$\langle \mathsf{dm}, \mathsf{dn} \rangle \hookrightarrow \mathit{T330}([\mathit{Stat11}, \cap]) \Rightarrow\ \mathit{Stat13}: \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} \neq [\emptyset, \emptyset]$

$\langle \mathsf{dk}, \mathsf{dn} \rangle \hookrightarrow \mathit{T330}([\mathit{Stat11}, \cap]) \Rightarrow\ \mathit{Stat14}: \mathsf{dn} *_{\mathbb{Z}} \mathsf{dk} \neq [\emptyset, \emptyset]$

$\langle \mathsf{dk}, \mathsf{dm} \rangle \hookrightarrow \mathit{T330}([\mathit{Stat11}, \cap]) \Rightarrow\ \mathit{Stat15}: \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk} \neq [\emptyset, \emptyset]$

Use_def$(+_{\mathbb{Q}})$ ⇒ $\mathit{Stat16}: \mathsf{m} +_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[2]} +_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}, \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[2]}\right])$

Use_def$(*_{\mathbb{Q}})$ ⇒ $\mathit{Stat17}: \mathsf{n} *_{\mathbb{Q}} \mathsf{m} = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}\right])$

Use_def$(*_{\mathbb{Q}})$ ⇒ $\mathit{Stat18}: \mathsf{n} *_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{k})^{[2]}\right])$

EQUAL ⇒ $\mathsf{m} +_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}])$

EQUAL ⇒ $\mathsf{n} *_{\mathbb{Q}} \mathsf{m} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}])$

EQUAL ⇒ $\mathsf{n} *_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{ak}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dk}])$

EQUAL ⇒ $\mathit{Stat19}: \mathsf{n} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{k}) = \mathsf{n} *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}])$

ALGEBRA ⇒ $\mathit{Stat20}: \mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm} \in \mathbb{Z}$

$\langle \mathsf{n}, \mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk} \rangle \hookrightarrow \mathit{T359}(\langle \mathit{Stat1}, \mathit{Stat19}, \mathit{Stat20}, \mathit{Stat12}, \mathit{Stat15} \rangle) \Rightarrow\ \mathit{Stat21}: \mathsf{n} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{k}) =$

$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}), \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})\right])$

EQUAL $\langle \mathit{Stat21}, \mathit{Stat5}, \mathit{Stat8} \rangle \Rightarrow\ \mathsf{n} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{k}) = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})])$

$\langle \mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{an} *_{\mathbb{Z}} \mathsf{ak}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dk} \rangle \hookrightarrow \mathit{T367}(\langle \mathit{Stat12}, \mathit{Stat13}, \mathit{Stat14}, \mathit{Stat15} \rangle) \Rightarrow\ \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]) +_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{ak}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dk}]) =$

$\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dk}) +_{\mathbb{Z}} \mathsf{an} *_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dk})])$

EQUAL ⇒ $\mathsf{n} *_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dk}) +_{\mathbb{Z}} \mathsf{an} *_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dk})])$

ALGEBRA ⇒ $\mathsf{n} *_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}\left([\mathsf{dn} *_{\mathbb{Z}} (\mathsf{an} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm})), \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}))]\right)$

Loc_def ⇒ $\mathit{Stat22}: \mathsf{aa} = \mathsf{an} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm})$

Loc_def $\Rightarrow$ $\quad$ *Stat23* : $\mathsf{dd} = \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk})$

-- This brings us to the following expressions for the quantities with which we are concerned, which must be different if our theorem fails:

EQUAL $\Rightarrow$ $\quad$ *Stat24* : $\mathsf{n} *_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} \mathsf{k} = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}])$
EQUAL $\Rightarrow$ $\quad$ *Stat25* : $\mathsf{n} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{k}) = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{aa}, \mathsf{dd}])$
Use_def($\approx_{\mathsf{Fr}}$) $\Rightarrow$ $\quad$ *Stat26* :
$\quad [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}] \approx_{\mathsf{Fr}} [\mathsf{aa}, \mathsf{dd}] \leftrightarrow [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}]^{[1]} *_{\mathbb{Z}} [\mathsf{aa}, \mathsf{dd}]^{[2]} = [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}]^{[2]} *_{\mathbb{Z}} [\mathsf{aa}, \mathsf{dd}]^{[1]}$
$\langle$*Stat26*$\rangle$ ELEM $\Rightarrow$ $\quad [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}]^{[1]} = \mathsf{dn} *_{\mathbb{Z}} \mathsf{aa} \ \& \ [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}]^{[2]} = \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}$
$\langle$*Stat26*$\rangle$ ELEM $\Rightarrow$ $\quad [\mathsf{aa}, \mathsf{dd}]^{[1]} = \mathsf{aa} \ \& \ [\mathsf{aa}, \mathsf{dd}]^{[2]} = \mathsf{dd}$
EQUAL $\langle$*Stat26*$\rangle$ $\Rightarrow$ $\quad$ *Stat27* : $[\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}] \approx_{\mathsf{Fr}} [\mathsf{aa}, \mathsf{dd}] \leftrightarrow \mathsf{dn} *_{\mathbb{Z}} \mathsf{aa} *_{\mathbb{Z}} \mathsf{dd} = \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd} *_{\mathbb{Z}} \mathsf{aa}$
ALGEBRA $\Rightarrow$ $\quad$ *Stat28* : $\mathsf{an} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dk} +_{\mathbb{Z}} \mathsf{ak} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}) \in \mathbb{Z}$
$\langle \mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}, \mathsf{dn} \rangle \hookrightarrow$ *T330*($\langle$*Stat11, Stat15, Stat12*, $*\rangle$) $\Rightarrow$ $\quad$ *Stat29* : $\mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dk}) \neq [\emptyset, \emptyset]$
EQUAL $\Rightarrow$ $\quad$ *Stat30* : $\mathsf{aa}, \mathsf{dd} \in \mathbb{Z} \ \& \ \mathsf{dd} \neq [\emptyset, \emptyset]$
ALGEBRA $\langle$*Stat11*$\rangle$ $\Rightarrow$ $\quad \mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd} \in \mathbb{Z}$
$\langle [\mathsf{aa}, \mathsf{dd}] \rangle \hookrightarrow$ *T352*($\langle$*Stat30*$\rangle$) $\Rightarrow$ $\quad$ *Stat31* : $[\mathsf{aa}, \mathsf{dd}] \in \mathsf{Fr}$
ALGEBRA $\langle$*Stat11*$\rangle$ $\Rightarrow$ $\quad \mathsf{dn} *_{\mathbb{Z}} \mathsf{aa} *_{\mathbb{Z}} \mathsf{dd} = \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd} *_{\mathbb{Z}} \mathsf{aa}$
$\langle \mathsf{dd}, \mathsf{dn} \rangle \hookrightarrow$ *T330*($\langle$*Stat11, Stat30*, $*\rangle$) $\Rightarrow$ $\quad \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd} \neq [\emptyset, \emptyset]$
$\langle [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}] \rangle \hookrightarrow$ *T352*($\langle$*Stat30*$\rangle$) $\Rightarrow$ $\quad$ *Stat32* : $[\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}] \in \mathsf{Fr}$
$\langle$*Stat27*$\rangle$ ELEM $\Rightarrow$ $\quad$ *Stat33* : $[\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}] \approx_{\mathsf{Fr}} [\mathsf{aa}, \mathsf{dd}]$
$\langle [\mathsf{dn} *_{\mathbb{Z}} \mathsf{aa}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dd}], [\mathsf{aa}, \mathsf{dd}] \rangle \hookrightarrow$ *T345*($\langle$*Stat1, Stat24, Stat25, Stat33, Stat31, Stat32*$\rangle$) $\Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ QED

-- The following result, which is only a slight variant of what has gone before, rexpresses the condition that the rational number derived from a fraction should be non-negative in terms of the fraction's numerator and denominator.

Theorem 524 (377) $\mathsf{X}, \mathsf{Y} \in \mathbb{Z} \ \& \ \mathsf{Y} \neq [\emptyset, \emptyset] \rightarrow \left(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{X}, \mathsf{Y}])) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{X} *_{\mathbb{Z}} \mathsf{Y})\right)$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\quad \mathsf{n}, \mathsf{m} \in \mathbb{Z} \ \& \ \mathsf{m} \neq [\emptyset, \emptyset] \ \& \ \neg\left(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{n}, \mathsf{m}])) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m})\right)$
$\langle [\mathsf{n}, \mathsf{m}] \rangle \hookrightarrow$ *T352* $\Rightarrow$ $\quad [\mathsf{n}, \mathsf{m}] \in \mathsf{Fr}$
$\langle [\mathsf{n}, \mathsf{m}] \rangle \hookrightarrow$ *T344* $\Rightarrow$ $\quad \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{n}, \mathsf{m}]) \in \mathbb{Q}$
$\langle \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{n}, \mathsf{m}]) \rangle \hookrightarrow$ *T346* $\Rightarrow$ $\quad \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{n}, \mathsf{m}]) \in \mathsf{Fr}$
$\langle [\mathsf{n}, \mathsf{m}] \rangle \hookrightarrow$ *T344* $\Rightarrow$ $\quad [\mathsf{n}, \mathsf{m}] \approx_{\mathsf{Fr}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{n}, \mathsf{m}])$
$\langle [\mathsf{n}, \mathsf{m}], \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{n}, \mathsf{m}]) \rangle \hookrightarrow$ *T362* $\Rightarrow$
$\quad \mathsf{is\_nonneg}_{\mathbb{N}}\left(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{n}, \mathsf{m}]) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{n}, \mathsf{m}])\right) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}([\mathsf{n}, \mathsf{m}]^{[1]} *_{\mathbb{Z}} [\mathsf{n}, \mathsf{m}]^{[2]})$
ELEM $\Rightarrow$ $\quad [\mathsf{n}, \mathsf{m}]^{[1]} = \mathsf{n} \ \& \ [\mathsf{n}, \mathsf{m}]^{[2]} = \mathsf{m}$
EQUAL $\Rightarrow$ $\quad \mathsf{is\_nonneg}_{\mathbb{N}}\left(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{n}, \mathsf{m}]) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{n}, \mathsf{m}])\right) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m})$

Use_def (is_nonneg$_{\mathbb{Q}}$) $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(\text{Fr\_to\_}\mathbb{Q}([n,m])) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([n,m]) *_{\mathbb{Z}} \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([n,m]))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we not that various utility constants are signed integers (or fractions): the zero and unity signed integers, likewise the zero and unity fractions.

Theorem 525 (378) $[1,\emptyset], [\emptyset,\emptyset] \in \mathbb{Z} \,\&\, [1,\emptyset] \neq [\emptyset,\emptyset] \,\&\, [[\emptyset,\emptyset],[1,\emptyset]] \in \text{Fr} \,\&\, [[1,\emptyset],[1,\emptyset]] \in \text{Fr}$. PROOF:

Suppose_not $\Rightarrow$
$\neg([1,\emptyset], [\emptyset,\emptyset] \in \mathbb{Z} \,\&\, [1,\emptyset] \neq [\emptyset,\emptyset] \,\&\, [[\emptyset,\emptyset],[1,\emptyset]], [[1,\emptyset],[1,\emptyset]] \in \text{Fr})$
T183 $\Rightarrow$ $\emptyset, 1 \in \mathbb{N} \,\&\, \emptyset \neq 1$
Suppose $\Rightarrow$ $[1,\emptyset] \notin \mathbb{Z}$
Use_def ($\mathbb{Z}$) $\Rightarrow$ *Stat1* : $[1,\emptyset] \notin \{[x,y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle 1,\emptyset \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ $[1,\emptyset] \in \mathbb{Z}$
Suppose $\Rightarrow$ $[\emptyset,\emptyset] \notin \mathbb{Z}$
Use_def ($\mathbb{Z}$) $\Rightarrow$ *Stat2* : $[\emptyset,\emptyset] \notin \{[x,y] : x \in \mathbb{N}, y \in \mathbb{N} \mid x = \emptyset \vee y = \emptyset\}$
$\langle \emptyset,\emptyset \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $[\emptyset,\emptyset] \in \mathbb{Z}$
ELEM $\Rightarrow$ $[1,\emptyset] \neq [\emptyset,\emptyset]$
$\langle [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[\emptyset,\emptyset],[1,\emptyset]] \in \text{Fr}$
$\langle [[1,\emptyset],[1,\emptyset]] \rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[1,\emptyset],[1,\emptyset]] \in \text{Fr}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next, quite elementary, result simply state that the unit rational number is the multiplicative unit for rationals.

Theorem 526 (379) $M \in \mathbb{Q} \rightarrow M = M *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$. PROOF:

Suppose_not (m) $\Rightarrow$ $m \in \mathbb{Q} \,\&\, m \neq m *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
$\langle m \rangle \hookrightarrow$ *T346* $\Rightarrow$ $\mathbf{arb}(m) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(m)) = m$
$\langle \mathbf{arb}(m) \rangle \hookrightarrow$ *T352* $\Rightarrow$ $\mathbf{arb}(m) = \left[\mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]}\right] \,\&\, \mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(m)^{[2]} \neq [\emptyset,\emptyset]$
Loc_def $\Rightarrow$ $am = \mathbf{arb}(m)^{[1]}$
Loc_def $\Rightarrow$ $dm = \mathbf{arb}(m)^{[2]}$
Use_def ($\mathbf{1}_{\mathbb{Q}}$) $\Rightarrow$ $m \neq m *_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])$
T378 $\Rightarrow$ $[1,\emptyset] \in \mathbb{Z} \,\&\, [1,\emptyset] \neq [\emptyset,\emptyset]$
$\langle m, [1,\emptyset], [1,\emptyset] \rangle \hookrightarrow$ *T359* $\Rightarrow$
$m \neq \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} [1,\emptyset], \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} [1,\emptyset]\right])$
EQUAL $\Rightarrow$ $m \neq \text{Fr\_to\_}\mathbb{Q}([am *_{\mathbb{Z}} [1,\emptyset], dm *_{\mathbb{Z}} [1,\emptyset]])$

$\langle \mathsf{am} \rangle \hookrightarrow T325 \Rightarrow$ $\mathsf{am} *_{\mathbb{Z}} [1, \emptyset] = \mathsf{am}$
$\langle \mathsf{dm} \rangle \hookrightarrow T325 \Rightarrow$ $\mathsf{dm} *_{\mathbb{Z}} [1, \emptyset] = \mathsf{dm}$
EQUAL $\Rightarrow$ $\mathsf{m} \neq \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am}, \mathsf{dm}])$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that if m is a nonzero rational, its reciprocal Recip (m) is also a rational, and is the multiplicative inverse of m. This tells us that the rational numbers form an algebraic 'field'.

Theorem 527 (380) $\mathsf{M} \in \mathbb{Q} \,\&\, \mathsf{M} \neq \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Recip}_{\mathbb{Q}}(\mathsf{M}) \in \mathbb{Q} \,\&\, \mathsf{M} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{M}) = \mathbf{1}_{\mathbb{Q}}$. PROOF:

Suppose_not(m) $\Rightarrow$ *Stat1* : $\mathsf{m} \in \mathbb{Q} \,\&\, \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \notin \mathbb{Q} \vee \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \neq \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\mathsf{m}) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{m})) = \mathsf{m}$
$\langle \mathbf{arb}(\mathsf{m}) \rangle \hookrightarrow T352 \Rightarrow$ $\mathbf{arb}(\mathsf{m}) = \left[\mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]}\right] \,\&\, \mathbf{arb}(\mathsf{m})^{[1]}, \mathbf{arb}(\mathsf{m})^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(\mathsf{m})^{[2]} \neq [\emptyset, \emptyset]$
Loc_def $\Rightarrow$ $\mathsf{am} = \mathbf{arb}(\mathsf{m})^{[1]}$
Loc_def $\Rightarrow$ $\mathsf{dm} = \mathbf{arb}(\mathsf{m})^{[2]}$
EQUAL $\Rightarrow$ *Stat2* : $\mathbf{arb}(\mathsf{m}) = [\mathsf{am}, \mathsf{dm}] \,\&\, \mathsf{am}, \mathsf{dm} \in \mathbb{Z} \,\&\, \mathsf{dm} \neq [\emptyset, \emptyset]$
EQUAL $\Rightarrow$ $[\mathsf{am}, \mathsf{dm}] \in \mathsf{Fr}$
T378 $\Rightarrow$ *Stat3* :
$[1, \emptyset], [\emptyset, \emptyset] \in \mathbb{Z} \,\&\, [1, \emptyset] \neq [\emptyset, \emptyset] \,\&\, [[\emptyset, \emptyset], [1, \emptyset]] \in \mathsf{Fr} \,\&$
$[[1, \emptyset], [1, \emptyset]] \in \mathsf{Fr}$
Suppose $\Rightarrow$ $\mathsf{am} = [\emptyset, \emptyset]$
EQUAL $\Rightarrow$ $\mathsf{am} *_{\mathbb{Z}} [1, \emptyset] = [\emptyset, \emptyset] *_{\mathbb{Z}} [1, \emptyset]$
$\langle \mathsf{dm} \rangle \hookrightarrow T324 \Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{dm} = [\emptyset, \emptyset]$
$\langle [1, \emptyset] \rangle \hookrightarrow T324 \Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathsf{dm} = [\emptyset, \emptyset]$
ALGEBRA $\Rightarrow$ $\mathsf{dm} *_{\mathbb{Z}} [\emptyset, \emptyset] = [\emptyset, \emptyset]$
ELEM $\Rightarrow$ $[\mathsf{am}, \mathsf{dm}]^{[1]} = \mathsf{am} \,\&\, [\mathsf{am}, \mathsf{dm}]^{[2]} = \mathsf{dm} \,\&$
$[[\emptyset, \emptyset], [1, \emptyset]]^{[1]} = [\emptyset, \emptyset] \,\&\, [[\emptyset, \emptyset], [1, \emptyset]]^{[2]} = [1, \emptyset]$
EQUAL $\Rightarrow$ $[\mathsf{am}, \mathsf{dm}]^{[1]} *_{\mathbb{Z}} [[\emptyset, \emptyset], [1, \emptyset]]^{[2]} = [\mathsf{am}, \mathsf{dm}]^{[2]} *_{\mathbb{Z}} [[\emptyset, \emptyset], [1, \emptyset]]^{[1]}$
Use_def($\approx_{\mathsf{Fr}}$) $\Rightarrow$ $[\mathsf{am}, \mathsf{dm}] \approx_{\mathsf{Fr}} [[\emptyset, \emptyset], [1, \emptyset]]$
$\langle [\mathsf{am}, \mathsf{dm}], [[\emptyset, \emptyset], [1, \emptyset]] \rangle \hookrightarrow T345 \Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am}, \mathsf{dm}]) =$
$\mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], [1, \emptyset]])$
EQUAL $\Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am}, \mathsf{dm}]) = \mathsf{m}$
Use_def($\mathbf{0}_{\mathbb{Q}}$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat4* : $\mathsf{am} \neq [\emptyset, \emptyset]$
Use_def($\mathsf{Recip}_{\mathbb{Q}}$) $\Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[2]}, \mathbf{arb}(\mathsf{m})^{[1]}\right])$
$\langle [\mathsf{dm}, \mathsf{am}] \rangle \hookrightarrow T352 \Rightarrow$ $[\mathsf{dm}, \mathsf{am}] \in \mathsf{Fr}$
ALGEBRA $\Rightarrow$ $\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am} \in \mathbb{Z}$
$\langle \mathsf{am}, \mathsf{dm} \rangle \hookrightarrow T330(\langle Stat2, Stat4, * \rangle) \Rightarrow$ $\mathsf{dm} *_{\mathbb{Z}} \mathsf{am} \neq [\emptyset, \emptyset]$

$\langle\, [\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}]\,\rangle \hookrightarrow T352 \Rightarrow \quad Stat5: \ [\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}] \in \mathsf{Fr}$
EQUAL $\Rightarrow \quad Stat6: \ \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{dm}, \mathsf{am}])$
$\langle\, [\mathsf{dm}, \mathsf{am}]\,\rangle \hookrightarrow T344 \Rightarrow \quad Stat7: \ \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q}$
EQUAL $\langle Stat7, Stat1, Stat6\rangle \Rightarrow \quad Stat8: \ \mathsf{m} *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{dm}, \mathsf{am}]) \neq \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{m}, \mathsf{dm}, \mathsf{am}\rangle \hookrightarrow T359(\langle Stat1, Stat8, Stat2, Stat4\rangle) \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathsf{dm}, \mathbf{arb}(\mathsf{m})^{[2]} *_{\mathbb{Z}} \mathsf{am}\right]) \neq \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}]) \neq \mathbf{1}_{\mathbb{Q}}$
Use_def $(\mathbf{1}_{\mathbb{Q}}) \Rightarrow \quad Stat9: \ \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}]) \neq \mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [1, \emptyset]])$
$\langle\, [\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}], [[1, \emptyset], [1, \emptyset]]\,\rangle \hookrightarrow T345(\langle Stat9, Stat3, Stat5\rangle) \Rightarrow$
$\neg\, [\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}] \approx_{\mathsf{Fr}} [[1, \emptyset], [1, \emptyset]]$
Use_def $(\approx_{\mathsf{Fr}}) \Rightarrow \quad Stat10: \ [\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}]^{[1]} *_{\mathbb{Z}} [[1, \emptyset], [1, \emptyset]]^{[2]} \neq$
$[\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{am}]^{[2]} *_{\mathbb{Z}} [[1, \emptyset], [1, \emptyset]]^{[1]}$
$\langle Stat10\rangle$ ELEM $\Rightarrow \quad \mathsf{am} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} [1, \emptyset] \neq \mathsf{dm} *_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} [1, \emptyset]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following elementary extension of Theorem 380 gives the inverse relationship between rational multiplication and division.

Theorem 528 (381) $\mathsf{N}, \mathsf{M} \in \mathbb{Q} \ \& \ \mathsf{M} \neq \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{N} = \mathsf{M} *_{\mathbb{Q}} (\mathsf{N} /_{\mathbb{Q}} \mathsf{M})$. PROOF:

Suppose_not $(\mathsf{n}, \mathsf{m}) \Rightarrow \quad \mathsf{n}, \mathsf{m} \in \mathbb{Q} \ \& \ \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}} \ \& \ \mathsf{n} \neq \mathsf{m} *_{\mathbb{Q}} (\mathsf{n} /_{\mathbb{Q}} \mathsf{m})$
$\langle \mathsf{m}\rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q}$
Use_def $(/_{\mathbb{Q}}) \Rightarrow \quad \mathsf{n} /_{\mathbb{Q}} \mathsf{m} = \mathsf{n} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})$
EQUAL $\Rightarrow \quad \mathsf{n} \neq \mathsf{m} *_{\mathbb{Q}} \left(\mathsf{n} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\right)$
$\langle \mathsf{m}, \mathsf{n} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{n} \neq \mathsf{m} *_{\mathbb{Q}} \left(\mathsf{n} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\right)$
$\langle \mathsf{n}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{n} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{n}$
EQUAL $\Rightarrow \quad \mathsf{n} \neq \mathsf{m} *_{\mathbb{Q}} \left(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{n}\right)$
$\langle \mathsf{n}, \mathsf{m}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow T374 \Rightarrow \quad \mathsf{n} \neq \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{n}$
$\langle \mathsf{m}\rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad \mathsf{n} \neq \mathbf{1}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{n}$
$T371 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathsf{n}, \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{n} \neq \mathsf{n} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{n}\rangle \hookrightarrow T379 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 529 (382) $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}})$. PROOF:

$\text{Suppose\_not} \Rightarrow \quad \neg\big(\text{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}})\big)$

$\mathit{T291} \Rightarrow \quad \mathit{Stat1}:\ [\emptyset,\emptyset]\,,[1,\emptyset]\in\mathbb{Z}$

$\mathit{T183} \Rightarrow \quad \mathit{Stat2}:\ 1\neq\emptyset$

$\big\langle \mathit{Stat1}\big\rangle\ \text{ELEM} \Rightarrow$

$$[[\emptyset,\emptyset],[1,\emptyset]] = \Big[[[\emptyset,\emptyset],[1,\emptyset]]^{[1]},[[\emptyset,\emptyset],[1,\emptyset]]^{[2]}\Big]\ \&\ [[\emptyset,\emptyset],[1,\emptyset]]^{[1]}\in\mathbb{Z}\ \&$$
$$[[\emptyset,\emptyset],[1,\emptyset]]^{[2]}\in\mathbb{Z}\ \&\ [[\emptyset,\emptyset],[1,\emptyset]]^{[2]}\neq[\emptyset,\emptyset]$$

$\big\langle [[\emptyset,\emptyset],[1,\emptyset]]\big\rangle\hookrightarrow \mathit{T352} \Rightarrow \quad [[\emptyset,\emptyset],[1,\emptyset]]\in\text{Fr}$

$\big\langle [[1,\emptyset],[1,\emptyset]]\big\rangle\hookrightarrow \mathit{T352} \Rightarrow \quad [[1,\emptyset],[1,\emptyset]]\in\text{Fr}$

$\big\langle [[\emptyset,\emptyset],[1,\emptyset]]\big\rangle\hookrightarrow \mathit{T344} \Rightarrow \quad \text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])\in\mathbb{Q}$

$\big\langle [[1,\emptyset],[1,\emptyset]]\big\rangle\hookrightarrow \mathit{T344} \Rightarrow \quad \text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])\in\mathbb{Q}$

$\big\langle \text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])\big\rangle\hookrightarrow \mathit{T346} \Rightarrow \quad \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})\,([[\emptyset,\emptyset],[1,\emptyset]])\in\text{Fr}$

$\big\langle \text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])\big\rangle\hookrightarrow \mathit{T346} \Rightarrow \quad \mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})\,([[1,\emptyset],[1,\emptyset]])\in\text{Fr}$

$\text{Use\_def}(\mathbf{0}_{\mathbb{Q}}) \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} = \text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$

$\text{Use\_def}(\mathbf{1}_{\mathbb{Q}}) \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} = \text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])$

$\text{EQUAL} \Rightarrow \quad \neg\Big(\text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])\big)\ \&\ \text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])\big)\Big)$

$\mathit{T378} \Rightarrow \quad [\emptyset,\emptyset]\,,[1,\emptyset]\in\mathbb{Z}\ \&\ [1,\emptyset]\neq[\emptyset,\emptyset]$

$\text{Use\_def}(\text{is\_nonneg}_{\mathbb{Q}}) \Rightarrow$

$$\Big(\text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}\big(\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([[\emptyset,\emptyset],[1,\emptyset]])*_{\mathbb{Z}}\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([[\emptyset,\emptyset],[1,\emptyset]])\big)\Big)\ \&$$
$$\text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}\big(\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([[1,\emptyset],[1,\emptyset]])*_{\mathbb{Z}}\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([[1,\emptyset],[1,\emptyset]])\big)$$

$\text{ELEM} \Rightarrow \quad [[\emptyset,\emptyset],[1,\emptyset]]^{[1]} = [\emptyset,\emptyset]\ \&\ [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} = [1,\emptyset]$

$\text{ELEM} \Rightarrow \quad [[1,\emptyset],[1,\emptyset]]^{[1]} = [1,\emptyset]\ \&\ [[1,\emptyset],[1,\emptyset]]^{[2]} = [1,\emptyset]$

$\big\langle [[1,\emptyset],[1,\emptyset]]\big\rangle\hookrightarrow \mathit{T344} \Rightarrow$

$$[[1,\emptyset],[1,\emptyset]]\approx_{\text{Fr}}\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})\,([[1,\emptyset],[1,\emptyset]])$$

$\big\langle [[\emptyset,\emptyset],[1,\emptyset]]\big\rangle\hookrightarrow \mathit{T344} \Rightarrow$

$$[[\emptyset,\emptyset],[1,\emptyset]]\approx_{\text{Fr}}\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})\,([[\emptyset,\emptyset],[1,\emptyset]])$$

$\big\langle [[1,\emptyset],[1,\emptyset]]\,,\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})\,([[1,\emptyset],[1,\emptyset]])\big\rangle\hookrightarrow \mathit{T362} \Rightarrow$

$$\text{is\_nonneg}_{\mathbb{N}}\big([[1,\emptyset],[1,\emptyset]]^{[1]}*_{\mathbb{Z}}[[1,\emptyset],[1,\emptyset]]^{[2]}\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}\big(\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([[1,\emptyset],[1,\emptyset]])*_{\mathbb{Z}}\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([[1,\emptyset],[1,\emptyset]])\big)$$

$\big\langle [[\emptyset,\emptyset],[1,\emptyset]]\,,\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})\,([[\emptyset,\emptyset],[1,\emptyset]])\big\rangle\hookrightarrow \mathit{T362} \Rightarrow$

$$\text{is\_nonneg}_{\mathbb{N}}\big([[\emptyset,\emptyset],[1,\emptyset]]^{[1]}*_{\mathbb{Z}}[[\emptyset,\emptyset],[1,\emptyset]]^{[2]}\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}\big(\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[1]}([[\emptyset,\emptyset],[1,\emptyset]])*_{\mathbb{Z}}\mathbf{arb}(\text{Fr\_to\_}\mathbb{Q})^{[2]}([[\emptyset,\emptyset],[1,\emptyset]])\big)$$

$\text{EQUAL} \Rightarrow \quad \Big(\text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}([\emptyset,\emptyset]*_{\mathbb{Z}}[1,\emptyset])\Big)\ \&$

$$\text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}([1,\emptyset]*_{\mathbb{Z}}[1,\emptyset])$$

$\big\langle [\emptyset,\emptyset]\big\rangle\hookrightarrow \mathit{T325} \Rightarrow \quad [\emptyset,\emptyset]*_{\mathbb{Z}}[1,\emptyset] = [\emptyset,\emptyset]$

$\big\langle [1,\emptyset]\big\rangle\hookrightarrow \mathit{T325} \Rightarrow \quad [1,\emptyset]*_{\mathbb{Z}}[1,\emptyset] = [1,\emptyset]$

$\text{EQUAL} \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}([\emptyset,\emptyset])$

$\text{EQUAL} \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\big(\text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]])\big)\leftrightarrow\text{is\_nonneg}_{\mathbb{N}}([1,\emptyset])$

Use_def(is_nonneg$_{\mathbb{N}}$) $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(\text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]]))\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(\text{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]]))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that either a rational number n or its reverse $\text{Rev}_{\mathbb{Q}}(\text{n})$ must be non-negative, and that if both are non-negative n must be zero.

Theorem 530 (383) $X \in \mathbb{Q} \rightarrow \text{is\_nonneg}_{\mathbb{Q}}(X) \vee \text{is\_nonneg}_{\mathbb{Q}}(\text{Rev}_{\mathbb{Q}}(X))\ \&\ \left(\text{is\_nonneg}_{\mathbb{Q}}(X)\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(\text{Rev}_{\mathbb{Q}}(X)) \rightarrow X = \mathbf{0}_{\mathbb{Q}}\right)$. PROOF:

Suppose_not(n) $\Rightarrow$ $\text{n} \in \mathbb{Q}\ \&\ \neg\left(\text{is\_nonneg}_{\mathbb{Q}}(\text{n}) \vee \text{is\_nonneg}_{\mathbb{Q}}(\text{Rev}_{\mathbb{Q}}(\text{n}))\right)\ \&\ \left(\text{is\_nonneg}_{\mathbb{Q}}(\text{n})\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(\text{Rev}_{\mathbb{Q}}(\text{n})) \rightarrow \text{n} = \mathbf{0}_{\mathbb{Q}}\right)$
$\langle \text{n} \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\text{n}) \in \text{Fr}\ \&\ \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\text{n})) = \text{n}$
$\langle \mathbf{arb}(\text{n}) \rangle \hookrightarrow T352 \Rightarrow$ $\mathbf{arb}(\text{n}) = \left[\mathbf{arb}(\text{n})^{[1]}, \mathbf{arb}(\text{n})^{[2]}\right]\ \&\ \mathbf{arb}(\text{n})^{[1]}, \mathbf{arb}(\text{n})^{[2]} \in \mathbb{Z}\ \&\ \mathbf{arb}(\text{n})^{[2]} \neq [\emptyset,\emptyset]$
Loc_def $\Rightarrow$ $\text{an} = \mathbf{arb}(\text{n})^{[1]}$
Loc_def $\Rightarrow$ $\text{dn} = \mathbf{arb}(\text{n})^{[2]}$
Loc_def $\Rightarrow$ $\text{av} = \mathbf{arb}(\text{Rev}_{\mathbb{Q}})^{[1]}(\text{n})$
Loc_def $\Rightarrow$ $\text{dv} = \mathbf{arb}(\text{Rev}_{\mathbb{Q}})^{[2]}(\text{n})$
Use_def(Rev$_{\mathbb{Q}}$) $\Rightarrow$ $\text{Rev}_{\mathbb{Q}}(\text{n}) = \text{Fr\_to\_}\mathbb{Q}([\text{Rev}_{\mathbb{Z}}(\text{an}), \text{dn}])$
EQUAL $\Rightarrow$ $\text{n} = \text{Fr\_to\_}\mathbb{Q}([\text{an},\text{dn}])\ \&\ \text{an},\text{dn} \in \mathbb{Z}\ \&\ \text{dn} \neq [\emptyset,\emptyset]$
$\langle \text{an} \rangle \hookrightarrow T314 \Rightarrow$ $\text{Rev}_{\mathbb{Z}}(\text{an}) \in \mathbb{Z}$
$\langle \text{Rev}_{\mathbb{Z}}(\text{an}), \text{dn} \rangle \hookrightarrow T377 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}\left(\text{Fr\_to\_}\mathbb{Q}([\text{Rev}_{\mathbb{Z}}(\text{an}), \text{dn}])\right) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(\text{Rev}_{\mathbb{Z}}(\text{an}) *_{\mathbb{Z}} \text{dn})$
EQUAL $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(\text{Rev}_{\mathbb{Q}}(\text{n})) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(\text{Rev}_{\mathbb{Z}}(\text{an}) *_{\mathbb{Z}} \text{dn})$
$\langle \text{an}, \text{dn} \rangle \hookrightarrow T377 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(\text{Fr\_to\_}\mathbb{Q}([\text{an},\text{dn}])) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(\text{an} *_{\mathbb{Z}} \text{dn})$
EQUAL $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(\text{n}) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(\text{an} *_{\mathbb{Z}} \text{dn})$
ALGEBRA $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(\text{Rev}_{\mathbb{Q}}(\text{n})) \leftrightarrow \text{is\_nonneg}_{\mathbb{N}}(\text{Rev}_{\mathbb{Z}}(\text{an} *_{\mathbb{Z}} \text{dn}))$
ALGEBRA $\Rightarrow$ $\text{an} *_{\mathbb{Z}} \text{dn} \in \mathbb{Z}$
$\langle \text{an} *_{\mathbb{Z}} \text{dn} \rangle \hookrightarrow T347 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{N}}(\text{an} *_{\mathbb{Z}} \text{dn}) \vee \text{is\_nonneg}_{\mathbb{N}}(\text{an} *_{\mathbb{Z}} \text{dn})\ \&\ \left(\text{is\_nonneg}_{\mathbb{N}}(\text{an} *_{\mathbb{Z}} \text{dn})\ \&\ \text{is\_nonneg}_{\mathbb{N}}(\text{an} *_{\mathbb{Z}} \text{dn}) \rightarrow \text{an} *_{\mathbb{Z}} \text{dn} = [\emptyset,\emptyset]\right)$
ELEM $\Rightarrow$ $\text{an} *_{\mathbb{Z}} \text{dn} = [\emptyset,\emptyset]$
$\langle \text{an}, \text{dn} \rangle \hookrightarrow T330 \Rightarrow$ $\text{n} = \text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],\text{dn}])$
Suppose $\Rightarrow$ $\neg\,[[\emptyset,\emptyset],\text{dn}] \approx_{\text{Fr}} [[\emptyset,\emptyset],[1,\emptyset]]$
Use_def($\approx_{\text{Fr}}$) $\Rightarrow$ $[\emptyset,\emptyset] *_{\mathbb{Z}} \text{dn} \neq [\emptyset,\emptyset] *_{\mathbb{Z}} [1,\emptyset]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $[[\emptyset,\emptyset],\text{dn}] \approx_{\text{Fr}} [[\emptyset,\emptyset],[1,\emptyset]]$
$\langle [[\emptyset,\emptyset],\text{dn}], [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow T345 \Rightarrow$
$\text{n} = \text{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$
Use_def($\mathbf{0}_{\mathbb{Q}}$) $\Rightarrow$ QED

APPLY $\langle \succcurlyeq_\Theta : \geqslant_{\mathbb{Q}}, \preccurlyeq_\Theta : \leqslant_{\mathbb{Q}}, \succ_\Theta : >_{\mathbb{Q}}, \prec_\Theta : <_{\mathbb{Q}} \rangle$ Ordered_add$(\text{g} \mapsto \mathbb{Q}, \text{e} \mapsto \mathbf{0}_{\mathbb{Q}}, \oplus \mapsto +_{\mathbb{Q}}, \text{minz} \mapsto -_{\mathbb{Q}}, \text{rvz} \mapsto \text{Rev}_{\mathbb{Q}}, \text{nneg} \mapsto \text{is_nonneg}_{\mathbb{Q}}) \Rightarrow$

Theorem 531 (384*a*)

$\langle \forall x, y \mid x \geqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) \rangle \,\&\, \langle \forall x, y \mid x \leqslant_{\mathbb{Q}} y \leftrightarrow y \geqslant_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x, y \mid x >_{\mathbb{Q}} y \leftrightarrow x \geqslant_{\mathbb{Q}} y \,\&\, x \neq y \rangle \,\&\, \langle \forall x, y \mid x <_{\mathbb{Q}} y \leftrightarrow y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x, y \mid x \leqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}$

$\langle \forall x, y \mid y, y \in \mathbb{Q} \rightarrow (x >_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x -_{\mathbb{Q}} y) \,\&\, x \neq y) \rangle \,\&\, \langle \forall x, y \mid x, y \in \mathbb{Q} \,\&\, x = y \vee \neg x \geqslant_{\mathbb{Q}} y \rightarrow y \geqslant_{\mathbb{Q}} x \rangle.$

Theorem 532 (384)

$\big(X \geqslant_{\mathbb{Q}} Y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(X +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(Y))\big) \,\&\, (X \leqslant_{\mathbb{Q}} Y \leftrightarrow Y \geqslant_{\mathbb{Q}} X) \,\&\, (X >_{\mathbb{Q}} Y \leftrightarrow X \geqslant_{\mathbb{Q}} Y \,\&\, X \neq Y) \,\&\, (X <_{\mathbb{Q}} Y \leftrightarrow Y >_{\mathbb{Q}} X) \,\&\, \big(X \leqslant_{\mathbb{Q}} Y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(Y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(X))\big) \,\&\, \big(X, Y \in \mathbb{Q} \rightarrow$

$\big(X, Y \in \mathbb{Q} \rightarrow (X >_{\mathbb{Q}} Y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(X -_{\mathbb{Q}} Y) \,\&\, X \neq Y)\big) \,\&\, (X, Y \in \mathbb{Q} \,\&\, X = Y \vee \neg X \geqslant_{\mathbb{Q}} Y \rightarrow Y \geqslant_{\mathbb{Q}} X).$ Proof:

Suppose_not$(x, y) \Rightarrow$

$\neg\Big(\big(x \geqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y))\big) \,\&\, (x \leqslant_{\mathbb{Q}} y \leftrightarrow y \geqslant_{\mathbb{Q}} x) \,\&\, (x >_{\mathbb{Q}} y \leftrightarrow x \geqslant_{\mathbb{Q}} y \,\&\, x \neq y) \,\&\, (x <_{\mathbb{Q}} y \leftrightarrow y >_{\mathbb{Q}} x) \,\&\, \big(x \leqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(x))\big) \,\&\, \big(x, y \in$

Suppose $\Rightarrow$ $\neg\big(x \geqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y))\big)$

T384a $\Rightarrow$ *Stat1* : $\langle \forall x, y \mid x \geqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ $x \geqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y))$

Suppose $\Rightarrow$ $\neg(x \leqslant_{\mathbb{Q}} y \leftrightarrow y \geqslant_{\mathbb{Q}} x)$

T384a $\Rightarrow$ *Stat2* : $\langle \forall x, y \mid x \leqslant_{\mathbb{Q}} y \leftrightarrow y \geqslant_{\mathbb{Q}} x \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $x \leqslant_{\mathbb{Q}} y \leftrightarrow y \geqslant_{\mathbb{Q}} x$

Suppose $\Rightarrow$ $\neg(x >_{\mathbb{Q}} y \leftrightarrow x \geqslant_{\mathbb{Q}} y \,\&\, x \neq y)$

T384a $\Rightarrow$ *Stat3* : $\langle \forall x, y \mid x >_{\mathbb{Q}} y \leftrightarrow x \geqslant_{\mathbb{Q}} y \,\&\, x \neq y \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ $x >_{\mathbb{Q}} y \leftrightarrow x \geqslant_{\mathbb{Q}} y \,\&\, x \neq y$

Suppose $\Rightarrow$ $\neg(x <_{\mathbb{Q}} y \leftrightarrow y >_{\mathbb{Q}} x)$

T384a $\Rightarrow$ *Stat4* : $\langle \forall x, y \mid x <_{\mathbb{Q}} y \leftrightarrow y >_{\mathbb{Q}} x \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$ $x <_{\mathbb{Q}} y \leftrightarrow y >_{\mathbb{Q}} x$

Suppose $\Rightarrow$ $\neg\big(x \leqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(x))\big)$

T384a $\Rightarrow$ *Stat5* : $\langle \forall x, y \mid x \leqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(x)) \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ false; Discharge $\Rightarrow$ $x \leqslant_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(x))$

Suppose $\Rightarrow$ $\neg\Big(x, y \in \mathbb{Q} \rightarrow \big(x >_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) \,\&\, x \neq y\big)\Big)$

T384a $\Rightarrow$ *Stat6* : $\langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow \big(x >_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) \,\&\, x \neq y\big) \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ false; Discharge $\Rightarrow$ $x, y \in \mathbb{Q} \rightarrow \big(x >_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) \,\&\, x \neq y\big)$

Suppose $\Rightarrow$ $\neg\big(x, y \in \mathbb{Q} \rightarrow (x >_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x -_{\mathbb{Q}} y) \,\&\, x \neq y)\big)$

T384a $\Rightarrow$ *Stat7* : $\langle \forall x, y \mid y, y \in \mathbb{Q} \rightarrow (x >_{\mathbb{Q}} y \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(x -_{\mathbb{Q}} y) \,\&\, x \neq y) \rangle$

$\langle x, y \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg(X, Y \in \mathbb{Q} \,\&\, X = Y \vee \neg X \geqslant_{\mathbb{Q}} Y \rightarrow Y \geqslant_{\mathbb{Q}} X)$

T384a $\Rightarrow$ *Stat8* : $\langle \forall x, y \mid x, y \in \mathbb{Q} \,\&\, x = y \vee \neg x \geqslant_{\mathbb{Q}} y \rightarrow y \geqslant_{\mathbb{Q}} x \rangle$
$\langle x, y \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 533 (385) $X \in \mathbb{Q} \rightarrow X = X *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$. PROOF:

Suppose_not(n) $\Rightarrow$ $n \in \mathbb{Q} \,\&\, n \neq n *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
Use_def($\mathbf{1}_{\mathbb{Q}}$) $\Rightarrow$ $n \neq n *_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [1, \emptyset]])$
T291 $\Rightarrow$ $[1, \emptyset] \in \mathbb{Z} \,\&\, [1, \emptyset] \neq [\emptyset, \emptyset]$
$\langle n \rangle \hookrightarrow$ *T346* $\Rightarrow$ $\mathbf{arb}(n) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$
$\langle n, [1, \emptyset], [1, \emptyset] \rangle \hookrightarrow$ *T359* $\Rightarrow$ $n *_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [1, \emptyset]]) =$
$\text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [1, \emptyset], \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} [1, \emptyset]\right])$
$\langle \mathbf{arb}(n) \rangle \hookrightarrow$ *T352* $\Rightarrow$ $\mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \,\&\, \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$
$\langle \mathbf{arb}(n)^{[1]} \rangle \hookrightarrow$ *T325* $\Rightarrow$ $\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [1, \emptyset] = \mathbf{arb}(n)^{[1]}$
$\langle \mathbf{arb}(n)^{[2]} \rangle \hookrightarrow$ *T325* $\Rightarrow$ $\mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} [1, \emptyset] = \mathbf{arb}(n)^{[2]}$
EQUAL $\Rightarrow$ $n *_{\mathbb{Q}} \text{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [1, \emptyset]]) = \text{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right])$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that a rational number is zero if and only if the denominator of any one of its expressions as a fraction is zero.

Theorem 534 (386) $X \in \mathbb{Q} \rightarrow (X = \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathbf{arb}(X)^{[1]} = [\emptyset, \emptyset])$. PROOF:

Suppose_not(n) $\Rightarrow$ $n \in \mathbb{Q} \,\&\, \neg(n = \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathbf{arb}(n)^{[1]} = [\emptyset, \emptyset])$
Use_def($\mathbf{0}_{\mathbb{Q}}$) $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} = \text{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], [1, \emptyset]])$
T291 $\Rightarrow$ *Stat1* : $[\emptyset, \emptyset], [1, \emptyset] \in \mathbb{Z} \,\&\, [1, \emptyset] \neq [\emptyset, \emptyset]$
$\langle [[\emptyset, \emptyset], [1, \emptyset]] \rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[\emptyset, \emptyset], [1, \emptyset]] \in \text{Fr}$
ELEM $\Rightarrow$ $[[\emptyset, \emptyset], [1, \emptyset]]^{[2]} = [1, \emptyset] \,\&\, [[\emptyset, \emptyset], [1, \emptyset]]^{[1]} = [\emptyset, \emptyset]$
$\langle n \rangle \hookrightarrow$ *T346* $\Rightarrow$ $\mathbf{arb}(n) \in \text{Fr} \,\&\, \text{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$
$\langle \mathbf{arb}(n) \rangle \hookrightarrow$ *T352* $\Rightarrow$ $\mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \,\&\, \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$
Suppose $\Rightarrow$ $\mathbf{arb}(n)^{[1]} = [\emptyset, \emptyset]$
Suppose $\Rightarrow$ $\neg \mathbf{arb}(n) \approx_{\text{Fr}} [[\emptyset, \emptyset], [1, \emptyset]]$
Use_def($\approx_{\text{Fr}}$) $\Rightarrow$ $\mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} [[\emptyset, \emptyset], [1, \emptyset]]^{[1]} \neq \mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [[\emptyset, \emptyset], [1, \emptyset]]^{[2]}$
EQUAL $\Rightarrow$ $\mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} [\emptyset, \emptyset] \neq [\emptyset, \emptyset] *_{\mathbb{Z}} [1, \emptyset]$
ALGEBRA $\Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset] *_{\mathbb{Z}} [1, \emptyset]$
$\langle \mathbf{arb}(n)^{[2]} \rangle \hookrightarrow$ *T324* $\Rightarrow$ $[\emptyset, \emptyset] *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} = [\emptyset, \emptyset]$

$\langle [1,\emptyset] \rangle \hookrightarrow T324 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{arb}(n) \approx_{Fr} [[\emptyset,\emptyset],[1,\emptyset]]$
$\langle \mathbf{arb}(n), [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow T345 \Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) =$
$\mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{arb}(n)^{[1]} \neq [\emptyset,\emptyset] \ \& \ n = \mathbf{0}_{\mathbb{Q}}$
ELEM $\Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$
$\langle \mathbf{arb}(n), [[\emptyset,\emptyset],[1,\emptyset]] \rangle \hookrightarrow T345 \Rightarrow$ $\mathbf{arb}(n) \approx_{Fr} [[\emptyset,\emptyset],[1,\emptyset]]$
Use_def$(\approx_{Fr}) \Rightarrow$ $\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} = \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[1]}$
EQUAL $\Rightarrow$ $\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [1,\emptyset] = \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} [\emptyset,\emptyset]$
ALGEBRA $\Rightarrow$ $\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [1,\emptyset] = [\emptyset,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}$
$\langle \mathbf{arb}(n)^{[2]} \rangle \hookrightarrow T324 \Rightarrow$ $\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} [1,\emptyset] = [\emptyset,\emptyset]$
ALGEBRA $\Rightarrow$ $[1,\emptyset] *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]} = [\emptyset,\emptyset]$
$\langle \mathbf{arb}(n)^{[1]} \rangle \hookrightarrow T324 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next theorem states that the product and sum of any two non-negative rational numbers is non-negative.

Theorem 535 (387) $X, Y \in \mathbb{Q} \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(X) \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(Y) \rightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(X +_{\mathbb{Q}} Y) \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(X *_{\mathbb{Q}} Y)$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n, m \in \mathbb{Q} \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(n) \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(m) \ \& \ \neg\big(\mathsf{is\_nonneg}_{\mathbb{Q}}(n +_{\mathbb{Q}} m) \ \& \ \mathsf{is\_nonneg}_{\mathbb{Q}}(n *_{\mathbb{Q}} m)\big)$
$\langle n \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(n) \in \mathsf{Fr} \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(n)) = n$
$\langle m \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(m) \in \mathsf{Fr} \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(m)) = m$
$\langle \mathbf{arb}(n) \rangle \hookrightarrow T352 \Rightarrow$ $\mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \ \& \ \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \ \& \ \mathbf{arb}(n)^{[2]} \neq [\emptyset,\emptyset]$
$\langle \mathbf{arb}(m) \rangle \hookrightarrow T352 \Rightarrow$ $\mathbf{arb}(m) = \left[\mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]}\right] \ \& \ \mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]} \in \mathbb{Z} \ \& \ \mathbf{arb}(m)^{[2]} \neq [\emptyset,\emptyset]$
Loc_def $\Rightarrow$ $an = \mathbf{arb}(n)^{[1]}$
Loc_def $\Rightarrow$ $dn = \mathbf{arb}(n)^{[2]}$
Loc_def $\Rightarrow$ $am = \mathbf{arb}(m)^{[1]}$
Loc_def $\Rightarrow$ $dm = \mathbf{arb}(m)^{[2]}$
EQUAL $\Rightarrow$ $Stat1: \ \mathbf{arb}(n) = [an, dn] \ \& \ an, dn \in \mathbb{Z} \ \& \ dn \neq [\emptyset,\emptyset]$
EQUAL $\Rightarrow$ $Stat2: \ \mathbf{arb}(m) = [am, dm] \ \& \ am, dm \in \mathbb{Z} \ \& \ dm \neq [\emptyset,\emptyset]$
$\langle dm, dn \rangle \hookrightarrow T330(\langle Stat1, Stat2, * \rangle) \Rightarrow$ $Stat3: \ dn *_{\mathbb{Z}} dm \neq [\emptyset,\emptyset]$
Use_def$(\mathsf{is_nonneg}_{\mathbb{Q}}) \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{N}}(\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}) \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}(\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]})$
EQUAL $\Rightarrow$ $Stat4: \ \mathsf{is\_nonneg}_{\mathbb{N}}(an *_{\mathbb{Z}} dn) \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}(am *_{\mathbb{Z}} dm)$
ALGEBRA $\Rightarrow$ $Stat5: \ an *_{\mathbb{Z}} am, dn *_{\mathbb{Z}} dm \in \mathbb{Z}$
ALGEBRA $\Rightarrow$ $Stat6: \ an *_{\mathbb{Z}} dn, am *_{\mathbb{Z}} dm \in \mathbb{Z}$
Suppose $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{Q}}(n *_{\mathbb{Q}} m)$
Use_def$(*_{\mathbb{Q}}) \Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{Q}}\left(\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(n)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(m)^{[2]}\right])\right)$

EQUAL $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]))$
$\langle \mathsf{an} *_{\mathbb{Z}} \mathsf{am}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} \rangle \hookrightarrow T377 \Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{an} *_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}))$
ALGEBRA $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{an} *_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{am} *_{\mathbb{Z}} \mathsf{dm}))$
$\langle \mathsf{an} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{am} *_{\mathbb{Z}} \mathsf{dm} \rangle \hookrightarrow T348 \Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n} +_{\mathbb{Q}} \mathsf{m})$
Use_def $(+_{\mathbb{Q}}) \Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]} +_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}, \mathbf{arb}(\mathsf{n})^{[2]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{m})^{[2]}\right]))$
EQUAL $\Rightarrow$ $Stat7:$ $\neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}]))$
ALGEBRA $\Rightarrow$ $Stat8:$ $\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn} \in \mathbb{Z}$
$\langle \mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dm} \rangle \hookrightarrow T377(\langle Stat7, Stat8, Stat5, Stat3 \rangle) \Rightarrow$
$\neg\mathsf{is\_nonneg}_{\mathbb{N}}((\mathsf{an} *_{\mathbb{Z}} \mathsf{dm} +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dn}) *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dm}))$
ALGEBRA $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{an} *_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}) +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dn}))$
$\langle \mathsf{dn} \rangle \hookrightarrow T350 \Rightarrow$ $Stat9:$ $\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{dn} *_{\mathbb{Z}} \mathsf{dn})$
$\langle \mathsf{dm} \rangle \hookrightarrow T350 \Rightarrow$ $Stat10:$ $\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm})$
ALGEBRA $\Rightarrow$ $Stat11:$ $\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dn} \in \mathbb{Z}$
$\langle \mathsf{an} *_{\mathbb{Z}} \mathsf{dn}, \mathsf{dm} *_{\mathbb{Z}} \mathsf{dm} \rangle \hookrightarrow T348(\langle Stat5, Stat4, Stat9, Stat10, Stat6, Stat11 \rangle) \Rightarrow$ $Stat12:$
$\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{an} *_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}))$
$\langle \mathsf{am} *_{\mathbb{Z}} \mathsf{dm}, \mathsf{dn} *_{\mathbb{Z}} \mathsf{dn} \rangle \hookrightarrow T348(\langle Stat5, Stat4, Stat9, Stat10, Stat6, Stat11 \rangle) \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{am} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dn}))$
ALGEBRA $\Rightarrow$ $\mathsf{an} *_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}) \in \mathsf{SI}$ & $\mathsf{am} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dn}) \in \mathbb{Z}$
$\langle \mathsf{an} *_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}), \mathsf{am} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dn}) \rangle \hookrightarrow T348(\langle Stat12 \rangle) \Rightarrow$
$\mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{an} *_{\mathbb{Z}} \mathsf{dn} *_{\mathbb{Z}} (\mathsf{dm} *_{\mathbb{Z}} \mathsf{dm}) +_{\mathbb{Z}} \mathsf{am} *_{\mathbb{Z}} \mathsf{dm} *_{\mathbb{Z}} (\mathsf{dn} *_{\mathbb{Z}} \mathsf{dn}))$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemma simply states that the zero rational is its own negative and that the unit rational is positive.

Theorem 536 (388) $\mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$ & $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$ & $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$. PROOF:

Suppose_not $\Rightarrow$ $Stat0:$ $\mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) \neq \mathbf{0}_{\mathbb{Q}} \vee \mathbf{1}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} \vee \neg \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

-- The first part of this assertion has the following elemenary algebraic proof.

$T371 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T372 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$ & $\mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) \in \mathbb{Q}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) \rangle \hookrightarrow T371 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) \rangle \hookrightarrow T365 \Rightarrow$ $Stat10:$ $\mathbf{0}_{\mathbb{Q}} = \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$

-- Similarly, to prove that $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$, we have only to use the definitions of the quantities and operators involved, and do a bit of algebra.

Suppose $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
Use_def ($\mathbf{1}_{\mathbb{Q}}$) $\Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]]) = \mathbf{0}_{\mathbb{Q}}$
Use_def ($\mathbf{0}_{\mathbb{Q}}$) $\Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[1,\emptyset]]) = \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$
T291 $\Rightarrow$ *Stat1* : $[\emptyset,\emptyset], [1,\emptyset] \in \mathbb{Z}$ & $[1,\emptyset] \neq [\emptyset,\emptyset]$
$\langle$ *Stat1* $\rangle$ ELEM $\Rightarrow$ $[[1,\emptyset],[1,\emptyset]]^{[1]}, [[1,\emptyset],[1,\emptyset]]^{[2]} \in \mathbb{Z}$ &
$[[1,\emptyset],[1,\emptyset]]^{[2]} \neq [\emptyset,\emptyset]$
$\langle\, [[1,\emptyset],[1,\emptyset]] \,\rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[1,\emptyset],[1,\emptyset]] \in \mathsf{Fr}$
$\langle$ *Stat1* $\rangle$ ELEM $\Rightarrow$ $[[\emptyset,\emptyset],[1,\emptyset]]^{[1]}, [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} \in \mathbb{Z}$ &
$[[\emptyset,\emptyset],[1,\emptyset]]^{[2]} \neq [\emptyset,\emptyset]$
$\langle\, [[\emptyset,\emptyset],[1,\emptyset]] \,\rangle \hookrightarrow$ *T352* $\Rightarrow$ $[[\emptyset,\emptyset],[1,\emptyset]] \in \mathsf{Fr}$
$\langle\, [[1,\emptyset],[1,\emptyset]], [[\emptyset,\emptyset],[1,\emptyset]] \,\rangle \hookrightarrow$ *T345* $\Rightarrow$
$[[1,\emptyset],[1,\emptyset]] \approx_{\mathsf{Fr}} [[\emptyset,\emptyset],[1,\emptyset]]$
Use_def ($\approx_{\mathsf{Fr}}$) $\Rightarrow$ $[[1,\emptyset],[1,\emptyset]]^{[1]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} =$
$[[1,\emptyset],[1,\emptyset]]^{[2]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[1]}$
TELEM $\Rightarrow$ $[[1,\emptyset],[1,\emptyset]]^{[1]} = [1,\emptyset]$ & $[[1,\emptyset],[1,\emptyset]]^{[2]} = [1,\emptyset]$
TELEM $\Rightarrow$ $[[\emptyset,\emptyset],[1,\emptyset]]^{[1]} = [\emptyset,\emptyset]$ & $[[\emptyset,\emptyset],[1,\emptyset]]^{[2]} = [1,\emptyset]$
EQUAL $\Rightarrow$ $[1,\emptyset] *_{\mathbb{Z}} [1,\emptyset] = [1,\emptyset] *_{\mathbb{Z}} [\emptyset,\emptyset]$
$\langle\, [1,\emptyset] \,\rangle \hookrightarrow$ *T325* $\Rightarrow$ $[1,\emptyset] = [1,\emptyset] *_{\mathbb{Z}} [\emptyset,\emptyset]$
$\langle\, [\emptyset,\emptyset] \,\rangle \hookrightarrow$ *T324* $\Rightarrow$ $[1,\emptyset] = [\emptyset,\emptyset]$
T291 $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat20* : $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$

$\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow$ *T384* $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}\left(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\right)$ & $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$ & $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow$ *T371* $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}})$ & $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
ALGEBRA $\Rightarrow$ *Stat30* : $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}})$ & $\mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
T382 $\Rightarrow$ *Stat40* : $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}})$
$\langle$ *Stat10, Stat20, Stat30, Stat40, Stat0,* $*\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next two theorems give the distributive law for subtraction, first in its simplest form, and then generally. We begin with two preliminary lemmas, which give the corresponding rule for fractions.

Theorem 537 (389) $\mathsf{X},\mathsf{Y},\mathsf{XP},\mathsf{YP} \in \mathbb{Z}$ & $[\mathsf{X},\mathsf{Y}] \approx_{\mathsf{Fr}} [\mathsf{XP},\mathsf{YP}] \rightarrow [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{X}),\mathsf{Y}] \approx_{\mathsf{Fr}} [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{XP}),\mathsf{YP}]$. PROOF:

Suppose_not ($\mathsf{m},\mathsf{n},\mathsf{m}',\mathsf{n}'$) $\Rightarrow$ $\mathsf{m},\mathsf{n},\mathsf{m}',\mathsf{n}' \in \mathbb{Z}$ & $[\mathsf{m},\mathsf{n}] \approx_{\mathsf{Fr}} [\mathsf{m}',\mathsf{n}']$ & $\neg\, [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}] \approx_{\mathsf{Fr}} [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}'),\mathsf{n}']$
Use_def ($\approx_{\mathsf{Fr}}$) $\Rightarrow$ $[\mathsf{m},\mathsf{n}]^{[1]} *_{\mathbb{Z}} [\mathsf{m}',\mathsf{n}']^{[2]} = [\mathsf{m},\mathsf{n}]^{[2]} *_{\mathbb{Z}} [\mathsf{m}',\mathsf{n}']^{[1]}$ &
$[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}]^{[1]} *_{\mathbb{Z}} [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}'),\mathsf{n}']^{[2]} \neq [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}]^{[2]} *_{\mathbb{Z}} [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}'),\mathsf{n}']^{[1]}$

ELEM $\Rightarrow$ $[\mathsf{m},\mathsf{n}]^{[1]} = \mathsf{m}$ & $[\mathsf{m},\mathsf{n}]^{[2]} = \mathsf{n}$ & $[\mathsf{m}',\mathsf{n}']^{[1]} = \mathsf{m}'$ & $[\mathsf{m}',\mathsf{n}']^{[2]} = \mathsf{n}'$
ELEM $\Rightarrow$ $[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}]^{[1]} = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m})$ & $[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}]^{[2]} = \mathsf{n}$ & $[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}'),\mathsf{n}']^{[1]} = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}')$ & $[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}'),\mathsf{n}']^{[2]} = \mathsf{n}'$
EQUAL $\Rightarrow$ $\mathsf{m} *_{\mathbb{Z}} \mathsf{n}' = \mathsf{n} *_{\mathbb{Z}} \mathsf{m}'$ & $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) *_{\mathbb{Z}} \mathsf{n}' \neq \mathsf{n} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}')$
$\langle \mathsf{m} \rangle \hookrightarrow T314 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) \in \mathbb{Z}$
ALGEBRA $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) *_{\mathbb{Z}} \mathsf{n}' = \mathsf{n}' *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m})$
$\langle \mathsf{n}', \mathsf{m} \rangle \hookrightarrow T313 \Rightarrow$ $\mathsf{n}' *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n}' *_{\mathbb{Z}} \mathsf{m})$
$\langle \mathsf{n}, \mathsf{m}' \rangle \hookrightarrow T313 \Rightarrow$ $\mathsf{n} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}') = \mathsf{Rev}_{\mathbb{Z}}(\mathsf{n} *_{\mathbb{Z}} \mathsf{m}')$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 538 (390) $\mathsf{X},\mathsf{Y} \in \mathbb{Z}$ & $\mathsf{Y} \neq [\emptyset,\emptyset] \to \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{X},\mathsf{Y}])) = \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{Rev}_{\mathbb{Z}}(\mathsf{X}),\mathsf{Y}])$. PROOF:

Suppose_not$(\mathsf{m},\mathsf{n}) \Rightarrow$ $(\mathsf{m},\mathsf{n} \in \mathbb{Z}$ & $\mathsf{n} \neq [\emptyset,\emptyset])$ & $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{m},\mathsf{n}])) \neq \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}])$
Use_def$(\mathsf{Rev}_{\mathbb{Q}}) \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{m},\mathsf{n}])) = \mathsf{Fr\_to\_}\mathbb{Q}\Big(\Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}])), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big]\Big)$
$\langle [\mathsf{m},\mathsf{n}] \rangle \hookrightarrow T352 \Rightarrow$ $[\mathsf{m},\mathsf{n}] \in \mathsf{Fr}$
$\langle \mathsf{m} \rangle \hookrightarrow T314 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}) \in \mathbb{Z}$
$\langle [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}] \rangle \hookrightarrow T352 \Rightarrow$ $[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}] \in \mathsf{Fr}$
$\langle [\mathsf{m},\mathsf{n}] \rangle \hookrightarrow T344 \Rightarrow$ $\mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{m},\mathsf{n}]) \in \mathbb{Q}$ & $[\mathsf{m},\mathsf{n}] \approx_{\mathsf{Fr}} \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{m},\mathsf{n}])$
$\langle \mathsf{Fr\_to\_}\mathbb{Q}([\mathsf{m},\mathsf{n}]) \rangle \hookrightarrow T346 \Rightarrow$ $\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{m},\mathsf{n}]) \in \mathsf{Fr}$
$\langle \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{m},\mathsf{n}]) \rangle \hookrightarrow T352 \Rightarrow$

$\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})([\mathsf{m},\mathsf{n}]) = \Big[\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}]), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big]$ & $\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}]) \in \mathbb{Z}$ &
$\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}]) \in \mathbb{Z}$ & $\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}]) \neq [\emptyset,\emptyset]$

$\langle \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}]) \rangle \hookrightarrow T314 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}])) \in \mathbb{Z}$
$\Big\langle \Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}])), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big] \Big\rangle \hookrightarrow T352 \Rightarrow$

$\Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}])), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big] \in \mathsf{Fr}$

EQUAL $\Rightarrow$ $[\mathsf{m},\mathsf{n}] \approx_{\mathsf{Fr}} \Big[\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}]), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big]$
EQUAL $\Rightarrow$ $\Big[\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}]), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big] \in \mathsf{Fr}$
$\langle \mathsf{m},\mathsf{n}, \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}]), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}]) \rangle \hookrightarrow T389 \Rightarrow$

$[\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}] \approx_{\mathsf{Fr}} \Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}])), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big]$

$\Big\langle [\mathsf{Rev}_{\mathbb{Z}}(\mathsf{m}),\mathsf{n}], \Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[1]}([\mathsf{m},\mathsf{n}])), \mathbf{arb}(\mathsf{Fr\_to\_}\mathbb{Q})^{[2]}([\mathsf{m},\mathsf{n}])\Big] \Big\rangle \hookrightarrow T345 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 539 (391) $\mathsf{X},\mathsf{Y} \in \mathbb{Q} \to \mathsf{X} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{X} *_{\mathbb{Q}} \mathsf{Y})$. PROOF:

Suppose_not$(\mathsf{m},\mathsf{n}) \Rightarrow$ *Stat0*: $\mathsf{m},\mathsf{n} \in \mathbb{Q}$ & $\mathsf{m} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \neq \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow T346 \Rightarrow$ $\mathsf{n} = \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{n}))$ & $\mathbf{arb}(\mathsf{n}) \in \mathsf{Fr}$

$\langle \mathbf{arb}(n) \rangle \hookrightarrow T352 \Rightarrow \quad \mathbf{arb}(n) = \left[\mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]}\right] \;\&\; \mathbf{arb}(n)^{[1]}, \mathbf{arb}(n)^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$

$\langle m \rangle \hookrightarrow T346 \Rightarrow \quad m = \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(m)) \;\&\; \mathbf{arb}(m) \in \mathsf{Fr}$

$\langle \mathbf{arb}(m) \rangle \hookrightarrow T352 \Rightarrow \quad \mathbf{arb}(m) = \left[\mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]}\right] \;\&\; \mathbf{arb}(m)^{[1]}, \mathbf{arb}(m)^{[2]} \in \mathbb{Z} \;\&\; \mathbf{arb}(m)^{[2]} \neq [\emptyset, \emptyset]$

$\mathsf{Use\_def}(\mathsf{Rev}_{\mathbb{Q}}) \Rightarrow \quad m *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) = m *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}), \mathbf{arb}(n)^{[2]}\right])$

$\langle \mathbf{arb}(n)^{[1]} \rangle \hookrightarrow T314 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}) \in \mathbb{Z}$

$\langle m, \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}), \mathbf{arb}(n)^{[2]} \rangle \hookrightarrow T359 \Rightarrow \quad m *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}), \mathbf{arb}(n)^{[2]}\right]) =$

$\quad \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}), \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}\right])$

$\langle \mathbf{arb}(m)^{[1]}, \mathbf{arb}(n)^{[1]} \rangle \hookrightarrow T313 \Rightarrow \quad \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}) = \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]})$

$\mathsf{EQUAL} \Rightarrow \quad m *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}), \mathbf{arb}(n)^{[2]}\right]) = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]}), \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}\right])$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} \in \mathbb{Z}$

$\langle \mathbf{arb}(n)^{[2]}, \mathbf{arb}(m)^{[2]} \rangle \hookrightarrow T330([Stat0, \cap]) \Rightarrow \quad \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} \neq [\emptyset, \emptyset]$

$\langle \mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]} \rangle \hookrightarrow T390 \Rightarrow \quad m *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(n)^{[1]}), \mathbf{arb}(n)^{[2]}\right]) =$

$\quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}\right]))$

$\mathsf{Use\_def}(*_{\mathbb{Q}}) \Rightarrow \quad m *_{\mathbb{Q}} n = \mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(m)^{[1]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[1]}, \mathbf{arb}(m)^{[2]} *_{\mathbb{Z}} \mathbf{arb}(n)^{[2]}\right])$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 540 (392) $X, Y, ZZ \in \mathbb{Q} \to X *_{\mathbb{Q}} (Y -_{\mathbb{Q}} ZZ) = X *_{\mathbb{Q}} Y -_{\mathbb{Q}} X *_{\mathbb{Q}} ZZ$. PROOF:

$\mathsf{Suppose\_not}(m, n, k) \Rightarrow \quad m, n, k \in \mathbb{Q} \;\&\; m *_{\mathbb{Q}} (n -_{\mathbb{Q}} k) \neq m *_{\mathbb{Q}} n -_{\mathbb{Q}} m *_{\mathbb{Q}} k$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad m *_{\mathbb{Q}} (n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k)) \neq m *_{\mathbb{Q}} n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(m *_{\mathbb{Q}} k)$

$\langle k \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(k) \in \mathbb{Q}$

$\langle \mathsf{Rev}_{\mathbb{Q}}(k), m, n \rangle \hookrightarrow T376 \Rightarrow \quad m *_{\mathbb{Q}} n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(m *_{\mathbb{Q}} k) \neq m *_{\mathbb{Q}} n +_{\mathbb{Q}} m *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k)$

$\langle m, k \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m *_{\mathbb{Q}} k) = m *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k)$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Now we prove that multiplication of two rationals m, n satisfying $m >_{\mathbb{Q}} n$ by a common strictly positive fraction produces two products satisfying the same condition.

Theorem 541 (393) $X, Y, X_1 \in \mathbb{Q} \;\&\; X >_{\mathbb{Q}} Y \;\&\; X_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \to X *_{\mathbb{Q}} X_1 >_{\mathbb{Q}} Y *_{\mathbb{Q}} X_1$. PROOF:

$\mathsf{Suppose\_not}(m, n, k) \Rightarrow \quad m, n, k \in \mathbb{Q} \;\&\; m >_{\mathbb{Q}} n \;\&\; k >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \neg m *_{\mathbb{Q}} k >_{\mathbb{Q}} n *_{\mathbb{Q}} k$

$\langle m, n \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(m -_{\mathbb{Q}} n) \;\&\; m \neq n$

$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle \mathsf{k}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{k} -_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \;\&\; \mathsf{k} \neq \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})) \;\&\; \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{k} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$

$\langle \mathsf{m}, \mathsf{k} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{k} \in \mathbb{Q}$

$\langle \mathsf{n}, \mathsf{k} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{n} *_{\mathbb{Q}} \mathsf{k} \in \mathbb{Q}$

$\langle \mathsf{m} *_{\mathbb{Q}} \mathsf{k}, \mathsf{n} *_{\mathbb{Q}} \mathsf{k} \rangle \hookrightarrow T384 \Rightarrow \quad \neg(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathsf{k} -_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} \mathsf{k}) \;\&\; \mathsf{m} *_{\mathbb{Q}} \mathsf{k} \neq \mathsf{n} *_{\mathbb{Q}} \mathsf{k})$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \neg\Big(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathsf{k} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{k})) \;\&\; \mathsf{m} *_{\mathbb{Q}} \mathsf{k} \neq \mathsf{n} *_{\mathbb{Q}} \mathsf{k}\Big)$

$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{k} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$

$\langle \mathsf{k} \rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{k} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{k}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{k})$

$\langle \mathsf{n} \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \in \mathbb{Q}$

$\langle \mathsf{m}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \in \mathbb{Q}$

$\langle \mathsf{k}, \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\Big(\mathsf{k} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))\Big)$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{k}, \mathsf{m} \rangle \hookrightarrow T376 \Rightarrow \quad \mathsf{k} *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})) = \mathsf{k} *_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{k} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})$

$\langle \mathsf{k}, \mathsf{n} \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{k} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{k} *_{\mathbb{Q}} \mathsf{n})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{k} *_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{k} *_{\mathbb{Q}} \mathsf{n}))$

$\langle \mathsf{k}, \mathsf{m} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{k} *_{\mathbb{Q}} \mathsf{m} = \mathsf{m} *_{\mathbb{Q}} \mathsf{k}$

$\langle \mathsf{k}, \mathsf{n} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{k} *_{\mathbb{Q}} \mathsf{n} = \mathsf{n} *_{\mathbb{Q}} \mathsf{k}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathsf{k} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{k}))$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{k} = \mathsf{n} *_{\mathbb{Q}} \mathsf{k}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}) = \mathsf{n} *_{\mathbb{Q}} \mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k})$

$\langle \mathsf{k} \rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}) \in \mathbb{Q} \;\&\; \mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}) = \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}), \mathsf{m}, \mathsf{k} \rangle \hookrightarrow T374 \Rightarrow \quad (\mathsf{m} *_{\mathbb{Q}} \mathsf{k}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}) = \mathsf{m} *_{\mathbb{Q}} (\mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}))$

$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}), \mathsf{n}, \mathsf{k} \rangle \hookrightarrow T374 \Rightarrow \quad (\mathsf{n} *_{\mathbb{Q}} \mathsf{k}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}) = \mathsf{n} *_{\mathbb{Q}} (\mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}))$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} (\mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k})) = \mathsf{n} *_{\mathbb{Q}} (\mathsf{k} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{k}))$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{n} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{m} \rangle \hookrightarrow T379 \Rightarrow \quad \mathsf{m} = \mathsf{m} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{n} \rangle \hookrightarrow T379 \Rightarrow \quad \mathsf{n} = \mathsf{n} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The following lemma states that the product of any rational by zero is zero.

Theorem 542 (394) $\mathsf{X} \in \mathbb{Q} \rightarrow \mathsf{X} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m}) \Rightarrow \quad \mathsf{m} \in \mathbb{Q} \;\&\; \mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$

$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$

$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathsf{m}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathsf{m}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$
$\langle \mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T383 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \vee \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})) \;\&\; \Big(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \;\&\; \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})) \rightarrow \mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}\Big)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the reciprocal of a positive rational is positive.

Theorem 543 (395) $\mathsf{X} \in \mathbb{Q} \;\&\; \mathsf{X} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Recip}_{\mathbb{Q}}(\mathsf{X}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$. PROOF:

Suppose_not(m) $\Rightarrow \quad \mathsf{m} \in \mathbb{Q} \;\&\; \mathsf{m} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \neg\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{m}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})) \;\&\; \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q} \;\&\; \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \in \mathbb{Q}$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})) \vee \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$
$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \;\&\; \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}} \;\&\; \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \vee \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow T371 \Rightarrow \quad Stat1: \; \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q} \;\&\; \mathsf{m} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{m}$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})$
EQUAL $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m}) \;\&\; \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}} \;\&\; \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \vee \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$
$T388 \Rightarrow \quad Stat2: \; \mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
Suppose $\Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow T394 \Rightarrow$ false; Discharge $\Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T383 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\Big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))\Big)$
$\langle \mathsf{m}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\Big(\mathsf{m} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))\Big)$
$\langle \mathsf{m}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$
$\langle \mathsf{m} \rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad Stat3: \; \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}}))$
$T382 \Rightarrow \quad Stat4: \; \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}})$
$\langle \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow T383(\langle Stat1, Stat2, Stat3, Stat4 \rangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is important to know that sign reversal commutes with rational addition.

Theorem 544 (396) $X, Y \in \mathbb{Q} \rightarrow \text{Rev}_{\mathbb{Q}}(X +_{\mathbb{Q}} Y) = \text{Rev}_{\mathbb{Q}}(X) +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(Y)$. Proof:

Suppose_not$(m, n) \Rightarrow$ $m, n \in \mathbb{Q} \;\&\; \text{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} n) \neq \text{Rev}_{\mathbb{Q}}(m) +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n)$
ALGEBRA $\Rightarrow$ $m +_{\mathbb{Q}} n \in \mathbb{Q}$
$\langle n \rangle \hookrightarrow T372 \Rightarrow$ $\text{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$
$\langle m \rangle \hookrightarrow T372 \Rightarrow$ $\text{Rev}_{\mathbb{Q}}(m) \in \mathbb{Q}$
$\langle m +_{\mathbb{Q}} n \rangle \hookrightarrow T372 \Rightarrow$ $\text{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} n) \in \mathbb{Q}$
Loc_def $\Rightarrow$ $rm = \text{Rev}_{\mathbb{Q}}(m)$
Loc_def $\Rightarrow$ $rn = \text{Rev}_{\mathbb{Q}}(n)$
Loc_def $\Rightarrow$ $rmpn = \text{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} n)$
EQUAL $\Rightarrow$ $rm, rn, rmpn \in \mathbb{Q}$
EQUAL $\Rightarrow$ $rmpn \neq rm +_{\mathbb{Q}} rn$
$\langle m \rangle \hookrightarrow T372 \Rightarrow$ $m +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(m) = \mathbf{0}_{\mathbb{Q}}$
$\langle n \rangle \hookrightarrow T372 \Rightarrow$ $n +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n) = \mathbf{0}_{\mathbb{Q}}$
$\langle m +_{\mathbb{Q}} n \rangle \hookrightarrow T372 \Rightarrow$ $m +_{\mathbb{Q}} n +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} n) = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $m +_{\mathbb{Q}} rm = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $n +_{\mathbb{Q}} rn = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $m +_{\mathbb{Q}} n +_{\mathbb{Q}} rmpn = \mathbf{0}_{\mathbb{Q}}$
ALGEBRA $\Rightarrow$ $rn +_{\mathbb{Q}} rm +_{\mathbb{Q}} (m +_{\mathbb{Q}} n +_{\mathbb{Q}} rmpn) = n +_{\mathbb{Q}} rn +_{\mathbb{Q}} (m +_{\mathbb{Q}} rm) +_{\mathbb{Q}} rmpn$
EQUAL $\Rightarrow$ $rn +_{\mathbb{Q}} rm +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} rmpn$
ALGEBRA $\Rightarrow$ $rm +_{\mathbb{Q}} rn = rmpn$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ Qed

-- Our next three lemmas assert the monotonicity and strict monotonicity of rational addition. We prove monotonicity first.

Theorem 545 (397) $X, Y, XP, YP \in \mathbb{Q} \;\&\; X \geqslant_{\mathbb{Q}} Y \;\&\; XP \geqslant_{\mathbb{Q}} YP \rightarrow X +_{\mathbb{Q}} XP \geqslant_{\mathbb{Q}} Y +_{\mathbb{Q}} YP$. Proof:

Suppose_not$(m, n, m', n') \Rightarrow$ $m, n, m', n' \in \mathbb{Q} \;\&\; m \geqslant_{\mathbb{Q}} n \;\&\; m' \geqslant_{\mathbb{Q}} n' \;\&\; \neg m +_{\mathbb{Q}} m' \geqslant_{\mathbb{Q}} n +_{\mathbb{Q}} n'$
$\langle n \rangle \hookrightarrow T372 \Rightarrow$ $\text{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$
$\langle n' \rangle \hookrightarrow T372 \Rightarrow$ $\text{Rev}_{\mathbb{Q}}(n') \in \mathbb{Q}$
Loc_def $\Rightarrow$ $rn = \text{Rev}_{\mathbb{Q}}(n)$
Loc_def $\Rightarrow$ $rn' = \text{Rev}_{\mathbb{Q}}(n')$
ALGEBRA $\Rightarrow$ $m +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n), m' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n') \in \mathbb{Q}$
$\langle m, n \rangle \hookrightarrow T384 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n))$
$\langle m', n' \rangle \hookrightarrow T384 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(m' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n'))$
$\langle m +_{\mathbb{Q}} m', n +_{\mathbb{Q}} n' \rangle \hookrightarrow T384 \Rightarrow$ $\neg\text{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} m' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n +_{\mathbb{Q}} n'))$
EQUAL $\Rightarrow$ $rn, rn', m +_{\mathbb{Q}} rn, m' +_{\mathbb{Q}} rn' \in \mathbb{Q}$
EQUAL $\Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} rn)$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m}' +_{\mathbb{Q}} \mathsf{rn}')$

$\langle \mathsf{n}, \mathsf{n}' \rangle \hookrightarrow T396 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}') = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} +_{\mathbb{Q}} \mathsf{n}')$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{rn} +_{\mathbb{Q}} \mathsf{rn}' = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} +_{\mathbb{Q}} \mathsf{n}')$

$\mathsf{EQUAL} \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{m}' +_{\mathbb{Q}} (\mathsf{rn} +_{\mathbb{Q}} \mathsf{rn}'))$

$\langle \mathsf{m} +_{\mathbb{Q}} \mathsf{rn}, \mathsf{m}' +_{\mathbb{Q}} \mathsf{rn}' \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn} +_{\mathbb{Q}} (\mathsf{m}' +_{\mathbb{Q}} \mathsf{rn}'))$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we prove the familiar facts that the reverse of the reverse of a rational n is n, and that the reciprocal of the reciprocal of non-zero rational n is n.

Theorem 546 (398) $\mathsf{X} \in \mathbb{Q} \rightarrow \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})) = \mathsf{x}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m}) \Rightarrow \quad \mathsf{m} \in \mathbb{Q} \,\&\, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) \neq \mathsf{m}$

$\langle \mathsf{m} \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q} \,\&\, \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) \in \mathbb{Q} \,\&\, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}))$

$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) \rangle \hookrightarrow T365 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) \rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}))$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})), \mathsf{m}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T370 \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \Big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}))\Big) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}))$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}))$

$\langle \mathsf{m} \rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Given a rational number r, either it or its reverse is non-negative, and if both are non-negative r is zero. This is the rational analog of Theorem 347 for signed integers.

Theorem 547 (399) $\mathsf{X} \in \mathbb{Q} \rightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{X}) \vee \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{X})) \,\&\, \Big(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{X}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{X})) \longrightarrow \mathsf{X} = \mathbf{0}_{\mathbb{Q}}\Big)$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}) \Rightarrow \quad \mathsf{n} \in \mathbb{Q} \,\&\, \neg\Big(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n}) \vee \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))\Big) \vee \Big(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})) \,\&\, \mathsf{n} \neq \mathbf{0}_{\mathbb{Q}}\Big)$

$\mathsf{Use\_def}(\mathsf{Rev}_{\mathbb{Q}}) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathsf{Fr\_to\_}\mathbb{Q}(\Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}), \mathbf{arb}(\mathsf{n})^{[2]}\Big])$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}\Big(\mathsf{Fr\_to\_}\mathbb{Q}(\Big[\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}), \mathbf{arb}(\mathsf{n})^{[2]}\Big])\Big)$

$\langle \mathsf{n} \rangle \hookrightarrow T346 \Rightarrow \quad Stat1: \ \mathbf{arb}(\mathsf{n}) \in \mathsf{Fr} \,\&\, \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{n})) = \mathsf{n}$

$\langle \mathbf{arb}(\mathsf{n}) \rangle \hookrightarrow T352 \Rightarrow \quad \mathbf{arb}(\mathsf{n}) = \Big[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\Big] \,\&\, \mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]} \in \mathbb{Z} \,\&\, \mathbf{arb}(\mathsf{n})^{[2]} \neq [\emptyset, \emptyset]$

$\mathsf{EQUAL} \Rightarrow \quad \Big[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\Big] \in \mathsf{Fr}$

$\langle \mathbf{arb}(\mathsf{n})^{[1]} \rangle \hookrightarrow T314 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) \in \mathbb{Z}$
$\langle \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}), \mathbf{arb}(\mathsf{n})^{[2]} \rangle \hookrightarrow T377 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})) \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]})$
ELEM $\Rightarrow \quad \neg\left(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n}) \vee \mathsf{is\_nonneg}_{\mathbb{N}}\left(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}\right)\right) \vee \left(\left(\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n}) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}\left(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}\right)\right) \;\&\; \mathsf{n} \neq \mathbf{0}_{\mathbb{Q}}\right)$
Use_def$(\mathsf{is_nonneg}_{\mathbb{Q}}) \Rightarrow \quad \neg\left(\mathsf{is_nonneg}_{\mathbb{N}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}) \vee \mathsf{is_nonneg}_{\mathbb{N}}\left(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}\right)\right) \vee \left(\mathsf{is_nonneg}_{\mathbb{N}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}) \;\&\; \mathsf{is_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) *_{\mathbb{Z}} \mathbf{ar}$
ALGEBRA $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]}) *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} = \mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]})$
EQUAL $\Rightarrow \quad \neg\left(\mathsf{is\_nonneg}_{\mathbb{N}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}) \vee \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}))\right) \vee \left(\mathsf{is\_nonneg}_{\mathbb{N}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]}) \;\&\; \mathsf{is\_nonneg}_{\mathbb{N}}(\mathsf{Rev}_{\mathbb{Z}}(\mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]})) \;\&\; \mathsf{r}$
ALGEBRA $\Rightarrow \quad \mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} \in \mathbb{Z}$
$\langle \mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} \rangle \hookrightarrow T347 \Rightarrow \quad Stat2: \; \mathbf{arb}(\mathsf{n})^{[1]} *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} = [\emptyset, \emptyset] \;\&\; \mathsf{n} \neq \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{arb}(\mathsf{n})^{[2]}, \mathbf{arb}(\mathsf{n})^{[1]} \rangle \hookrightarrow T330 \Rightarrow \quad \mathbf{arb}(\mathsf{n})^{[1]} = [\emptyset, \emptyset]$
Use_def$(\mathbf{0}_{\mathbb{Q}}) \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} = \mathsf{Fr_to_}\mathbb{Q}([[\emptyset, \emptyset], [1, \emptyset]])$
Use_def$(\approx_{\mathsf{Fr}}) \Rightarrow \quad Stat3:$

$$[[\emptyset, \emptyset], [1, \emptyset]] \approx_{\mathsf{Fr}} \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right] \leftrightarrow [[\emptyset, \emptyset], [1, \emptyset]]^{[1]} *_{\mathbb{Z}} \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right]^{[2]} = [[\emptyset, \emptyset], [1, \emptyset]]^{[2]} *_{\mathbb{Z}} \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right]^{[1]}$$

$\langle Stat3 \rangle$ ELEM $\Rightarrow \quad [[\emptyset, \emptyset], [1, \emptyset]] \approx_{\mathsf{Fr}} \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right] \leftrightarrow [\emptyset, \emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} = [1, \emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[1]}$
EQUAL $\Rightarrow \quad Stat4: \; [[\emptyset, \emptyset], [1, \emptyset]] \approx_{\mathsf{Fr}} \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right] \leftrightarrow [\emptyset, \emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} = [1, \emptyset] *_{\mathbb{Z}} [\emptyset, \emptyset]$
$\langle \mathbf{arb}(\mathsf{n})^{[2]} \rangle \hookrightarrow T324 \Rightarrow \quad [\emptyset, \emptyset] *_{\mathbb{Z}} \mathbf{arb}(\mathsf{n})^{[2]} = [\emptyset, \emptyset]$
$T291 \Rightarrow \quad [\emptyset, \emptyset] \in \mathbb{Z}$
$\langle [\emptyset, \emptyset] \rangle \hookrightarrow T324 \Rightarrow \quad [1, \emptyset] *_{\mathbb{Z}} [\emptyset, \emptyset] = [\emptyset, \emptyset]$
$T291 \Rightarrow \quad Stat5: \; [\emptyset, \emptyset], [1, \emptyset] \in \mathbb{Z}$
$T183 \Rightarrow \quad Stat6: \; 1 \neq \emptyset$
$\langle Stat5 \rangle$ ELEM $\Rightarrow$

$$[[\emptyset, \emptyset], [1, \emptyset]] = \left[[[\emptyset, \emptyset], [1, \emptyset]]^{[1]}, [[\emptyset, \emptyset], [1, \emptyset]]^{[2]}\right] \;\&\; [[\emptyset, \emptyset], [1, \emptyset]]^{[1]} \in \mathbb{Z} \;\&\; [[\emptyset, \emptyset], [1, \emptyset]]^{[2]} \in \mathbb{Z} \;\&\; [[\emptyset, \emptyset], [1, \emptyset]]^{[2]} \neq [\emptyset, \emptyset]$$

$\langle [[\emptyset, \emptyset], [1, \emptyset]] \rangle \hookrightarrow T352 \Rightarrow \quad [[\emptyset, \emptyset], [1, \emptyset]] \in \mathsf{Fr}$
$\langle Stat4 \rangle$ ELEM $\Rightarrow \quad [[\emptyset, \emptyset], [1, \emptyset]] \approx_{\mathsf{Fr}} \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right]$
$\langle [[\emptyset, \emptyset], [1, \emptyset]], \left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right] \rangle \hookrightarrow T345 \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset, \emptyset], [1, \emptyset]]) =$

$$\mathsf{Fr\_to\_}\mathbb{Q}(\left[\mathbf{arb}(\mathsf{n})^{[1]}, \mathbf{arb}(\mathsf{n})^{[2]}\right])$$

EQUAL $\Rightarrow \quad Stat7: \; \mathbf{0}_{\mathbb{Q}} = \mathsf{Fr\_to\_}\mathbb{Q}(\mathbf{arb}(\mathsf{n}))$
$\langle Stat7, Stat2, Stat1 \rangle$ ELEM $\Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad$ QED

-- The following elementary consequence of Lemma 358a a is sometimes more convenient.

Theorem 548 (400) $X, Y \in \mathbb{Q} \rightarrow X \geqslant_{\mathbb{Q}} Y \vee Y \geqslant_{\mathbb{Q}} X \;\&\; (X \geqslant_{\mathbb{Q}} Y \;\&\; Y \geqslant_{\mathbb{Q}} X \rightarrow X = Y)$. PROOF:

$\mathsf{Suppose\_not}(m, n) \Rightarrow \quad m, n \in \mathbb{Q} \;\&\; \neg(m \geqslant_{\mathbb{Q}} n \vee n \geqslant_{\mathbb{Q}} m) \vee ((m \geqslant_{\mathbb{Q}} n \;\&\; n \geqslant_{\mathbb{Q}} m) \;\&\; m \neq n)$

$\langle m, n \rangle \hookrightarrow T384 \Rightarrow \quad m \geqslant_{\mathbb{Q}} n \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n))$

$\langle n, m \rangle \hookrightarrow T384 \Rightarrow \quad n \geqslant_{\mathbb{Q}} m \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}(n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(m))$

$\langle n \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$

$\langle m \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m) \in \mathbb{Q}$

$\langle m, \mathsf{Rev}_{\mathbb{Q}}(n) \rangle \hookrightarrow T396 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)) = \mathsf{Rev}_{\mathbb{Q}}(m) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(n))$

$\langle n \rangle \hookrightarrow T398 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(n)) = n$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)) = \mathsf{Rev}_{\mathbb{Q}}(m) +_{\mathbb{Q}} n$

$\langle \mathsf{Rev}_{\mathbb{Q}}(m), n \rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m) +_{\mathbb{Q}} n = n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(m)$

$\mathsf{EQUAL} \Rightarrow \quad n \geqslant_{\mathbb{Q}} m \leftrightarrow \mathsf{is\_nonneg}_{\mathbb{Q}}\left(\mathsf{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n))\right)$

$\mathsf{ELEM} \Rightarrow \quad \neg\left(\mathsf{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)) \vee \mathsf{is\_nonneg}_{\mathbb{Q}}\left(\mathsf{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n))\right)\right) \vee \left(\mathsf{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)) \;\&\; \mathsf{is\_nonneg}_{\mathbb{Q}}\left(\mathsf{Rev}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n))\right) \;\&\; m \neq n\right)$

$\mathsf{ALGEBRA} \Rightarrow \quad m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$

$\langle m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) \rangle \hookrightarrow T399 \Rightarrow \quad Stat1 : \; m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) = \mathbf{0}_{\mathbb{Q}} \;\&\; m \neq n$

$\langle m, \mathsf{Rev}_{\mathbb{Q}}(n) \rangle \hookrightarrow T365 \Rightarrow \quad m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) = \mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} m$

$\mathsf{EQUAL} \Rightarrow \quad n +_{\mathbb{Q}} (\mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} m) = n +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle n \rangle \hookrightarrow T371 \Rightarrow \quad n +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = n$

$\langle Stat1 \rangle \; \mathsf{ELEM} \Rightarrow \quad n +_{\mathbb{Q}} (\mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} m) = n$

$\langle m, n, \mathsf{Rev}_{\mathbb{Q}}(n) \rangle \hookrightarrow T370 \Rightarrow \quad n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} m = n$

$\langle n \rangle \hookrightarrow T372 \Rightarrow \quad n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} m = n$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Next we prove that rational reversal is a monotone decreasing function.

Theorem 549 (401) $X, Y \in \mathbb{Q} \;\&\; X \geqslant_{\mathbb{Q}} Y \rightarrow \mathsf{Rev}_{\mathbb{Q}}(Y) \geqslant_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(X)$. PROOF:

$\mathsf{Suppose\_not}(m, n) \Rightarrow \quad m, n \in \mathbb{Q} \;\&\; m \geqslant_{\mathbb{Q}} n \;\&\; \neg\mathsf{Rev}_{\mathbb{Q}}(n) \geqslant_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(m)$

$\langle n \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$

$\langle m \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m) \in \mathbb{Q}$

$\langle \mathsf{Rev}_{\mathbb{Q}}(n), \mathsf{Rev}_{\mathbb{Q}}(m) \rangle \hookrightarrow T400 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m) \geqslant_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)$

$\langle m, n \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n))$

$\langle \mathsf{Rev}_{\mathbb{Q}}(n), \mathsf{Rev}_{\mathbb{Q}}(m) \rangle \hookrightarrow T384 \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}\left(\mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(m))\right)$

$\langle m \rangle \hookrightarrow T398 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(m)) = m$

EQUAL $\Rightarrow$ $\neg \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) +_{\mathbb{Q}} \mathsf{m})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{m}\rangle \hookrightarrow T365 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) +_{\mathbb{Q}} \mathsf{m} = \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- An equally familiar fact is that the reverse of y is smaller than the reverse of x when x, y are rational numbers and x is smaller than y.

Theorem 550 (401*a*) $\mathsf{X}, \mathsf{Y} \in \mathbb{Q} \,\&\, \mathsf{X} >_{\mathbb{Q}} \mathsf{Y} \rightarrow \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Y}) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{X})$. PROOF:

Suppose_not$(\mathsf{x}, \mathsf{y}) \Rightarrow$ $\mathsf{x}, \mathsf{y} \in \mathbb{Q} \,\&\, \mathsf{x} >_{\mathbb{Q}} \mathsf{y} \,\&\, \neg \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$
$\langle \mathsf{x}, \mathsf{y}\rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})) \,\&\, \mathsf{x} \neq \mathsf{y}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}), \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\rangle \hookrightarrow T384 \Rightarrow$ $\neg \mathsf{is\_nonneg}_{\mathbb{Q}}\Big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}))\Big) \vee \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$
$\langle \mathsf{x}\rangle \hookrightarrow T398 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})) = \mathsf{x}$
$\langle \mathsf{y}\rangle \hookrightarrow T398 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})) = \mathsf{y}$
$\langle \mathsf{y}\rangle \hookrightarrow T372 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) \in \mathbb{Q}$
$\langle \mathsf{x}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})\rangle \hookrightarrow T365 \Rightarrow$ $\mathsf{x} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) +_{\mathbb{Q}} \mathsf{x}$
EQUAL $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{Q}}\Big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}))\Big)$
ELEM $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We now combine the preceding lemmas to prove strict monotonicity of rational addition.

Theorem 551 (402) $\mathsf{X}, \mathsf{Y}, \mathsf{XP}, \mathsf{YP} \in \mathbb{Q} \,\&\, \mathsf{X} \geqslant_{\mathbb{Q}} \mathsf{Y} \,\&\, \mathsf{XP} >_{\mathbb{Q}} \mathsf{YP} \rightarrow \mathsf{X} +_{\mathbb{Q}} \mathsf{XP} >_{\mathbb{Q}} \mathsf{Y} +_{\mathbb{Q}} \mathsf{YP}$. PROOF:

Suppose_not$(\mathsf{m}, \mathsf{n}, \mathsf{m}', \mathsf{n}') \Rightarrow$ $\mathsf{m}, \mathsf{n}, \mathsf{m}', \mathsf{n}' \in \mathbb{Q} \,\&\, \mathsf{m} \geqslant_{\mathbb{Q}} \mathsf{n} \,\&\, \mathsf{m}' >_{\mathbb{Q}} \mathsf{n}' \,\&\, \neg \mathsf{m} +_{\mathbb{Q}} \mathsf{m}' >_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathsf{n}'$

-- Suppose that m, n, m', and n' form a counterexample to our theorem. Since addition is known to be monotone, we have at least $\mathsf{m} +_{\mathbb{Q}} \mathsf{m}' \geqslant_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathsf{n}'$, and since the corresponding strict inequality is false we must have $\mathsf{m} +_{\mathbb{Q}} \mathsf{m}' = \mathsf{n} +_{\mathbb{Q}} \mathsf{n}'$, so that $\mathsf{n}' +_{\mathbb{Q}} \mathsf{n} \geqslant_{\mathbb{Q}} \mathsf{m}' +_{\mathbb{Q}} \mathsf{m}$ is also true.

$\langle \mathsf{m}', \mathsf{n}'\rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{m}' \geqslant_{\mathbb{Q}} \mathsf{n}'$
$\langle \mathsf{m}, \mathsf{n}, \mathsf{m}', \mathsf{n}'\rangle \hookrightarrow T397 \Rightarrow$ $\mathsf{m} +_{\mathbb{Q}} \mathsf{m}' \geqslant_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathsf{n}'$
$\langle \mathsf{m} +_{\mathbb{Q}} \mathsf{m}', \mathsf{n} +_{\mathbb{Q}} \mathsf{n}'\rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{m} +_{\mathbb{Q}} \mathsf{m}' = \mathsf{n} +_{\mathbb{Q}} \mathsf{n}'$
ALGEBRA $\Rightarrow$ $\mathsf{n} +_{\mathbb{Q}} \mathsf{n}' \geqslant_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{m}'$
ALGEBRA $\Rightarrow$ $\mathsf{n}' +_{\mathbb{Q}} \mathsf{n} \geqslant_{\mathbb{Q}} \mathsf{m}' +_{\mathbb{Q}} \mathsf{m}$

-- By theorem 401, we have $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \geqslant_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})$, and then adding these last two inequalities and reassociating we find that $\mathsf{n}' \geqslant_{\mathbb{Q}} \mathsf{m}'$, so that $\mathsf{n}' = \mathsf{m}'$, contradicting $\mathsf{m}' >_{\mathbb{Q}} \mathsf{n}'$ an so proving our theorem.

$\langle \mathsf{m}, \mathsf{n} \rangle \hookrightarrow T401 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \geqslant_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})$

$\langle \mathsf{n} \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \in \mathbb{Q} \;\&\; \mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m} \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q} \;\&\; \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m}' +_{\mathbb{Q}} \mathsf{m}, \mathsf{n}' +_{\mathbb{Q}} \mathsf{n} \in \mathbb{Q}$

$\langle \mathsf{n}' +_{\mathbb{Q}} \mathsf{n}, \mathsf{m}' +_{\mathbb{Q}} \mathsf{m}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T397 \Rightarrow \quad \mathsf{n}' +_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \geqslant_{\mathbb{Q}} \mathsf{m}' +_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m})$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{n}', \mathsf{n} \rangle \hookrightarrow T370 \Rightarrow \quad (\mathsf{n}' +_{\mathbb{Q}} \mathsf{n}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathsf{n}' +_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}), \mathsf{m}', \mathsf{m} \rangle \hookrightarrow T370 \Rightarrow \quad (\mathsf{m}' +_{\mathbb{Q}} \mathsf{m}) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{m}' +_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{m}))$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n}' +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \mathsf{m}' +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{n}' \rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{n}' +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{n}'$

$\langle \mathsf{m}' \rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{m}' +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{m}'$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n}' \geqslant_{\mathbb{Q}} \mathsf{m}'$

$\langle \mathsf{m}', \mathsf{n}' \rangle \hookrightarrow T400 \Rightarrow \quad \mathsf{m}' = \mathsf{n}'$

$\langle \mathsf{m}', \mathsf{n}' \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathrm{Q{\small ED}}$

Theorem 552 (403) $\mathsf{X} \in \mathbb{Q} \;\&\; \mathsf{X} \neq \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Recip}_{\mathbb{Q}}(\mathsf{X}) \neq \mathbf{0}_{\mathbb{Q}} \;\&\; \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{X})) = \mathsf{X}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m}) \Rightarrow \quad \mathsf{m} \in \mathbb{Q} \;\&\; \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}} \;\&\; \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}} \vee \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \neq \mathsf{m}$

$\langle \mathsf{m} \rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q} \;\&\; \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{1}_{\mathbb{Q}}$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{m} \rangle \hookrightarrow T394 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} = \mathbf{1}_{\mathbb{Q}}$

$T388 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \neq \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \in \mathbb{Q} \;\&\; \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) = \mathbf{1}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) = \mathbf{1}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$

$T371 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle \mathbf{1}_{\mathbb{Q}}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \rangle \hookrightarrow T368 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) \rangle \hookrightarrow T379 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$

$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})), \mathsf{m}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T374 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \Big(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))\Big) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}))$

$\langle \mathsf{m} \rangle \hookrightarrow T379 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathrm{Q{\small ED}}$

-- The following lemmas supply the evident facts that the strict and nonstrict rational comparisons are both transitive relationships. We begin with the nonstrict version.

Theorem 553 (404) $X, Y, ZZ \in \mathbb{Q} \,\&\, X \geqslant_{\mathbb{Q}} Y \,\&\, Y \geqslant_{\mathbb{Q}} ZZ \rightarrow X \geqslant_{\mathbb{Q}} ZZ$. PROOF:

$\mathsf{Suppose\_not}(m, n, j) \Rightarrow \quad m, n, j \in \mathbb{Q} \,\&\, m \geqslant_{\mathbb{Q}} n \,\&\, n \geqslant_{\mathbb{Q}} j \,\&\, \neg m \geqslant_{\mathbb{Q}} j$
$\langle m, n\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)\right)$
$\langle n, j\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\left(n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j)\right)$
$\langle m, j\rangle \hookrightarrow T384 \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j)\right)$
$\langle m\rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(m) \in \mathbb{Q}$
$\langle n\rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$
$\langle j\rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(j) \in \mathbb{Q}$
$\mathsf{ALGEBRA} \Rightarrow \quad m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n), n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j) \in \mathbb{Q}$
$\langle m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n), n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j)\rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} \left(n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j)\right)\right)$
$\langle n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j), m, \mathsf{Rev}_{\mathbb{Q}}(n)\rangle \hookrightarrow T370 \Rightarrow \quad m +_{\mathbb{Q}} (\mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} (n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j))) = (m +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)) +_{\mathbb{Q}} (n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j))$
$\langle \mathsf{Rev}_{\mathbb{Q}}(j), \mathsf{Rev}_{\mathbb{Q}}(n), n\rangle \hookrightarrow T370 \Rightarrow \quad (\mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} n) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j) = \mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} (n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j))$
$\langle \mathsf{Rev}_{\mathbb{Q}}(n), n\rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} n = n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n)$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} (n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j))\right)$
$\langle n\rangle \hookrightarrow T372 \Rightarrow \quad n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(n) = \mathbf{0}_{\mathbb{Q}}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} (\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j))\right)$
$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Rev}_{\mathbb{Q}}(j)\rangle \hookrightarrow T365 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(j) = \mathsf{Rev}_{\mathbb{Q}}(j) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(j)\rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(j) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{Rev}_{\mathbb{Q}}(j)$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- The preceding lemma can be used to prove the following 'strict' version of itself.

Theorem 554 (405) $X, Y, ZZ \in \mathbb{Q} \,\&\, X >_{\mathbb{Q}} Y \,\&\, Y \geqslant_{\mathbb{Q}} ZZ \rightarrow X >_{\mathbb{Q}} ZZ$. PROOF:

$\mathsf{Suppose\_not}(m, n, j) \Rightarrow \quad m, n, j \in \mathbb{Q} \,\&\, m >_{\mathbb{Q}} n \,\&\, n \geqslant_{\mathbb{Q}} j \,\&\, \neg m >_{\mathbb{Q}} j$
$\langle m, n\rangle \hookrightarrow T384 \Rightarrow \quad m \geqslant_{\mathbb{Q}} n$
$\langle m, n, j\rangle \hookrightarrow T404 \Rightarrow \quad m \geqslant_{\mathbb{Q}} j$
$\langle m, j\rangle \hookrightarrow T384 \Rightarrow \quad m = j$
$\mathsf{EQUAL} \Rightarrow \quad j >_{\mathbb{Q}} n$
$\langle j, n\rangle \hookrightarrow T384 \Rightarrow \quad j \geqslant_{\mathbb{Q}} n$

$\langle n, j \rangle \hookrightarrow T400 \Rightarrow \quad n = j$
$\langle j, n \rangle \hookrightarrow T384 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- It is sometimes convenient to use the following slight variant of the preceding lemma.

Theorem 555 (406) $X, Y, ZZ \in \mathbb{Q} \,\&\, X \geqslant_{\mathbb{Q}} Y \,\&\, Y >_{\mathbb{Q}} ZZ \rightarrow X >_{\mathbb{Q}} ZZ$. PROOF:

Suppose_not$(m, n, j) \Rightarrow \quad m, n, j \in \mathbb{Q} \,\&\, m \geqslant_{\mathbb{Q}} n \,\&\, n >_{\mathbb{Q}} j \,\&\, \neg m >_{\mathbb{Q}} j$
$\langle n, j \rangle \hookrightarrow T384 \Rightarrow \quad n \geqslant_{\mathbb{Q}} j$
$\langle m, n, j \rangle \hookrightarrow T404 \Rightarrow \quad m \geqslant_{\mathbb{Q}} j$
$\langle m, j \rangle \hookrightarrow T384 \Rightarrow \quad m = j$
EQUAL $\Rightarrow \quad j \geqslant_{\mathbb{Q}} n$
$\langle n, j \rangle \hookrightarrow T384 \Rightarrow \quad n \geqslant_{\mathbb{Q}} j$
$\langle n, j \rangle \hookrightarrow T400 \Rightarrow \quad n = j$
$\langle n, j \rangle \hookrightarrow T384 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

Theorem 556 (10041a) $X, Y, ZZ \in \mathbb{Q} \,\&\, X >_{\mathbb{Q}} Y \,\&\, Y >_{\mathbb{Q}} ZZ \rightarrow X >_{\mathbb{Q}} ZZ$. PROOF:

Suppose_not$(x, y, w) \Rightarrow \quad x, y, w \in \mathbb{Q} \,\&\, x >_{\mathbb{Q}} y \,\&\, y >_{\mathbb{Q}} w \,\&\, \neg x >_{\mathbb{Q}} w$
$\langle y, w \rangle \hookrightarrow T384 \Rightarrow \quad y \geqslant_{\mathbb{Q}} w$
$\langle x, y, w \rangle \hookrightarrow T405 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

-- The properties of $>_{\mathbb{Q}}$ are mirrored by properties of $<_{\mathbb{Q}}$ and hence enable us to exploit, in connection with $\mathbb{Q}$, the theory of linear orderings.

Theorem 557 (10041) $\langle \forall x \in \mathbb{Q}, y \in \mathbb{Q}, z \in \mathbb{Q} \mid x <_{\mathbb{Q}} y \,\&\, y <_{\mathbb{Q}} z \longrightarrow x <_{\mathbb{Q}} z \rangle$. PROOF:

Suppose_not$(x, y, w) \Rightarrow \quad x, y, w \in \mathbb{Q} \,\&\, x <_{\mathbb{Q}} y \,\&\, y <_{\mathbb{Q}} w \,\&\, \neg x <_{\mathbb{Q}} w$
$\langle x, y \rangle \hookrightarrow T384 \Rightarrow \quad y >_{\mathbb{Q}} x$
$\langle y, w \rangle \hookrightarrow T384 \Rightarrow \quad w >_{\mathbb{Q}} y$
$\langle w, y \rangle \hookrightarrow T384 \Rightarrow \quad w \geqslant_{\mathbb{Q}} y$
$\langle w, y, x \rangle \hookrightarrow T406 \Rightarrow \quad w >_{\mathbb{Q}} x$
$\langle x, w \rangle \hookrightarrow T384 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

Theorem 558 (10042) $\langle \forall x \in \mathbb{Q} \,|\, \neg x <_{\mathbb{Q}} x \rangle$. PROOF:

Suppose_not$(x) \Rightarrow$ $x \in \mathbb{Q} \;\&\; x <_{\mathbb{Q}} x$
$\langle x, x \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 559 (10043) $\langle \forall x \in \mathbb{Q}, y \in \mathbb{Q} \,|\, x <_{\mathbb{Q}} y \vee x = y \vee y <_{\mathbb{Q}} x \rangle$. PROOF:

Suppose_not$(x, y) \Rightarrow$ $x, y \in \mathbb{Q} \;\&\; \neg(x <_{\mathbb{Q}} y \vee x = y \vee y <_{\mathbb{Q}} x)$
$\langle y, x \rangle \hookrightarrow T400 \Rightarrow$ $y \geqslant_{\mathbb{Q}} x \vee x \geqslant_{\mathbb{Q}} y$
Suppose $\Rightarrow$ $y \geqslant_{\mathbb{Q}} x$
$\langle y, x \rangle \hookrightarrow T384 \Rightarrow$ $y >_{\mathbb{Q}} x$
$\langle x, y \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $x \geqslant_{\mathbb{Q}} y \;\&\; x \neq y$
$\langle x, y \rangle \hookrightarrow T384 \Rightarrow$ $x >_{\mathbb{Q}} y$
$\langle y, x \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle \text{smaller}_\Theta : \text{smaller\_Ra}, \text{ubs}_\Theta : \text{ubs\_Ra}, \text{max}_\Theta : \text{max\_Ra}, \text{lub}_\Theta : \text{lub\_Ra} \rangle$ linear_order$(s \mapsto \mathbb{Q}, X \lhd Y \mapsto X <_{\mathbb{Q}} Y) \Rightarrow$

Theorem 560 (10044)
$\langle \forall x, y \,|\, \text{smaller\_Ra}(x, y) = \textbf{if } x \notin \mathbb{Q} \vee y \notin \mathbb{Q} \textbf{ then } \mathbb{Q} \textbf{ else if } x <_{\mathbb{Q}} y \textbf{ then } x \textbf{ else } y \textbf{ fi fi} \rangle \;\&\; \langle \forall x, y \,|\, \text{smaller\_Ra}(x, y) = \text{smaller\_Ra}(y, x) \rangle \;\&\; \langle \forall x \,|\, \text{smaller\_Ra}(x, \mathbb{Q}) = \mathbb{Q} \;\&\; \text{smaller\_Ra}(\mathbb{Q}, x) =$
$\langle \forall x, y, z \,|\, \text{smaller\_Ra}(x, (\text{smaller\_Ra})(y, z)) = \text{smaller\_Ra}((\text{smaller\_Ra})(x, y), z) \rangle \;\&\; \langle \forall x \in \mathbb{Q} \cup \{\mathbb{Q}\}, y \in \mathbb{Q} \cup \{\mathbb{Q}\} \,|\, \text{smaller\_Ra}(x, y) \in \mathbb{Q} \cup \{\mathbb{Q}\} \rangle \;\&\; \langle \forall t \,|\, \text{ubs\_Ra}(t) = \{x \in \mathbb{Q} \,|\, \langle \forall y$

-- For positive rationals, multiplication is also monotone and strictly monotone. We prove the nonstrict version of this assertion first.

Theorem 561 (407) $X, Y, XP, YP \in \mathbb{Q} \;\&\; X \geqslant_{\mathbb{Q}} Y \;\&\; Y \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; YP \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; XP \geqslant_{\mathbb{Q}} YP \rightarrow X *_{\mathbb{Q}} XP \geqslant_{\mathbb{Q}} Y *_{\mathbb{Q}} YP$. PROOF:

Suppose_not$(m, n, m', n') \Rightarrow$ $m, n, m', n' \in \mathbb{Q} \;\&\; m \geqslant_{\mathbb{Q}} n \;\&\; m' \geqslant_{\mathbb{Q}} n' \;\&\; n \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; n' \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \neg m *_{\mathbb{Q}} m' \geqslant_{\mathbb{Q}} n *_{\mathbb{Q}} n'$
$\langle m, n \rangle \hookrightarrow T384 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n))$
$\langle m', n' \rangle \hookrightarrow T384 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{Q}}(m' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n'))$
$\langle m *_{\mathbb{Q}} m', n *_{\mathbb{Q}} n' \rangle \hookrightarrow T384 \Rightarrow$ $\neg\text{is\_nonneg}_{\mathbb{Q}}(m *_{\mathbb{Q}} m' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n *_{\mathbb{Q}} n'))$
$\langle n', n \rangle \hookrightarrow T368 \Rightarrow$ $n *_{\mathbb{Q}} n' = n' *_{\mathbb{Q}} n$
$\langle n', n \rangle \hookrightarrow T391 \Rightarrow$ $n' *_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n) = \text{Rev}_{\mathbb{Q}}(n' *_{\mathbb{Q}} n)$
$T371 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle n', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $n' \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m, n, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T404 \Rightarrow$ $m \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle m, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad m \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\right)$
$\langle n', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(n' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\right)$
$T388 \Rightarrow \quad \text{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(n' +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$
ALGEBRA $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}(m)\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(n')$
$\langle n \rangle \hookrightarrow T372 \Rightarrow \quad \text{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}$
$\langle n' \rangle \hookrightarrow T372 \Rightarrow \quad \text{Rev}_{\mathbb{Q}}(n') \in \mathbb{Q}\ \&\ n' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n') = \mathbf{0}_{\mathbb{Q}}$
Loc_def $\Rightarrow \quad rn = \text{Rev}_{\mathbb{Q}}(n)$
Loc_def $\Rightarrow \quad rn' = \text{Rev}_{\mathbb{Q}}(n')$
EQUAL $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}(m +_{\mathbb{Q}} rn)\ \&\ \text{is\_nonneg}_{\mathbb{Q}}(m' +_{\mathbb{Q}} rn')\ \&\ rn, rn' \in \mathbb{Q}$
EQUAL $\Rightarrow \quad n' +_{\mathbb{Q}} rn' = \mathbf{0}_{\mathbb{Q}}\ \&\ \neg\text{is\_nonneg}_{\mathbb{Q}}(m *_{\mathbb{Q}} m' +_{\mathbb{Q}} n' *_{\mathbb{Q}} rn)$
ALGEBRA $\Rightarrow \quad m +_{\mathbb{Q}} rn, m' +_{\mathbb{Q}} rn', m *_{\mathbb{Q}} (m' +_{\mathbb{Q}} rn'), n' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn) \in \mathbb{Q}$
$\langle m, m' +_{\mathbb{Q}} rn' \rangle \hookrightarrow T387 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(m *_{\mathbb{Q}} (m' +_{\mathbb{Q}} rn')\right)$
$\langle n', m +_{\mathbb{Q}} rn \rangle \hookrightarrow T387 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(n' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn)\right)$
$\langle m *_{\mathbb{Q}} (m' +_{\mathbb{Q}} rn'), n' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn) \rangle \hookrightarrow T387 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(m *_{\mathbb{Q}} (m' +_{\mathbb{Q}} rn') +_{\mathbb{Q}} n' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn)\right)$
ALGEBRA $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(m *_{\mathbb{Q}} m' +_{\mathbb{Q}} n' *_{\mathbb{Q}} rn +_{\mathbb{Q}} m *_{\mathbb{Q}} (n' +_{\mathbb{Q}} rn')\right)$
EQUAL $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}(m *_{\mathbb{Q}} m' +_{\mathbb{Q}} n' *_{\mathbb{Q}} rn +_{\mathbb{Q}} m *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$
ALGEBRA $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}(m *_{\mathbb{Q}} m' +_{\mathbb{Q}} n' *_{\mathbb{Q}} rn)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following lemma is the 'strict' variant of the preceding.

Theorem 562 (408) $X, Y, XP, YP \in \mathbb{Q}\ \&\ X >_{\mathbb{Q}} Y\ \&\ Y >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ YP >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ XP \geqslant_{\mathbb{Q}} YP \rightarrow X *_{\mathbb{Q}} XP >_{\mathbb{Q}} Y *_{\mathbb{Q}} YP$. PROOF:

Suppose_not$(m, n, m', n') \Rightarrow \quad m, n, m', n' \in \mathbb{Q}\ \&\ m >_{\mathbb{Q}} n\ \&\ m' \geqslant_{\mathbb{Q}} n'\ \&\ n >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ n' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ \neg m *_{\mathbb{Q}} m' >_{\mathbb{Q}} n *_{\mathbb{Q}} n'$
$\langle n \rangle \hookrightarrow T372 \Rightarrow \quad \text{Rev}_{\mathbb{Q}}(n) \in \mathbb{Q}\ \&\ n +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n) = \mathbf{0}_{\mathbb{Q}}$
$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
EQUAL $\Rightarrow \quad m +_{\mathbb{Q}} \left(n +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n)\right) = m +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m \rangle \hookrightarrow T371 \Rightarrow \quad m +_{\mathbb{Q}} \left(n +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n)\right) = m$
$\langle m, n \rangle \hookrightarrow T384 \Rightarrow \quad m \geqslant_{\mathbb{Q}} n\ \&\ m \neq n$
$\langle n, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad n \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ n \neq \mathbf{0}_{\mathbb{Q}}$
$\langle n', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad n' \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ n' \neq \mathbf{0}_{\mathbb{Q}}$
$\langle m, n \rangle \hookrightarrow T384 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(m +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(n)\right)$
$\langle m', n', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T406 \Rightarrow \quad m' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad m' \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ m' \neq \mathbf{0}_{\mathbb{Q}}$
$\langle m', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{Q}}\left(m' +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\right)$

$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m}' +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m}')$

$\mathsf{Loc\_def} \Rightarrow \quad \mathsf{rn} = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})$

$\mathsf{EQUAL} \Rightarrow \quad Stat1:\ \mathsf{m} +_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} \mathsf{rn}) = \mathsf{m}\ \&\ \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn})\ \&\ \mathsf{rn} \in \mathbb{Q}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{n} +_{\mathbb{Q}} \mathsf{rn} = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{rn}, \mathsf{n} +_{\mathbb{Q}} \mathsf{rn} \in \mathbb{Q}$

$\langle \mathsf{m}', \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}))$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) = \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL}\ \langle Stat1 \rangle \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} \mathsf{m} = \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} \mathsf{rn}))$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} \mathsf{rn}) = \mathsf{n} +_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn})$

$\mathsf{EQUAL}\ \langle Stat1 \rangle \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} \mathsf{m} = \mathsf{m}' *_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}))$

$\langle \mathsf{m} +_{\mathbb{Q}} \mathsf{rn}, \mathsf{m}', \mathsf{n} \rangle \hookrightarrow T376 \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn})) = \mathsf{m}' *_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn})$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} \in \mathbb{Q}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} \mathsf{n}, \mathsf{n}' *_{\mathbb{Q}} \mathsf{n}, \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) \in \mathbb{Q}$

$\langle \mathsf{n} \rangle \hookrightarrow T372 \Rightarrow \quad Stat2:\ \mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathbf{0}_{\mathbb{Q}}$

$T382 \Rightarrow \quad Stat3:\ \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$

$\mathsf{EQUAL}\ \langle Stat2 \rangle \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))$

$\langle \mathsf{n}, \mathsf{n} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{n} \geqslant_{\mathbb{Q}} \mathsf{n}$

$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T372 \Rightarrow \quad Stat4:\ \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL}\ \langle Stat4, Stat3 \rangle \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$

$\langle \mathbf{0}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}', \mathsf{n}', \mathsf{n}, \mathsf{n} \rangle \hookrightarrow T407 \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} \mathsf{n} \geqslant_{\mathbb{Q}} \mathsf{n}' *_{\mathbb{Q}} \mathsf{n}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$

$\langle \mathsf{m} +_{\mathbb{Q}} \mathsf{rn}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}', \mathbf{0}_{\mathbb{Q}}, \mathsf{m} +_{\mathbb{Q}} \mathsf{rn}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T407 \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T394 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Suppose} \Rightarrow \quad Stat5:\ \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL}\ \langle Stat5 \rangle \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} +_{\mathbb{Q}} \mathsf{n} = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{n}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} (\mathsf{n} +_{\mathbb{Q}} \mathsf{rn}) = \mathsf{n}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{n}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} \neq \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad Stat6:\ \mathsf{m}' *_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) = \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn})$

$\langle \mathsf{m} +_{\mathbb{Q}} \mathsf{rn} \rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) \in \mathbb{Q}\ \&\ (\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m} +_{\mathbb{Q}} \mathsf{rn}) = \mathbf{1}_{\mathbb{Q}}$

ALGEBRA $\Rightarrow$ $m' *_{\mathbb{Q}} ((m +_{\mathbb{Q}} rn) *_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(m +_{\mathbb{Q}} rn)) = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $m' *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $m' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn) \neq \mathbf{0}_{\mathbb{Q}}$
$\langle m' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $m' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m' *_{\mathbb{Q}} n, n' *_{\mathbb{Q}} n, m' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T402 \Rightarrow$ $m' *_{\mathbb{Q}} n +_{\mathbb{Q}} m' *_{\mathbb{Q}} (m +_{\mathbb{Q}} rn) >_{\mathbb{Q}} n' *_{\mathbb{Q}} n +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
ALGEBRA $\Rightarrow$ $m' *_{\mathbb{Q}} (n +_{\mathbb{Q}} (m +_{\mathbb{Q}} rn)) >_{\mathbb{Q}} n' *_{\mathbb{Q}} n$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Next we show that the mapping of a positive rational to its reciprocal is strictly monotone decreasing.

Theorem 563 (409) $X, Y \in \mathbb{Q} \ \& \ X >_{\mathbb{Q}} Y \ \& \ Y >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathrm{Recip}_{\mathbb{Q}}(Y) >_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(X)$. PROOF:

Suppose_not(m, n, m', n') $\Rightarrow$ $m, n \in \mathbb{Q} \ \& \ m >_{\mathbb{Q}} n \ \& \ n >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \neg \mathrm{Recip}_{\mathbb{Q}}(n) >_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(m)$
$\langle m, n \rangle \hookrightarrow T384 \Rightarrow$ $m \geqslant_{\mathbb{Q}} n$
$T371 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle m, n, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T406 \Rightarrow$ $m >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle n, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $n \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ n \neq \mathbf{0}_{\mathbb{Q}}$
$\langle m, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $m \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ m \neq \mathbf{0}_{\mathbb{Q}}$
$\langle m \rangle \hookrightarrow T380 \Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(m) \in \mathbb{Q}$
$\langle n \rangle \hookrightarrow T380 \Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(n) \in \mathbb{Q}$
$\langle \mathrm{Recip}_{\mathbb{Q}}(n), \mathrm{Recip}_{\mathbb{Q}}(m) \rangle \hookrightarrow T384 \Rightarrow$ $\neg \mathrm{Recip}_{\mathbb{Q}}(n) \geqslant_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(m) \vee \mathrm{Recip}_{\mathbb{Q}}(n) = \mathrm{Recip}_{\mathbb{Q}}(m)$
Suppose $\Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(n) = \mathrm{Recip}_{\mathbb{Q}}(m)$
EQUAL $\Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(\mathrm{Recip}_{\mathbb{Q}}(n)) = \mathrm{Recip}_{\mathbb{Q}}(\mathrm{Recip}_{\mathbb{Q}}(m))$
$\langle m \rangle \hookrightarrow T403 \Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(\mathrm{Recip}_{\mathbb{Q}}(m)) = m$
$\langle n \rangle \hookrightarrow T403 \Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(\mathrm{Recip}_{\mathbb{Q}}(n)) = n$
ELEM $\Rightarrow$ $m = n$
$\langle m, n \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\neg \mathrm{Recip}_{\mathbb{Q}}(n) \geqslant_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(m)$
$\langle \mathrm{Recip}_{\mathbb{Q}}(n), \mathrm{Recip}_{\mathbb{Q}}(m) \rangle \hookrightarrow T400 \Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(m) \geqslant_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(n)$
$\langle n \rangle \hookrightarrow T395 \Rightarrow$ $\mathrm{Recip}_{\mathbb{Q}}(n) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m, n, \mathrm{Recip}_{\mathbb{Q}}(m), \mathrm{Recip}_{\mathbb{Q}}(n) \rangle \hookrightarrow T408 \Rightarrow$ $m *_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(m) >_{\mathbb{Q}} n *_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(n)$
$\langle m \rangle \hookrightarrow T380 \Rightarrow$ $m *_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(m) = \mathbf{1}_{\mathbb{Q}}$
$\langle n \rangle \hookrightarrow T380 \Rightarrow$ $n *_{\mathbb{Q}} \mathrm{Recip}_{\mathbb{Q}}(n) = \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following result gives us a fact about rationals that will be significant when we come to use them for defining real numbers: between any two distinct rational numbers there exists a third (their average) which lies between the larger and the smaller of the two.

Theorem 564 (410) $\mathsf{X}, \mathsf{Y} \in \mathbb{Q} \,\&\, \mathsf{X} >_{\mathbb{Q}} \mathsf{Y} \rightarrow \mathsf{X} >_{\mathbb{Q}} (\mathsf{X} +_{\mathbb{Q}} \mathsf{Y}) /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \,\&\, (\mathsf{X} +_{\mathbb{Q}} \mathsf{Y}) /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{Y}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{m}, \mathsf{n}) \Rightarrow \quad \mathsf{m}, \mathsf{n} \in \mathbb{Q} \,\&\, \mathsf{m} >_{\mathbb{Q}} \mathsf{n} \,\&\, \neg\big(\mathsf{m} >_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \,\&\, (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{n}\big)$

$\big\langle \mathsf{m}, \mathsf{m} \big\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{m} \geqslant_{\mathbb{Q}} \mathsf{m}$

$\big\langle \mathsf{n}, \mathsf{n} \big\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{n} \geqslant_{\mathbb{Q}} \mathsf{n}$

$\big\langle \mathsf{n}, \mathsf{n}, \mathsf{m}, \mathsf{n} \big\rangle \hookrightarrow T402 \Rightarrow \quad \mathsf{n} +_{\mathbb{Q}} \mathsf{m} >_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathsf{n}$

$\big\langle \mathsf{m}, \mathsf{m}, \mathsf{m}, \mathsf{n} \big\rangle \hookrightarrow T402 \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{m} >_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{n}$

$T371 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{n} >_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{m} +_{\mathbb{Q}} \mathsf{n}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} +_{\mathbb{Q}} \mathsf{n}, \mathsf{n} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{n} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{n} \,\&\, \mathsf{m} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{m}$

$T388 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\big\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \big\rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\big\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \big\rangle \hookrightarrow T402 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\big\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \big\rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$

$\big\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \big\rangle \hookrightarrow T380 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$

$\big\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \big\rangle \hookrightarrow T395 \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\big\langle \mathsf{m} +_{\mathbb{Q}} \mathsf{n}, \mathsf{n} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}), \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \big\rangle \hookrightarrow T393 \Rightarrow \quad (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$

$\mathsf{ALGEBRA} \Rightarrow \quad (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{n} *_{\mathbb{Q}} \big((\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})\big)$

$\big\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \big\rangle \hookrightarrow T380 \Rightarrow \quad (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) = \mathbf{1}_{\mathbb{Q}}$

$\mathsf{EQUAL} \Rightarrow \quad (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{n}$

$\mathsf{Use\_def}(/_{\mathbb{Q}}) \Rightarrow \quad (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{n}$

$\big\langle \mathsf{m} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}), \mathsf{m} +_{\mathbb{Q}} \mathsf{n}, \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \big\rangle \hookrightarrow T393 \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \big((\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})\big) >_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{m} >_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$

$\mathsf{Use\_def}(/_{\mathbb{Q}}) \Rightarrow \quad \mathsf{m} >_{\mathbb{Q}} (\mathsf{m} +_{\mathbb{Q}} \mathsf{n}) /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 565 (10013) $\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q} \,\&\, \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q} \,\&\, \big(\mathsf{X} \in \mathbb{Q} \rightarrow \mathsf{X} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}\big)$. PROOF:

Suppose_not(x) $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \notin \mathbb{Q} \vee \neg \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \notin \mathbb{Q} \vee \big(\mathsf{x} \in \mathbb{Q} \,\&\, \mathsf{x} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \notin \mathbb{Q}\big)$
$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T371 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q} \,\&\, \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow T365 \Rightarrow$ $Stat1: \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
$T388 \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T402 \Rightarrow$ $Stat2: \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T371 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
EQUAL $\langle Stat2 \rangle \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $Stat3: \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow T380([Stat1, Stat3]) \Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$
ELEM $\Rightarrow$ $\mathsf{x} \in \mathbb{Q} \,\&\, \mathsf{x} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \notin \mathbb{Q}$
Use_def($/_{\mathbb{Q}}$) $\Rightarrow$ $\mathsf{x} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \notin \mathbb{Q}$
$\langle \mathsf{x}, \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \rangle \hookrightarrow T368 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The next theorem asserts that every rational number x can be obtained as the sum of its half plus its half. Its proof derives this general statement from the specific case when $\mathsf{x} = 1$.

Theorem 566 (10014) $\mathsf{X} \in \mathbb{Q} \rightarrow \mathsf{X} = \mathsf{X} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{X} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$. PROOF:

Suppose_not $\Rightarrow$ $\mathsf{x} \in \mathbb{Q} \,\&\, \mathsf{x} \neq \mathsf{x} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{x} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$T10013 \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q} \,\&\, \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$
Suppose $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} \neq \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \neq \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow T380 \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} = (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$T371 \Rightarrow$ $\mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \rangle \hookrightarrow T368 \Rightarrow$ $(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) = \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$\langle \mathbf{1}_{\mathbb{Q}}, \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}), \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow T376 \Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) = \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \rangle \hookrightarrow T379 \Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} = \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
Use_def($/_{\mathbb{Q}}$) $\Rightarrow$ $\mathsf{x} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{x} /_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) = \mathsf{x} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{x} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}), \mathsf{x}, \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \rangle \hookrightarrow T376 \Rightarrow$ $\mathsf{x} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{x} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) = \mathsf{x} *_{\mathbb{Q}} \big(\mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})\big)$
$\langle \mathsf{x} \rangle \hookrightarrow T379 \Rightarrow$ $\mathsf{x} *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{x}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- The following theorem shows that every positive rational number x exceeds the sum of two smaller, positive and distinct rational numbers.

Theorem 567 (10015) $X \in \mathbb{Q} \,\&\, X >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \langle \exists e \in \mathbb{Q}, e' \in \mathbb{Q} \,|\, X >_{\mathbb{Q}} e \,\&\, e >_{\mathbb{Q}} e' \,\&\, e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, e >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, X >_{\mathbb{Q}} e +_{\mathbb{Q}} e' \rangle$. PROOF:

Suppose_not$(x) \Rightarrow$ $(x \in \mathbb{Q} \,\&\, x >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \,\&\, \mathit{Stat0}: \neg\langle \exists e \in \mathbb{Q}, e' \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} e \,\&\, e >_{\mathbb{Q}} e' \,\&\, e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, e >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, x >_{\mathbb{Q}} e +_{\mathbb{Q}} e' \rangle$

-- For, if there could be a counterexample x, then we could take e to be one half of x and e' to be a half of e; and with these values, exploiting previously proved lemmas, we would easily come to a contradiction.

Loc_def $\Rightarrow$ $e = x \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$\langle x \rangle \hookrightarrow \mathit{T371} \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \in \mathbb{Q} \,\&\, x = x +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle x, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T410} \Rightarrow$ $x >_{\mathbb{Q}} (x +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \,\&\, (x +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle x \rangle \hookrightarrow \mathit{T10013} \Rightarrow$ $x \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$
EQUAL $\Rightarrow$ $x >_{\mathbb{Q}} e \,\&\, e >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, e \in \mathbb{Q}$
$\langle e, e \rangle \hookrightarrow \mathit{T384} \Rightarrow$ $e \geqslant_{\mathbb{Q}} e$
$\langle e, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T384} \Rightarrow$ $e \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
Loc_def $\Rightarrow$ $e' = e \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
$\langle e \rangle \hookrightarrow \mathit{T371} \Rightarrow$ $e = e +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle e, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T410} \Rightarrow$ $e >_{\mathbb{Q}} (e +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \,\&\, (e +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle e \rangle \hookrightarrow \mathit{T10013} \Rightarrow$ $e \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) \in \mathbb{Q}$
EQUAL $\Rightarrow$ $e >_{\mathbb{Q}} e' \,\&\, e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, e' \in \mathbb{Q}$
$\langle x \rangle \hookrightarrow \mathit{T10014} \Rightarrow$ $x = x \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}) +_{\mathbb{Q}} x \,/_{\mathbb{Q}} (\mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}})$
EQUAL $\Rightarrow$ $x = e +_{\mathbb{Q}} e$
$\langle e, e, e, e' \rangle \hookrightarrow \mathit{T402} \Rightarrow$ $e +_{\mathbb{Q}} e >_{\mathbb{Q}} e +_{\mathbb{Q}} e'$
EQUAL $\Rightarrow$ $x >_{\mathbb{Q}} e +_{\mathbb{Q}} e'$
$\langle e, \mathbf{0}_{\mathbb{Q}}, e', \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T402} \Rightarrow$ $e +_{\mathbb{Q}} e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T371} \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $e +_{\mathbb{Q}} e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle e, e' \rangle \hookrightarrow \mathit{Stat0} \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is also useful to know that the square of any rational number is non-negative.

Theorem 568 (411) $X \in \mathbb{Q} \rightarrow X *_{\mathbb{Q}} X \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$. PROOF:

Suppose_not$(n) \Rightarrow$ $n \in \mathbb{Q} \,\&\, \neg n *_{\mathbb{Q}} n \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle n *_{\mathbb{Q}} n, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T384} \Rightarrow$ $\neg \mathsf{is\_nonneg}_{\mathbb{Q}}(n *_{\mathbb{Q}} n +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$

$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{n} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$
ALGEBRA $\Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{n})$
$\langle \mathsf{n} \rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \in \mathbb{Q}$
$\langle \mathsf{n} \rangle \hookrightarrow T399 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n}) \vee \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))$
Suppose $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{n})$
$\langle \mathsf{n}, \mathsf{n} \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}))$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{n} \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) *_{\mathbb{Q}} \mathsf{n})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}), \mathsf{n} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) *_{\mathbb{Q}} \mathsf{n} = \mathsf{n} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})$
$\langle \mathsf{n}, \mathsf{n} \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{n}) = \mathsf{n} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n})$
EQUAL $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{n}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{n}))$
ALGEBRA $\Rightarrow \quad \mathsf{n} *_{\mathbb{Q}} \mathsf{n} \in \mathbb{Q}$
$\langle \mathsf{n} *_{\mathbb{Q}} \mathsf{n} \rangle \hookrightarrow T398 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{n} *_{\mathbb{Q}} \mathsf{n})) = \mathsf{n} *_{\mathbb{Q}} \mathsf{n}$
EQUAL $\Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow$ QED

15 Additional theorems under development

DEF 10001. $\mathsf{divides}(\mathsf{X}, \mathsf{Y}) \quad \leftrightarrow_{\mathrm{Def}} \quad \langle \exists \mathsf{j} \mid \mathsf{Y} = \mathsf{X} * \mathsf{j} \rangle$

-- Primality criterion for unsigned integers
DEF 10002. $\mathsf{Is\_prime}(\mathsf{X}) \quad \leftrightarrow_{\mathrm{Def}} \quad 1 \in \mathsf{X} \,\&\, \neg\langle \exists \mathsf{k} \in \mathsf{X} \mid 1 \in \mathsf{k} \,\&\, \mathsf{divides}(\mathsf{k}, \mathsf{X}) \rangle$

-- Smallest factor of an unsigned integer
DEF 10003. $\mathsf{smallest\_factor}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{\mathsf{k} \in \mathsf{next}(\mathsf{X}) \mid 1 \in \mathsf{k} \,\&\, \mathsf{divides}(\mathsf{k}, \mathsf{X})\})$

-- Nondecreasing sequence of unsigned prime integers factorizing an unsigned integer
DEF 10004. $\mathsf{standard\_factorization}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathsf{concat}(\{[\emptyset, \mathsf{smallest\_factor}(\mathsf{X})]\}, \mathbf{arb}(\{\mathsf{standard\_factorization}(\mathsf{m}) : \mathsf{m} \in \mathsf{X} \mid \mathsf{m} * \mathsf{smallest\_factor}(\mathsf{X}) = \mathsf{X}\}))$

Theorem 569 (10005) $\mathcal{O}(\mathsf{K}) \rightarrow \mathsf{divides}(\mathsf{K}, \emptyset)$. PROOF:

Suppose_not(k) $\Rightarrow \quad \mathcal{O}(\mathsf{k}) \,\&\, \neg\mathsf{divides}(\mathsf{k}, \emptyset)$
Use_def(divides) $\Rightarrow \quad Stat1: \neg\langle \exists \mathsf{j} \mid \emptyset = \mathsf{k} * \mathsf{j} \rangle$
$\langle \mathsf{k} \rangle \hookrightarrow T209 \Rightarrow \quad \emptyset = \mathsf{k} * \emptyset$

$\langle\emptyset\rangle\hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 570 (10006) $\mathsf{Card}(M) \vee M \in \mathbb{N} \rightarrow \mathsf{divides}(1, M) \;\&\; \mathsf{divides}(M, M)$. PROOF:

Suppose_not(m) $\Rightarrow$ $\mathsf{Card}(m) \vee m \in \mathbb{N} \;\&\; \neg\mathsf{divides}(1, m) \vee \neg\mathsf{divides}(m, m)$
$\langle m\rangle\hookrightarrow T179 \Rightarrow$ $\mathsf{Card}(m)$
$\langle m\rangle\hookrightarrow T138 \Rightarrow$ $m = \#m$
Suppose $\Rightarrow$ $\neg\mathsf{divides}(1, m)$
Use_def(divides) $\Rightarrow$ $Stat1: \neg\langle\exists j \mid m = 1 * j\rangle$
$\langle m\rangle\hookrightarrow T212 \Rightarrow$ $m = 1 * m$
$\langle m\rangle\hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{divides}(m, m)$
Use_def(divides) $\Rightarrow$ $Stat2: \neg\langle\exists j \mid m = m * j\rangle$
$\langle m\rangle\hookrightarrow T213 \Rightarrow$ $m = m * 1$
$\langle 1\rangle\hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 571 (10007) $\mathsf{divides}(K, I) \;\&\; \mathsf{divides}(I, N) \rightarrow \mathsf{divides}(K, N)$. PROOF:

Suppose_not(k, i, n) $\Rightarrow$ $\mathsf{divides}(k, i) \;\&\; \mathsf{divides}(i, n) \;\&\; \neg\mathsf{divides}(k, n)$
Use_def(divides) $\Rightarrow$ $Stat1: \langle\exists j \mid i = k * j\rangle \;\&\; Stat2: \langle\exists j \mid n = i * j\rangle \;\&\; Stat3: \neg\langle\exists j \mid n = k * j\rangle$
$\langle j\rangle\hookrightarrow Stat1 \Rightarrow$ $i = k * j$
$\langle j'\rangle\hookrightarrow Stat2 \Rightarrow$ $n = i * j'$
EQUAL $\Rightarrow$ $n = k * j * j'$
$\langle k, j, j'\rangle\hookrightarrow T222 \Rightarrow$ $n = k * (j * j')$
$\langle j * j'\rangle\hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Theorem 10008: ((M = 0) & (M in Z) & (D in Z) & divides (D, M)) imp (D in next (M)) Proof: Suppose_not (m, d) $\Rightarrow$ (m = 0) & (m in Z) & (d in Z) & divides (d, m) & (d notin next (m)) TOBECOMPLETED $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 572 (10009) $M \in \mathbb{N} \;\&\; 1 \in M \rightarrow \mathsf{Is\_prime}(\mathsf{smallest\_factor}(M)) \;\&\; \mathsf{divides}(\mathsf{smallest\_factor}(M), M)$. PROOF:

Suppose_not(m) $\Rightarrow$ $m \in \mathbb{N} \;\&\; 1 \in m \;\&\; \neg\mathsf{Is\_prime}(\mathsf{smallest\_factor}(m)) \vee \neg\mathsf{divides}(\mathsf{smallest\_factor}(m), m)$
Use_def(next) $\Rightarrow$ $m \in \mathsf{next}(m)$
$\langle m\rangle\hookrightarrow T10006 \Rightarrow$ $\mathsf{divides}(m, m)$
Suppose $\Rightarrow$ $Stat1: \{k : k \in \mathsf{next}(m) \mid 1 \in k \;\&\; \mathsf{divides}(k, m)\} = \emptyset$
$\langle m\rangle\hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$
$\mathbf{arb}(\{k : k \in \mathsf{next}(m) \mid 1 \in k \;\&\; \mathsf{divides}(k, m)\}) \in \{k : k \in \mathsf{next}(m) \mid 1 \in k \;\&\; \mathsf{divides}(k, m)\} \;\&\; \mathbf{arb}(\{k : k \in \mathsf{next}(m) \mid 1 \in k \;\&\; \mathsf{divides}(k, m)\}) \cap \{k : k \in \mathsf{next}(m) \mid 1 \in k \;\&\; \mathsf{di}$

Use_def (smallest_factor) $\Rightarrow$ $\mathit{Stat2}$: $\mathsf{smallest\_factor}(m) \in \{k : k \in \mathsf{next}(m) \mid 1 \in k \ \& \ \mathsf{divides}(k, m)\} \ \& \ \mathsf{smallest\_factor}(m) \cap \{k : k \in \mathsf{next}(m) \mid 1 \in k \ \& \ \mathsf{divides}(k, m)\} = \emptyset$

$\langle k_0 \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $k_0 = \mathsf{smallest\_factor}(m) \ \& \ k_0 \in \mathsf{next}(m) \ \& \ 1 \in k_0 \ \& \ \mathsf{divides}(k_0, m)$

EQUAL $\Rightarrow$ $\mathsf{smallest\_factor}(m) \in \mathsf{next}(m) \ \& \ 1 \in \mathsf{smallest\_factor}(m) \ \& \ \mathsf{divides}\big(\mathsf{smallest\_factor}(m), m\big)$

ELEM $\Rightarrow$ $\neg \mathsf{Is\_prime}\big(\mathsf{smallest\_factor}(m)\big)$

Use_def (Is_prime) $\Rightarrow$ $\mathit{Stat3}$: $\big\langle \exists k \in \mathsf{smallest\_factor}(m) \mid 1 \in k \ \& \ \mathsf{divides}\big(k, \mathsf{smallest\_factor}(m)\big)\big\rangle$

$\langle k \rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $k \in \mathsf{smallest\_factor}(m) \ \& \ 1 \in k \ \& \ \mathsf{divides}\big(k, \mathsf{smallest\_factor}(m)\big)$

$\langle k, \mathsf{smallest\_factor}(m), m \rangle \hookrightarrow \mathit{T10007} \Rightarrow$ $\mathsf{divides}(k, m)$

Suppose $\Rightarrow$ $\mathit{Stat4}$: $k \notin \{k : k \in \mathsf{next}(m) \mid 1 \in k \ \& \ \mathsf{divides}(k, m)\}$

$\langle k \rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $k \notin \mathsf{next}(m)$

$\mathit{T179} \Rightarrow$ $\mathcal{O}(\mathbb{N})$

$\langle \mathbb{N}, m \rangle \hookrightarrow \mathit{T11} \Rightarrow$ $\mathcal{O}(m)$

$\langle m \rangle \hookrightarrow \mathit{T29} \Rightarrow$ $\mathcal{O}\big(\mathsf{next}(m)\big)$

$\langle \mathsf{next}(m), \mathsf{smallest\_factor}(m) \rangle \hookrightarrow \mathit{T12} \Rightarrow$ false; Discharge $\Rightarrow$ $k \in \{k : k \in \mathsf{next}(m) \mid 1 \in k \ \& \ \mathsf{divides}(k, m)\}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Theorem 10010: ((M in Z) & (not Is_prime (M))) imp (standard_factorization (M) = concat ({[0, smallest_factor (M)]}, standard_factorization (M OVER smallest_factor (M)))) Proof: TOBECOMPLETED $\Rightarrow$ QED

Theorem 573 (10011) $[2, \emptyset] \in \mathbb{Z} \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}([2, \emptyset]) \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}([1, \emptyset]) \ \& \ \mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [2, \emptyset]]) \in \mathbb{Q} \ \&$ $\mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [2, \emptyset]]) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$. PROOF:

Suppose_not $\Rightarrow$ $\mathit{Stat0}$:

$[2, \emptyset] \notin \mathbb{Z} \ \vee$

$\neg \mathsf{is\_nonneg}_{\mathbb{N}}([2, \emptyset]) \vee \neg \mathsf{is\_nonneg}_{\mathbb{N}}([1, \emptyset]) \vee \mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [2, \emptyset]]) \notin \mathbb{Q} \vee \neg \mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [2, \emptyset]]) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee \neg(X >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}) *_{\mathbb{Q}} \mathsf{Fr\_to\_}\mathbb{Q}([[1, \emptyset], [2, \emptyset]]) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\mathit{T378} \Rightarrow$ $[1, \emptyset] \in \mathbb{Z}$

Use_def ($\mathsf{is\_nonneg}_{\mathbb{N}}$) $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{N}}([1, \emptyset])$

$\langle [1, \emptyset], [1, \emptyset] \rangle \hookrightarrow \mathit{T348} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{N}}([1, \emptyset] +_{\mathbb{Z}} [1, \emptyset])$

Use_def ($+_{\mathbb{Z}}$) $\Rightarrow$ $[1, \emptyset] +_{\mathbb{Z}} [1, \emptyset] = \mathsf{Red}\Big(\Big[[1, \emptyset]^{[1]} + [1, \emptyset]^{[1]}, [1, \emptyset]^{[2]} + [1, \emptyset]^{[2]}\Big]\Big)$

$\mathit{T182} \Rightarrow$ $\emptyset, 1, 2 \in \mathbb{N} \ \& \ \mathsf{Card}(\emptyset) \ \& \ \mathsf{Card}(1)$

$\langle \emptyset \rangle \hookrightarrow \mathit{T138} \Rightarrow$ $\emptyset = \#\emptyset$

$\langle 1 \rangle \hookrightarrow \mathit{T138} \Rightarrow$ $1 = \#1$

$\langle \emptyset \rangle \hookrightarrow \mathit{T211} \Rightarrow$ $\#\emptyset + \emptyset = \emptyset$

$\langle 1 \rangle \hookrightarrow \mathit{T265} \Rightarrow$ $1 + 1 = \mathsf{next}(1)$

Use_def (2) $\Rightarrow$ $1 + 1 = 2$

$\mathit{T183} \Rightarrow$ $\mathit{Stat1}$: $1 \neq \emptyset \ \& \ 2 \neq \emptyset$

$\langle \mathit{Stat1} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat1a}$: $[1, \emptyset]^{[1]} = 1 \ \& \ [1, \emptyset]^{[2]} = \emptyset \ \& \ [2, \emptyset] \neq [\emptyset, \emptyset] \ \& \ [1, \emptyset] \neq [\emptyset, \emptyset]$

$\langle 2 \rangle \hookrightarrow \mathit{T310} \Rightarrow$ $\mathsf{Red}([2, \emptyset]) = [2, \emptyset]$

$\mathsf{EQUAL} \Rightarrow \quad [1,\emptyset] +_{\mathbb{Z}} [1,\emptyset] = [2,\emptyset]$

$\langle \emptyset \rangle \hookrightarrow T290 \Rightarrow \quad Stat2: \ [\emptyset,\emptyset] \in \mathbb{Z}$

$\langle 1 \rangle \hookrightarrow T290 \Rightarrow \quad Stat3: \ [1,\emptyset] \in \mathbb{Z}$

$\langle 2 \rangle \hookrightarrow T290 \Rightarrow \quad Stat4: \ [2,\emptyset] \in \mathbb{Z}$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}([2,\emptyset]) \ \& \ \neg\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$TELEM \Rightarrow \quad [[1,\emptyset],[2,\emptyset]] \in \mathsf{Fr}$

$\langle\, [[1,\emptyset],[2,\emptyset]] \,\rangle \hookrightarrow T344 \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) \in \mathbb{Q}$

$\langle \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) -_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\big) \vee$
$\quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) = \mathbf{0}_{\mathbb{Q}}$

$\langle\, [[1,\emptyset],[2,\emptyset]] \,\rangle \hookrightarrow T352(\langle Stat1a, Stat3, Stat4, * \rangle) \Rightarrow$
$\quad [[1,\emptyset],[2,\emptyset]] \in \mathsf{Fr}$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) \notin \mathbb{Q}$

$\langle\, [[1,\emptyset],[2,\emptyset]] \,\rangle \hookrightarrow T344([Stat0, \cap]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) \in \mathbb{Q}$

$\mathsf{Suppose} \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) -_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\big)$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\big)$

$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) \rangle \hookrightarrow T371(\langle \cap \rangle) \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} =$
$\quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]])$

$\mathsf{EQUAL} \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]])\big)$

$\langle\, [1,\emptyset], [2,\emptyset] \,\rangle \hookrightarrow T348([Stat0, \cap]) \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{N}}([1,\emptyset] *_{\mathbb{Z}} [2,\emptyset])$

$\langle\, [1,\emptyset], [2,\emptyset] \,\rangle \hookrightarrow T377([Stat0, \cap]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Use\_def}(\mathbf{0}_{\mathbb{Q}}) \Rightarrow \quad \mathsf{Fr\_to\_}\mathbb{Q}([[1,\emptyset],[2,\emptyset]]) = \mathsf{Fr\_to\_}\mathbb{Q}([[\emptyset,\emptyset],[1,\emptyset]])$

$T343 \Rightarrow \quad Stat7: \ \big\langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x}, \mathsf{y} \in \mathsf{Fr} \rightarrow \big(\mathsf{x} \approx_{\mathsf{Fr}} \mathsf{y} \leftrightarrow \mathsf{Fr\_to\_}\mathbb{Q}(\mathsf{x}) = \mathsf{Fr\_to\_}\mathbb{Q}(\mathsf{y})\big)\big\rangle$

$\langle\, [[\emptyset,\emptyset],[1,\emptyset]] \,\rangle \hookrightarrow T352(\langle Stat1a, Stat2, Stat3, * \rangle) \Rightarrow$
$\quad [[\emptyset,\emptyset],[1,\emptyset]] \in \mathsf{Fr}$

$\langle\, [[1,\emptyset],[2,\emptyset]], [[\emptyset,\emptyset],[1,\emptyset]] \,\rangle \hookrightarrow Stat7([Stat4, \cap]) \Rightarrow$
$\quad [[1,\emptyset],[2,\emptyset]] \approx_{\mathsf{Fr}} [[\emptyset,\emptyset],[1,\emptyset]]$

$\mathsf{Use\_def}(\approx_{\mathsf{Fr}}) \Rightarrow \quad Stat8:$
$\quad [[1,\emptyset],[2,\emptyset]]^{[1]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} =$
$\qquad [[1,\emptyset],[2,\emptyset]]^{[2]} *_{\mathbb{Z}} [[\emptyset,\emptyset],[1,\emptyset]]^{[1]}$

$TELEM \Rightarrow$
$\quad [[1,\emptyset],[2,\emptyset]]^{[1]} = [1,\emptyset] \ \& \ [[\emptyset,\emptyset],[1,\emptyset]]^{[2]} = [1,\emptyset] \ \&$
$\qquad [[1,\emptyset],[2,\emptyset]]^{[2]} = [2,\emptyset] \ \& \ [[\emptyset,\emptyset],[1,\emptyset]]^{[1]} = [\emptyset,\emptyset]$

$\mathsf{EQUAL} \ \langle Stat8 \rangle \Rightarrow \quad Stat10: \ [1,\emptyset] *_{\mathbb{Z}} [1,\emptyset] = [2,\emptyset] *_{\mathbb{Z}} [\emptyset,\emptyset]$

$\langle\, [1,\emptyset] \,\rangle \hookrightarrow T324([Stat3, Stat3]) \Rightarrow \quad Stat11: \ [1,\emptyset] *_{\mathbb{Z}} [1,\emptyset] = [1,\emptyset]$

$\langle\, [2,\emptyset] \,\rangle \hookrightarrow T324([Stat4, Stat4]) \Rightarrow \quad Stat12: \ [\emptyset,\emptyset] *_{\mathbb{Z}} [2,\emptyset] = [\emptyset,\emptyset]$

$\langle\, [\emptyset,\emptyset], [2,\emptyset] \,\rangle \hookrightarrow T307([Stat2, Stat4]) \Rightarrow \quad Stat13: \ [\emptyset,\emptyset] *_{\mathbb{Z}} [2,\emptyset] =$

$[2, \emptyset] *_{z} [\emptyset, \emptyset]$

$\langle$ *Stat10, Stat11, Stat12, Stat13, Stat1a,* $*\rangle$ ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Theorem 10012: ((X in Ra) & (X Ra_GT Ra_0)) imp (((X Ra_GT Ra_0) Ra_TIMES Fr_to_Ra ([[1, 0], [2, 0]])) Ra_GT Ra_0) Proof: Suppose_not (x) $\Rightarrow$ (x in Ra) & (x Ra_GT Ra_0) & (not (((x Ra_GT Ra_0) Ra_TIMES Fr_to_Ra ([[1, 0], [2, 0]])) Ra_GT Ra_0)) TO_BE_CONTINUED $\Rightarrow$ QED

THEORY setformer_meet_join$(s, t, h(u, v), P(u, v), Q(u, v))$
END setformer_meet_join

ENTER_THEORY setformer_meet_join

Theorem 574 (setformer_meet_join · 1) $\{h(u,v) : u \in s, v \in t \mid P(u,v) \vee Q(u,v)\} = \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cup \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$. PROOF:

Suppose_not $\Rightarrow$ *Stat0* : $\{h(u,v) : u \in s, v \in t \mid P(u,v) \vee Q(u,v)\} \neq \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cup \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$

$\langle c\rangle \hookrightarrow$ *Stat0* $\Rightarrow$
$c \in \{h(u,v) : u \in s, v \in t \mid P(u,v) \vee Q(u,v)\} \leftrightarrow c \notin \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cup \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$

Suppose $\Rightarrow$ $c \notin \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cup \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$ & *Stat1* : $c \in \{h(u,v) : u \in s, v \in t \mid P(u,v) \vee Q(u,v)\}$

ELEM $\Rightarrow$ *Stat2* : $c \notin \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$ & $c \notin \{h(u,v) : u \in s, v \in t \mid P(u,v)\}$

$\langle u, v\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $c = h(u,v)$ & $u \in s$ & $v \in t$ & $P(u,v) \vee Q(u,v)$

$\langle u, v, u, v\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $c \in \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cup \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$ & *Stat3* : $c \notin \{h(u,v) : u \in s, v \in t \mid P(u,v) \vee Q(u,v)\}$

ELEM $\Rightarrow$ $c \in \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \vee c \in \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$

Suppose $\Rightarrow$ *Stat4* : $c \in \{h(u,v) : u \in s, v \in t \mid P(u,v)\}$

$\langle u', v'\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $c = h(u', v')$ & $u' \in s$ & $v' \in t$ & $P(u', v')$

$\langle u', v'\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat5* : $c \in \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$

$\langle uq, vq\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $c = h(uq, vq)$ & $uq \in s$ & $vq \in t$ & $Q(uq, vq)$

$\langle uq, vq\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 575 (setformer_meet_join · 2) $\{h(u,v) : u \in s, v \in t \mid P(u,v) \,\&\, Q(u,v)\} \subseteq \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cap \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$. PROOF:

Suppose_not $\Rightarrow$ *Stat0* : $\neg$
$\{h(u,v) : u \in s, v \in t \mid P(u,v) \,\&\, Q(u,v)\} \subseteq \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cap \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$

$\langle c\rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $c \notin \{h(u,v) : u \in s, v \in t \mid P(u,v)\} \cap \{h(u,v) : u \in s, v \in t \mid Q(u,v)\}$ & *Stat1* : $c \in \{h(u,v) : u \in s, v \in t \mid P(u,v) \,\&\, Q(u,v)\}$

ELEM $\Rightarrow$ *Stat2* : $c \notin \{h(u,v) : u \in s, v \in t \mid Q(u,v)\} \vee c \notin \{h(u,v) : u \in s, v \in t \mid P(u,v)\}$

$\langle u,v\rangle \hookrightarrow Stat1 \Rightarrow$ $c = h(u,v)$ & $u \in s$ & $v \in t$ & $P(u,v)$ & $Q(u,v)$
Suppose $\Rightarrow$ $Stat3:\ c \notin \{h(u,v):\ u \in s, v \in t \mid Q(u,v)\}$
$\langle u,v\rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat4:\ c \notin \{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$
$\langle u,v\rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 576 (setformer_meet_join · 3) $\langle \forall u \in s, v \in t \mid Q(u,v) \rightarrow P(u,v)\rangle \rightarrow \{h(u,v):\ u \in s, v \in t \mid Q(u,v)\} \subseteq \{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$. PROOF:

Suppose_not $\Rightarrow$ $Stat0:$
$\langle \forall u \in s, v \in t \mid Q(u,v) \rightarrow P(u,v)\rangle$ & $Stat1:\ \{h(u,v):\ u \in s, v \in t \mid Q(u,v)\} \not\subseteq \{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$
$\langle u,v\rangle \hookrightarrow Stat1 \Rightarrow$ $u \in s$ & $v \in t$ & $Q(u,v)$ & $\neg P(u,v)$
$\langle u,v\rangle \hookrightarrow Stat0 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 577 (setformer_meet_join · 4) $\#\{[u,v]:\ u \in s, v \in t \mid P(u,v)\} \supseteq \#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$. PROOF:

Suppose_not $\Rightarrow$ $\#\{[u,v]:\ u \in s, v \in t \mid P(u,v)\} \not\supseteq \#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$

-- APPLY () Must_be_svm_2 (b (u, v) $\mapsto$ h (u, v), s $\mapsto$ s, t $\mapsto$ t, P (u, v) $\mapsto$ P (u, v))
$\Rightarrow$ Svm ({[[u, v], h (u, v)]: u in s, v in t |P (u, v)})

TELEM $\Rightarrow$ $\mathsf{Svm}(\{[[u,v],h(u,v)]:\ u \in s, v \in t \mid P(u,v)\})$
TELEM $\Rightarrow$
$\mathbf{domain}(\{[[u,v],h(u,v)]:\ u \in s, v \in t \mid P(u,v)\}) = \{[u,v]:\ u \in s, v \in t \mid P(u,v)\}$ &
$\mathbf{range}(\{[[u,v],h(u,v)]:\ u \in s, v \in t \mid P(u,v)\}) = \{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$
$\langle \{[[u,v],h(u,v)]:\ u \in s, v \in t \mid P(u,v)\}\rangle \hookrightarrow T145 \Rightarrow$
$\#\mathbf{domain}(\{[[u,v],h(u,v)]:\ u \in s, v \in t \mid P(u,v)\}) \supseteq \#\mathbf{range}(\{[[u,v],h(u,v)]:\ u \in s, v \in t \mid P(u,v)\})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 578 (setformer_meet_join · 5) $\mathsf{Finite}(\{[u,v]:\ u \in s, v \in t \mid P(u,v)\}) \rightarrow \mathsf{Finite}(\{h(u,v):\ u \in s, v \in t \mid P(u,v)\})$. PROOF:

Suppose_not $\Rightarrow$ $\mathsf{Finite}(\{[u,v]:\ u \in s, v \in t \mid P(u,v)\})$ & $\neg\mathsf{Finite}(\{h(u,v):\ u \in s, v \in t \mid P(u,v)\})$
Tsetformer_meet_join · 4 $\Rightarrow$ $\#\{[u,v]:\ u \in s, v \in t \mid P(u,v)\} \supseteq \#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$
$\langle \{[u,v]:\ u \in s, v \in t \mid P(u,v)\}\rangle \hookrightarrow T166 \Rightarrow$
$\mathsf{Finite}(\#\{[u,v]:\ u \in s, v \in t \mid P(u,v)\})$
$\langle \#\{[u,v]:\ u \in s, v \in t \mid P(u,v)\}, \#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\}\rangle \hookrightarrow T162 \Rightarrow$
$\mathsf{Finite}(\#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\})$
$\langle \#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\}\rangle \hookrightarrow T166 \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY setformer_meet_join

THEORY setformer_meet_join$(s, t, h(u,v), P(u,v), Q(u,v))$
$\Rightarrow$

$\{h(u,v):\ u \in s, v \in t \mid P(u,v) \vee Q(u,v)\} = \{h(u,v):\ u \in s, v \in t \mid P(u,v)\} \cup \{h(u,v):\ u \in s, v \in t \mid Q(u,v)\}$
$\{h(u,v):\ u \in s, v \in t \mid P(u,v) \,\&\, Q(u,v)\} \subseteq \{h(u,v):\ u \in s, v \in t \mid P(u,v)\} \cap \{h(u,v):\ u \in s, v \in t \mid Q(u,v)\}$
$\langle \forall u \in s, v \in t \mid Q(u,v) \rightarrow P(u,v)\rangle \rightarrow \{h(u,v):\ u \in s, v \in t \mid Q(u,v)\} \subseteq \{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$
$\#\{[u,v]:\ u \in s, v \in t \mid P(u,v)\} \supseteq \#\{h(u,v):\ u \in s, v \in t \mid P(u,v)\}$
$\mathrm{Finite}(\{[u,v]:\ u \in s, v \in t \mid P(u,v)\}) \rightarrow \mathrm{Finite}(\{h(u,v):\ u \in s, v \in t \mid P(u,v)\})$
END setformer_meet_join

16 Real numbers

APPLY $\langle \mathrm{abs}_\Theta :\ \mathrm{Ra\_ABS}\rangle$ orderedGroups$(\mathrm{ln_domain}(x) \mapsto x \in \mathbb{Q}, x \oplus y \mapsto x +_{\mathbb{Q}} y, e \mapsto \mathbf{0}_{\mathbb{Q}}, \mathrm{rvz}(x) \mapsto \mathrm{Rev}_{\mathbb{Q}}(x), \mathrm{nneg}(x) \mapsto \mathrm{is_nonneg}_{\mathbb{Q}}(x), \mathrm{leq}(x,y) \mapsto x \leqslant_{\mathbb{Q}} y) \Rightarrow$

Theorem 579 (10050)
$\langle \forall x \mid \mathrm{Ra\_ABS}(x) = \mathbf{if}\ \mathrm{is\_nonneg}_{\mathbb{Q}}(x)\ \mathbf{then}\ x\ \mathbf{else}\ \mathrm{Rev}_{\mathbb{Q}}(x)\ \mathbf{fi}\rangle \,\&\, \langle \forall x, y, z \mid x, y, z \in \mathbb{Q} \rightarrow x +_{\mathbb{Q}} y = x +_{\mathbb{Q}} z \rightarrow y = z\rangle \,\&\, \langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow \mathrm{Rev}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y)) = y +_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(x)\rangle \,\&\, \langle \forall x, y \mid$

$\langle \forall x, y, z \mid x, y, z \in \mathbb{Q} \rightarrow x \leqslant_{\mathbb{Q}} y \,\&\, x \neq y \,\&\, y \leqslant_{\mathbb{Q}} z \rightarrow x \neq z\rangle \,\&\, \langle \forall x, y, z \mid x, y, z \in \mathbb{Q} \rightarrow x \leqslant_{\mathbb{Q}} y \,\&\, y \leqslant_{\mathbb{Q}} z \,\&\, y \neq z \rightarrow x \neq z\rangle \,\&\, \langle \forall x, y, z \mid x, y, z \in \mathbb{Q} \rightarrow x \leqslant_{\mathbb{Q}} y \rightarrow x +_{\mathbb{Q}} z \leqslant_{\mathbb{Q}} y +_{\mathbb{Q}} z\rangle \,\&$

-- Def 45a: [Absolute value of a rational number] Ra_ABS (X) := if Ra_is_nonneg (X) then X else Ra_Rev (X) end if

-- The set of rational sequences
DEF 46. $\mathrm{RaSeq} =_{\mathrm{Def}} \{f:\ f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \,\&\, \mathrm{Svm}(f)\}$
-- The constant rational sequence 0
DEF 47. $\mathrm{RaSeq}_0 =_{\mathrm{Def}} \mathbb{N} \times \{\mathbf{0}_{\mathbb{Q}}\}$
-- The constant rational sequence 1
DEF 48. $\mathrm{RaSeq}_1 =_{\mathrm{Def}} \mathbb{N} \times \{\mathbf{1}_{\mathbb{Q}}\}$
-- Sum of rational sequences
DEF 49. $X +_{\mathbb{QS}} Y =_{\mathrm{Def}} \{[p^{[1]}, p^{[2]} +_{\mathbb{Q}} Y{\upharpoonright}p^{[1]}]:\ p \in X\}$
-- Inverse of rational sequence

DEF 50. $\mathsf{Ras\_Rev}(X) \;=_{\text{Def}}\; \{[p^{[1]}, \mathsf{Rev}_{\mathbb{Q}}(p^{[2]})] : p \in X\}$

-- Absolute values of rational sequence

DEF 51. $\mathsf{Ras\_ABS}(X) \;=_{\text{Def}}\; \{[p^{[1]}, \mathsf{Ra\_ABS}(p^{[2]})] : p \in X\}$

-- Difference of rational sequences

DEF 52. $X -_{\mathbb{Q}S} Y \;=_{\text{Def}}\; X +_{\mathbb{Q}S} \mathsf{Ras\_Rev}(Y)$

-- Product of rational sequences

DEF 53. $X *_{\mathbb{Q}S} Y \;=_{\text{Def}}\; \{[p^{[1]}, p^{[2]} *_{\mathbb{Q}} Y \restriction p^{[1]}] : p \in X\}$

-- Def 54: [Reciprocal of rational sequence] Ras_Recip (X) := {[# {j: j in i |(X [j]) = Ra_0}, Ra_Recip (X [i])]: i in Z |(X [i]) = Ra_0}

-- Reciprocal of rational sequence

DEF 54. $\mathsf{Ras\_Recip}(X) \;=_{\text{Def}}\; \mathsf{Shifted\_seq}(\{[i, \mathsf{Ra\_Recip}(X \restriction i)] : i \in \mathbb{N}\}, \mathbf{arb}(\{h \in \mathbb{N} \mid \langle \forall i \in \mathbb{N} \backslash h \mid X \restriction i \neq \mathbf{0}_{\mathbb{Q}} \rangle\}))$

-- Quotient of rational sequences

DEF 55. $X /_{\mathbb{Q}S} Y \;=_{\text{Def}}\; X *_{\mathbb{Q}S} \mathsf{Ras\_Recip}(Y)$

-- Rational Cauchy sequences

DEF 56. $\mathsf{RaCauchy} \;=_{\text{Def}}\; \{f : f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f \restriction i -_{\mathbb{Q}} f \restriction j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$

-- Equivalence of rational sequences

DEF 57. $\mathsf{Ra\_eqseq}(X, Y) \;\leftrightarrow_{\text{Def}}\; \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{x : x \in \mathbf{domain}(X) \mid \mathsf{Ra\_ABS}(X \restriction x -_{\mathbb{Q}} Y \restriction x) >_{\mathbb{Q}} \varepsilon\})\rangle$

THEORY $\mathsf{pointwiseU}(f, h, d, r, \mathsf{uo}'(X))$

$\mathsf{Svm}(f)$ & $\mathbf{domain}(f) = d$ & $\mathbf{range}(f) \subseteq r$

$\langle \forall x \in r \mid \mathsf{uo}'(x) \in r \rangle$

$h = \{[p^{[1]}, \mathsf{uo}'(p^{[2]})] : p \in f\}$

END pointwiseU

ENTER_THEORY pointwiseU

Theorem 580 (pointwiseU · 1) $h = \{[u, \mathsf{uo}'(f \restriction u)] : u \in d\}$. PROOF:

Suppose_not ⇒ $h \neq \{[u, \mathsf{uo}'(f \restriction u)] : u \in d\}$

Assump ⇒ $h = \{[p^{[1]}, \mathsf{uo}'(p^{[2]})] : p \in f\}$

ELEM ⇒ $\{[p^{[1]}, \mathsf{uo}'(p^{[2]})] : p \in f\} \neq \{[u, \mathsf{uo}'(f \restriction u)] : u \in d\}$

Assump ⇒ $\mathsf{Svm}(f)$ & $\mathbf{domain}(f) = d$ & $\mathbf{range}(f) \subseteq r$

$\langle f \rangle \hookrightarrow T65$ ⇒ $f = \{[w, f \restriction w] : w \in \mathbf{domain}(f)\}$

EQUAL ⇒ $\{[p^{[1]}, \mathsf{uo}'(p^{[2]})] : p \in f\} = \{[p^{[1]}, \mathsf{uo}'(p^{[2]})] : p \in \{[w, f \restriction w] : w \in d\}\}$

SIMPLF ⇒ *Stat1* : $\left\{\left[[w, f \restriction w]^{[1]}, \mathsf{uo}'([w, f \restriction w]^{[2]})\right] : w \in d\right\} \neq \{[u, \mathsf{uo}'(f \restriction u)] : u \in d\}$

$\langle c \rangle \hookrightarrow Stat1$ ⇒ $c \in \left\{\left[[w, f \restriction w]^{[1]}, \mathsf{uo}'([w, f \restriction w]^{[2]})\right] : w \in d\right\} \leftrightarrow c \notin \{[u, \mathsf{uo}'(f \restriction u)] : u \in d\}$

Suppose ⇒ *Stat2* : $c \in \left\{\left[[w, f \restriction w]^{[1]}, \mathsf{uo}'([w, f \restriction w]^{[2]})\right] : w \in d\right\}$ & $c \notin \{[u, \mathsf{uo}'(f \restriction u)] : u \in d\}$

$\langle \mathsf{w},\mathsf{w}\rangle \hookrightarrow Stat2(\langle Stat2\rangle) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat3:\ \mathsf{c} \in \{[\mathsf{u}, \mathsf{uo}'(\mathsf{f}\restriction\mathsf{u})] :\ \mathsf{u} \in \mathsf{d}\}\ \&\ \mathsf{c} \notin \left\{\left[[\mathsf{w}, \mathsf{f}\restriction\mathsf{w}]^{[1]}, \mathsf{uo}'([\mathsf{w}, \mathsf{f}\restriction\mathsf{w}]^{[2]})\right] :\ \mathsf{w} \in \mathsf{d}\right\}$

$\langle \mathsf{u},\mathsf{u}\rangle \hookrightarrow Stat3(\langle Stat3\rangle) \Rightarrow$ fail

Discharge $\Rightarrow$ QED

Theorem 581 (pointwiseU · 2) $\mathsf{Svm}(\mathsf{h})\ \&\ \mathbf{domain}(\mathsf{h}) = \mathsf{d}\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathsf{r}$. PROOF:

Suppose_not $\Rightarrow$ $\neg\mathsf{Svm}(\mathsf{h}) \vee \mathbf{domain}(\mathsf{h}) \neq \mathsf{d} \vee \mathbf{range}(\mathsf{h}) \nsubseteq \mathsf{r}$

$TpointwiseU \cdot 1 \Rightarrow$ $\mathsf{h} = \{[\mathsf{u}, \mathsf{uo}'(\mathsf{f}\restriction\mathsf{u})] :\ \mathsf{u} \in \mathsf{d}\}$

APPLY $\langle\ \rangle$ fcn_symbol $(\mathsf{f}(\mathsf{u}) \mapsto \mathsf{uo}'(\mathsf{f}\restriction\mathsf{u}), \mathsf{g} \mapsto \mathsf{h}, \mathsf{s} \mapsto \mathsf{d}) \Rightarrow$

$\mathbf{domain}(\mathsf{h}) = \mathsf{d}\ \&\ \mathbf{range}(\mathsf{h}) = \{\mathsf{uo}'(\mathsf{f}\restriction\mathsf{u}) :\ \mathsf{u} \in \mathsf{d}\}\ \&\ \mathsf{Svm}(\mathsf{h})$

ELEM $\Rightarrow$ $Stat1:\ \{\mathsf{uo}'(\mathsf{f}\restriction\mathsf{u}) :\ \mathsf{u} \in \mathsf{d}\} \nsubseteq \mathsf{r}$

$\langle \mathsf{c}\rangle \hookrightarrow Stat1 \Rightarrow$ $Stat2:\ \mathsf{c} \in \{\mathsf{uo}'(\mathsf{f}\restriction\mathsf{u}) :\ \mathsf{u} \in \mathsf{d}\}\ \&\ \mathsf{c} \notin \mathsf{r}$

$\langle \mathsf{x}\rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{x} \in \mathsf{d}\ \&\ \mathsf{c} = \mathsf{uo}'(\mathsf{f}\restriction\mathsf{x})$

Assump $\Rightarrow$ $\mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{d}\ \&\ \mathbf{range}(\mathsf{f}) \subseteq \mathsf{r}$

$\langle \mathsf{f}\rangle \hookrightarrow T65 \Rightarrow$ $\mathsf{f} = \{[\mathsf{w}, \mathsf{f}\restriction\mathsf{w}] :\ \mathsf{w} \in \mathbf{domain}(\mathsf{f})\}$

$TELEM \Rightarrow$ $\mathbf{range}(\{[\mathsf{w}, \mathsf{f}\restriction\mathsf{w}] :\ \mathsf{w} \in \mathbf{domain}(\mathsf{f})\}) = \{\mathsf{f}\restriction\mathsf{w} :\ \mathsf{w} \in \mathbf{domain}(\mathsf{f})\}$

EQUAL $\Rightarrow$ $\mathbf{range}(\mathsf{f}) = \{\mathsf{f}\restriction\mathsf{w} :\ \mathsf{w} \in \mathbf{domain}(\mathsf{f})\}$

EQUAL $\Rightarrow$ $Stat3:\ \mathbf{range}(\mathsf{f}) = \{\mathsf{f}\restriction\mathsf{w} :\ \mathsf{w} \in \mathsf{d}\}$

Suppose $\Rightarrow$ $Stat3a:\ \mathsf{f}\restriction\mathsf{x} \notin \mathbf{range}(\mathsf{f})$

$\langle Stat3, Stat3a\rangle$ ELEM $\Rightarrow$ $Stat5:\ \mathsf{f}\restriction\mathsf{x} \notin \{\mathsf{f}\restriction\mathsf{w} :\ \mathsf{w} \in \mathsf{d}\}$

$\langle \mathsf{x}\rangle \hookrightarrow Stat5 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{f}\restriction\mathsf{x} \in \mathsf{r}$

Assump $\Rightarrow$ $Stat7:\ \langle \forall \mathsf{x} \in \mathsf{r} \mid \mathsf{uo}'(\mathsf{x}) \in \mathsf{r}\rangle$

$\langle \mathsf{f}\restriction\mathsf{x}\rangle \hookrightarrow Stat7 \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY pointwiseU

THEORY pointwiseU $(\mathsf{f}, \mathsf{h}, \mathsf{d}, \mathsf{r}, \mathsf{uo}'(\mathsf{X}))$

$\mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{domain}(\mathsf{f}) = \mathsf{d}\ \&\ \mathbf{range}(\mathsf{f}) \subseteq \mathsf{r}$

$\langle \forall \mathsf{x} \in \mathsf{r}, \mathsf{y} \in \mathsf{r} \mid \mathsf{bo}'(\mathsf{x}, \mathsf{y}) \in \mathsf{r}\rangle$

$\mathsf{h} = \{[\mathsf{p}^{[1]}, \mathsf{uo}'(\mathsf{p}^{[2]})] :\ \mathsf{p} \in \mathsf{f}\}$

$\Rightarrow$

$\mathsf{h} = \{[\mathsf{u}, \mathsf{uo}'(\mathsf{f}\restriction\mathsf{u})] :\ \mathsf{u} \in \mathsf{d}\}$

$\mathsf{Svm}(\mathsf{h})\ \&\ \mathbf{domain}(\mathsf{h}) = \mathsf{d}\ \&\ \mathbf{range}(\mathsf{h}) \subseteq \mathsf{r}$

END pointwiseU

THEORY pointwise $(\mathsf{f}, \mathsf{f}', \mathsf{h}, \mathsf{d}, \mathsf{r}, \mathsf{bo}'(\mathsf{X}, \mathsf{Y}))$

$Svm(f) \,\&\, \mathbf{domain}(f) = d \,\&\, \mathbf{range}(f) \subseteq r$
$Svm(f') \,\&\, \mathbf{domain}(f') = d \,\&\, \mathbf{range}(f') \subseteq r$
$\langle \forall x \in r, y \in r \mid bo'(x,y) \in r \rangle$
$h = \{[p^{[1]}, bo'(p^{[2]}, f' \restriction p^{[1]})] : p \in f\}$

END pointwise

ENTER_THEORY pointwise

Theorem 582 (pointwise · 1) $h = \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$. PROOF:

Suppose_not ⇒ $h \neq \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$
Assump ⇒ $h = \{[p^{[1]}, bo'(p^{[2]}, f' \restriction p^{[1]})] : p \in f\}$
ELEM ⇒ $\{[p^{[1]}, bo'(p^{[2]}, f' \restriction p^{[1]})] : p \in f\} \neq \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$
Assump ⇒ $Svm(f) \,\&\, \mathbf{domain}(f) = d \,\&\, \mathbf{range}(f) \subseteq r$
$\langle f \rangle \hookrightarrow T65$ ⇒ $f = \{[w, f \restriction w] : w \in \mathbf{domain}(f)\}$
EQUAL ⇒ $\{[p^{[1]}, bo'(p^{[2]}, f' \restriction p^{[1]})] : p \in f\} = \{[p^{[1]}, bo'(p^{[2]}, f' \restriction p^{[1]})] : p \in \{[w, f \restriction w] : w \in d\}\}$
SIMPLF ⇒ *Stat1* :
$\left\{\left[[w, f \restriction w]^{[1]}, bo'([w, f \restriction w]^{[2]}, f' \restriction [w, f \restriction w]^{[1]})\right] : w \in d\right\} \neq \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$
$\langle c \rangle \hookrightarrow Stat1$ ⇒
$c \in \left\{\left[[w, f \restriction w]^{[1]}, bo'([w, f \restriction w]^{[2]}, f' \restriction [w, f \restriction w]^{[1]})\right] : w \in d\right\} \leftrightarrow c \notin \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$
Suppose ⇒ *Stat2* :
$c \in \left\{\left[[w, f \restriction w]^{[1]}, bo'([w, f \restriction w]^{[2]}, f' \restriction [w, f \restriction w]^{[1]})\right] : w \in d\right\} \,\&\, c \notin \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$
$\langle w, w \rangle \hookrightarrow Stat2(\langle Stat2 \rangle)$ ⇒ false; Discharge ⇒
Stat3 : $c \in \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\} \,\&\, c \notin \left\{\left[[w, f \restriction w]^{[1]}, bo'([w, f \restriction w]^{[2]}, f' \restriction [w, f \restriction w]^{[1]})\right] : w \in d\right\}$
$\langle u, u \rangle \hookrightarrow Stat3(\langle Stat3 \rangle)$ ⇒ fail
Discharge ⇒ QED

Theorem 583 (pointwise · 2) $Svm(h) \,\&\, \mathbf{domain}(h) = d \,\&\, \mathbf{range}(h) \subseteq r$. PROOF:

Suppose_not ⇒ $\neg Svm(h) \vee \mathbf{domain}(h) \neq d \vee \mathbf{range}(h) \not\subseteq r$
Tpointwise · 1 ⇒ $h = \{[u, bo'(f \restriction u, f' \restriction u)] : u \in d\}$
APPLY $\langle\,\rangle$ fcn_symbol$(f(u) \mapsto bo'(f \restriction u, f' \restriction u), g \mapsto h, s \mapsto d)$ ⇒
$\mathbf{domain}(h) = d \,\&\, \mathbf{range}(h) = \{bo'(f \restriction u, f' \restriction u) : u \in d\} \,\&\, Svm(h)$

ELEM $\Rightarrow$ $\mathit{Stat1}: \left\{ \mathrm{bo}'(f\upharpoonright u, f'\upharpoonright u) : u \in d \right\} \not\subseteq r$

$\langle c \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}: c \in \left\{ \mathrm{bo}'(f\upharpoonright u, f'\upharpoonright u) : u \in d \right\} \;\&\; c \notin r$

$\langle x \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $x \in d \;\&\; c = \mathrm{bo}'(f\upharpoonright x, f'\upharpoonright x)$

Assump $\Rightarrow$ $\mathrm{Svm}(f) \;\&\; \mathbf{domain}(f) = d \;\&\; \mathbf{range}(f) \subseteq r$

Assump $\Rightarrow$ $\mathrm{Svm}(f') \;\&\; \mathbf{domain}(f') = d \;\&\; \mathbf{range}(f') \subseteq r$

$\langle f \rangle \hookrightarrow \mathit{T65} \Rightarrow$ $f = \{[w, f\upharpoonright w] : w \in \mathbf{domain}(f)\}$

$\mathit{TELEM} \Rightarrow$ $\mathbf{range}(\{[w, f\upharpoonright w] : w \in \mathbf{domain}(f)\}) = \{f\upharpoonright w : w \in \mathbf{domain}(f)\}$

EQUAL $\Rightarrow$ $\mathbf{range}(f) = \{f\upharpoonright w : w \in \mathbf{domain}(f)\}$

$\langle f' \rangle \hookrightarrow \mathit{T65} \Rightarrow$ $f' = \{[w, f'\upharpoonright w] : w \in \mathbf{domain}(f')\}$

$\mathit{TELEM} \Rightarrow$ $\mathbf{range}(\{[w, f'\upharpoonright w] : w \in \mathbf{domain}(f')\}) = \{f'\upharpoonright w : w \in \mathbf{domain}(f')\}$

EQUAL $\Rightarrow$ $\mathbf{range}(f') = \{f'\upharpoonright w : w \in \mathbf{domain}(f')\}$

EQUAL $\Rightarrow$ $\mathit{Stat3}: \mathbf{range}(f) = \{f\upharpoonright w : w \in d\}$

EQUAL $\Rightarrow$ $\mathit{Stat4}: \mathbf{range}(f') = \{f'\upharpoonright w : w \in d\}$

-- ?? x $\hookrightarrow$Stat3 $\Rightarrow$ (f [x]) in range (f) ?? x $\hookrightarrow$Stat4 $\Rightarrow$ (fp [x]) in range (fp)

Suppose $\Rightarrow$ $\mathit{Stat3a}: f\upharpoonright x \notin \mathbf{range}(f)$

$\langle \mathit{Stat3}, \mathit{Stat3a} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat5}: f\upharpoonright x \notin \{f\upharpoonright w : w \in d\}$

$\langle x \rangle \hookrightarrow \mathit{Stat5} \Rightarrow$ false; Discharge $\Rightarrow$ $f\upharpoonright x \in r$

Suppose $\Rightarrow$ $\mathit{Stat4a}: f'\upharpoonright x \notin \mathbf{range}(f')$

$\langle \mathit{Stat4}, \mathit{Stat4a} \rangle$ ELEM $\Rightarrow$ $\mathit{Stat6}: f'\upharpoonright x \notin \{f'\upharpoonright w : w \in d\}$

$\langle x \rangle \hookrightarrow \mathit{Stat6} \Rightarrow$ false; Discharge $\Rightarrow$ $f'\upharpoonright x \in r$

Assump $\Rightarrow$ $\mathit{Stat7}: \langle \forall x \in r, y \in r \mid \mathrm{bo}'(x, y) \in r \rangle$

$\langle f\upharpoonright x, f'\upharpoonright x \rangle \hookrightarrow \mathit{Stat7} \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY pointwise

THEORY pointwise$\left(f, f', h, d, r, \mathrm{bo}'(X, Y)\right)$

$\mathrm{Svm}(f) \;\&\; \mathbf{domain}(f) = d \;\&\; \mathbf{range}(f) \subseteq r$

$\mathrm{Svm}(f') \;\&\; \mathbf{domain}(f') = d \;\&\; \mathbf{range}(f') \subseteq r$

$\langle \forall x \in r, y \in r \mid \mathrm{bo}'(x, y) \in r \rangle$

$h = \left\{ \left[p^{[1]}, \mathrm{bo}'(p^{[2]}, f'\upharpoonright p^{[1]}) \right] : p \in f \right\}$

$\Rightarrow$

$h = \left\{ \left[u, \mathrm{bo}'(f\upharpoonright u, f'\upharpoonright u) \right] : u \in d \right\}$

$\mathrm{Svm}(h) \;\&\; \mathbf{domain}(h) = d \;\&\; \mathbf{range}(h) \subseteq r$

END pointwise

-- It is useful to keep at hand the facts that the absolute value of a rational number is a rational number and that the absolute value of the product of two rational numbers equals the product of the absolute values of the operands.

Theorem 584 (10045) $X \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(X) \in \mathbb{Q} \;\&\; \mathsf{Ra\_ABS}(X) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$. PROOF:

Suppose_not(x) ⇒ $x \in \mathbb{Q} \;\&\; \mathsf{Ra\_ABS}(x) \notin \mathbb{Q} \vee \neg \mathsf{Ra\_ABS}(x) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

T10050 ⇒ *Stat0* : $\langle \forall x \mid \mathsf{Ra\_ABS}(x) = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{Q}}(x) \textbf{ then } x \textbf{ else } \mathsf{Rev}_{\mathbb{Q}}(x) \textbf{ fi} \rangle$

$\langle x \rangle \hookrightarrow$ *Stat0* ⇒ $\mathsf{Ra\_ABS}(x) = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{Q}}(x) \textbf{ then } x \textbf{ else } \mathsf{Rev}_{\mathbb{Q}}(x) \textbf{ fi}$

Suppose ⇒ $\mathsf{Ra\_ABS}(x) \notin \mathbb{Q}$

$\langle x \rangle \hookrightarrow$ *T372* ⇒ false; Discharge ⇒ $\mathsf{Ra\_ABS}(x) \in \mathbb{Q} \;\&\; \neg \mathsf{Ra\_ABS}(x) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Ra\_ABS}(x) \rangle \hookrightarrow$ *T371* ⇒ $\mathsf{Ra\_ABS}(x) = \mathsf{Ra\_ABS}(x) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

T388 ⇒ $\mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$

Suppose ⇒ $\mathsf{is\_nonneg}_{\mathbb{Q}}(x)$

ELEM ⇒ $\mathsf{Ra\_ABS}(x) = x$

EQUAL ⇒ $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Ra\_ABS}(x) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$

$\langle \mathsf{Ra\_ABS}(x), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow$ *T384* ⇒ false; Discharge ⇒ $\neg \mathsf{is\_nonneg}_{\mathbb{Q}}(x) \;\&\; \mathsf{Ra\_ABS}(x) = \mathsf{Rev}_{\mathbb{Q}}(x)$

$\langle x \rangle \hookrightarrow$ *T383* ⇒ $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(x))$

EQUAL ⇒ $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Ra\_ABS}(x) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$

$\langle \mathsf{Ra\_ABS}(x), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow$ *T384* ⇒ false; Discharge ⇒ QED

Theorem 585 (10046) $X, Y \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(X *_{\mathbb{Q}} Y) = \mathsf{Ra\_ABS}(X) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(Y)$. PROOF:

Suppose_not(x, y) ⇒ $x, y \in \mathbb{Q} \;\&\; \mathsf{Ra\_ABS}(x *_{\mathbb{Q}} y) \neq \mathsf{Ra\_ABS}(x) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(y)$

-- For, assuming x, y to be a counterexample to the desired statement, we will reach a contradiction in each one of the possible cases, which are: $x = \mathbf{0}_{\mathbb{Q}}$ or $y = \mathbf{0}_{\mathbb{Q}}$; x and y both positive; one of x, y positive, the other one negative; both of x and y negative. We begin by collecting various elementary algebraic consequences of our hypothesis.

ALGEBRA ⇒ $x *_{\mathbb{Q}} y, \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle x \rangle \hookrightarrow$ *T372* ⇒ $\mathsf{Rev}_{\mathbb{Q}}(x) \in \mathbb{Q}$

$\langle y \rangle \hookrightarrow$ *T372* ⇒ $\mathsf{Rev}_{\mathbb{Q}}(y) \in \mathbb{Q}$

$\langle x \rangle \hookrightarrow$ *T10045* ⇒ $\mathsf{Ra\_ABS}(x) \in \mathbb{Q}$

$\langle y \rangle \hookrightarrow$ *T10045* ⇒ $\mathsf{Ra\_ABS}(y) \in \mathbb{Q}$

$\langle x, y \rangle \hookrightarrow$ *T368* ⇒ $x *_{\mathbb{Q}} y = y *_{\mathbb{Q}} x$

$\langle x \rangle \hookrightarrow$ *T371* ⇒ $x = x +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle y \rangle \hookrightarrow$ *T371* ⇒ $y = y +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})\rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$T388 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y}\rangle \hookrightarrow T372 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}) \in \mathbb{Q}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})\rangle \hookrightarrow T371 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T365 \Rightarrow \quad Stat1: \ \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$
$\langle \mathsf{x}, \mathsf{y}\rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{x} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$
$\langle \mathsf{y}, \mathsf{x}\rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y} *_{\mathbb{Q}} \mathsf{x})$
$\langle \mathsf{y}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}$
$T10050 \Rightarrow \quad Stat50: \ \langle \forall \mathsf{x} \mid \mathsf{Ra\_ABS}(\mathsf{x}) = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x}) \textbf{ then } \mathsf{x} \textbf{ else } \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) \textbf{ fi}\rangle \ \& \ Stat51: \ \langle \forall \mathsf{x} \mid \mathsf{x} \in \mathbb{Q} \rightarrow \big(\mathsf{Ra\_ABS}(\mathsf{x}) = \mathbf{0}_{\mathbb{Q}} \leftrightarrow \mathsf{x} = \mathbf{0}_{\mathbb{Q}}\big)\rangle \ \& \ Stat52: \ \langle \forall \mathsf{x} \mid \mathsf{x} \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\big) = \mathsf{Ra\_ABS}(\mathsf{x})\rangle$

-- The case when either x or y is zero readily leads to a contradiction, because one of $\mathsf{Ra\_ABS}(\mathsf{x}), \mathsf{Ra\_ABS}(\mathsf{y})$ is zero in this case, and hence the product of x, y, its absolute value, and the product of $\mathsf{Ra\_ABS}(\mathsf{x}), \mathsf{Ra\_ABS}(\mathsf{y})$ coincide.

$\mathsf{Suppose} \Rightarrow \quad \mathsf{x} = \mathbf{0}_{\mathbb{Q}} \vee \mathsf{y} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{x}\rangle \hookrightarrow T394 \Rightarrow \quad \mathsf{x} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{y}\rangle \hookrightarrow T394 \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \neq \mathbf{0}_{\mathbb{Q}} \vee \mathsf{Ra\_ABS}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}) \neq \mathbf{0}_{\mathbb{Q}}$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{x} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{x}\rangle \hookrightarrow Stat51 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x}) = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Ra\_ABS}(\mathsf{y})\rangle \hookrightarrow T394 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{y}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Ra\_ABS}(\mathsf{y}), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{y}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{y} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{y}\rangle \hookrightarrow Stat51 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{y}) = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Ra\_ABS}(\mathsf{x})\rangle \hookrightarrow T394 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{x} *_{\mathbb{Q}} \mathsf{y} = \mathbf{0}_{\mathbb{Q}} \ \& \ \mathsf{Ra\_ABS}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}) = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y}\rangle \hookrightarrow Stat51 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{x} \neq \mathbf{0}_{\mathbb{Q}} \ \& \ \mathsf{y} \neq \mathbf{0}_{\mathbb{Q}}$

-- The case when x and y are positive is settled with equal ease, because in this case the absolute value of $\mathsf{x} *_{\mathbb{Q}} \mathsf{y}$, coinciding with $\mathsf{x} *_{\mathbb{Q}} \mathsf{y}$, equals the product of x and y, which in their turn are the absolute values of x and y.

$\mathsf{Suppose} \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{x} \ \& \ \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{y}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{x}\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{x} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{x}, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{x} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})\big)$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x})$

$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{y} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{y} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{y}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{y} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$
EQUAL $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{y})$

$\langle \mathsf{x}, \mathsf{y} \rangle \hookrightarrow T387 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$
$\langle \mathsf{x} \rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x}) = \mathsf{x}$
$\langle \mathsf{y} \rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{y}) = \mathsf{y}$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}) = \mathsf{x} *_{\mathbb{Q}} \mathsf{y}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow \quad \neg(\mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{x} \,\&\, \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{y})$

-- The case when x is positive and y is negative is but slightly more complicated. The product of x and y is negative, and hence the absolute value of this product is the reverse of it. This equals the product of x with the reverse of y, which turns out to be the product of the absolute values of x and y.

Suppose $\Rightarrow \quad \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{x} \,\&\, \mathsf{y} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{x} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{x} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{x}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$
EQUAL $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x})$
$\langle \mathsf{x} \rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x}) = \mathsf{x}$

$\langle \mathsf{y}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{y}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{y} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}))$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) \rangle \hookrightarrow T365 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})$
EQUAL $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}))$
$\langle \mathsf{y} \rangle \hookrightarrow Stat52 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})) = \mathsf{Ra\_ABS}(\mathsf{y})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) \rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})$

EQUAL $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}) = \mathsf{Ra\_ABS}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y})$

Suppose $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$

-- Under this assumption, since $\mathsf{y} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$ yields $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$ and moreover x is positive, the product $\mathsf{x} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})$ would turn out to be positive too; and consequently $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}))$, which equals $\mathsf{x} *_{\mathbb{Q}} \mathsf{y}$, would be negative.

$\langle \mathsf{y}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{y}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{y} \rangle \hookrightarrow T401a \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$
EQUAL $\Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle x, \mathbf{0}_{\mathbb{Q}}, \mathrm{Rev}_{\mathbb{Q}}(y)\rangle \hookrightarrow T393 \Rightarrow \quad x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y)$
ALGEBRA $\Rightarrow \quad x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y), \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y) \in \mathbb{Q}$
$\langle x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y), \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y)\rangle \hookrightarrow T401a \Rightarrow \quad \mathrm{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y)) >_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y))$
$\langle \mathrm{Rev}_{\mathbb{Q}}(y)\rangle \hookrightarrow T394 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Q}}(y) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathrm{Rev}_{\mathbb{Q}}(y), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T368 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Q}}(y) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y)$
$\langle x *_{\mathbb{Q}} y\rangle \hookrightarrow T398 \Rightarrow \quad \mathrm{Rev}_{\mathbb{Q}}(\mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} y)) = x *_{\mathbb{Q}} y$
EQUAL $\Rightarrow \quad x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T401a \Rightarrow \quad \mathrm{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y))$
$\langle x, y\rangle \hookrightarrow T391 \Rightarrow \quad x *_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(y) = \mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} y)$
EQUAL $\Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x *_{\mathbb{Q}} y$
$\langle \mathbf{0}_{\mathbb{Q}}, x *_{\mathbb{Q}} y\rangle \hookrightarrow T384 \Rightarrow \quad Stat2: \ \mathrm{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} y))$
EQUAL $\langle Stat1, Stat2\rangle \Rightarrow \quad \mathrm{is\_nonneg}_{\mathbb{Q}}(\mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} y))$
$\langle x *_{\mathbb{Q}} y\rangle \hookrightarrow T383 \Rightarrow \quad x *_{\mathbb{Q}} y = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{0}_{\mathbb{Q}}, x *_{\mathbb{Q}} y\rangle \hookrightarrow T384 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad \neg \mathrm{is\_nonneg}_{\mathbb{Q}}(x *_{\mathbb{Q}} y)$
$\langle x *_{\mathbb{Q}} y\rangle \hookrightarrow Stat50 \Rightarrow \quad \mathrm{Ra\_ABS}(x *_{\mathbb{Q}} y) = \mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} y)$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow \quad \neg(\mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} x \ \& \ y <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}})$

-- The case when y is positive and x is negative is entirely analogous to the one just discussed.

Suppose $\Rightarrow \quad x <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} y$
$\langle \mathbf{0}_{\mathbb{Q}}, y\rangle \hookrightarrow T384 \Rightarrow \quad y >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle y, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow \quad \mathrm{is\_nonneg}_{\mathbb{Q}}(y +_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}))$
EQUAL $\Rightarrow \quad \mathrm{is\_nonneg}_{\mathbb{Q}}(y)$
$\langle y\rangle \hookrightarrow Stat50 \Rightarrow \quad \mathrm{Ra\_ABS}(y) = y$

$\langle x, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x$
$\langle \mathbf{0}_{\mathbb{Q}}, x\rangle \hookrightarrow T384 \Rightarrow \quad \mathrm{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(x))$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathrm{Rev}_{\mathbb{Q}}(x)\rangle \hookrightarrow T365 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathrm{Rev}_{\mathbb{Q}}(x) = \mathrm{Rev}_{\mathbb{Q}}(x)$
EQUAL $\Rightarrow \quad \mathrm{is\_nonneg}_{\mathbb{Q}}(\mathrm{Rev}_{\mathbb{Q}}(x))$
$\langle x\rangle \hookrightarrow Stat52 \Rightarrow \quad \mathrm{Ra\_ABS}(\mathrm{Rev}_{\mathbb{Q}}(x)) = \mathrm{Ra\_ABS}(x)$
$\langle \mathrm{Rev}_{\mathbb{Q}}(x)\rangle \hookrightarrow Stat50 \Rightarrow \quad \mathrm{Ra\_ABS}(x) = \mathrm{Rev}_{\mathbb{Q}}(x)$

EQUAL $\Rightarrow \quad \mathrm{Rev}_{\mathbb{Q}}(x *_{\mathbb{Q}} y) = \mathrm{Ra\_ABS}(x) *_{\mathbb{Q}} \mathrm{Ra\_ABS}(y)$

Suppose $\Rightarrow \quad \mathrm{is\_nonneg}_{\mathbb{Q}}(x *_{\mathbb{Q}} y)$

-- Under this assumption, since $\mathsf{x} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$ yields $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$ and moreover y is positive, the product $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}$ would turn out to be positive too; and consequently $\mathsf{Rev}_{\mathbb{Q}}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}\big)$, which equals $\mathsf{x} *_{\mathbb{Q}} \mathsf{y}$, would be negative.

$\langle \mathsf{x}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{x}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{x} \rangle \hookrightarrow T401a \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}), \mathbf{0}_{\mathbb{Q}}, \mathsf{y} \rangle \hookrightarrow T393 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{y}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}, \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{y} \in \mathbb{Q}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}, \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow T401a \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{y}) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}\big)$
$\langle \mathsf{y} \rangle \hookrightarrow T394 \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{y}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{y}$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow T398 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})) = \mathsf{x} *_{\mathbb{Q}} \mathsf{y}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T401a \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}}) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y}\big)$
$\langle \mathsf{y}, \mathsf{x} \rangle \hookrightarrow T391 \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y} *_{\mathbb{Q}} \mathsf{x})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}), \mathsf{y} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y} = \mathsf{y} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$
$\mathsf{EQUAL} \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{x} *_{\mathbb{Q}} \mathsf{y}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow T384 \Rightarrow \quad Stat3: \ \mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})\big)$
$\mathsf{EQUAL}\ \langle Stat1, Stat3 \rangle \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})\big)$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow T383 \Rightarrow \quad \mathsf{x} *_{\mathbb{Q}} \mathsf{y} = \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \neg \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y} \rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \neg(\mathsf{x} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{y})$

-- The only remaining case is the one in which x, y are negative. In this case the absolute value of $\mathsf{x} *_{\mathbb{Q}} \mathsf{y}$ coincides with $\mathsf{x} *_{\mathbb{Q}} \mathsf{y}$ and equals the product of the reverses of x and y, which in their turn are the absolute values of x and y.

$T10043 \Rightarrow \quad Stat43: \ \langle \forall \mathsf{x} \in \mathbb{Q}, \mathsf{y} \in \mathbb{Q} \mid \mathsf{x} <_{\mathbb{Q}} \mathsf{y} \vee \mathsf{x} = \mathsf{y} \vee \mathsf{y} <_{\mathbb{Q}} \mathsf{x} \rangle$
$\langle \mathsf{x}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow Stat43 \Rightarrow \quad \mathsf{x} = \mathbf{0}_{\mathbb{Q}} \vee \mathsf{x} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{x}$
$\langle \mathsf{y}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow Stat43 \Rightarrow \quad \mathsf{y} = \mathbf{0}_{\mathbb{Q}} \vee \mathsf{y} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee \mathbf{0}_{\mathbb{Q}} <_{\mathbb{Q}} \mathsf{y}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{x} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \mathsf{y} <_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{x}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{x}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{x} \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\big)$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) \rangle \hookrightarrow T365 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\big)$
$\langle \mathsf{x} \rangle \hookrightarrow Stat52 \Rightarrow \quad \mathsf{Ra\_ABS}\big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\big) = \mathsf{Ra\_ABS}(\mathsf{x})$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})\rangle \hookrightarrow \mathit{Stat50} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$

$\langle \mathsf{y}, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{y}$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{y}\rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}))$
$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})\rangle \hookrightarrow \mathit{T365} \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})$
EQUAL $\Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}))$
$\langle \mathsf{y}\rangle \hookrightarrow \mathit{Stat52} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})) = \mathsf{Ra\_ABS}(\mathsf{y})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})\rangle \hookrightarrow \mathit{Stat50} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y})$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}), \mathsf{y}\rangle \hookrightarrow \mathit{T391} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y})$
$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}), \mathsf{y}\rangle \hookrightarrow \mathit{T368} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{y} = \mathsf{y} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})$
$\langle \mathsf{y}, \mathsf{x}\rangle \hookrightarrow \mathit{T391} \Rightarrow \quad \mathsf{y} *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{y} *_{\mathbb{Q}} \mathsf{x})$
$\langle \mathsf{x} *_{\mathbb{Q}} \mathsf{y}\rangle \hookrightarrow \mathit{T398} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x} *_{\mathbb{Q}} \mathsf{y})) = \mathsf{x} *_{\mathbb{Q}} \mathsf{y}$

-- This yields the conclusion

$$\mathsf{x} *_{\mathbb{Q}} \mathsf{y} = \mathsf{Ra\_ABS}(\mathsf{x}) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}),$$

leading to a contradiction. Since all possible cases have been analyzed and then discarded, this terminates our proof.

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We begin by proving that every constant sequence whose image belongs to the rationals is a Cauchy sequence.

Theorem 586 (*412a*) $\mathsf{X} \in \mathbb{Q} \rightarrow \mathbb{N} \times \{\mathsf{X}\} \in \mathsf{RaCauchy}$. PROOF:

Suppose_not$(\mathsf{x}_2) \Rightarrow \quad \mathsf{x}_2 \in \mathbb{Q} \,\&\, \mathbb{N} \times \{\mathsf{x}_2\} \notin \mathsf{RaCauchy}$
Use_def(RaCauchy) $\Rightarrow \quad \mathit{Stat0}: \mathbb{N} \times \{\mathsf{x}_2\} \notin \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}$
Use_def($\times$) $\Rightarrow \quad \mathbb{N} \times \{\mathsf{x}_2\} = \{[\mathsf{yy}, \mathsf{x}] : \mathsf{yy} \in \mathbb{N}, \mathsf{x} \in \{\mathsf{x}_2\}\}$
SIMPLF $\Rightarrow \quad \mathbb{N} \times \{\mathsf{x}_2\} = \{[\mathsf{yy}, \mathsf{x}_2] : \mathsf{yy} \in \mathbb{N}\}$
APPLY $\langle\,\rangle$ fcn_symbol$(\mathsf{f}(\mathsf{yy}) \mapsto \mathsf{x}_2, \mathsf{g} \mapsto \mathbb{N} \times \{\mathsf{x}_2\}, \mathsf{s} \mapsto \mathbb{N}) \Rightarrow$
$\quad \mathbf{domain}(\mathbb{N} \times \{\mathsf{x}_2\}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathbb{N} \times \{\mathsf{x}_2\}) \,\&\, \mathit{Stat1}: \langle \forall \mathsf{yy} \mid (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{yy} = \mathbf{if}\ \mathsf{yy} \in \mathbb{N}\ \mathbf{then}\ \mathsf{x}_2\ \mathbf{else}\ \emptyset\ \mathbf{fi}\rangle$
Suppose $\Rightarrow \quad \mathbb{N} \times \{\mathsf{x}_2\} \notin \mathsf{RaSeq}$
$\langle \mathbb{N}, \mathbb{N}, \{\mathsf{x}_2\}, \mathbb{Q}\rangle \hookrightarrow \mathit{T219} \Rightarrow \quad \mathbb{N} \times \{\mathsf{x}_2\} \subseteq \mathbb{N} \times \mathbb{Q}$
Use_def(RaSeq) $\Rightarrow \quad \mathit{Stat2}: \mathbb{N} \times \{\mathsf{x}_2\} \notin \{\mathsf{f} \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(\mathsf{f}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f})\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow \quad \mathbb{N} \times \{\mathsf{x}_2\} \in \mathsf{RaSeq}$
$\langle\,\rangle \hookrightarrow \mathit{Stat0} \Rightarrow \quad \mathit{Stat3}: \neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}\Big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}((\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{i} -_{\mathbb{Q}} (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{j}) >_{\mathbb{Q}} \varepsilon\}\Big)\rangle$

$\langle\varepsilon\rangle\hookrightarrow Stat3 \Rightarrow \quad \varepsilon \in \mathbb{Q} \,\&\, \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \neg\mathsf{Finite}\Big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{i} -_{\mathbb{Q}} (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{j}\big) >_{\mathbb{Q}} \varepsilon\}\Big)$
$T161 \Rightarrow \quad \mathsf{Finite}(\emptyset)$
$\mathsf{EQUAL} \Rightarrow \quad Stat4: \emptyset \neq \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{i} -_{\mathbb{Q}} (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{j}\big) >_{\mathbb{Q}} \varepsilon\}$
$\langle\mathsf{i},\mathsf{j}\rangle\hookrightarrow Stat4 \Rightarrow \quad Stat5: \mathsf{i},\mathsf{j} \in \mathbb{N} \,\&\, \mathsf{Ra\_ABS}\big((\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{i} -_{\mathbb{Q}} (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{j}\big) >_{\mathbb{Q}} \varepsilon$
$\langle\mathsf{i}\rangle\hookrightarrow Stat1 \Rightarrow \quad (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{i} = \mathsf{x}_2$
$\langle\mathsf{j}\rangle\hookrightarrow Stat1 \Rightarrow \quad (\mathbb{N} \times \{\mathsf{x}_2\})\restriction\mathsf{j} = \mathsf{x}_2$
$\langle\mathsf{x}_2\rangle\hookrightarrow T372 \Rightarrow \quad \mathsf{x}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}_2) = \mathbf{0}_{\mathbb{Q}}$
$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \mathsf{x}_2 -_{\mathbb{Q}} \mathsf{x}_2 = \mathbf{0}_{\mathbb{Q}}$
$T382 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$
$T10050 \Rightarrow \quad Stat6: \langle\forall \mathsf{x} \mid \mathsf{Ra\_ABS}(\mathsf{x}) = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{Q}}(\mathsf{x}) \textbf{ then } \mathsf{x} \textbf{ else } \mathsf{Rev}_{\mathbb{Q}}(\mathsf{x}) \textbf{ fi}\rangle$
$\langle\mathbf{0}_{\mathbb{Q}}\rangle\hookrightarrow Stat6 \Rightarrow \quad \mathsf{Ra\_ABS}(\mathbf{0}_{\mathbb{Q}}) = \mathbf{0}_{\mathbb{Q}}$
$\mathsf{EQUAL}\ \langle Stat5\rangle \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \varepsilon$
$\langle\mathbf{0}_{\mathbb{Q}},\varepsilon\rangle\hookrightarrow T384 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \varepsilon \,\&\, \mathbf{0}_{\mathbb{Q}} \neq \varepsilon$
$\langle\varepsilon,\mathbf{0}_{\mathbb{Q}}\rangle\hookrightarrow T384 \Rightarrow \quad \varepsilon \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle\varepsilon,\mathbf{0}_{\mathbb{Q}}\rangle\hookrightarrow T400 \Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow$ QED

-- As a corollary, the sequence which has constant value $\mathbf{0}_{\mathbb{Q}}$, and the one which has constant value $\mathbf{1}_{\mathbb{Q}}$, are Cauchy sequences.

Theorem 587 (412) $\{\mathsf{RaSeq}_0, \mathsf{RaSeq}_1\} \subseteq \mathsf{RaCauchy}$. PROOF:

$\mathsf{Suppose\_not} \Rightarrow \quad \{\mathsf{RaSeq}_0, \mathsf{RaSeq}_1\} \not\subseteq \mathsf{RaCauchy}$
$\mathsf{Use\_def}(\mathsf{RaSeq}_0) \Rightarrow \quad \mathsf{RaSeq}_0 = \mathbb{N} \times \{\mathbf{0}_{\mathbb{Q}}\}$
$\mathsf{Use\_def}(\mathsf{RaSeq}_1) \Rightarrow \quad \mathsf{RaSeq}_1 = \mathbb{N} \times \{\mathbf{1}_{\mathbb{Q}}\}$
$T371 \Rightarrow \quad \mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle\mathbf{0}_{\mathbb{Q}}\rangle\hookrightarrow T412a \Rightarrow \quad \mathbb{N} \times \{\mathbf{0}_{\mathbb{Q}}\} \in \mathsf{RaCauchy}$
$\langle\mathbf{1}_{\mathbb{Q}}\rangle\hookrightarrow T412a \Rightarrow \quad \mathsf{false};$ Discharge $\Rightarrow$ QED

-- Every sequence of rational numbers is equivalent to itself:

Theorem 588 (413a) $\mathsf{F} \in \mathsf{RaSeq} \rightarrow \mathbf{domain}(\mathsf{F}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{F}) \,\&\, \mathbf{range}(\mathsf{F}) \subseteq \mathbb{Q} \,\&\, \mathsf{Ra\_eqseq}(\mathsf{F},\mathsf{F})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{f}) \Rightarrow \quad \mathsf{f} \in \mathsf{RaSeq} \,\&\, \neg\big(\mathbf{domain}(\mathsf{f}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f}) \,\&\, \mathbf{range}(\mathsf{f}) \subseteq \mathbb{Q} \,\&\, \mathsf{Ra\_eqseq}(\mathsf{f},\mathsf{f})\big)$
$\mathsf{Use\_def}(\mathsf{RaSeq}) \Rightarrow \quad Stat2: \mathsf{f} \in \{\mathsf{f} \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(\mathsf{f}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f})\}$
$\langle\ \rangle\hookrightarrow Stat2 \Rightarrow \quad \mathsf{f} \subseteq \mathbb{N} \times \mathbb{Q} \,\&\, \mathbf{domain}(\mathsf{f}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f})$
$\langle\mathsf{f},\mathbb{N},\mathbb{Q}\rangle\hookrightarrow T116 \Rightarrow \quad \mathbf{range}(\mathsf{f}) \subseteq \mathbb{Q} \,\&\, \neg\mathsf{Ra\_eqseq}(\mathsf{f},\mathsf{f})$

Use_def(Ra_eqseq) $\Rightarrow$ $\mathit{Stat3}: \neg\langle\forall\varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{x : x \in \mathbf{domain}(f) \mid \mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) >_{\mathbb{Q}} \varepsilon\})\rangle$

$\langle e\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $e \in \mathbb{Q} \;\&\; e >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \neg\mathsf{Finite}(\{x : x \in \mathbf{domain}(f) \mid \mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) >_{\mathbb{Q}} e\})$

$\mathit{T161} \Rightarrow$ $\mathsf{Finite}(\emptyset)$

$\langle \mathit{Stat3}\rangle$ ELEM $\Rightarrow$ $\mathit{Stat4}: \{x : x \in \mathbf{domain}(f) \mid \mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) >_{\mathbb{Q}} e\} \neq \emptyset$

$\langle x\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $x \in \mathbf{domain}(f) \;\&\; \mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) >_{\mathbb{Q}} e$

$\langle \mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x), e\rangle \hookrightarrow \mathit{T384} \Rightarrow$ $\mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) \geqslant_{\mathbb{Q}} e$

$\langle f\rangle \hookrightarrow \mathit{T66} \Rightarrow$ $\mathit{Stat5}: \mathbf{range}(f) = \{f{\upharpoonright}x : x \in \mathbf{domain}(f)\}$

Suppose $\Rightarrow$ $f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x \neq \mathbf{0}_{\mathbb{Q}}$

Use_def($-_{\mathbb{Q}}$) $\Rightarrow$ $f{\upharpoonright}x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\upharpoonright}x) \neq \mathbf{0}_{\mathbb{Q}}$

$\langle f{\upharpoonright}x\rangle \hookrightarrow \mathit{T372} \Rightarrow$ $\mathit{Stat6}: f{\upharpoonright}x \notin \{f{\upharpoonright}x : x \in \mathbf{domain}(f)\}$

$\langle x\rangle \hookrightarrow \mathit{Stat6} \Rightarrow$ false; Discharge $\Rightarrow$ $f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x = \mathbf{0}_{\mathbb{Q}}$

$\mathit{T382} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$

EQUAL $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{Q}}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x)$

$\mathit{T10050} \Rightarrow$ $\mathit{Stat7}: \langle\forall x \mid \mathsf{Ra\_ABS}(x) = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{Q}}(x) \textbf{ then } x \textbf{ else } \mathsf{Rev}_{\mathbb{Q}}(x) \textbf{ fi}\rangle$

$\langle f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x\rangle \hookrightarrow \mathit{Stat7} \Rightarrow$ $\mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) = \mathbf{0}_{\mathbb{Q}}$

$\mathit{T371} \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x), e, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow \mathit{T406} \Rightarrow$ $\mathsf{Ra\_ABS}(f{\upharpoonright}x -_{\mathbb{Q}} f{\upharpoonright}x) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

Use_def($>_{\mathbb{Q}}$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- More generally, in the case of a rational Cauchy sequence f, every infinite subsequence g of f is a rational Cauchy sequence equivalent to f.

Theorem 589 (10090) $F \in \mathsf{RaCauchy} \;\&\; G \in \mathsf{Subseqs}(F) \;\&\; \neg\mathsf{Finite}(G) \rightarrow G \in \mathsf{RaCauchy} \;\&\; \mathsf{Ra\_eqseq}(F, G)$. Proof:

Suppose_not(f, g) $\Rightarrow$ $f \in \mathsf{RaCauchy} \;\&\; g \in \mathsf{Subseqs}(f) \;\&\; \neg\mathsf{Finite}(g) \;\&\; g \notin \mathsf{RaCauchy} \vee \neg\mathsf{Ra\_eqseq}(f, g)$

Use_def(RaCauchy) $\Rightarrow$ $\mathit{Stat1}: f \in \{f \in \mathsf{RaSeq} \mid \langle\forall\varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$

$\langle\,\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $f \in \mathsf{RaSeq} \;\&\; \mathit{Stat2}: \langle\forall\varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \varepsilon\})\rangle$

Use_def(RaSeq) $\Rightarrow$ $\mathit{Stat3}: f \in \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathsf{Svm}(f)\}$

$\langle\,\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $f \subseteq \mathbb{N} \times \mathbb{Q} \;\&\; \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathsf{Svm}(f)$

Use_def(next) $\Rightarrow$ $\mathsf{next}(\mathbb{N}) = \mathbb{N} \cup \{\mathbb{N}\}$

APPLY $\langle h_\Theta : h\rangle$ subseq($g \mapsto g, f \mapsto f$) $\Rightarrow$

$g = f \bullet h \;\&\; 1\text{–}1(h) \;\&\; \mathbf{domain}(h) \in \mathsf{next}(\mathbb{N}) \;\&\; \mathbf{range}(h) \subseteq \mathbf{domain}(f) \;\&\; g \subseteq \mathbf{domain}(f) \times \mathbf{range}(f) \;\&\; \mathsf{Svm}(g) \;\&\; \mathbf{domain}(g) \in \mathsf{next}(\mathbb{N}) \cap \mathsf{next}(\mathbf{domain}(f)) \;\&\; \{i \in \mathbf{domai}$

Use_def(1–1) $\Rightarrow$ $\mathsf{Svm}(h)$

$\langle f, \mathbb{N}, \mathbb{Q}\rangle \hookrightarrow \mathit{T116} \Rightarrow$ $\mathbf{domain}(f) \subseteq \mathbb{N} \;\&\; \mathbf{range}(f) \subseteq \mathbb{Q}$

$\langle \mathbf{domain}(f), \mathbb{N}, \mathbf{range}(f), \mathbb{Q}\rangle \hookrightarrow \mathit{T219} \Rightarrow$ $g \subseteq \mathbb{N} \times \mathbb{Q}$

Suppose $\Rightarrow$ $\mathbf{domain}(g) \neq \mathbb{N} \vee \mathbf{domain}(h) \neq \mathbb{N}$

$\langle h, f\rangle \hookrightarrow \mathit{T85} \Rightarrow$ $\mathbf{domain}(f \bullet h) = \mathbf{domain}(h)$

EQUAL $\Rightarrow$ $\mathbf{domain}(g) \in \mathsf{next}(\mathbb{N}) \;\&\; \mathbf{domain}(g) = \mathbf{domain}(h)$

Use_def(next) $\Rightarrow$ $\mathbf{domain}(g) \in \mathbb{N}$

$\langle \mathbf{domain}(g) \rangle \hookrightarrow T179 \Rightarrow \quad \mathsf{Finite}(\#\mathbf{domain}(g))$
$\langle g \rangle \hookrightarrow T148 \Rightarrow \quad \#\mathbf{domain}(g) = \#g$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Finite}(\#g)$
$\langle g \rangle \hookrightarrow T166 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat4a:\ \mathbf{domain}(g) = \mathbb{N}\ \&\ \mathbf{domain}(h) = \mathbb{N}$
$\mathsf{Suppose} \Rightarrow \quad g \notin \mathsf{RaCauchy}$
$\mathsf{Use\_def}(\mathsf{RaCauchy}) \Rightarrow \quad Stat5:\ g \notin \{f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$
$\langle\ \rangle \hookrightarrow Stat5 \Rightarrow \quad g \notin \mathsf{RaSeq} \vee \neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \varepsilon\})\rangle$
$\mathsf{Suppose} \Rightarrow \quad g \notin \mathsf{RaSeq}$
$\mathsf{Use\_def}(\mathsf{RaSeq}) \Rightarrow \quad Stat6:\ g \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N}\ \&\ \mathsf{Svm}(f)\}$
$\langle\ \rangle \hookrightarrow Stat6 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat7:\ \neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \varepsilon\})\rangle$
$\langle \mathsf{eps}_0 \rangle \hookrightarrow Stat7 \Rightarrow \quad \mathsf{eps}_0 \in \mathbb{Q}\ \&\ \mathsf{eps}_0 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ \neg\mathsf{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\})$
$\langle \mathsf{eps}_0 \rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\})$
$\langle h, \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\} \rangle \hookrightarrow T53 \Rightarrow$
$\quad 1\text{–}1\left(h_{|\{i \cap j:\, i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}}\right)$
$\mathsf{Suppose} \Rightarrow \quad \mathbf{domain}(h_{|\{i \cap j:\, i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}}) \neq$
$\quad \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}$
$\langle h, \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\} \rangle \hookrightarrow T84 \Rightarrow \quad Stat8:\ \neg$
$\quad \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\} \subseteq \mathbb{N}$
$\langle c \rangle \hookrightarrow Stat8 \Rightarrow \quad Stat9:\ c \in \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}\ \&\ c \notin \mathbb{N}$
$\langle i', j' \rangle \hookrightarrow Stat9 \Rightarrow \quad i', j' \in \mathbb{N}\ \&\ i' \cap j' \notin \mathbb{N}$
$\langle i', j' \rangle \hookrightarrow T289 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathbf{domain}(h_{|\{i \cap j:\, i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}}) = \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}$
$\mathsf{Suppose} \Rightarrow \quad Stat10:$
$\qquad \mathbf{range}(h_{|\{i \cap j:\, i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}}) \not\subseteq \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}$
$\langle d \rangle \hookrightarrow Stat10 \Rightarrow \quad d \in \mathbf{range}(h_{|\{i \cap j:\, i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}})\ \&\ Stat11:$
$\qquad d \notin \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}$
$\langle h, \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\} \rangle \hookrightarrow T101 \Rightarrow \quad Stat11:$
$\quad d \in \{h{\upharpoonright}x :\ x \in \mathbf{domain}(h) \mid x \in \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}\}$
$\langle iq \rangle \hookrightarrow Stat11 \Rightarrow \quad Stat12:$
$\quad iq \in \mathbb{N}\ \&\ iq \in \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g{\upharpoonright}i -_{\mathbb{Q}} g{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}\ \&$
$\qquad h{\upharpoonright}iq \notin \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\upharpoonright}i -_{\mathbb{Q}} f{\upharpoonright}j) >_{\mathbb{Q}} \mathsf{eps}_0\}$
$\langle i_0, j_0, h{\upharpoonright}i_0, h{\upharpoonright}j_0 \rangle \hookrightarrow Stat12 \Rightarrow \quad Stat12a:$
$\quad i_0, j_0 \in \mathbb{N}\ \&\ iq = i_0 \cap j_0\ \&\ \mathsf{Ra\_ABS}(g{\upharpoonright}i_0 -_{\mathbb{Q}} g{\upharpoonright}j_0) >_{\mathbb{Q}} \mathsf{eps}_0\ \&$
$\qquad h{\upharpoonright}iq \neq h{\upharpoonright}i_0 \cap h{\upharpoonright}j_0 \vee h{\upharpoonright}i_0 \notin \mathbb{N} \vee h{\upharpoonright}j_0 \notin \mathbb{N} \vee \neg\mathsf{Ra\_ABS}(f{\upharpoonright}(h{\upharpoonright}i_0) -_{\mathbb{Q}} f{\upharpoonright}(h{\upharpoonright}j_0)) >_{\mathbb{Q}} \mathsf{eps}_0$

-- ? (i0, j0, h [i0], h [j0]) $\hookrightarrow$Stat12 $\Rightarrow$ Stat12a: (i0 in Z) & (j0 in Z) & (iq = (i0 * j0)) & (Ra_ABS ((g [i0]) Ra_MINUS (g [j0])) Ra_GT eps0) & (((h [iq]) = (h [i0]) * (h [j0])) or ((h [i0]) notin Z) or ((h [j0]) notin Z) or (not (Ra_ABS ((f [h [i0]]) Ra_MINUS (f [h [j0]])) Ra_GT eps0)))

$\langle f, h, i_0\rangle \hookrightarrow T104 \Rightarrow \quad f{\bullet}h \upharpoonright i_0 = f\upharpoonright(h\upharpoonright i_0)$
$\langle f, h, j_0\rangle \hookrightarrow T104 \Rightarrow \quad f{\bullet}h \upharpoonright j_0 = f\upharpoonright(h\upharpoonright j_0)$
EQUAL $\Rightarrow$ $\quad \mathsf{Ra\_ABS}\big(f\upharpoonright(h\upharpoonright i_0) -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright j_0)\big) >_{\mathbb{Q}} eps_0$
$\langle i_0, h\rangle \hookrightarrow T64([Stat3, \cap]) \Rightarrow \quad h\upharpoonright i_0 \in \mathbb{N}$
$\langle j_0, h\rangle \hookrightarrow T64([Stat3, \cap]) \Rightarrow \quad h\upharpoonright j_0 \in \mathbb{N}$
EQUAL $\Rightarrow$ $\quad h\upharpoonright(i_0 \cap j_0) \neq h\upharpoonright i_0 \cap h\upharpoonright j_0$
$T179 \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\langle \mathbb{N}, i_0\rangle \hookrightarrow T11([Stat12, \cap]) \Rightarrow \quad \mathcal{O}(i_0)$
$\langle \mathbb{N}, j_0\rangle \hookrightarrow T11([Stat12, \cap]) \Rightarrow \quad \mathcal{O}(j_0)$
Suppose $\Rightarrow$ $\quad i_0 = j_0$
$\langle Stat12, *\rangle$ ELEM $\Rightarrow$ $\quad i_0 \cap j_0 = i_0$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad i_0 \neq j_0$
$\langle i_0, j_0\rangle \hookrightarrow T28([Stat12, \cap]) \Rightarrow \quad i_0 \in j_0 \vee j_0 \in i_0$
Suppose $\Rightarrow$ $\quad Stat13: \; i_0 \in j_0$
$\langle j_0, i_0\rangle \hookrightarrow T12([Stat12, \cap]) \Rightarrow \quad i_0 \cap j_0 = i_0$
$\langle i_0, j_0\rangle \hookrightarrow Stat4(\langle Stat4a, Stat12a, Stat13\rangle) \Rightarrow \quad h\upharpoonright i_0 \in h\upharpoonright j_0$

-- ?? (i0, j0) $\hookrightarrow$Stat4 (Stat4a) $\Rightarrow$ (h [i0]) in (h [j0])

$\langle \mathbb{N}, h\upharpoonright j_0\rangle \hookrightarrow T11([Stat12, \cap]) \Rightarrow \quad \mathcal{O}(h\upharpoonright j_0)$
$\langle h\upharpoonright j_0, h\upharpoonright i_0\rangle \hookrightarrow T12([Stat12, \cap]) \Rightarrow \quad h\upharpoonright i_0 \cap h\upharpoonright j_0 = h\upharpoonright i_0$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad Stat14: \; j_0 \in i_0$
$\langle i_0, j_0\rangle \hookrightarrow T12([Stat12, \cap]) \Rightarrow \quad i_0 \cap j_0 = j_0$
$\langle j_0, i_0\rangle \hookrightarrow Stat4(\langle Stat4a, Stat12a, Stat14\rangle) \Rightarrow \quad h\upharpoonright j_0 \in h\upharpoonright i_0$
$\langle \mathbb{N}, h\upharpoonright i_0\rangle \hookrightarrow T11([Stat12, \cap]) \Rightarrow \quad \mathcal{O}(h\upharpoonright i_0)$
$\langle h\upharpoonright i_0, h\upharpoonright j_0\rangle \hookrightarrow T12([Stat12, \cap]) \Rightarrow \quad h\upharpoonright i_0 \cap h\upharpoonright j_0 = h\upharpoonright j_0$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad \mathbf{range}(h_{|\{i \cap j:\, i\in\mathbb{N}, j\in\mathbb{N} \,|\, \mathsf{Ra\_ABS}(g\upharpoonright i -_{\mathbb{Q}} g\upharpoonright j) >_{\mathbb{Q}} eps_0\}}) \subseteq \{i \cap j : \; i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} eps_0\}$
$\big\langle \{i \cap j : \; i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} eps_0\}, \mathbf{range}(h_{|\{i \cap j:\, i\in\mathbb{N}, j\in\mathbb{N} \,|\, \mathsf{Ra\_ABS}(g\upharpoonright i -_{\mathbb{Q}} g\upharpoonright j) >_{\mathbb{Q}} eps_0\}})\big\rangle \hookrightarrow T162 \Rightarrow$
$\quad \mathsf{Finite}\big(\mathbf{range}(h_{|\{i \cap j:\, i\in\mathbb{N}, j\in\mathbb{N} \,|\, \mathsf{Ra\_ABS}(g\upharpoonright i -_{\mathbb{Q}} g\upharpoonright j) >_{\mathbb{Q}} eps_0\}})\big)$
$\langle h_{|\{i \cap j:\, i\in\mathbb{N}, j\in\mathbb{N} \,|\, \mathsf{Ra\_ABS}(g\upharpoonright i -_{\mathbb{Q}} g\upharpoonright j) >_{\mathbb{Q}} eps_0\}}\rangle \hookrightarrow T164 \Rightarrow$
$\quad \mathsf{Finite}\big(\mathbf{domain}(h_{|\{i \cap j:\, i\in\mathbb{N}, j\in\mathbb{N} \,|\, \mathsf{Ra\_ABS}(g\upharpoonright i -_{\mathbb{Q}} g\upharpoonright j) >_{\mathbb{Q}} eps_0\}})\big)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad \neg\mathsf{Ra\_eqseq}(f, g)$

Use_def(Ra_eqseq) $\Rightarrow$ $\mathit{Stat50}: \neg\langle\forall\varepsilon \in \mathbb{Q} \,|\, \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{x : x \in \mathbf{domain}(f) \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \varepsilon\})\rangle$

$\langle \mathsf{eps}_1\rangle \hookrightarrow \mathit{Stat50} \Rightarrow$ $\mathsf{eps}_1 \in \mathbb{Q} \;\&\; \mathsf{eps}_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \neg\mathsf{Finite}(\{x \in \mathbf{domain}(f) \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\})$

Suppose $\Rightarrow$ $\mathsf{Finite}\Big(\{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x)) >_{\mathbb{Q}} \mathsf{eps}_1\}\Big)$

EQUAL $\Rightarrow$ $\mathit{Stat51}: \{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x)) >_{\mathbb{Q}} \mathsf{eps}_1\} \neq \{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\bullet h\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\}$

$\langle x'\rangle \hookrightarrow \mathit{Stat51} \Rightarrow$ $x' \in \mathbb{N} \;\&\; \neg\Big(\mathsf{Ra\_ABS}(f\upharpoonright x' -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x')) >_{\mathbb{Q}} \mathsf{eps}_1 \leftrightarrow \mathsf{Ra\_ABS}(f\upharpoonright x' -_{\mathbb{Q}} f\bullet h\upharpoonright x') >_{\mathbb{Q}} \mathsf{eps}_1\Big)$

$\langle f, h, x'\rangle \hookrightarrow \mathit{T104} \Rightarrow$ $f\bullet h\upharpoonright x' = f\upharpoonright(h\upharpoonright x')$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{Finite}\Big(\{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x)) >_{\mathbb{Q}} \mathsf{eps}_1\}\Big)$

$\langle \mathsf{eps}_1\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_1\})$

$\langle \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_1\}, \{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x)) >_{\mathbb{Q}} \mathsf{eps}_1\}\rangle \hookrightarrow \mathit{T162} \Rightarrow$ $\mathit{Stat52}:$
$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_1\} \not\supseteq \{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x)) >_{\mathbb{Q}} \mathsf{eps}_1\}$

$\langle i_1\rangle \hookrightarrow \mathit{Stat52} \Rightarrow$ $\mathit{Stat53}:$
$i_1 \in \{x \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} f\upharpoonright(h\upharpoonright x)) >_{\mathbb{Q}} \mathsf{eps}_1\} \;\&\; i_1 \notin \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_1\}$

$\langle i_1, h\upharpoonright i_1\rangle \hookrightarrow \mathit{Stat53}([\mathit{Stat3}, \cap]) \Rightarrow$ $i_1 \in \mathbb{N} \;\&\; i_1 \cap h\upharpoonright i_1 \neq i_1 \vee h\upharpoonright i_1 \notin \mathbb{N}$

$\langle i_1, h\rangle \hookrightarrow \mathit{T64}([\mathit{Stat3}, \cap]) \Rightarrow$ $i_1 \cap h\upharpoonright i_1 \neq i_1$

ELEM $\Rightarrow$ $\mathit{Stat55}: i_1 \notin \{i \in \mathbf{domain}(h) \,|\, i \not\subseteq h\upharpoonright i\}$

$\langle\,\rangle \hookrightarrow \mathit{Stat55}([\mathit{Stat3}, \cap]) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Every Cauchy sequence, when applied to an element of $\mathbb{N}$, returns an element of $\mathbb{Q}$.

Theorem 590 (10059) $F \in \mathsf{RaSeq} \vee F \in \mathsf{RaCauchy} \rightarrow F \in \mathsf{RaSeq} \;\&\; \langle\forall h \in \mathbb{N} \,|\, F\upharpoonright h \in \mathbb{Q}\rangle$. PROOF:

Suppose_not(f) $\Rightarrow$ $f \in \mathsf{RaSeq} \vee f \in \mathsf{RaCauchy} \;\&\; F \notin \mathsf{RaSeq} \vee \neg\langle\forall h \in \mathbb{N} \,|\, f\upharpoonright h \in \mathbb{Q}\rangle$

Suppose $\Rightarrow$ $f \notin \mathsf{RaSeq}$

Use_def(RaCauchy) $\Rightarrow$ $\mathit{Stat1}: f \in \{f \in \mathsf{RaSeq} \,|\, \langle\forall\varepsilon \in \mathbb{Q} \,|\, \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$

$\langle\,\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ false; Discharge $\Rightarrow$ $f \in \mathsf{RaSeq} \;\&\; \mathit{Stat0}: \neg\langle\forall h \in \mathbb{N} \,|\, f\upharpoonright h \in \mathbb{Q}\rangle$

$\langle f\rangle \hookrightarrow \mathit{T413a} \Rightarrow$ $\mathbf{domain}(f) = \mathbb{N} \;\&\; \mathsf{Svm}(f) \;\&\; \mathbf{range}(f) \subseteq \mathbb{Q}$

$\langle f\rangle \hookrightarrow \mathit{T66} \Rightarrow$ $\{f\upharpoonright i : i \in \mathbf{domain}(f)\} \subseteq \mathbb{Q}$

$\langle h\rangle \hookrightarrow \mathit{Stat0} \Rightarrow$ $\mathit{Stat35}: h \in \mathbf{domain}(f) \;\&\; \mathit{Stat36}: f\upharpoonright h \notin \{f\upharpoonright i : i \in \mathbf{domain}(f)\}$

$\langle h\rangle \hookrightarrow \mathit{Stat36}(\langle\mathit{Stat35}\rangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- For every Cauchy sequence f, there is a subscript k past which the distance between any two images $f\upharpoonright i, f\upharpoonright j$ is smaller than any fixed positive rational.

Theorem 591 (10060) $\mathsf{Eps} \in \mathbb{Q} \;\&\; \mathsf{Eps} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; F \in \mathsf{RaCauchy} \rightarrow \langle\exists k \in \mathbb{N} \,|\, k \neq \emptyset \;\&\; \langle\forall i \in \mathbb{N}, j \in \mathbb{N} \,|\, i \notin k \;\&\; j \notin k \longrightarrow \mathsf{Eps} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j)\rangle\rangle$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{f},\varepsilon) \Rightarrow \quad \mathsf{f} \in \mathsf{RaCauchy} \;\&\; \varepsilon \in \mathbb{Q} \;\&\; \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \mathit{Stat0}: \neg$
$\quad \langle \exists \mathsf{k} \in \mathbb{N} \mid \mathsf{k} \neq \emptyset \;\&\; \langle \forall \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{i} \notin \mathsf{k} \;\&\; \mathsf{j} \notin \mathsf{k} \rightarrow \varepsilon >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j})\rangle\rangle$
$\langle \varepsilon \rangle \hookrightarrow \mathit{T10015} \Rightarrow \quad \mathit{Stat0a}: \langle \exists \mathsf{e} \in \mathbb{Q}, \mathsf{e}' \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathsf{e} \;\&\; \mathsf{e} >_{\mathbb{Q}} \mathsf{e}' \;\&\; \mathsf{e}' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \mathsf{e} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \varepsilon >_{\mathbb{Q}} \mathsf{e} +_{\mathbb{Q}} \mathsf{e}'\rangle$
$\langle \mathsf{eps}_1, \mathsf{eps}_2 \rangle \hookrightarrow \mathit{Stat0a} \Rightarrow \quad \mathsf{eps}_1 \in \mathbb{Q} \;\&\; \varepsilon >_{\mathbb{Q}} \mathsf{eps}_1 \;\&\; \mathsf{eps}_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

-- For, assuming by contradiction the negative of the statement, we can consider the immediate successor i_0 of the maximum of the finite set

$$\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \;,$$

where eps_1 is positive and smaller than ε, and then proceed to show that it meets the property

$$\langle \forall \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{i} \notin \mathsf{i}_0 \;\&\; \mathsf{j} \notin \mathsf{i}_0 \rightarrow \varepsilon >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j})\rangle \,.$$

The existence of the said maximum follows from the very definition of Cauchy sequence:

$\mathsf{Use\_def}(\mathsf{RaCauchy}) \Rightarrow \quad \mathit{Stat1}: \mathsf{f} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}$
$\langle\ \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathit{Stat2}: \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle$
$\langle \mathsf{eps}_1 \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\})$
$\mathsf{Loc\_def} \Rightarrow \quad \mathsf{i}_0 = \mathsf{next}(\bigcup\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\})$
$\mathsf{Use\_def}(\mathsf{next}) \Rightarrow \quad \mathit{Stat29}: \mathsf{i}_0 \neq \emptyset$

-- Having defined i_0 in the way just seen, we can in fact easily check that it belongs to $\mathbb{N}$.

$\mathit{T179} \Rightarrow \quad \mathcal{O}(\mathbb{N})$
$\mathsf{Suppose} \Rightarrow \quad \mathsf{i}_0 \notin \mathbb{N}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat3}: \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \not\subseteq \mathbb{N}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathit{Stat31}: \mathsf{c} \in \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \;\&\; \mathsf{c} \notin \mathbb{N}$
$\langle \mathsf{i}_1, \mathsf{j}_1 \rangle \hookrightarrow \mathit{Stat31} \Rightarrow \quad \mathsf{i}_1, \mathsf{j}_1 \in \mathbb{N} \;\&\; \mathsf{i}_1 \cap \mathsf{j}_1 \notin \mathbb{N}$
$\langle \mathbb{N}, \mathsf{i}_1 \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(\mathsf{i}_1)$
$\langle \mathbb{N}, \mathsf{j}_1 \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(\mathsf{j}_1)$
$\langle \mathsf{i}_1, \mathsf{j}_1 \rangle \hookrightarrow \mathit{T26} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \subseteq \mathbb{N}$
$\langle \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \rangle \hookrightarrow \mathit{T266} \Rightarrow$
$\quad \bigcup\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \in \mathbb{N}$
$\langle \bigcup\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \rangle \hookrightarrow \mathit{T265} \Rightarrow$
$\quad \bigcup\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} + 1 = \mathsf{i}_0$
$\mathit{T182} \Rightarrow \quad 1 \in \mathbb{N}$
$\mathsf{ALGEBRA} \Rightarrow \quad \bigcup\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} + 1 \in \mathbb{N}$
$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat30}: \mathsf{i}_0 \in \mathbb{N}$

-- Recall that f has domain $\mathbb{N}$ and is single-valued; moreover, its range $\{f{\restriction}i : i \in \textbf{domain}(f)\}$ is included in $\mathbb{Q}$.

$\langle f \rangle \hookrightarrow T10059 \Rightarrow \quad Stat40: \langle \forall h \in \mathbb{N} \mid f{\restriction}h \in \mathbb{Q} \rangle$

-- In order to see that i_0 has the desired property, conflicting with the initial hypothesis, we proceed as follows. We begin with assuming that i_2, j_2 make a counter-example to the desired property.

Suppose $\Rightarrow \quad Stat5: \neg\langle \forall i \in \mathbb{N}, j \in \mathbb{N} \mid i \notin i_0 \;\&\; j \notin i_0 \rightarrow \neg \mathrm{Ra\_ABS}(f{\restriction}i -_{\mathbb{Q}} f{\restriction}j) \geqslant_{\mathbb{Q}} \varepsilon \rangle$
$\langle i_2, j_2 \rangle \hookrightarrow Stat5 \Rightarrow \quad Stat50: i_2, j_2 \in \mathbb{N} \;\&\; i_2 \notin i_0 \;\&\; j_2 \notin i_0 \;\&\; \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) \geqslant_{\mathbb{Q}} \varepsilon$

-- It turns out that these i_2, j_2 are ordinals greater than or equal to i_0.

$\langle \mathbb{N}, i_2 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(i_2)$
$\langle \mathbb{N}, j_2 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(j_2)$
Suppose $\Rightarrow \quad i_2 \cap j_2 \in i_0$
$\langle i_2, j_2 \rangle \hookrightarrow T28 \Rightarrow \quad j_2 \in i_2 \vee i_2 \in j_2$
$\langle i_2, j_2 \rangle \hookrightarrow T12 \Rightarrow \quad i_2 \in j_2$
$\langle j_2, i_2 \rangle \hookrightarrow T12 \Rightarrow \quad$ false; Discharge $\Rightarrow \quad i_2 \cap j_2 \notin i_0$

-- The intersection $i_2 \cap j_2$, which obviously equals the smaller of i_2, j_2 and therefore is an ordinal, must belong to the set

$$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(f{\restriction}i -_{\mathbb{Q}} f{\restriction}j) >_{\mathbb{Q}} \mathrm{eps}_1\},$$

and hence is included in its unionset, and in the successor of its unionset, which is i_0.

Suppose $\Rightarrow \quad Stat51: i_2 \cap j_2 \notin \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(f{\restriction}i -_{\mathbb{Q}} f{\restriction}j) >_{\mathbb{Q}} \mathrm{eps}_1\}$
$\langle i_2, j_2 \rangle \hookrightarrow Stat51([Stat50, Stat51]) \Rightarrow \quad \neg \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) >_{\mathbb{Q}} \mathrm{eps}_1$
$\langle \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2), \mathrm{eps}_1 \rangle \hookrightarrow T384 \Rightarrow \quad \neg \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) \geqslant_{\mathbb{Q}} \mathrm{eps}_1 \vee$
$\quad \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) = \mathrm{eps}_1$
$\langle i_2 \rangle \hookrightarrow Stat40 \Rightarrow \quad f{\restriction}i_2 \in \mathbb{Q}$
$\langle j_2 \rangle \hookrightarrow Stat40 \Rightarrow \quad f{\restriction}j_2 \in \mathbb{Q}$
ALGEBRA $\Rightarrow \quad f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2 \in \mathbb{Q}$
$\langle f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2 \rangle \hookrightarrow T10045 \Rightarrow \quad \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) \in \mathbb{Q}$
$\langle \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2), \mathrm{eps}_1 \rangle \hookrightarrow T384 \Rightarrow \quad \mathrm{eps}_1 \geqslant_{\mathbb{Q}} \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2)$
$\langle \varepsilon, \mathrm{eps}_1, \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) \rangle \hookrightarrow T405 \Rightarrow \quad \varepsilon >_{\mathbb{Q}} \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2)$
$\langle \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2), \varepsilon, \mathrm{eps}_1 \rangle \hookrightarrow T406 \Rightarrow$
$\quad \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) >_{\mathbb{Q}} \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2)$
$\langle \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2), \mathrm{Ra\_ABS}(f{\restriction}i_2 -_{\mathbb{Q}} f{\restriction}j_2) \rangle \hookrightarrow T384 \Rightarrow$

$\mathsf{Ra\_ABS}(f\restriction i_2 -_{\mathbb{Q}} f\restriction j_2) <_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\restriction i_2 -_{\mathbb{Q}} f\restriction j_2)$

$T10042 \Rightarrow \quad Stat56: \ \langle \forall x \in \mathbb{Q} \mid \neg x <_{\mathbb{Q}} x \rangle$

$\langle \mathsf{Ra\_ABS}(f\restriction i_2 -_{\mathbb{Q}} f\restriction j_2) \rangle \hookrightarrow Stat56 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad i_2 \cap j_2 \in \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \}$

$\mathsf{Suppose} \Rightarrow \quad i_2 \cap j_2 \neq j_2 \ \& \ i_2 \cap j_2 \neq i_2$

$\langle i_2, j_2 \rangle \hookrightarrow T28 \Rightarrow \quad j_2 \in i_2 \vee i_2 \in j_2$

$\mathsf{Suppose} \Rightarrow \quad j_2 \in i_2$

$\langle i_2, j_2 \rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad i_2 \in j_2$

$\langle j_2, i_2 \rangle \hookrightarrow T12 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad i_2 \cap j_2 = j_2 \vee i_2 \cap j_2 = i_2$

$\mathsf{Suppose} \Rightarrow \quad \neg \mathcal{O}(i_2 \cap j_2)$

$\mathsf{Suppose} \Rightarrow \quad i_2 \cap j_2 = j_2$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad i_2 \cap j_2 = i_2$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathcal{O}(i_2 \cap j_2)$

$\langle \mathbb{N}, i_0 \rangle \hookrightarrow T11 \Rightarrow \quad \mathcal{O}(i_0)$

$\langle \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \} \rangle \hookrightarrow T235 \Rightarrow \quad Stat6:$

$\quad \langle \forall x \in \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \} \mid x \subseteq \bigcup \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \} \rangle$

$\langle i_2 \cap j_2 \rangle \hookrightarrow Stat6 \Rightarrow \quad i_2 \cap j_2 \subseteq \bigcup \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \}$

$\mathsf{Use\_def}(\mathsf{next}) \Rightarrow \quad i_2 \cap j_2 \subseteq \mathsf{next}(\bigcup \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \})$

$\mathsf{EQUAL} \Rightarrow \quad i_2 \cap j_2 \subseteq i_0$

$\langle i_2 \cap j_2, i_0 \rangle \hookrightarrow T31 \Rightarrow \quad i_0 \notin i_2 \cap j_2$

$\mathsf{Use\_def}(\mathsf{next}) \Rightarrow \quad i_2 \cap j_2 \neq \mathsf{next}(\bigcup \{ i \cap j : \ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \mathsf{eps}_1 \})$

$\mathsf{EQUAL} \Rightarrow \quad i_2 \cap j_2 \neq i_0$

$\langle i_0, i_2 \cap j_2 \rangle \hookrightarrow T31 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat7: \ \langle \forall i \in \mathbb{N}, j \in \mathbb{N} \mid i \notin i_0 \ \& \ j \notin i_0 \rightarrow \neg \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) \geqslant_{\mathbb{Q}} \varepsilon \rangle$

$\langle i_0 \rangle \hookrightarrow Stat0([Stat29, Stat30]) \Rightarrow \quad Stat8: \ \neg \langle \forall i \in \mathbb{N}, j \in \mathbb{N} \mid i \notin i_0 \ \& \ j \notin i_0 \rightarrow \varepsilon >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) \rangle$

$\langle i_3, j_3 \rangle \hookrightarrow Stat8 \Rightarrow \quad i_3, j_3 \in \mathbb{N} \ \& \ i_3 \notin i_0 \ \& \ j_3 \notin i_0 \ \& \ \neg \varepsilon >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3)$

$\langle i_3, j_3 \rangle \hookrightarrow Stat7 \Rightarrow \quad \neg \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3) \geqslant_{\mathbb{Q}} \varepsilon$

$\langle i_3 \rangle \hookrightarrow Stat40 \Rightarrow \quad f\restriction i_3 \in \mathbb{Q}$

$\langle j_3 \rangle \hookrightarrow Stat40 \Rightarrow \quad f\restriction j_3 \in \mathbb{Q}$

$\mathsf{ALGEBRA} \Rightarrow \quad f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3 \in \mathbb{Q}$

$\langle f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3 \rangle \hookrightarrow T10045 \Rightarrow \quad \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3) \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3), \varepsilon \rangle \hookrightarrow T384 \Rightarrow \quad \varepsilon \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3)$

$\langle \varepsilon, \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3) \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3) = \varepsilon$

-- Having thus reached a contradiction, we draw the desired conclusion.

$\langle \varepsilon, \mathsf{Ra\_ABS}(f\restriction i_3 -_{\mathbb{Q}} f\restriction j_3) \rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- We next show that the relation $\mathsf{Ra\_eqseq}(\mathsf{F}, \mathsf{G})$ between rational sequences is an equivalence relation. We check first a property which is a hybrid of transitivity and symmetry, as follows:

Theorem 592 (10063) $\{\mathsf{F}, \mathsf{G}, \mathsf{H}\} \subseteq \mathsf{RaSeq} \;\&\; \mathsf{Ra\_eqseq}(\mathsf{F}, \mathsf{G}) \;\&\; \mathsf{Ra\_eqseq}(\mathsf{G}, \mathsf{H}) \rightarrow \mathsf{Ra\_eqseq}(\mathsf{H}, \mathsf{F})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{f}, \mathsf{g}, \mathsf{k}) \Rightarrow \quad \mathsf{f}, \mathsf{g}, \mathsf{k} \in \mathsf{RaSeq} \;\&\; \mathsf{Ra\_eqseq}(\mathsf{f}, \mathsf{g}) \;\&\; \mathsf{Ra\_eqseq}(\mathsf{g}, \mathsf{k}) \;\&\; \neg\mathsf{Ra\_eqseq}(\mathsf{k}, \mathsf{f})$

-- For, assuming $\mathsf{f}, \mathsf{g}, \mathsf{k}$ to be a counterexample to the desired conclusion, we will derive a contradiction from the fact that the set

$$\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{k}) \mid \mathsf{Ra\_ABS}(\mathsf{k}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{f}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\}$$

can be infinite for some positive value of ε, unlike the corresponding sets which have f, g and g, k in place of k, f respectively.

$\mathsf{Use\_def}(\mathsf{Ra\_eqseq}) \Rightarrow$
$\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \longrightarrow \mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}) \mid \mathsf{Ra\_ABS}(\mathsf{f}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{g}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\})\rangle \;\&\; \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \longrightarrow \mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{g}) \mid \mathsf{Ra\_ABS}(\mathsf{g}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{k}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\})\rangle \;\&$
$\neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \longrightarrow \mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{k}) \mid \mathsf{Ra\_ABS}(\mathsf{k}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{f}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\})\rangle$

-- Noting that $\mathsf{f}, \mathsf{g}, \mathsf{k}$ have $\mathbb{N}$ as their common domain, and have all of their images in $\mathbb{Q}$, we can restate our hypothesis more simply.

$\langle \mathsf{f} \rangle \hookrightarrow T413a \Rightarrow \quad \mathbf{domain}(\mathsf{f}) = \mathbb{N} \;\&\; \mathsf{Svm}(\mathsf{f})$
$\langle \mathsf{g} \rangle \hookrightarrow T413a \Rightarrow \quad \mathbf{domain}(\mathsf{g}) = \mathbb{N} \;\&\; \mathsf{Svm}(\mathsf{g})$
$\langle \mathsf{k} \rangle \hookrightarrow T413a \Rightarrow \quad \mathbf{domain}(\mathsf{k}) = \mathbb{N} \;\&\; \mathsf{Svm}(\mathsf{k})$
$\langle \mathsf{f} \rangle \hookrightarrow T10059 \Rightarrow \quad Stat41 : \langle \forall \mathsf{n} \in \mathbb{N} \mid \mathsf{f}{\upharpoonright}\mathsf{n} \in \mathbb{Q}\rangle$
$\langle \mathsf{g} \rangle \hookrightarrow T10059 \Rightarrow \quad Stat42 : \langle \forall \mathsf{n} \in \mathbb{N} \mid \mathsf{g}{\upharpoonright}\mathsf{n} \in \mathbb{Q}\rangle$
$\langle \mathsf{k} \rangle \hookrightarrow T10059 \Rightarrow \quad Stat43 : \langle \forall \mathsf{n} \in \mathbb{N} \mid \mathsf{k}{\upharpoonright}\mathsf{n} \in \mathbb{Q}\rangle$
$\mathsf{EQUAL} \Rightarrow \quad Stat1 :$
$\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \longrightarrow \mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{g}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\})\rangle \;\&\; Stat2 :$
$\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \longrightarrow \mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{g}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{k}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\})\rangle \;\&\; Stat3 : \neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \longrightarrow \mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{k}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{f}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \varepsilon\})\rangle$

-- Let eps_0 be a positive value for ε which makes the set indicated above infinite; moreover, let $\mathsf{eps}_1, \mathsf{eps}_2$ be smaller positive values whose addition is smaller than eps_0 (e. g., one could take $\mathsf{eps}_1 = \mathsf{eps}_2$ to be one half or one third of eps_0). We have assumed the finiteness of the two sets

$$\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{g}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\}\ ,$$

$$\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{g}{\upharpoonright}\mathsf{x} -_{\mathbb{Q}} \mathsf{k}{\upharpoonright}\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\}\ .$$

$\langle \mathsf{eps}_0 \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathsf{eps}_0 \in \mathbb{Q} \,\&\, \mathsf{eps}_0 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \neg\mathsf{Finite}\big(\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0\}\big)$
$\langle \mathsf{eps}_0 \rangle \hookrightarrow \mathit{T10015} \Rightarrow \quad \mathit{Stat4} : \langle \exists \mathsf{e} \in \mathbb{Q}, \mathsf{e}' \in \mathbb{Q} \mid \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{e} \,\&\, \mathsf{e} >_{\mathbb{Q}} \mathsf{e}' \,\&\, \mathsf{e}' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{e} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{e} +_{\mathbb{Q}} \mathsf{e}' \rangle$
$\langle \mathsf{eps}_1, \mathsf{eps}_2 \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad \mathsf{eps}_1, \mathsf{eps}_2 \in \mathbb{Q} \,\&\, \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_1 \,\&\, \mathsf{eps}_1 >_{\mathbb{Q}} \mathsf{eps}_2 \,\&\, \mathsf{eps}_2 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{eps}_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2$
$\langle \mathsf{eps}_1 \rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{Finite}\big(\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\}\big)$
$\langle \mathsf{eps}_2 \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{Finite}\big(\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\}\big)$

-- Due to the inequality

$$\mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}),$$

it turns out that the set

$$\{\mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0\}$$

is included in the union of the two sets

$$\{\mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\},$$

$$\{\mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\},$$

both of which must be finite, in consequence of the assumptions made.

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat5} :$
$\quad \{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0\} \not\subseteq \{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\}$
$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad \mathit{Stat6} :$
$\quad \mathsf{x} \in \{\mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0\} \,\&\, \mathit{Stat7} :$
$\qquad \mathsf{x} \notin \{\mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\} \,\&\, \mathit{Stat8} : \mathsf{x} \notin \{\mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad \mathsf{x} \in \mathbb{N} \,\&\, \mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0$
$\langle\,\rangle \hookrightarrow \mathit{Stat7} \Rightarrow \quad \neg\mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1$
$\langle\,\rangle \hookrightarrow \mathit{Stat8} \Rightarrow \quad \neg\mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2$
$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat41} \Rightarrow \quad \mathsf{f}\restriction\mathsf{x} \in \mathbb{Q}$
$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat42} \Rightarrow \quad \mathsf{g}\restriction\mathsf{x} \in \mathbb{Q}$
$\langle \mathsf{x} \rangle \hookrightarrow \mathit{Stat43} \Rightarrow \quad \mathsf{k}\restriction\mathsf{x} \in \mathbb{Q}$
$\mathsf{ALGEBRA} \Rightarrow$
$\quad \mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}, \mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}, \mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}, \mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x} \in \mathbb{Q} \,\&$
$\qquad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \in \mathbb{Q}$
$\langle \mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x} \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) \in \mathbb{Q}$
$\langle \mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x} \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) \in \mathbb{Q}$
$\langle \mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x} \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{k}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{x}) \in \mathbb{Q}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{g}\restriction\mathsf{x}) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{g}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{k}\restriction\mathsf{x}) \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x), \mathsf{eps}_1\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{eps}_1 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x)$

$\langle \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x), \mathsf{eps}_2\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x)$

$T10050 \Rightarrow \quad Stat50:$

$\langle \forall x, y, z \mid x, y, z \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}\big(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(z)\big) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)\big) +_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(z)\big)\rangle \;\&\; Stat51:$

$\langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow \mathsf{Rev}_{\mathbb{Q}}\big(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)\big) = y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(x)\rangle \;\&\; Stat52: \;\langle \forall x \mid x \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}\big(\mathsf{Rev}_{\mathbb{Q}}(x)\big) = \mathsf{Ra\_ABS}(x)\rangle$

$\langle f\upharpoonright x, g\upharpoonright x, k\upharpoonright x\rangle \hookrightarrow Stat50 \Rightarrow$

$\mathsf{Ra\_ABS}\big(f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x)\big) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(g\upharpoonright x)\big) +_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(g\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x)\big)$

$\langle f\upharpoonright x, k\upharpoonright x\rangle \hookrightarrow Stat51 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}\big(f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x)\big) = k\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f\upharpoonright x)$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x) \in \mathbb{Q}$

$\langle f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x)\rangle \hookrightarrow Stat52 \Rightarrow \quad \mathsf{Ra\_ABS}\Big(\mathsf{Rev}_{\mathbb{Q}}\big(f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x)\big)\Big) =$

$\mathsf{Ra\_ABS}\big(f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x)\big)$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow$

$f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x) = f\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x \;\&\; f\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(g\upharpoonright x) = f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x \;\&$

$g\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(k\upharpoonright x) = g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x \;\&\; k\upharpoonright x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f\upharpoonright x) = k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x)$

$\langle \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x), \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x)\rangle \hookrightarrow T384 \Rightarrow$

$\mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x)$

$\langle \mathsf{eps}_1, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x), \mathsf{eps}_2, \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x)\rangle \hookrightarrow T397 \Rightarrow$

$\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x)$

$\langle \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x), \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x)\rangle \hookrightarrow T404 \Rightarrow$

$\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x)$

$\langle \mathsf{eps}_0, \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x)\rangle \hookrightarrow T405 \Rightarrow$

$\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x)$

$\langle \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x), \mathsf{eps}_0\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x) \geqslant_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_0, \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x), \mathsf{eps}_0\rangle \hookrightarrow T405 \Rightarrow \quad \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_0, \mathsf{eps}_0\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$

$\{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\upharpoonright x -_{\mathbb{Q}} f\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_0\} \subseteq \{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_2\}$

$TELEM \Rightarrow$

$\mathsf{Svm}(\{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\}) \;\&\; \mathbf{range}(\{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\}) = \{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\} \;\&\; \mathbf{domain}(\{x : x \in \mathbb{N}$

$\mathbf{domain}(\{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_2\}) = \{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_2\}$

$\langle \{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\}\rangle \hookrightarrow T165 \Rightarrow$

$\mathsf{Finite}(\{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\upharpoonright x -_{\mathbb{Q}} g\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_1\})$

$\langle \{x : x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_2\}\rangle \hookrightarrow T165 \Rightarrow$

$\mathsf{Finite}(\{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\upharpoonright x -_{\mathbb{Q}} k\upharpoonright x) >_{\mathbb{Q}} \mathsf{eps}_2\})$

-- Accordingly, since the sets

$$\{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}\ ,$$

$$\{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}$$

have the same cardinality, they must both be finite, which contradicts our assumption that the former of them is infinite. This contradiction leads to the desired conclusion.

$\langle\, \{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction x -_{\mathbb{Q}} g\restriction x) >_{\mathbb{Q}} \mathsf{eps}_1\}, \{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\restriction x -_{\mathbb{Q}} k\restriction x) >_{\mathbb{Q}} \mathsf{eps}_2\}\,\rangle \hookrightarrow T205 \Rightarrow$
$\quad \mathsf{Finite}(\{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction x -_{\mathbb{Q}} g\restriction x) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\restriction x -_{\mathbb{Q}} k\restriction x) >_{\mathbb{Q}} \mathsf{eps}_2\})$

$\langle\, \{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction x -_{\mathbb{Q}} g\restriction x) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(g\restriction x -_{\mathbb{Q}} k\restriction x) >_{\mathbb{Q}} \mathsf{eps}_2\}, \{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}\,\rangle \hookrightarrow T162 \Rightarrow$
$\quad \mathsf{Finite}(\{x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\})$

$TELEM \Rightarrow$
$\quad \mathsf{Svm}(\{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\})\ \&\ \mathbf{domain}(\{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}) = \{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}\ \&$
$\quad\quad \mathbf{range}(\{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}) = \{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}$

$\langle\, \{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\}\,\rangle \hookrightarrow T165 \Rightarrow$
$\quad \mathsf{Finite}(\{x:\ x \in \mathbb{N} \mid \mathsf{Ra\_ABS}(k\restriction x -_{\mathbb{Q}} f\restriction x) >_{\mathbb{Q}} \mathsf{eps}_0\})$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- We are now ready to state the properties of the relation $\mathsf{Ra\_eqseq}(F, G)$ between rational Cauchy sequences in such a way that we can apply the THEORY equivalence_classes

Theorem 593 (10064) $\langle \forall f \in \mathsf{RaCauchy}, g \in \mathsf{RaCauchy} \mid (\mathsf{Ra\_eqseq}(f,g) \leftrightarrow \mathsf{Ra\_eqseq}(g,f))\ \&\ \mathsf{Ra\_eqseq}(f,f)\rangle$. PROOF:

Suppose_not$(f,g) \Rightarrow$ $f, g \in \mathsf{RaCauchy}\ \&\ (\mathsf{Ra\_eqseq}(f,g) \leftrightarrow \neg\mathsf{Ra\_eqseq}(g,f)) \vee \neg\mathsf{Ra\_eqseq}(f,f)$
Suppose $\Rightarrow$ $f \notin \mathsf{RaSeq} \vee g \notin \mathsf{RaSeq} \vee \neg\mathsf{Ra\_eqseq}(f,f) \vee \neg\mathsf{Ra\_eqseq}(g,g)$
Use_def(RaCauchy) $\Rightarrow$ *Stat1* :
$\quad f \in \{f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \to \mathsf{Finite}(\{i \cap j:\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \varepsilon\})\rangle\}\ \&\ Stat2 :$
$\quad\quad g \in \{f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \to \mathsf{Finite}(\{i \cap j:\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f\restriction i -_{\mathbb{Q}} f\restriction j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$
$\langle\,\rangle \hookrightarrow Stat2 \Rightarrow$ $g \in \mathsf{RaSeq}$
$\langle\,\rangle \hookrightarrow Stat1 \Rightarrow$ $f \in \mathsf{RaSeq}$
$\langle f\rangle \hookrightarrow T413a \Rightarrow$ $\neg\mathsf{Ra\_eqseq}(g,g)$
$\langle g\rangle \hookrightarrow T413a \Rightarrow$ false; Discharge $\Rightarrow$ $f, g \in \mathsf{RaSeq}\ \&\ \mathsf{Ra\_eqseq}(f,f)\ \&\ \mathsf{Ra\_eqseq}(g,g)\ \&\ (\mathsf{Ra\_eqseq}(f,g) \leftrightarrow \neg\mathsf{Ra\_eqseq}(g,f))$
$\langle f,f,g\rangle \hookrightarrow T10063 \Rightarrow$ $\mathsf{Ra\_eqseq}(g,f)$
$\langle g,g,f\rangle \hookrightarrow T10063 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 594 (10065) $\langle \forall f \in \mathsf{RaCauchy}, g \in \mathsf{RaCauchy}, h \in \mathsf{RaCauchy} \mid \mathsf{Ra\_eqseq}(f,g)\ \&\ \mathsf{Ra\_eqseq}(g,h) \to \mathsf{Ra\_eqseq}(f,h)\rangle$. PROOF:

$\mathsf{Suppose\_not}(f,g,h) \Rightarrow \quad f,g,h \in \mathsf{RaCauchy} \,\&\, \mathsf{Ra\_eqseq}(f,g) \,\&\, \mathsf{Ra\_eqseq}(g,h) \,\&\, \neg\mathsf{Ra\_eqseq}(f,h)$

$\mathsf{Suppose} \Rightarrow \quad f \notin \mathsf{RaSeq} \vee g \notin \mathsf{RaSeq} \vee h \notin \mathsf{RaSeq}$

$\mathsf{Use\_def}(\mathsf{RaCauchy}) \Rightarrow \quad \mathit{Stat1}:$

$f \in \{f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\restriction}i -_{\mathbb{Q}} f{\restriction}j) >_{\mathbb{Q}} \varepsilon\})\rangle\} \,\&\, \mathit{Stat2}:$

$g \in \{f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(f{\restriction}i -_{\mathbb{Q}} f{\restriction}j) >_{\mathbb{Q}} \varepsilon\})\rangle\} \,\&\, \mathit{Stat3}:\; h \in \{f \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j$

$\langle\,\rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad h \in \mathsf{RaSeq}$

$\langle\,\rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad g \in \mathsf{RaSeq}$

$\langle\,\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad f,g,h \in \mathsf{RaSeq}$

$\langle f,g,h\rangle \hookrightarrow \mathit{T10063} \Rightarrow \quad \mathsf{Ra\_eqseq}(h,f)$

$\langle h\rangle \hookrightarrow \mathit{T413a} \Rightarrow \quad \mathsf{Ra\_eqseq}(h,h)$

$\langle h,h,f\rangle \hookrightarrow \mathit{T10063} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Now that we know that Ra_eqseq is an equivalence relationship, we can apply the equivalence_classes theory to it, to derive

APPLY $\langle \mathsf{Eqc}_{\Theta} : \mathbb{R}, \mathsf{f}_{\Theta} : \mathsf{Cauchy\_to\_Re}\rangle$ $\mathsf{equivalence\_classes}(\mathsf{P}(x,y) \mapsto \mathsf{Ra\_eqseq}(x,y), s \mapsto \mathsf{RaCauchy}) \Rightarrow$

Theorem 595 (10066) $\langle \forall x,y \mid x,y \in \mathsf{RaCauchy} \rightarrow (\mathsf{Ra\_eqseq}(x,y) \leftrightarrow \mathsf{Cauchy\_to\_Re}(x) = \mathsf{Cauchy\_to\_Re}(y))\rangle \,\&\, \langle \forall x \mid x \in \mathbb{R} \rightarrow \mathbf{arb}(x) \in \mathsf{RaCauchy} \,\&\, \mathsf{Cauchy\_to\_Re}(\mathbf{arb}(x)) = x\rangle$
$\langle \forall x \mid x \in \mathsf{RaCauchy} \rightarrow \mathsf{Cauchy\_to\_Re}(x) \in \mathbb{R}\rangle \,\&\, \langle \forall x \mid x \in \mathsf{RaCauchy} \rightarrow \mathsf{Ra\_eqseq}(x, \mathbf{arb}(\mathsf{Cauchy\_to\_Re})(x))\rangle$.

-- In sight of showing that the set of rational Cauchy sequences is closed under the pointwise algebraic operations introduced above, we prove that when such operations are applied to rational sequences, the results are rational sequences.

Theorem 596 (10062) $\{F,G\} \subseteq \mathsf{RaSeq} \rightarrow$
$F +_{\mathbb{QS}} G, \mathsf{Ras\_ABS}(F), \mathsf{Ras\_Rev}(F), F *_{\mathbb{QS}} G \in \mathsf{RaSeq} \,\&\, F +_{\mathbb{QS}} G = \{[u, F{\restriction}u +_{\mathbb{Q}} G{\restriction}u] : u \in \mathbb{N}\} \,\&\, \mathsf{Ras\_ABS}(F) = \{[u, \mathsf{Ra\_ABS}(F{\restriction}u)] : u \in \mathbb{N}\} \,\&\, \mathsf{Ras\_Rev}(F) = \{[u, \mathsf{Rev}_{\mathbb{Q}}(F{\restriction}u)] : u$

$\mathsf{Suppose\_not}(fq,f') \Rightarrow$

$\{fq,f'\} \subseteq \mathsf{RaSeq} \,\&$

$fq +_{\mathbb{QS}} f' \notin \mathsf{RaSeq} \vee \mathsf{Ras\_ABS}(f') \notin \mathsf{RaSeq} \vee \mathsf{Ras\_Rev}(f') \notin \mathsf{RaSeq} \vee fq *_{\mathbb{QS}} f' \notin \mathsf{RaSeq} \vee fq +_{\mathbb{QS}} f' \neq \{[u, fq{\restriction}u +_{\mathbb{Q}} f'{\restriction}u] : u \in \mathbb{N}\} \vee \mathsf{Ras\_ABS}(f') \neq \{[u, \mathsf{Ra\_ABS}(f'{\restriction}u)] : u \in$

-- Reasoning by contradiction, assume that fq, f′ form a counterexample to the desired statement.

$\langle fq\rangle \hookrightarrow \mathit{T413a} \Rightarrow \quad \mathit{Stat4}:\; \mathbf{domain}(fq) = \mathbb{N} \,\&\, \mathsf{Svm}(fq) \,\&\, \mathbf{range}(fq) \subseteq \mathbb{Q}$

$\langle f'\rangle \hookrightarrow \mathit{T413a} \Rightarrow \quad \mathit{Stat5}:\; \mathbf{domain}(f') = \mathbb{N} \,\&\, \mathsf{Svm}(f') \,\&\, \mathbf{range}(f') \subseteq \mathbb{Q}$

$\mathsf{Use\_def}(+_{\mathbb{QS}}) \Rightarrow \quad fq +_{\mathbb{QS}} f' = \{[p^{[1]}, p^{[2]} +_{\mathbb{Q}} f'{\restriction}p^{[1]}] : p \in fq\}$

$\mathsf{Use\_def}(\mathsf{Ras\_ABS}) \Rightarrow \quad \mathsf{Ras\_ABS}(f') = \{[p^{[1]}, \mathsf{Ra\_ABS}(p^{[2]})] : p \in f'\}$

$\mathsf{Use\_def}(\mathsf{Ras\_Rev}) \Rightarrow \quad \mathsf{Ras\_Rev}(f') = \{[p^{[1]}, \mathsf{Rev}_{\mathbb{Q}}(p^{[2]})] : p \in f'\}$

Use_def$(*_{QS})\Rightarrow$ $fq *_{QS} f' = \{[p^{[1]}, p^{[2]} *_{Q} f' \restriction p^{[1]}] : p \in fq\}$

-- After unfolding the definitions which are directly involved, we recall that $\mathbb{Q}$ is closed under addition, multiplication, sign inversion, and absolute value operation. This allows us to invoke the THEORY 'pointwise' ('pointwiseU' in the monadic case) for each one of these four operations, thereby leading to the desired contradiction.

Suppose $\Rightarrow$ *Stat6* : $\neg\langle\forall x \in \mathbb{Q}, y \in \mathbb{Q} \mid x +_{Q} y \in \mathbb{Q}\rangle$

$\langle x_1, y_1\rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $x_1, y_1 \in \mathbb{Q} \;\&\; x_1 +_{Q} y_1 \notin \mathbb{Q}$

$\langle x_1, y_1\rangle \hookrightarrow$ *T365* $\Rightarrow$ false; Discharge $\Rightarrow$ $\langle\forall x \in \mathbb{Q}, y \in \mathbb{Q} \mid x +_{Q} y \in \mathbb{Q}\rangle$

Suppose $\Rightarrow$ *Stat7* : $\neg\langle\forall x \in \mathbb{Q}, y \in \mathbb{Q} \mid x *_{Q} y \in \mathbb{Q}\rangle$

$\langle x_2, y_2\rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $x_2, y_2 \in \mathbb{Q} \;\&\; x_2 *_{Q} y_2 \notin \mathbb{Q}$

$\langle x_2, y_2\rangle \hookrightarrow$ *T368* $\Rightarrow$ false; Discharge $\Rightarrow$ $\langle\forall x \in \mathbb{Q}, y \in \mathbb{Q} \mid x *_{Q} y \in \mathbb{Q}\rangle$

Suppose $\Rightarrow$ *Stat8* : $\neg\langle\forall x \in \mathbb{Q} \mid \mathrm{Ra\_ABS}(x) \in \mathbb{Q}\rangle$

$\langle x_3\rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $x_3 \in \mathbb{Q} \;\&\; \mathrm{Ra\_ABS}(x_3) \notin \mathbb{Q}$

$\langle x_3\rangle \hookrightarrow$ *T10045* $\Rightarrow$ false; Discharge $\Rightarrow$ $\langle\forall x \in \mathbb{Q} \mid \mathrm{Ra\_ABS}(x) \in \mathbb{Q}\rangle$

Suppose $\Rightarrow$ *Stat9* : $\neg\langle\forall x \in \mathbb{Q} \mid \mathrm{Rev}_{Q}(x) \in \mathbb{Q}\rangle$

$\langle x_4\rangle \hookrightarrow$ *Stat9* $\Rightarrow$ $x_4 \in \mathbb{Q} \;\&\; \mathrm{Rev}_{Q}(x_4) \notin \mathbb{Q}$

$\langle x_4\rangle \hookrightarrow$ *T372* $\Rightarrow$ false; Discharge $\Rightarrow$ $\langle\forall x \in \mathbb{Q} \mid \mathrm{Rev}_{Q}(x) \in \mathbb{Q}\rangle$

APPLY $\langle\,\rangle$ pointwise$(f \mapsto fq, f' \mapsto f', h \mapsto fq +_{QS} f', d \mapsto \mathbb{N}, r \mapsto \mathbb{Q}, bo'(x,y) \mapsto x +_{Q} y) \Rightarrow$

$fq +_{QS} f' = \{[u, fq\restriction u +_{Q} f'\restriction u] : u \in \mathbb{N}\} \;\&\; \mathrm{Svm}(fq +_{QS} f') \;\&\; \mathbf{domain}(fq +_{QS} f') = \mathbb{N} \;\&\; \mathbf{range}(fq +_{QS} f') \subseteq \mathbb{Q}$

APPLY $\langle\,\rangle$ pointwise$(f \mapsto fq, f' \mapsto f', h \mapsto fq *_{QS} f', d \mapsto \mathbb{N}, r \mapsto \mathbb{Q}, bo'(x,y) \mapsto x *_{Q} y) \Rightarrow$

$fq *_{QS} f' = \{[u, fq\restriction u *_{Q} f'\restriction u] : u \in \mathbb{N}\} \;\&\; \mathrm{Svm}(fq *_{QS} f') \;\&\; \mathbf{domain}(fq *_{QS} f') = \mathbb{N} \;\&\; \mathbf{range}(fq *_{QS} f') \subseteq \mathbb{Q}$

APPLY $\langle\,\rangle$ pointwiseU$(f \mapsto f', h \mapsto \mathrm{Ras_ABS}(f'), d \mapsto \mathbb{N}, r \mapsto \mathbb{Q}, uo'(x) \mapsto \mathrm{Ra_ABS}(x)) \Rightarrow$

$\mathrm{Ras\_ABS}(f') = \{[u, \mathrm{Ra\_ABS}(f'\restriction u)] : u \in \mathbb{N}\} \;\&\; \mathrm{Svm}(\mathrm{Ras\_ABS}(f')) \;\&\; \mathbf{domain}(\mathrm{Ras\_ABS})(f') = \mathbb{N} \;\&\; \mathbf{range}(\mathrm{Ras\_ABS})(f') \subseteq \mathbb{Q}$

APPLY $\langle\,\rangle$ pointwiseU$(f \mapsto f', h \mapsto \mathrm{Ras_Rev}(f'), d \mapsto \mathbb{N}, r \mapsto \mathbb{Q}, uo'(x) \mapsto \mathrm{Rev}_{Q}(x)) \Rightarrow$

$\mathrm{Ras\_Rev}(f') = \{[u, \mathrm{Rev}_{Q}(f'\restriction u)] : u \in \mathbb{N}\} \;\&\; \mathrm{Svm}(\mathrm{Ras\_Rev}(f')) \;\&\; \mathbf{domain}(\mathrm{Ras\_Rev})(f') = \mathbb{N} \;\&\; \mathbf{range}(\mathrm{Ras\_Rev})(f') \subseteq \mathbb{Q}$

Use_def(RaSeq) $\Rightarrow$

$fq +_{QS} f' \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\} \vee$

$fq *_{QS} f' \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\} \vee \mathrm{Ras\_ABS}(f') \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\} \vee \mathrm{Ras\_Rev}(f') \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\}$

Suppose $\Rightarrow$ *Stat11* : $\mathrm{Ras\_ABS}(f') \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\}$

Use_def(Svm) $\Rightarrow$ $\mathrm{Is\_map}(\mathrm{Ras\_ABS}(f'))$

$\langle \mathrm{Ras\_ABS}(f'), \mathbb{N}, \mathbb{Q}\rangle \hookrightarrow$ *T116* $\Rightarrow$ $\mathrm{Ras\_ABS}(f') \subseteq \mathbb{N} \times \mathbb{Q}$

$\langle\,\rangle \hookrightarrow$ *Stat11* $\Rightarrow$ false; Discharge $\Rightarrow$ $fq +_{QS} f' \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\} \vee fq *_{QS} f' \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\} \vee \mathrm{Ras\_Rev}(f') \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\}$

Suppose $\Rightarrow$ *Stat12* : $fq *_{QS} f' \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\}$

Use_def(Svm) $\Rightarrow$ $\mathrm{Is\_map}(fq *_{QS} f')$

$\langle fq *_{QS} f', \mathbb{N}, \mathbb{Q}\rangle \hookrightarrow$ *T116* $\Rightarrow$ $fq *_{QS} f' \subseteq \mathbb{N} \times \mathbb{Q}$

$\langle\,\rangle \hookrightarrow$ *Stat12* $\Rightarrow$ false; Discharge $\Rightarrow$ $fq +_{QS} f' \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\} \vee \mathrm{Ras\_Rev}(f') \notin \{f \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(f) = \mathbb{N} \;\&\; \mathrm{Svm}(f)\}$

Suppose $\Rightarrow$ $\quad Stat13:\ \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' \notin \{\mathsf{f} \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(\mathsf{f}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f})\}$

Use_def(Svm) $\Rightarrow$ $\quad \mathsf{Is\_map}(\mathsf{fq} +_{\mathsf{QS}} \mathsf{f}')$

$\langle \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}', \mathbb{N}, \mathbb{Q} \rangle \hookrightarrow T116 \Rightarrow$ $\quad \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' \subseteq \mathbb{N} \times \mathbb{Q}$

$\langle\ \rangle \hookrightarrow Stat13 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad Stat14:\ \mathsf{Ras\_Rev}(\mathsf{f}') \notin \{\mathsf{f} \subseteq \mathbb{N} \times \mathbb{Q} \mid \mathbf{domain}(\mathsf{f}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f})\}$

Use_def(Svm) $\Rightarrow$ $\quad \mathsf{Is\_map}\big(\mathsf{Ras\_Rev}(\mathsf{f}')\big)$

$\langle \mathsf{Ras\_Rev}(\mathsf{f}'), \mathbb{N}, \mathbb{Q} \rangle \hookrightarrow T116 \Rightarrow$ $\quad \mathsf{Ras\_Rev}(\mathsf{f}') \subseteq \mathbb{N} \times \mathbb{Q}$

$\langle\ \rangle \hookrightarrow Stat14 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

-- We next prove that the set of rational Cauchy sequences is closed under the pointwise algebraic operations introduced above.

Theorem 597 (413) $\quad \{\mathsf{F}, \mathsf{G}\} \subseteq \mathsf{RaCauchy} \to \mathsf{F} +_{\mathsf{QS}} \mathsf{G}, \mathsf{Ras\_ABS}(\mathsf{F}), \mathsf{Ras\_Rev}(\mathsf{F}) \in \mathsf{RaCauchy}$. PROOF:

Suppose_not(fq, f′) $\Rightarrow$ $\quad \{\mathsf{fq}, \mathsf{f}'\} \subseteq \mathsf{RaCauchy} \,\&\, \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' \notin \mathsf{RaCauchy} \vee \mathsf{Ras\_ABS}(\mathsf{fq}) \notin \mathsf{RaCauchy} \vee \mathsf{Ras\_Rev}(\mathsf{fq}) \notin \mathsf{RaCauchy}$

-- Reasoning by contradiction, assume that fq, f' form a counterexample to the desired statement.

Use_def(RaCauchy) $\Rightarrow$ $\quad Stat0:$

$\mathsf{fq} \in \big\{\mathsf{f} \in \mathsf{RaSeq} \mid \big\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathsf{Q}} \mathbf{0}_{\mathsf{Q}} \to \mathsf{Finite}\big(\,\{\mathsf{i} \cap \mathsf{j} :\ \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathsf{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathsf{Q}} \varepsilon\}\,\big)\big\rangle\big\} \,\&\, Stat1:$

$\mathsf{f}' \in \big\{\mathsf{f} \in \mathsf{RaSeq} \mid \big\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathsf{Q}} \mathbf{0}_{\mathsf{Q}} \to \mathsf{Finite}\big(\,\{\mathsf{i} \cap \mathsf{j} :\ \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathsf{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathsf{Q}} \varepsilon\}\,\big)\big\rangle\big\}$

$\langle\ \rangle \hookrightarrow Stat0 \Rightarrow$ $\quad \mathsf{fq} \in \mathsf{RaSeq} \,\&\, Stat2:\ \big\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathsf{Q}} \mathbf{0}_{\mathsf{Q}} \to \mathsf{Finite}\big(\,\{\mathsf{i} \cap \mathsf{j} :\ \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction\mathsf{i} -_{\mathsf{Q}} \mathsf{fq}\restriction\mathsf{j}) >_{\mathsf{Q}} \varepsilon\}\,\big)\big\rangle$

$\langle\ \rangle \hookrightarrow Stat1 \Rightarrow$ $\quad \mathsf{f}' \in \mathsf{RaSeq} \,\&\, Stat3:\ \big\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathsf{Q}} \mathbf{0}_{\mathsf{Q}} \to \mathsf{Finite}\big(\,\{\mathsf{i} \cap \mathsf{j} :\ \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}'\restriction\mathsf{i} -_{\mathsf{Q}} \mathsf{f}'\restriction\mathsf{j}) >_{\mathsf{Q}} \varepsilon\}\,\big)\big\rangle$

$\langle \mathsf{fq}, \mathsf{f}' \rangle \hookrightarrow T10062 \Rightarrow$ $\quad Stat14:$

$\mathsf{fq} +_{\mathsf{QS}} \mathsf{f}', \mathsf{Ras\_ABS}(\mathsf{fq}), \mathsf{Ras\_Rev}(\mathsf{fq}) \in \mathsf{RaSeq} \,\&\, Stat18:$

$\mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' = \big\{[\mathsf{u}, \mathsf{fq}\restriction\mathsf{u} +_{\mathsf{Q}} \mathsf{f}'\restriction\mathsf{u}] :\ \mathsf{u} \in \mathbb{N}\big\} \,\&\, Stat19:\ \mathsf{Ras\_ABS}(\mathsf{fq}) = \{[\mathsf{u}, \mathsf{Ra\_ABS}(\mathsf{fq}\restriction\mathsf{u})] :\ \mathsf{u} \in \mathbb{N}\} \,\&\, Stat10:\ \mathsf{Ras\_Rev}(\mathsf{fq}) = \big\{[\mathsf{u}, \mathsf{Rev}_{\mathsf{Q}}(\mathsf{fq}\restriction\mathsf{u})] :\ \mathsf{u} \in \mathbb{N}\big\}$

$\langle \mathsf{fq} \rangle \hookrightarrow T413a \Rightarrow$ $\quad Stat4:\ \mathbf{domain}(\mathsf{fq}) = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{fq}) \,\&\, \mathbf{range}(\mathsf{fq}) \subseteq \mathbb{Q}$

$\langle \mathsf{f}' \rangle \hookrightarrow T413a \Rightarrow$ $\quad Stat5:\ \mathbf{domain}(\mathsf{f}') = \mathbb{N} \,\&\, \mathsf{Svm}(\mathsf{f}') \,\&\, \mathbf{range}(\mathsf{f}') \subseteq \mathbb{Q}$

$\langle \mathsf{fq} \rangle \hookrightarrow T66 \Rightarrow$ $\quad \mathbf{range}(\mathsf{fq}) = \{\mathsf{fq}\restriction\mathsf{j} :\ \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$

-- After unfolding all definitions which are directly involved, and recalling that, by the preceding theorem, pointwise addition, sign inversion and absolutization of rational sequences produce rational sequences, we argue as follows:

Suppose $\Rightarrow$ $\quad \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' \notin \mathsf{RaCauchy}$

-- Assuming that $\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}'$ is not a Cauchy sequence (unlike fq and f'), there would exist a positive real eps_0 for which the set

$$\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}\big((fq +_{\mathbb{QS}} f')\upharpoonright i -_{\mathbb{Q}} (fq +_{\mathbb{QS}} f')\upharpoonright j\big) >_{\mathbb{Q}} eps_0\}$$

is infinite, unlike the analogous sets which have fq and f', respectively, in place of $\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}'$, and positive reals eps_1 and eps_2 smaller than one half of eps_0 in place of eps_0.

Use_def (RaCauchy) $\Rightarrow$ $\mathit{Stat15}:\ fq +_{\mathbb{QS}} f' \notin \{f \in \mathrm{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathrm{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(f\upharpoonright i -_{\mathbb{Q}} f\upharpoonright j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat15} \Rightarrow$ $\mathit{Stat16}:\ \neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathrm{Finite}\Big(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}\big((fq +_{\mathbb{QS}} f')\upharpoonright i -_{\mathbb{Q}} (fq +_{\mathbb{QS}} f')\upharpoonright j\big) >_{\mathbb{Q}} \varepsilon\}\Big)\rangle$

$\langle eps_0\rangle \hookrightarrow \mathit{Stat16}(\langle \mathit{Stat16}\rangle) \Rightarrow$ $\mathit{Stat17}:\ eps_0 \in \mathbb{Q}\ \&\ eps_0 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ \neg\mathrm{Finite}\Big(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}\big((fq +_{\mathbb{QS}} f')\upharpoonright i -_{\mathbb{Q}} (fq +_{\mathbb{QS}} f')\upharpoonright j\big) >_{\mathbb{Q}} eps_0\}\Big)$

$\langle eps_0\rangle \hookrightarrow T10015 \Rightarrow$ $\mathit{Stat20}:\ \langle \exists e \in \mathbb{Q}, e' \in \mathbb{Q} \mid eps_0 >_{\mathbb{Q}} e\ \&\ e >_{\mathbb{Q}} e'\ \&\ e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ e >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ eps_0 >_{\mathbb{Q}} e +_{\mathbb{Q}} e'\rangle$

$\langle eps_1, eps_2\rangle \hookrightarrow \mathit{Stat20} \Rightarrow$ $eps_1, eps_2 \in \mathbb{Q}\ \&\ eps_2 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ eps_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ eps_0 >_{\mathbb{Q}} eps_1 +_{\mathbb{Q}} eps_2$

$\langle eps_1\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathrm{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1\})$

$\langle eps_2\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $\mathrm{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2\})$

-- However, this assumption would lead to conflict with the inequality (holding for all i, j)

$$\mathrm{Ra\_ABS}\big((fq +_{\mathbb{QS}} f')\upharpoonright i -_{\mathbb{Q}} (fq +_{\mathbb{QS}} f')\upharpoonright j\big) \leqslant_{\mathbb{Q}} \mathrm{Ra\_ABS}\big(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j +_{\mathbb{QS}} (f'\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j)\big).$$

APPLY $\langle\ \rangle$ setformer_meet_join $(s \mapsto \mathbb{N}, t \mapsto \mathbb{N}, h(i,j) \mapsto i \cap j, P(i,j) \mapsto \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1, Q(i,j) \mapsto \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2) \Rightarrow$
$\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1 \vee \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2\} = \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1\} \cup \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}($

$\langle \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1\}, \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2\}\rangle \hookrightarrow T162 \Rightarrow$
$\mathrm{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1\} \cup \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2\})$

EQUAL $\Rightarrow$ $\mathrm{Finite}(\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1 \vee \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2\})$

Suppose $\Rightarrow$ $\mathit{Stat21}:$
$\{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}\big((fq +_{\mathbb{QS}} f')\upharpoonright i -_{\mathbb{Q}} (fq +_{\mathbb{QS}} f')\upharpoonright j\big) >_{\mathbb{Q}} eps_0\} \not\subseteq \{i \cap j :\ i \in \mathbb{N}, j \in \mathbb{N} \mid \mathrm{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_1 \vee \mathrm{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_2\}$

$\langle i_0, j_0\rangle \hookrightarrow \mathit{Stat21} \Rightarrow$ $\mathit{Stat21a}:$
$i_0, j_0 \in \mathbb{N}\ \&\ \mathrm{Ra\_ABS}\big((fq +_{\mathbb{QS}} f')\upharpoonright i_0 -_{\mathbb{Q}} (fq +_{\mathbb{QS}} f')\upharpoonright j_0\big) >_{\mathbb{Q}} eps_0\ \&\ \neg\mathrm{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) >_{\mathbb{Q}} eps_1\ \&\ \neg\mathrm{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) >_{\mathbb{Q}} eps_2$

APPLY $\langle\ \rangle$ fcn_symbol $(f(u) \mapsto fq\upharpoonright u +_{\mathbb{Q}} f'\upharpoonright u, g \mapsto fq +_{\mathbb{QS}} f', s \mapsto \mathbb{N}) \Rightarrow$
$\mathit{Stat22}:\ \langle \forall x \mid (fq +_{\mathbb{QS}} f')\upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } fq\upharpoonright x +_{\mathbb{Q}} f'\upharpoonright x \textbf{ else } \emptyset \textbf{ fi}\rangle$

$\langle i_0\rangle \hookrightarrow \mathit{Stat22} \Rightarrow$ $(fq +_{\mathbb{QS}} f')\upharpoonright i_0 = fq\upharpoonright i_0 +_{\mathbb{Q}} f'\upharpoonright i_0$

$\langle j_0\rangle \hookrightarrow \mathit{Stat22} \Rightarrow$ $(fq +_{\mathbb{QS}} f')\upharpoonright j_0 = fq\upharpoonright j_0 +_{\mathbb{Q}} f'\upharpoonright j_0$

EQUAL $\langle \mathit{Stat21}\rangle \Rightarrow$ $\mathrm{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} (fq\upharpoonright j_0 +_{\mathbb{Q}} f'\upharpoonright j_0)\big) >_{\mathbb{Q}} eps_0$

Suppose $\Rightarrow$ $fq\upharpoonright i_0 \notin \mathbb{Q}$

ELEM $\Rightarrow$ $\mathit{Stat24}:\ fq\upharpoonright i_0 \notin \{fq\upharpoonright j :\ j \in \mathbf{domain}(fq)\}$

$\langle i_0\rangle \hookrightarrow \mathit{Stat24}([\mathit{Stat4}, \mathit{Stat21a}]) \Rightarrow$ false; Discharge $\Rightarrow$ $fq\upharpoonright i_0 \in \mathbb{Q}$

$\text{Suppose} \Rightarrow \quad \mathsf{fq}\upharpoonright\mathsf{j}_0 \notin \mathbb{Q}$

$\text{ELEM} \Rightarrow \quad \mathit{Stat23}: \ \mathsf{fq}\upharpoonright\mathsf{j}_0 \notin \{\mathsf{fq}\upharpoonright\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$

$\langle \mathsf{j}_0 \rangle \hookrightarrow \mathit{Stat23}([\mathit{Stat4}, \mathit{Stat21a}]) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \mathsf{fq}\upharpoonright\mathsf{j}_0 \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright\mathsf{j}_0 \rangle \hookrightarrow \mathit{T372} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright\mathsf{i}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright\mathsf{j}_0) \rangle \hookrightarrow \mathit{T365} \Rightarrow \quad \mathsf{fq}\upharpoonright\mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\text{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{f}' \rangle \hookrightarrow \mathit{T66} \Rightarrow \quad \mathbf{range}(\mathsf{f}') = \{\mathsf{f}'\upharpoonright\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{f}')\}$

$\text{Suppose} \Rightarrow \quad \mathsf{f}'\upharpoonright\mathsf{i}_0 \notin \mathbb{Q}$

$\text{ELEM} \Rightarrow \quad \mathit{Stat24a}: \ \mathsf{f}'\upharpoonright\mathsf{i}_0 \notin \{\mathsf{f}'\upharpoonright\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{f}')\}$

$\langle \mathsf{i}_0 \rangle \hookrightarrow \mathit{Stat24a}([\mathit{Stat5}, \mathit{Stat21a}]) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \mathsf{f}'\upharpoonright\mathsf{i}_0 \in \mathbb{Q}$

$\text{Suppose} \Rightarrow \quad \mathsf{f}'\upharpoonright\mathsf{j}_0 \notin \mathbb{Q}$

$\text{ELEM} \Rightarrow \quad \mathit{Stat23a}: \ \mathsf{f}'\upharpoonright\mathsf{j}_0 \notin \{\mathsf{f}'\upharpoonright\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{f}')\}$

$\langle \mathsf{j}_0 \rangle \hookrightarrow \mathit{Stat23a}([\mathit{Stat5}, \mathit{Stat21a}]) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \mathsf{f}'\upharpoonright\mathsf{j}_0 \in \mathbb{Q}$

$\langle \mathsf{f}'\upharpoonright\mathsf{j}_0 \rangle \hookrightarrow \mathit{T372} \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{f}'\upharpoonright\mathsf{i}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\upharpoonright\mathsf{j}_0) \rangle \hookrightarrow \mathit{T365} \Rightarrow \quad \mathsf{f}'\upharpoonright\mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\text{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0 \in \mathbb{Q}$

$\langle \mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0 \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0), \mathsf{eps}_1 \rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{eps}_1 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0)$

$\langle \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0), \mathsf{eps}_2 \rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)$

$\langle \mathsf{eps}_1, \mathsf{eps}_2 \rangle \hookrightarrow \mathit{T365} \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0), \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \rangle \hookrightarrow \mathit{T365} \Rightarrow$

$\quad \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0, \mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0 \rangle \hookrightarrow \mathit{T365} \Rightarrow$

$\quad \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \rangle \hookrightarrow \mathit{T10045} \Rightarrow$

$\quad \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big) \in \mathbb{Q}$

$\langle \mathsf{eps}_1, \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0), \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \rangle \hookrightarrow \mathit{T397} \Rightarrow$

$\quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)$

$\mathit{T10050} \Rightarrow \quad \mathit{Stat25}: \ \langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x}, \mathsf{y} \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(\mathsf{x} +_{\mathbb{Q}} \mathsf{y}) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{x}) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}) \rangle$

$\langle \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0, \mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0 \rangle \hookrightarrow \mathit{Stat25} \Rightarrow$

$\quad \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)$

$\langle \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big), \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \rangle \hookrightarrow \mathit{T384} \Rightarrow$

$\qquad \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big)$

$\langle \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0), \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big) \rangle \hookrightarrow \mathit{T404} \Rightarrow$

$\quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big)$

$\text{ALGEBRA} \Rightarrow \quad \mathsf{fq}\upharpoonright\mathsf{i}_0 +_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} (\mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0) = \mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)$

$\text{EQUAL} \Rightarrow \quad \mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright\mathsf{j}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{j}_0)\big) >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright j_0), \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright j_0)), \mathsf{eps}_0\rangle \hookrightarrow T406 \Rightarrow$

$\mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright j_0) >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright i_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright j_0), \mathsf{eps}_0\rangle \hookrightarrow T406 \Rightarrow$

$\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{eps}_0\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_0, \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{eps}_0\rangle \hookrightarrow T405 \Rightarrow \quad \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_0, \mathsf{eps}_0\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$

$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}((\mathsf{fq} +_{\mathbb{Q}S} \mathsf{f}')\upharpoonright i -_{\mathbb{Q}} (\mathsf{fq} +_{\mathbb{Q}S} \mathsf{f}')\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_0\} \subseteq \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_1 \vee \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright i -_{\mathbb{Q}} \mathsf{f}'\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_2\}$

-- Since the inclusion just proves entails that the set on the left-hand side is finite, we have reached a contradiction, proving that the pointwise sum of rational Cauchy sequences is an alike sequence.

$\langle \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_1 \vee \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright i -_{\mathbb{Q}} \mathsf{f}'\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_2\}, \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}((\mathsf{fq} +_{\mathbb{Q}S} \mathsf{f}')\upharpoonright i -_{\mathbb{Q}} (\mathsf{fq} +_{\mathbb{Q}S} \mathsf{f}')\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_0\}\rangle \hookrightarrow T162(\langle Stat17\rangle) \Rightarrow$

false; Discharge $\Rightarrow$ $\mathsf{Ras\_ABS}(\mathsf{fq}) \notin \mathsf{RaCauchy} \vee \mathsf{Ras\_Rev}(\mathsf{fq}) \notin \mathsf{RaCauchy}$

-- Assuming next that $\mathsf{Ras\_ABS}(\mathsf{fq})$ is not a Cauchy sequence (unlike fq), there would exist a positive real eps_3 for which the set

$$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright i -_{\mathbb{Q}} \mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_3\}$$

is infinite, unlike the analogous set having fq in place of $\mathsf{Ras\_ABS}(\mathsf{fq})$. However, this would lead to a conflict with the inequality (holding for all i, j)

$$\mathsf{Ra\_ABS}(\mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright i -_{\mathbb{Q}} \mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright j) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j)\,.$$

Suppose $\Rightarrow$ $\mathsf{Ras\_ABS}(\mathsf{fq}) \notin \mathsf{RaCauchy}$

Use_def(RaCauchy) $\Rightarrow$ $Stat31: \ \mathsf{Ras\_ABS}(\mathsf{fq}) \notin \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright i -_{\mathbb{Q}} \mathsf{f}\upharpoonright j) >_{\mathbb{Q}} \varepsilon\})\rangle\}$

$\langle\,\rangle \hookrightarrow Stat31 \Rightarrow$ $Stat32: \ \neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright i -_{\mathbb{Q}} \mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright j) >_{\mathbb{Q}} \varepsilon\})\rangle$

$\langle \mathsf{eps}_3\rangle \hookrightarrow Stat32 \Rightarrow$ $\mathsf{eps}_3 \in \mathbb{Q} \ \& \ \mathsf{eps}_3 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \neg\mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright i -_{\mathbb{Q}} \mathsf{Ras\_ABS}(\mathsf{fq})\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_3\})$

$\langle \mathsf{eps}_3\rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{Finite}(\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright i -_{\mathbb{Q}} \mathsf{fq}\upharpoonright j) >_{\mathbb{Q}} \mathsf{eps}_3\})$

-- Indeed, since

$$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \text{Ra\_ABS}(fq \upharpoonright i -_{\mathbb{Q}} fq \upharpoonright j) >_{\mathbb{Q}} eps_3\}$$

is a superset of

$$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \text{Ra\_ABS}\big(\text{Ra\_ABS}(fq \upharpoonright i) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j)\big) >_{\mathbb{Q}} eps_3\} ,$$

the latter cannot be infinite when the former is finite.

Suppose $\Rightarrow$ *Stat33* : $\neg$
$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \text{Ra\_ABS}\big(\text{Ras\_ABS}(fq) \upharpoonright i -_{\mathbb{Q}} \text{Ras\_ABS}(fq) \upharpoonright j\big) >_{\mathbb{Q}} eps_3\} \subseteq \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \text{Ra\_ABS}(fq \upharpoonright i -_{\mathbb{Q}} fq \upharpoonright j) >_{\mathbb{Q}} eps_3\}$

$\langle i_1, j_1 \rangle \hookrightarrow$ *Stat33* $\Rightarrow$ *Stat34* : $i_1, j_1 \in \mathbb{N}\ \&\ \text{Ra\_ABS}\big(\text{Ras\_ABS}(fq) \upharpoonright i_1 -_{\mathbb{Q}} \text{Ras\_ABS}(fq) \upharpoonright j_1\big) >_{\mathbb{Q}} eps_3\ \&\ \neg \text{Ra\_ABS}(fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1) >_{\mathbb{Q}} eps_3$

APPLY $\langle\ \rangle$ fcn_symbol$\big(f(u) \mapsto \text{Ra_ABS}(fq \upharpoonright u), g \mapsto \text{Ras_ABS}(fq), s \mapsto \mathbb{N}\big) \Rightarrow$

Stat35 : $\langle \forall x \mid \text{Ras\_ABS}(fq) \upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } \text{Ra\_ABS}(fq \upharpoonright x) \textbf{ else } \emptyset \textbf{ fi} \rangle$

$\langle i_1 \rangle \hookrightarrow$ *Stat35* $\Rightarrow$ $\text{Ras\_ABS}(fq) \upharpoonright i_1 = \text{Ra\_ABS}(fq \upharpoonright i_1)$

$\langle j_1 \rangle \hookrightarrow$ *Stat35* $\Rightarrow$ $\text{Ras\_ABS}(fq) \upharpoonright j_1 = \text{Ra\_ABS}(fq \upharpoonright j_1)$

EQUAL $\langle$*Stat34*$\rangle \Rightarrow$ $\text{Ra\_ABS}\big(\text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1)\big) >_{\mathbb{Q}} eps_3$

Suppose $\Rightarrow$ $fq \upharpoonright i_1 \notin \mathbb{Q}$

ELEM $\Rightarrow$ *Stat36* : $fq \upharpoonright i_1 \notin \{fq \upharpoonright j : j \in \textbf{domain}(fq)\}$

$\langle i_1 \rangle \hookrightarrow$ *Stat36*([*Stat4*, *Stat34*]) $\Rightarrow$ false; Discharge $\Rightarrow$ $fq \upharpoonright i_1 \in \mathbb{Q}$

$\langle fq \upharpoonright i_1 \rangle \hookrightarrow$ *T10045* $\Rightarrow$ $\text{Ra\_ABS}(fq \upharpoonright i_1) \in \mathbb{Q}$

Suppose $\Rightarrow$ $fq \upharpoonright j_1 \notin \mathbb{Q}$

ELEM $\Rightarrow$ *Stat37* : $fq \upharpoonright j_1 \notin \{fq \upharpoonright j : j \in \textbf{domain}(fq)\}$

$\langle j_1 \rangle \hookrightarrow$ *Stat37*([*Stat4*, *Stat34*]) $\Rightarrow$ false; Discharge $\Rightarrow$ $fq \upharpoonright j_1 \in \mathbb{Q}$

$\langle fq \upharpoonright j_1 \rangle \hookrightarrow$ *T10045* $\Rightarrow$ $\text{Ra\_ABS}(fq \upharpoonright j_1) \in \mathbb{Q}$

$\langle fq \upharpoonright j_1 \rangle \hookrightarrow$ *T372* $\Rightarrow$ $\text{Rev}_{\mathbb{Q}}(fq \upharpoonright j_1) \in \mathbb{Q}$

$\langle \text{Ra\_ABS}(fq \upharpoonright j_1) \rangle \hookrightarrow$ *T372* $\Rightarrow$ $\text{Rev}_{\mathbb{Q}}\big(\text{Ra\_ABS}(fq \upharpoonright j_1)\big) \in \mathbb{Q}$

$\langle fq \upharpoonright i_1, \text{Rev}_{\mathbb{Q}}(fq \upharpoonright j_1) \rangle \hookrightarrow$ *T365* $\Rightarrow$ $fq \upharpoonright i_1 +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(fq \upharpoonright j_1) \in \mathbb{Q}$

$\langle \text{Ra\_ABS}(fq \upharpoonright i_1), \text{Rev}_{\mathbb{Q}}\big(\text{RA\_ABS}(fq \upharpoonright j_1)\big) \rangle \hookrightarrow$ *T365* $\Rightarrow$ $\text{Ra\_ABS}(fq \upharpoonright i_1) +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}\big(\text{Ra\_ABS}(fq \upharpoonright j_1)\big) \in \mathbb{Q}$

Use_def$(-_{\mathbb{Q}}) \Rightarrow$ $fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1, \text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1) \in \mathbb{Q}$

$\langle fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1 \rangle \hookrightarrow$ *T10045* $\Rightarrow$ $\text{Ra\_ABS}(fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1) \in \mathbb{Q}$

$\langle \text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1) \rangle \hookrightarrow$ *T10045* $\Rightarrow$ $\text{Ra\_ABS}\big(\text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1)\big) \in \mathbb{Q}$

T10050 $\Rightarrow$ *Stat38* : $\langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow \text{Ra\_ABS}\Big(\text{Ra\_ABS}(x) +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}\big(\text{Ra\_ABS}(y)\big)\Big) \leqslant_{\mathbb{Q}} \text{Ra\_ABS}\big(x +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(y)\big) \rangle$

$\langle fq \upharpoonright i_1, fq \upharpoonright j_1 \rangle \hookrightarrow$ *Stat38* $\Rightarrow$

$\text{Ra\_ABS}\Big(\text{Ra\_ABS}(fq \upharpoonright i_1) +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}\big(\text{Ra\_ABS}(fq \upharpoonright j_1)\big)\Big) \leqslant_{\mathbb{Q}} \text{Ra\_ABS}\big(fq \upharpoonright i_1 +_{\mathbb{Q}} \text{Rev}_{\mathbb{Q}}(fq \upharpoonright j_1)\big)$

Use_def$(-_{\mathbb{Q}}) \Rightarrow$ $\text{Ra\_ABS}\big(\text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1)\big) \leqslant_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1)$

$\langle \text{Ra\_ABS}\big(\text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1)\big), \text{Ra\_ABS}(fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1) \rangle \hookrightarrow$ *T384* $\Rightarrow$

$\text{Ra\_ABS}(fq \upharpoonright i_1 -_{\mathbb{Q}} fq \upharpoonright j_1) \geqslant_{\mathbb{Q}} \text{Ra\_ABS}\big(\text{Ra\_ABS}(fq \upharpoonright i_1) -_{\mathbb{Q}} \text{Ra\_ABS}(fq \upharpoonright j_1)\big)$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_1 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_1), \mathsf{Ra\_ABS}(\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_1) -_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{j}_1)), \mathsf{eps}_3\rangle \hookrightarrow T406 \Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_ABS}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_ABS}(\mathsf{fq})\restriction \mathsf{j}) >_{\mathbb{Q}}$ e

$\langle \{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_3\}, \{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_ABS}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_ABS}(\mathsf{fq})\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_3\}\rangle \hookrightarrow T162 \Rightarrow$ false; Discharge $\Rightarrow$

$\mathsf{Ras\_Rev}(\mathsf{fq}) \notin \mathsf{RaCauchy}$

-- Third and last, let us assume that $\mathsf{Ras\_Rev}(\mathsf{fq})$ is not a Cauchy sequence (unlike fq). Then there would exist a positive real eps_4 for which the set

$$\{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_4\}$$

is infinite, unlike the analogous set having fq in place of $\mathsf{Ras\_Rev}(\mathsf{fq})$. However, this would lead to a conflict with the equality (holding for all i, j)

$$\mathsf{Ra\_ABS}(\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}) = \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j})\,.$$

Use_def(RaCauchy) $\Rightarrow$ $Stat41:\ \mathsf{Ras\_Rev}(\mathsf{fq}) \notin \{\mathsf{f}\in\mathsf{RaSeq} \mid \langle\forall\varepsilon\in\mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction \mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}$

$\langle\,\rangle \hookrightarrow Stat41 \Rightarrow$ $Stat42:\ \neg\langle\forall\varepsilon\in\mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle$

$\langle \mathsf{eps}_4\rangle \hookrightarrow Stat42 \Rightarrow$ $Stat43:\ \mathsf{eps}_4 \in \mathbb{Q} \,\&\, \mathsf{eps}_4 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \,\&\, \neg\mathsf{Finite}(\{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_4\})$

$\langle \mathsf{eps}_4\rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{Finite}(\{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_4\})$

APPLY $\langle\,\rangle$ fcn_symbol$(\mathsf{f}(\mathsf{u}) \mapsto \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{u}), \mathsf{g} \mapsto \mathsf{Ras_Rev}(\mathsf{fq}), \mathsf{s} \mapsto \mathbb{N}) \Rightarrow$

$Stat44:\ \langle\forall \mathsf{x} \mid \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{x} = \textbf{if } \mathsf{x}\in\mathbb{N} \textbf{ then } \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{x}) \textbf{ else } \emptyset \textbf{ fi}\rangle$

EQUAL $\langle Stat43\rangle \Rightarrow$ $Stat45:\ \{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_4\} \neq \{\mathsf{i}\cap\mathsf{j} : \mathsf{i}\in\mathbb{N}, \mathsf{j}\in\mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_4\}$

$\langle \mathsf{i}_2, \mathsf{j}_2\rangle \hookrightarrow Stat45 \Rightarrow$ $Stat45a:\ \mathsf{i}_2, \mathsf{j}_2 \in \mathbb{N} \,\&\, (\mathsf{Ra\_ABS}(\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}_2) >_{\mathbb{Q}} \mathsf{eps}_4 \leftrightarrow \neg\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_2) >_{\mathbb{Q}} \mathsf{eps}_4)$

$\langle \mathsf{i}_2\rangle \hookrightarrow Stat44 \Rightarrow$ $\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{i}_2 = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{i}_2)$

$\langle \mathsf{j}_2\rangle \hookrightarrow Stat44 \Rightarrow$ $\mathsf{Ras\_Rev}(\mathsf{fq})\restriction \mathsf{j}_2 = \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{j}_2)$

EQUAL $\langle Stat45\rangle \Rightarrow$ $Stat46:\ \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{j}_2)) >_{\mathbb{Q}} \mathsf{eps}_4 \leftrightarrow \neg\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_2) >_{\mathbb{Q}} \mathsf{eps}_4$

Suppose $\Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{j}_2)) \neq \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_2)$

Use_def$(-_{\mathbb{Q}}) \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{j}_2)) = \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{i}_2) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{j}_2)))$

Suppose $\Rightarrow$ $\mathsf{fq}\restriction \mathsf{i}_2 \notin \mathbb{Q}$

ELEM $\Rightarrow$ $Stat47:\ \mathsf{fq}\restriction \mathsf{i}_2 \notin \{\mathsf{fq}\restriction \mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$

$\langle \mathsf{i}_2\rangle \hookrightarrow Stat47([Stat4, Stat45a]) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{fq}\restriction \mathsf{i}_2 \in \mathbb{Q}$

Suppose $\Rightarrow$ $\mathsf{fq}\restriction \mathsf{j}_2 \notin \mathbb{Q}$

ELEM $\Rightarrow$ $Stat48:\ \mathsf{fq}\restriction \mathsf{j}_2 \notin \{\mathsf{fq}\restriction \mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$

$\langle \mathsf{j}_2\rangle \hookrightarrow Stat48([Stat4, Stat45a]) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{fq}\restriction \mathsf{j}_2 \in \mathbb{Q}$

$\langle \mathsf{fq}\restriction \mathsf{i}_2\rangle \hookrightarrow T372 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{i}_2) \in \mathbb{Q}$

$\langle \mathsf{fq}\restriction \mathsf{j}_2\rangle \hookrightarrow T372 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\restriction \mathsf{j}_2) \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright \mathsf{j}_2\rangle \hookrightarrow T398 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2)) = \mathsf{fq}\upharpoonright \mathsf{j}_2$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2), \mathsf{fq}\upharpoonright \mathsf{j}_2\rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_2 =$
$\mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2)$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) = \mathsf{fq}\upharpoonright \mathsf{j}_2 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_2$

$\mathsf{EQUAL}\ \langle Stat46\rangle \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2)) = \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2))$

$T10050 \Rightarrow \quad Stat50 : \ \langle \forall \mathsf{x} \mid \mathsf{x} \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{x})) = \mathsf{Ra\_ABS}(\mathsf{x})\rangle$

$\langle \mathsf{fq}\upharpoonright \mathsf{j}_2, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2)\rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2)\rangle \hookrightarrow Stat50 \Rightarrow \quad \mathsf{Ra\_ABS}\Big(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2))\Big) =$
$\mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2))$

$\langle \mathsf{fq}\upharpoonright \mathsf{j}_2, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2)\rangle \hookrightarrow T396 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2)) =$
$\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2) +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2))$

$\langle \mathsf{fq}\upharpoonright \mathsf{i}_2\rangle \hookrightarrow T398 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2)) = \mathsf{fq}\upharpoonright \mathsf{i}_2$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2), \mathsf{fq}\upharpoonright \mathsf{i}_2\rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2) +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_2 =$
$\mathsf{fq}\upharpoonright \mathsf{i}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2)$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad \mathsf{fq}\upharpoonright \mathsf{i}_2 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2) = \mathsf{fq}\upharpoonright \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_2$

$\mathsf{EQUAL}\ \langle Stat46\rangle \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat49 : \ \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2)) = \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_2)$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2)) >_{\mathbb{Q}} \mathsf{eps}_4$

$\mathsf{EQUAL}\ \langle Stat49\rangle \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_2) >_{\mathbb{Q}} \mathsf{eps}_4$

$\langle Stat46\rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \neg \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_2) -_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_2)) >_{\mathbb{Q}} \mathsf{eps}_4$

$\mathsf{EQUAL}\ \langle Stat49\rangle \Rightarrow \quad \neg \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_2 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_2) >_{\mathbb{Q}} \mathsf{eps}_4$

$\langle Stat46\rangle\ \mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow$ QED

-- Every Cauchy sequence has an upper bound.

Theorem 598 (10061) $\mathsf{F} \in \mathsf{RaCauchy} \rightarrow \langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{F}) \mid \mathsf{y} \leqslant_{\mathbb{Q}} \mathsf{x}\rangle$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{f}) \Rightarrow \quad \mathsf{f} \in \mathsf{RaCauchy}\ \&\ Stat0 : \ \neg\langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{y} \leqslant_{\mathbb{Q}} \mathsf{x}\rangle$

-- Reasoning by contradiction, let f be a Cauchy sequence lacking an upper bound. Fix an unsigned integer i_0 past which the distance between any two components of f is always smaller than $\mathbf{1}_{\mathbb{Q}}$.

$\langle \mathsf{f}\rangle \hookrightarrow T10059 \Rightarrow \quad \mathsf{f} \in \mathsf{RaSeq}\ \&\ Stat3 : \ \langle \forall \mathsf{h} \in \mathbb{N} \mid \mathsf{f}\upharpoonright \mathsf{h} \in \mathbb{Q}\rangle$

$\langle \mathsf{f}\rangle \hookrightarrow T413a \Rightarrow \quad \mathbf{domain}(\mathsf{f}) = \mathbb{N}\ \&\ \mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) \subseteq \mathbb{Q}$

$T371 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$T388 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathbf{1}_{\mathbb{Q}}, \mathsf{f}\rangle \hookrightarrow T10060 \Rightarrow \quad Stat4 : \ \langle \exists \mathsf{k} \in \mathbb{N} \mid \mathsf{k} \neq \emptyset\ \&\ \langle \forall \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{i} \notin \mathsf{k}\ \&\ \mathsf{j} \notin \mathsf{k} \rightarrow \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}\upharpoonright \mathsf{j})\rangle\rangle$

$\langle i_0 \rangle \hookrightarrow \mathit{Stat4} \Rightarrow \quad i_0 \in \mathbb{N} \;\&\; i_0 \neq \emptyset \;\&\; \mathit{Stat5}: \; \langle \forall i \in \mathbb{N}, j \in \mathbb{N} \mid i \notin i_0 \;\&\; j \notin i_0 \rightarrow \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f{\restriction}i -_{\mathbb{Q}} f{\restriction}j) \rangle$

-- Then consider the largest among $f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$ and all components of f which precede the i-th component.

$\mathit{TELEM} \Rightarrow \quad \mathsf{Svm}(\{[h, f{\restriction}h] : \; h \in i_0\}) \;\&\; \mathbf{domain}(\{[h, f{\restriction}h] : \; h \in i_0\}) = i_0$
$\mathit{T179} \Rightarrow \quad \mathcal{O}(\mathbb{N})$
Suppose $\Rightarrow \quad \{f{\restriction}h : \; h \in i_0\} \not\subseteq \mathbb{Q}$
$\langle \mathbb{N}, i_0 \rangle \hookrightarrow \mathit{T12} \Rightarrow \quad i_0 \subseteq \mathbf{domain}(f)$
$\langle f \rangle \hookrightarrow \mathit{T66} \Rightarrow \quad \mathbf{range}(f) = \{f{\restriction}h : \; h \in \mathbf{domain}(f)\}$
Set_monot $\Rightarrow \quad \{f{\restriction}h : \; h \in i_0\} \subseteq \{f{\restriction}h : \; h \in \mathbf{domain}(f)\}$
ELEM $\Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \{f{\restriction}h : \; h \in i_0\} \subseteq \mathbb{Q}$
Loc_def $\Rightarrow \quad m = \textbf{if } \mathsf{max\_Ra}(\{f{\restriction}h : \; h \in i_0\}) <_{\mathbb{Q}} f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \textbf{ then } f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \textbf{ else } \mathsf{max\_Ra}(\{f{\restriction}h : \; h \in i_0\}) \textbf{ fi}$

-- Consider next a component $f{\restriction}j$ of f which exceeds m (since m cannot be an upper bound of f, due to our initial assumption, such a component must exists).

$\langle i_0 \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad f{\restriction}i_0 \in \mathbb{Q}$
ALGEBRA $\Rightarrow \quad f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
Suppose $\Rightarrow \quad \neg \mathsf{Finite}(\{f{\restriction}h : \; h \in i_0\}) \vee \{f{\restriction}h : \; h \in i_0\} = \emptyset$
Suppose $\Rightarrow \quad \mathit{Stat51}: \; f{\restriction}\mathbf{arb}(i_0) \notin \{f{\restriction}h : \; h \in i_0\}$
$\langle \mathbf{arb}(i_0) \rangle \hookrightarrow \mathit{Stat51} \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \mathit{Stat52}: \; \neg \mathsf{Finite}(\{f{\restriction}h : \; h \in i_0\})$
$\langle i_0 \rangle \hookrightarrow \mathit{T179} \Rightarrow \quad \mathsf{Finite}(\mathbf{domain}(\{[h, f{\restriction}h] : \; h \in i_0\}))$
$\langle \{[h, f{\restriction}h] : \; h \in i_0\} \rangle \hookrightarrow \mathit{T165}(\langle \mathit{Stat52} \rangle) \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \mathsf{Finite}(\{f{\restriction}h : \; h \in i_0\}) \;\&\; \mathbf{arb}(\{f{\restriction}h : \; h \in i_0\}) \in \{f{\restriction}h : \; h \in i_0\}$
$\mathit{T10044} \Rightarrow \quad \mathit{Stat11}: \; \langle \forall t, x \mid \mathsf{Finite}(t) \;\&\; x \in t \;\&\; t \subseteq \mathbb{Q} \rightarrow \mathsf{max\_Ra}(t) \in t \;\&\; x = \mathsf{max\_Ra}(t) \vee x <_{\mathbb{Q}} \mathsf{max\_Ra}(t) \rangle$
Suppose $\Rightarrow \quad \mathsf{max\_Ra}(\{f{\restriction}h : \; h \in i_0\}) \notin \mathbb{Q}$
$\langle \{f{\restriction}h : \; h \in i_0\}, \mathbf{arb}(\{f{\restriction}h : \; h \in i_0\}) \rangle \hookrightarrow \mathit{Stat11} \Rightarrow \quad$ false; $\qquad$ Discharge $\Rightarrow \quad \mathsf{max\_Ra}(\{f{\restriction}h : \; h \in i_0\}), m \in \mathbb{Q}$
$\langle m \rangle \hookrightarrow \mathit{Stat0} \Rightarrow \quad \mathit{Stat6}: \; \neg \langle \forall y \in \mathbf{range}(f) \mid y \leqslant_{\mathbb{Q}} m \rangle$
$\langle y_0 \rangle \hookrightarrow \mathit{Stat6} \Rightarrow \quad y_0 \in \mathbf{range}(f) \;\&\; \neg y_0 \leqslant_{\mathbb{Q}} m$
$\langle f \rangle \hookrightarrow \mathit{T66} \Rightarrow \quad \mathbf{range}(f) = \{f{\restriction}j : \; j \in \mathbf{domain}(f)\} \;\&\; \mathit{Stat7}: \; y_0 \in \{f{\restriction}j : \; j \in \mathbf{domain}(f)\}$
$\langle j \rangle \hookrightarrow \mathit{Stat7} \Rightarrow \quad y_0 = f{\restriction}j \;\&\; j \in \mathbf{domain}(f)$
EQUAL $\Rightarrow \quad j \in \mathbb{N} \;\&\; f{\restriction}j \in \mathbb{Q} \;\&\; \neg f{\restriction}j \leqslant_{\mathbb{Q}} m$
$\langle f{\restriction}j, m \rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \neg m \geqslant_{\mathbb{Q}} f{\restriction}j$

-- We will obtain a contradiction in each of the three cases $j \in i_0$, $j = i_0$, $i_0 \in j$, after observing that $m \geqslant_{\mathbb{Q}} f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$.

$\langle \mathbb{N}, i_0 \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(i_0)$
$\langle \mathbb{N}, j \rangle \hookrightarrow \mathit{T11} \Rightarrow \quad \mathcal{O}(j)$
$\langle i_0, j \rangle \hookrightarrow \mathit{T28} \Rightarrow \quad j \in i_0 \vee j = i_0 \vee i_0 \in j$
ALGEBRA $\Rightarrow \quad f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$

Suppose $\Rightarrow$ $\neg m \geqslant_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
Suppose $\Rightarrow$ $\text{max\_Ra}(\{f\restriction h : h \in i_0\}) <_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
ELEM $\Rightarrow$ $m = f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
$\langle f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, m\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $m = \text{max\_Ra}(\{f\restriction h : h \in i_0\})\ \&\ \neg\text{max\_Ra}(\{f\restriction h : h \in i_0\}) <_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $\neg m <_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
$\langle m, f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow$ $\neg f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} m$
$\langle f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, m\rangle \hookrightarrow T384 \Rightarrow$ $\neg f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} m \vee f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = m$
Suppose $\Rightarrow$ $\neg f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} m$
$\langle m, f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow$ $\neg m \leqslant_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
T10050 $\Rightarrow$ *Stat14* : $\langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow x \leqslant_{\mathbb{Q}} y \vee y \leqslant_{\mathbb{Q}} x\rangle$
$\langle m, f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow Stat14 \Rightarrow$ $f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \leqslant_{\mathbb{Q}} m$
$\langle f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, m\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = m$
$\langle f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, m\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $m \geqslant_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

-- Case $j \in i_0$: a contradiction ensues from the facts $f\restriction j \leqslant_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$, $\text{max\_Ra}(\{f\restriction h : h \in i_0\}) \leqslant_{\mathbb{Q}} m$.

Suppose $\Rightarrow$ $j \in i_0$
Suppose $\Rightarrow$ *Stat8* : $f\restriction j \notin \{f\restriction h : h \in i_0\}$
$\langle j\rangle \hookrightarrow Stat8 \Rightarrow$ false; Discharge $\Rightarrow$ $f\restriction j \in \{f\restriction h : h \in i_0\}$
$\langle \{f\restriction h : h \in i_0\}, f\restriction j\rangle \hookrightarrow Stat11 \Rightarrow$ $f\restriction j = \text{max\_Ra}(\{f\restriction h : h \in i_0\}) \vee$
$f\restriction j <_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
Suppose $\Rightarrow$ $\neg f\restriction j \leqslant_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
Suppose $\Rightarrow$ $f\restriction j = \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
$\langle f\restriction j, \text{max\_Ra}(\{f\restriction h : h \in i_0\})\rangle \hookrightarrow T384 \Rightarrow$
$\text{max\_Ra}(\{f\restriction h : h \in i_0\}) \geqslant_{\mathbb{Q}} f\restriction j$
$\langle f\restriction j, \text{max\_Ra}(\{f\restriction h : h \in i_0\})\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $f\restriction j <_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
$\langle f\restriction j, \text{max\_Ra}(\{f\restriction h : h \in i_0\})\rangle \hookrightarrow T384 \Rightarrow$
$\text{max\_Ra}(\{f\restriction h : h \in i_0\}) >_{\mathbb{Q}} f\restriction j$
$\langle \text{max\_Ra}(\{f\restriction h : h \in i_0\}), f\restriction j\rangle \hookrightarrow T384 \Rightarrow$
$\text{max\_Ra}(\{f\restriction h : h \in i_0\}) \geqslant_{\mathbb{Q}} f\restriction j$
$\langle f\restriction j, \text{max\_Ra}(\{f\restriction h : h \in i_0\})\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $f\restriction j \leqslant_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
Suppose $\Rightarrow$ $\neg\text{max\_Ra}(\{f\restriction h : h \in i_0\}) \leqslant_{\mathbb{Q}} m$
Suppose $\Rightarrow$ $\text{max\_Ra}(\{f\restriction h : h \in i_0\}) <_{\mathbb{Q}} f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
ELEM $\Rightarrow$ $m = f\restriction i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $\text{max\_Ra}(\{f\restriction h : h \in i_0\}) <_{\mathbb{Q}} m$
$\langle \text{max\_Ra}(\{f\restriction h : h \in i_0\}), m\rangle \hookrightarrow T384 \Rightarrow$ $m >_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
$\langle m, \text{max\_Ra}(\{f\restriction h : h \in i_0\})\rangle \hookrightarrow T384 \Rightarrow$ $m \geqslant_{\mathbb{Q}} \text{max\_Ra}(\{f\restriction h : h \in i_0\})$
$\langle \text{max\_Ra}(\{f\restriction h : h \in i_0\}), m\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $m = \text{max\_Ra}(\{f\restriction h : h \in i_0\})$

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{max\_Ra}(\{f{\restriction}h :\ h \in i_0\}) \leqslant_{\mathbb{Q}} m$

$\langle \mathsf{max\_Ra}(\{f{\restriction}h :\ h \in i_0\}), m\rangle \hookrightarrow T384 \Rightarrow \quad m \geqslant_{\mathbb{Q}} \mathsf{max\_Ra}(\{f{\restriction}h :\ h \in i_0\})$

$\langle f{\restriction}j, \mathsf{max\_Ra}(\{f{\restriction}h :\ h \in i_0\})\rangle \hookrightarrow T384 \Rightarrow$

$\quad \mathsf{max\_Ra}(\{f{\restriction}h :\ h \in i_0\}) \geqslant_{\mathbb{Q}} f{\restriction}j$

$\langle m, \mathsf{max\_Ra}(\{f{\restriction}h :\ h \in i_0\}), f{\restriction}j\rangle \hookrightarrow T404 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad j = i_0 \vee i_0 \in j$

-- Case $j = i_0$: a contradiction ensues from the facts $f{\restriction}j <_{\mathbb{Q}} f{\restriction}j +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$, $f{\restriction}j +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$, $f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} +_{\mathbb{Q}} m$.

$\mathsf{Suppose} \Rightarrow \quad j = i_0$

$\langle f{\restriction}j, f{\restriction}j\rangle \hookrightarrow T384 \Rightarrow \quad f{\restriction}j \geqslant_{\mathbb{Q}} f{\restriction}j$

$T388 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle f{\restriction}j, f{\restriction}j, \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T402 \Rightarrow \quad f{\restriction}j +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} f{\restriction}j +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle f{\restriction}j\rangle \hookrightarrow T371 \Rightarrow \quad f{\restriction}j +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = f{\restriction}j$

$\mathsf{EQUAL} \Rightarrow \quad f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} f{\restriction}j$

$\langle m, f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, f{\restriction}j\rangle \hookrightarrow T406 \Rightarrow \quad m >_{\mathbb{Q}} f{\restriction}j$

$\langle m, f{\restriction}j\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad i_0 \in j$

-- Case $i_0 \in j$: Since we have $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f{\restriction}j -_{\mathbb{Q}} f{\restriction}i_0)$ in this case, we get $f{\restriction}j \leqslant_{\mathbb{Q}} f{\restriction}j +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$, whence a contradiction follows, because $f{\restriction}j +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \leqslant_{\mathbb{Q}} m$. This completes our proof.

$\langle j, i_0\rangle \hookrightarrow Stat5 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f{\restriction}j -_{\mathbb{Q}} f{\restriction}i_0)$

$\langle \mathbf{1}_{\mathbb{Q}}, \mathsf{Ra\_ABS}(f{\restriction}j -_{\mathbb{Q}} f{\restriction}i_0)\rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f{\restriction}j -_{\mathbb{Q}} f{\restriction}i_0)$

$T10050 \Rightarrow \quad Stat15:$

$\quad \langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow \mathsf{Rev}_{\mathbb{Q}}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) = y +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(x)\rangle \ \&\ Stat16:$

$\qquad \langle \forall x \mid x \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(\mathsf{Rev}_{\mathbb{Q}}(x)) = \mathsf{Ra\_ABS}(x)\rangle \ \&\ Stat17:\ \langle \forall x, y, z \mid x, y, z \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(x +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(y)) \leqslant_{\mathbb{Q}} z \rightarrow y \leqslant_{\mathbb{Q}} x +_{\mathbb{Q}} z\rangle$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad f{\restriction}j -_{\mathbb{Q}} f{\restriction}i_0 = f{\restriction}j +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}i_0)$

$\mathsf{ALGEBRA} \Rightarrow \quad f{\restriction}j -_{\mathbb{Q}} f{\restriction}i_0 \in \mathbb{Q}$

$\mathsf{EQUAL} \Rightarrow \quad f{\restriction}j +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}i_0) \in \mathbb{Q}$

$\langle f{\restriction}j +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}i_0)\rangle \hookrightarrow Stat16 \Rightarrow \quad \mathsf{Ra\_ABS}\Big(\mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}i_0))\Big) =$

$\quad \mathsf{Ra\_ABS}(f{\restriction}j +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}i_0))$

$\langle f{\restriction}j, f{\restriction}i_0\rangle \hookrightarrow Stat15 \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}i_0)) = f{\restriction}i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j)$

$\mathsf{EQUAL} \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f{\restriction}i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j))$

$\langle \mathbf{1}_{\mathbb{Q}}, \mathsf{Ra\_ABS}(f{\restriction}i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j))\rangle \hookrightarrow T384 \Rightarrow \quad \mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f{\restriction}i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j))$

$\langle \mathsf{Ra\_ABS}(f{\restriction}i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j)), \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow \quad \mathsf{Ra\_ABS}(f{\restriction}i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f{\restriction}j)) \leqslant_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\langle f{\restriction}i_0, f{\restriction}j, \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow Stat17 \Rightarrow \quad f{\restriction}j \leqslant_{\mathbb{Q}} f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\langle f{\restriction}j, f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow \quad f{\restriction}i_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \geqslant_{\mathbb{Q}} f{\restriction}j$

$\langle \mathsf{m}, \mathsf{f}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}, \mathsf{f}\upharpoonright \mathsf{j}\rangle \hookrightarrow T404 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- As an immediate corollary, every Cauchy sequence has an upper bound for the absolute values of its components.

Theorem 599 (10061*a*) $\mathsf{F} \in \mathsf{RaCauchy} \rightarrow \langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{F}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{x}\rangle$. PROOF:

Suppose_not(f) $\Rightarrow$ $\mathsf{f} \in \mathsf{RaCauchy}\ \&\ Stat0: \neg\langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{x}\rangle$
$\langle \mathsf{f}\rangle \hookrightarrow T10059 \Rightarrow$ $\mathsf{f} \in \mathsf{RaSeq}$
$\langle \mathsf{f}\rangle \hookrightarrow T413a \Rightarrow$ $\mathbf{domain}(\mathsf{f}) = \mathbb{N}\ \&\ \mathsf{Svm}(\mathsf{f})\ \&\ \mathbf{range}(\mathsf{f}) \subseteq \mathbb{Q}$
$\langle \mathsf{f}, \mathsf{f}\rangle \hookrightarrow T413 \Rightarrow$ $\mathsf{Ras\_ABS}(\mathsf{f}) \in \mathsf{RaCauchy}$
$\langle \mathsf{f}, \mathsf{f}\rangle \hookrightarrow T10062 \Rightarrow$ $\mathsf{Ras\_ABS}(\mathsf{f}) = \{[\mathsf{u}, \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{u})] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{Ras\_ABS}(\mathsf{f})\rangle \hookrightarrow T10061 \Rightarrow$ $Stat4: \langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{Ras\_ABS})(\mathsf{f}) \mid \mathsf{y} \leqslant_{\mathbb{Q}} \mathsf{x}\rangle$
$\langle \mathsf{c}\rangle \hookrightarrow Stat4 \Rightarrow$ $\mathsf{c} \in \mathbb{Q}\ \&\ Stat5: \langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{Ras\_ABS})(\mathsf{f}) \mid \mathsf{y} \leqslant_{\mathbb{Q}} \mathsf{c}\rangle$
ALGEBRA $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}}, \mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} \in \mathbb{Q}$
$\langle \mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow Stat0 \Rightarrow$ $Stat6: \neg\langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle$
$\langle \mathsf{d}\rangle \hookrightarrow Stat6 \Rightarrow$ $\mathsf{d} \in \mathbf{range}(\mathsf{f})\ \&\ \neg \mathsf{Ra\_ABS}(\mathsf{d}) <_{\mathbb{Q}} \mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$
Suppose $\Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{d}) \notin \mathbf{range}(\mathsf{Ras\_ABS})(\mathsf{f})$
$\langle \mathsf{f}\rangle \hookrightarrow T66 \Rightarrow$ $Stat7: \mathsf{d} \in \{\mathsf{f}\upharpoonright \mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f})\}$
$\langle \mathsf{a}\rangle \hookrightarrow Stat7 \Rightarrow$ $\mathsf{d} = \mathsf{f}\upharpoonright \mathsf{a}\ \&\ \mathsf{a} \in \mathbb{N}$
TELEM $\Rightarrow$ $\mathbf{range}(\{[\mathsf{u}, \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{u})] : \mathsf{u} \in \mathbb{N}\}) = \{\mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{u}) : \mathsf{u} \in \mathbb{N}\}$
EQUAL $\Rightarrow$ $\mathbf{range}(\mathsf{Ras\_ABS})(\mathsf{f}) = \{\mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{u}) : \mathsf{u} \in \mathbb{N}\}\ \&\ \mathsf{Ra\_ABS}(\mathsf{d}) = \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{a})$
ELEM $\Rightarrow$ $Stat8: \mathsf{Ra\_ABS}(\mathsf{d}) \notin \{\mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright \mathsf{u}) : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{a}\rangle \hookrightarrow Stat8 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{d}) \in \mathbf{range}(\mathsf{Ras\_ABS})(\mathsf{f})$
$\langle \mathsf{Ra\_ABS}(\mathsf{d})\rangle \hookrightarrow Stat5 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{d}) \leqslant_{\mathbb{Q}} \mathsf{c}$
$\langle \mathsf{Ra\_ABS}(\mathsf{d}), \mathsf{c}\rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{c} \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{d})$
T388 $\Rightarrow$ $\mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{d}\rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{d}) \in \mathbb{Q}$
$\langle \mathsf{c}, \mathsf{Ra\_ABS}(\mathsf{d}), \mathbf{1}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T402 \Rightarrow$ $\mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{d}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
ALGEBRA $\Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{d}) +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathsf{Ra\_ABS}(\mathsf{d})$
EQUAL $\Rightarrow$ $\mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{d})$
$\langle \mathsf{Ra\_ABS}(\mathsf{d}), \mathsf{c} +_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 600 (414*a*) $\{\mathsf{F}, \mathsf{G}\} \subseteq \mathsf{RaCauchy} \rightarrow \mathsf{F} -_{\mathbb{QS}} \mathsf{G}, \mathsf{F} *_{\mathbb{QS}} \mathsf{G} \in \mathsf{RaCauchy}$. PROOF:

Suppose_not(fq, f′) $\Rightarrow$ $Stat99: \{\mathsf{fq}, \mathsf{f}'\} \subseteq \mathsf{RaCauchy}\ \&\ \mathsf{fq} -_{\mathbb{QS}} \mathsf{f}' \notin \mathsf{RaCauchy} \vee \mathsf{fq} *_{\mathbb{QS}} \mathsf{f}' \notin \mathsf{RaCauchy}$

-- Reasoning by contradiction, assume that fq, f' form a counterexample to the desired statement. We readily discard the possibility that $\mathsf{fq} -_{QS} \mathsf{f}' \notin \mathsf{RaCauchy}$. Only the possibility that $\mathsf{fq} *_{QS} \mathsf{f}' \notin \mathsf{RaCauchy}$ must be analyzed in detail, and we will reach a contradiction in this case also.

Suppose $\Rightarrow$ $\quad \mathsf{fq} -_{QS} \mathsf{f}' \notin \mathsf{RaCauchy}$

Use_def$(-_{QS}) \Rightarrow$ $\quad \mathsf{fq} +_{QS} \mathsf{Ras\_Rev}(\mathsf{f}') \notin \mathsf{RaCauchy}$

$\langle \mathsf{f}', \mathsf{f}' \rangle \hookrightarrow T413 \Rightarrow$ $\quad \mathsf{Ras\_Rev}(\mathsf{f}') \in \mathsf{RaCauchy}$

$\langle \mathsf{fq}, \mathsf{Ras\_Rev}(\mathsf{f}') \rangle \hookrightarrow T413 \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad \mathsf{fq} *_{QS} \mathsf{f}' \notin \mathsf{RaCauchy}$

-- Assuming that $\mathsf{fq} *_{QS} \mathsf{f}'$ is not a Cauchy sequence (unlike fq and f'), there would exist a positive real eps_0 for which the set

$$\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{QS} \mathsf{f}')\restriction\mathsf{i} -_{Q} (\mathsf{fq} *_{QS} \mathsf{f}')\restriction\mathsf{j}\big) >_{Q} \mathsf{eps}_0\}$$

is infinite, unlike the analogous sets which have f' and fq, respectively, in place of $\mathsf{fq} *_{QS} \mathsf{f}'$ and have, in place of eps_0, positive reals eps_1 and eps_2 smaller than one half of $\mathsf{eps}_0 /_{Q} \mathsf{m}$, where m is chosen to be larger than the absolute value of any component of fq and of any component of f'.

Use_def(RaCauchy) $\Rightarrow$ $\quad$ *Stat15* :

$\quad \mathsf{fq} *_{QS} \mathsf{f}' \notin \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{Q} \mathbf{0}_{Q} \rightarrow \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{Q} \mathsf{f}\restriction\mathsf{j}) >_{Q} \varepsilon\}\big)\rangle\}$ & *Stat0* :

$\qquad \mathsf{fq} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{Q} \mathbf{0}_{Q} \rightarrow \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{Q} \mathsf{f}\restriction\mathsf{j}) >_{Q} \varepsilon\}\big)\rangle\}$ & *Stat1* : $\mathsf{f}' \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{Q} \mathbf{0}_{Q} \rightarrow \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N},$

$\langle\ \rangle \hookrightarrow Stat0 \Rightarrow$ $\quad \mathsf{fq} \in \mathsf{RaSeq}$ & *Stat2* : $\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{Q} \mathbf{0}_{Q} \rightarrow \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction\mathsf{i} -_{Q} \mathsf{fq}\restriction\mathsf{j}) >_{Q} \varepsilon\}\big)\rangle$

$\langle\ \rangle \hookrightarrow Stat1 \Rightarrow$ $\quad \mathsf{f}' \in \mathsf{RaSeq}$ & *Stat3* : $\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{Q} \mathbf{0}_{Q} \rightarrow \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}'\restriction\mathsf{i} -_{Q} \mathsf{f}'\restriction\mathsf{j}) >_{Q} \varepsilon\}\big)\rangle$

$\langle \mathsf{fq} \rangle \hookrightarrow T413a \Rightarrow$ $\quad$ *Stat4* : $\mathbf{domain}(\mathsf{fq}) = \mathbb{N}$ & $\mathsf{Svm}(\mathsf{fq})$ & $\mathbf{range}(\mathsf{fq}) \subseteq \mathbb{Q}$

$\langle \mathsf{f}' \rangle \hookrightarrow T413a \Rightarrow$ $\quad$ *Stat5* : $\mathbf{domain}(\mathsf{f}') = \mathbb{N}$ & $\mathsf{Svm}(\mathsf{f}')$ & $\mathbf{range}(\mathsf{f}') \subseteq \mathbb{Q}$

$\langle \mathsf{fq}, \mathsf{f}' \rangle \hookrightarrow T10062 \Rightarrow$ $\quad$ *Stat14* : $\mathsf{fq} *_{QS} \mathsf{f}' \in \mathsf{RaSeq}$ & *Stat18* : $\mathsf{fq} *_{QS} \mathsf{f}' = \{[\mathsf{u}, \mathsf{fq}\restriction\mathsf{u} *_{Q} \mathsf{f}'\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$

$\langle\ \rangle \hookrightarrow Stat15(\langle Stat15 \rangle) \Rightarrow$ $\quad$ *Stat16* : $\neg\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{Q} \mathbf{0}_{Q} \rightarrow \mathsf{Finite}\Big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{QS} \mathsf{f}')\restriction\mathsf{i} -_{Q} (\mathsf{fq} *_{QS} \mathsf{f}')\restriction\mathsf{j}\big) >_{Q} \varepsilon\}\Big)\rangle$

$\langle \mathsf{eps}_0 \rangle \hookrightarrow Stat16(\langle Stat16 \rangle) \Rightarrow$ $\quad$ *Stat17* : $\mathsf{eps}_0 \in \mathbb{Q}$ & $\mathsf{eps}_0 >_{Q} \mathbf{0}_{Q}$ & $\neg\mathsf{Finite}\Big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{QS} \mathsf{f}')\restriction\mathsf{i} -_{Q} (\mathsf{fq} *_{QS} \mathsf{f}')\restriction\mathsf{j}\big) >_{Q} \mathsf{eps}_0\}\Big)$

$\langle \mathsf{fq} \rangle \hookrightarrow T10061a([Stat99, Stat99]) \Rightarrow$ $\quad$ *Stat18a* : $\langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{Q} \mathsf{x}\rangle$

$\langle \mathsf{mq} \rangle \hookrightarrow Stat18a(\langle Stat18a \rangle) \Rightarrow$ $\quad \mathsf{mq} \in \mathbb{Q}$ & *Stat33* : $\langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{Q} \mathsf{mq}\rangle$

$\langle \mathsf{f}' \rangle \hookrightarrow T10061a \Rightarrow$ $\quad$ *Stat19* : $\langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{Q} \mathsf{x}\rangle$

$\langle \mathsf{m}' \rangle \hookrightarrow Stat19(\langle Stat19 \rangle) \Rightarrow$ $\quad \mathsf{m}' \in \mathbb{Q}$ & *Stat44* : $\langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{Q} \mathsf{m}'\rangle$

Loc_def $\Rightarrow$ $\quad \mathsf{m} = \textbf{if}\ \mathsf{mq} >_{Q} \mathsf{m}'\ \textbf{then}\ \mathsf{mq}\ \textbf{else}\ \mathsf{m}'\ \textbf{fi}$

ELEM $\Rightarrow$ $\quad \mathsf{m} \in \mathbb{Q}$

Suppose $\Rightarrow$ $\quad$ *Stat66* : $\neg\langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{Q} \mathsf{m}\rangle$

$\langle \mathsf{y}_1 \rangle \hookrightarrow Stat66 \Rightarrow$ $\quad \mathsf{y}_1 \in \mathbf{range}(\mathsf{fq})$ & $\neg\mathsf{Ra\_ABS}(\mathsf{y}_1) <_{Q} \mathsf{m}$

$\langle \mathsf{y}_1 \rangle \hookrightarrow Stat33 \Rightarrow$ $\quad \mathsf{Ra\_ABS}(\mathsf{y}_1) <_{Q} \mathsf{mq}$

Suppose $\Rightarrow$ $\mathsf{m} = \mathsf{mq}$
EQUAL $\langle Stat66 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{m} \neq \mathsf{mq}$
ELEM $\Rightarrow$ $\neg \mathsf{mq} >_{\mathbb{Q}} \mathsf{m}'$
$\langle \mathsf{mq}, \mathsf{m}' \rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{m}' \geqslant_{\mathbb{Q}} \mathsf{mq} \;\&\; \mathsf{m} = \mathsf{m}'$
$\langle \mathsf{y}_1 \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{y}_1) \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(\mathsf{y}_1), \mathsf{mq} \rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{mq} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}_1)$
$\langle \mathsf{m}', \mathsf{mq}, \mathsf{Ra\_ABS}(\mathsf{y}_1) \rangle \hookrightarrow T406 \Rightarrow$ $\mathsf{m}' >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}_1)$
EQUAL $\langle Stat66 \rangle \Rightarrow$ $\mathsf{m} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}_1)$
$\langle \mathsf{Ra\_ABS}(\mathsf{y}_1), \mathsf{m} \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat55: \langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{m} \rangle$
Suppose $\Rightarrow$ $Stat7: \neg \langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{m} \rangle$
$\langle \mathsf{y}_2 \rangle \hookrightarrow Stat7 \Rightarrow$ $\mathsf{y}_2 \in \mathbf{range}(\mathsf{f}') \;\&\; \neg \mathsf{Ra\_ABS}(\mathsf{y}_2) <_{\mathbb{Q}} \mathsf{m}$
$\langle \mathsf{y}_2 \rangle \hookrightarrow Stat44 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{y}_2) <_{\mathbb{Q}} \mathsf{m}'$
Suppose $\Rightarrow$ $\mathsf{m} = \mathsf{m}'$
EQUAL $\langle Stat7 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{m} \neq \mathsf{m}'$
ELEM $\Rightarrow$ $\mathsf{mq} >_{\mathbb{Q}} \mathsf{m}' \;\&\; \mathsf{m} = \mathsf{mq}$
$\langle \mathsf{y}_2 \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{y}_2) \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(\mathsf{y}_2), \mathsf{m}' \rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{m}' >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}_2)$
$\langle \mathsf{mq}, \mathsf{m}', \mathsf{Ra\_ABS}(\mathsf{y}_2) \rangle \hookrightarrow T10041a \Rightarrow$ $\mathsf{mq} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}_2)$
EQUAL $\langle Stat7 \rangle \Rightarrow$ $\mathsf{m} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y}_2)$
$\langle \mathsf{Ra\_ABS}(\mathsf{y}_2), \mathsf{m} \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat77: \langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{m} \rangle$
ALGEBRA $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$
Suppose $\Rightarrow$ $\neg \mathsf{m} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$T182 \Rightarrow$ $\emptyset \in \mathbf{domain}(\mathsf{fq})$
$\langle \emptyset, \mathsf{fq} \rangle \hookrightarrow T64 \Rightarrow$ $\mathsf{fq} \upharpoonright \emptyset \in \mathbf{range}(\mathsf{fq})$
$\langle \mathsf{fq} \upharpoonright \emptyset \rangle \hookrightarrow Stat55 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \emptyset) <_{\mathbb{Q}} \mathsf{m}$
$\langle \mathsf{fq} \upharpoonright \emptyset \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \emptyset) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \emptyset), \mathsf{m} \rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{m} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \emptyset)$
$\langle \mathsf{fq} \upharpoonright \emptyset \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \emptyset) \in \mathbb{Q}$
$\langle \mathsf{m}, \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \emptyset), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T405 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{m} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{m}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{m} \neq \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow T380 \Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q} \;\&\; \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{1}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow T395 \Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{eps}_0, \mathbf{0}_{\mathbb{Q}}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T393 \Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})$
$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T394 \Rightarrow$ $\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$
ALGEBRA $\Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q} \;\&\; \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
EQUAL $\Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T10015 \Rightarrow$ $Stat20: \langle \exists \mathsf{e} \in \mathbb{Q}, \mathsf{e}' \in \mathbb{Q} \mid \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathsf{e} \;\&\; \mathsf{e} >_{\mathbb{Q}} \mathsf{e}' \;\&\; \mathsf{e}' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \mathsf{e} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathsf{e} +_{\mathbb{Q}} \mathsf{e}' \rangle$

$\langle \mathsf{eps}_1, \mathsf{eps}_2 \rangle \hookrightarrow \mathit{Stat20} \Rightarrow \quad \mathsf{eps}_1, \mathsf{eps}_2 \in \mathbb{Q} \ \& \ \mathsf{eps}_2 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \mathsf{eps}_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \ \& \ \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2$
$\langle \mathsf{eps}_1, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{eps}_1 \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{eps}_1, \mathbf{0}_{\mathbb{Q}}, \mathsf{eps}_2, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T402} \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T371} \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{m} \rangle \hookrightarrow \mathit{T395} \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \in \mathbb{Q}$
$\langle \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}), \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{m} \rangle \hookrightarrow \mathit{T393} \Rightarrow \quad \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{m} >_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) *_{\mathbb{Q}} \mathsf{m}$
$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{m} = \mathsf{eps}_0 *_{\mathbb{Q}} \big(\mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\big) \ \& \ (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) *_{\mathbb{Q}} \mathsf{m} = \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2)$
$\langle \mathsf{eps}_0 \rangle \hookrightarrow \mathit{T379} \Rightarrow \quad \mathsf{eps}_0 *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = \mathsf{eps}_0$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2)$

$\langle \mathsf{eps}_2 \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2\}\big)$
$\langle \mathsf{eps}_1 \rangle \hookrightarrow \mathit{Stat3} \Rightarrow \quad \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\}\big)$

-- However, this assumption would lead to a conflict with the inequality (holding for all $\mathsf{i}, \mathsf{j} \in \mathbb{N}$)

$$\mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{f}') \upharpoonright \mathsf{i} -_{\mathbb{Q}} (\mathsf{fq} *_{\mathbb{QS}} \mathsf{f}') \upharpoonright \mathsf{j}\big) \leqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j} +_{\mathbb{Q}} (\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j})\big),$$

as we are about to show.

$\mathsf{APPLY} \ \langle\,\rangle \ \mathsf{setformer\_meet\_join}\big(\mathsf{s} \mapsto \mathbb{N}, \mathsf{t} \mapsto \mathbb{N}, \mathsf{h}(\mathsf{i},\mathsf{j}) \mapsto \mathsf{i} \cap \mathsf{j}, \mathsf{P}(\mathsf{i},\mathsf{j}) \mapsto \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Q}(\mathsf{i},\mathsf{j}) \mapsto \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\big) \Rightarrow$
$\quad \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2 \vee \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} = \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2\} \cup \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}($
$\big\langle \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2\}, \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\} \big\rangle \hookrightarrow \mathit{T162} \Rightarrow$
$\quad \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2\} \cup \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\}\big)$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2 \vee \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\}\big)$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat21}:$
$\quad \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{f}') \upharpoonright \mathsf{i} -_{\mathbb{Q}} (\mathsf{fq} *_{\mathbb{QS}} \mathsf{f}') \upharpoonright \mathsf{j}\big) >_{\mathbb{Q}} \mathsf{eps}_0\} \not\subseteq \{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{fq} \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_2 \vee \mathsf{Ra\_ABS}(\mathsf{f}' \upharpoonright \mathsf{i} -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{j}) >_{\mathbb{Q}} \mathsf{eps}_1\}$

-- If we make the temporary assumption that

$$\neg\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}((fq *_{QS} f')\restriction i -_{Q} (fq *_{QS} f')\restriction j) >_{Q} eps_0\} \subseteq \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(fq\restriction i -_{Q} fq\restriction j) >_{Q} eps_2 \vee \mathsf{Ra\_ABS}(f'\restriction i -_{Q} f'\restriction j) >_{Q} eps_1\},$$

then, by proving the above-stated inequality, we reach a contradiction as follows. In the first place, observe that if i_0, j_0 are unsigned integers for which $i_0 \cap j_0$ belongs to the set appearing as left-hand side but does not belong to the right-hand side then

$$\mathsf{Ra\_ABS}(fq\restriction i_0 *_{Q} f'\restriction i_0 -_{Q} fq\restriction j_0 *_{Q} f'\restriction j_0) >_{Q} eps_0,$$

$$\mathsf{Ra\_ABS}(f'\restriction i_0 -_{Q} f'\restriction j_0) \leqslant_{Q} eps_1 \;\&\; \mathsf{Ra\_ABS}(fq\restriction i_0 -_{Q} fq\restriction j_0) \leqslant_{Q} eps_2.$$

$\langle i_0, j_0\rangle \hookrightarrow Stat21 \Rightarrow \quad Stat21a:$
$i_0, j_0 \in \mathbb{N} \;\&\; \mathsf{Ra\_ABS}((fq *_{QS} f')\restriction i_0 -_{Q} (fq *_{QS} f')\restriction j_0) >_{Q} eps_0 \;\&\; \neg\mathsf{Ra\_ABS}(fq\restriction i_0 -_{Q} fq\restriction j_0) >_{Q} eps_2 \;\&\; \neg\mathsf{Ra\_ABS}(f'\restriction i_0 -_{Q} f'\restriction j_0) >_{Q} eps_1$
APPLY $\langle\,\rangle$ fcn_symbol$(f(u) \mapsto fq\restriction u *_{Q} f'\restriction u, g \mapsto fq *_{QS} f', s \mapsto \mathbb{N}) \Rightarrow$
$Stat22: \langle \forall x \mid (fq *_{QS} f')\restriction x = \textbf{if } x \in \mathbb{N} \textbf{ then } fq\restriction x *_{Q} f'\restriction x \textbf{ else } \emptyset \textbf{ fi}\rangle$
$\langle i_0\rangle \hookrightarrow Stat22 \Rightarrow \quad (fq *_{QS} f')\restriction i_0 = fq\restriction i_0 *_{Q} f'\restriction i_0$
$\langle j_0\rangle \hookrightarrow Stat22 \Rightarrow \quad (fq *_{QS} f')\restriction j_0 = fq\restriction j_0 *_{Q} f'\restriction j_0$
EQUAL $\langle Stat21\rangle \Rightarrow \quad \mathsf{Ra\_ABS}(fq\restriction i_0 *_{Q} f'\restriction i_0 -_{Q} fq\restriction j_0 *_{Q} f'\restriction j_0) >_{Q} eps_0$

-- Secondly, we easily check that various quantities belong to $\mathbb{Q}$.

Suppose $\Rightarrow \quad fq\restriction i_0 \notin \mathbf{range}(fq)$
$\langle fq\rangle \hookrightarrow T66(\langle Stat4\rangle) \Rightarrow \quad Stat24: fq\restriction i_0 \notin \{fq\restriction j : j \in \mathbf{domain}(fq)\}$
$\langle i_0\rangle \hookrightarrow Stat24([Stat4, Stat21a]) \Rightarrow$ false; Discharge $\Rightarrow \quad fq\restriction i_0 \in \mathbf{range}(fq)$
Suppose $\Rightarrow \quad f'\restriction j_0 \notin \mathbf{range}(f')$
$\langle f'\rangle \hookrightarrow T66(\langle Stat5\rangle) \Rightarrow \quad Stat23a: f'\restriction j_0 \notin \{f'\restriction j : j \in \mathbf{domain}(f')\}$
$\langle j_0\rangle \hookrightarrow Stat23a([Stat5, Stat21a]) \Rightarrow$ false; Discharge $\Rightarrow \quad f'\restriction j_0 \in \mathbf{range}(f')$
ELEM $\Rightarrow \quad Stat88: fq\restriction i_0 \in \mathbb{Q}$
ELEM $\Rightarrow \quad Stat89: f'\restriction j_0 \in \mathbb{Q}$
$\langle fq\restriction i_0\rangle \hookrightarrow T10045(\langle Stat88\rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(fq\restriction i_0) \in \mathbb{Q}$
Suppose $\Rightarrow \quad fq\restriction j_0 \notin \mathbb{Q}$
$\langle fq\rangle \hookrightarrow T66(\langle Stat4\rangle) \Rightarrow \quad Stat23: fq\restriction j_0 \notin \{fq\restriction j : j \in \mathbf{domain}(fq)\}$
$\langle j_0\rangle \hookrightarrow Stat23([Stat4, Stat21a]) \Rightarrow$ false; Discharge $\Rightarrow \quad Stat86: fq\restriction j_0 \in \mathbb{Q}$
$\langle fq\restriction j_0\rangle \hookrightarrow T372(\langle Stat88\rangle) \Rightarrow \quad \mathsf{Rev}_{Q}(fq\restriction j_0) \in \mathbb{Q}$
$\langle fq\restriction i_0, \mathsf{Rev}_{Q}(fq\restriction j_0)\rangle \hookrightarrow T365(\langle Stat88\rangle) \Rightarrow \quad fq\restriction i_0 +_{Q} \mathsf{Rev}_{Q}(fq\restriction j_0) \in \mathbb{Q}$
Use_def$(-_{Q}) \Rightarrow \quad fq\restriction i_0 -_{Q} fq\restriction j_0 \in \mathbb{Q}$
$\langle fq\restriction i_0 -_{Q} fq\restriction j_0\rangle \hookrightarrow T10045(\langle Stat88\rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(fq\restriction i_0 -_{Q} fq\restriction j_0) \in \mathbb{Q}$
Suppose $\Rightarrow \quad f'\restriction i_0 \notin \mathbb{Q}$
$\langle f'\rangle \hookrightarrow T66(\langle Stat5\rangle) \Rightarrow \quad Stat24a: f'\restriction i_0 \notin \{f'\restriction j : j \in \mathbf{domain}(f')\}$

$\langle i_0 \rangle \hookrightarrow Stat24a([Stat5, Stat21a]) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad f' \upharpoonright i_0 \in \mathbb{Q}$
$\langle f' \upharpoonright j_0 \rangle \hookrightarrow T10045(\langle Stat88 \rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle f' \upharpoonright j_0 \rangle \hookrightarrow T372(\langle Stat88 \rangle) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle f' \upharpoonright i_0, \mathsf{Rev}_{\mathbb{Q}}(f' \upharpoonright j_0) \rangle \hookrightarrow T365(\langle Stat88 \rangle) \Rightarrow \quad f' \upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f' \upharpoonright j_0) \in \mathbb{Q}$
$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0 \in \mathbb{Q}$
$\langle f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0 \rangle \hookrightarrow T10045(\langle Stat88 \rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle m, eps_1 +_{\mathbb{Q}} eps_2 \rangle \hookrightarrow T368 \Rightarrow \quad m *_{\mathbb{Q}} (eps_1 +_{\mathbb{Q}} eps_2) \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0), \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \rangle \hookrightarrow T365(\langle Stat88 \rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0, f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0 \rangle \hookrightarrow T365(\langle Stat88 \rangle) \Rightarrow$
$\quad fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0 +_{\mathbb{Q}} (f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0 +_{\mathbb{Q}} (f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \rangle \hookrightarrow T10045(\langle Stat88 \rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}\big(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0 +_{\mathbb{Q}} (f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0)\big) \in \mathbb{Q}$
$\langle fq \upharpoonright i_0, f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0 \rangle \hookrightarrow T368(\langle Stat88 \rangle) \Rightarrow$
$\quad fq \upharpoonright i_0 *_{\mathbb{Q}} (f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle f' \upharpoonright j_0, fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0 \rangle \hookrightarrow T368(\langle Stat88 \rangle) \Rightarrow$
$\quad f' \upharpoonright j_0 *_{\mathbb{Q}} (fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q}$
$\langle fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0 \rangle \hookrightarrow T10045(\langle Stat88 \rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q} \;\&$
$\quad \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0 \rangle \hookrightarrow T10045(\langle Stat88 \rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q} \;\&$
$\quad \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle m, \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \rangle \hookrightarrow T368 \Rightarrow \quad m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q}$
$\langle m, \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \rangle \hookrightarrow T368 \Rightarrow \quad m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0), m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \rangle \hookrightarrow T365(\langle Stat88 \rangle) \Rightarrow$
$\quad m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) +_{\mathbb{Q}} m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(f' \upharpoonright j_0), \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \rangle \hookrightarrow T368(\langle Stat88 \rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}(f' \upharpoonright j_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(fq \upharpoonright i_0), \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \rangle \hookrightarrow T368(\langle Stat88 \rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}(fq \upharpoonright i_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0), \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \rangle \hookrightarrow T365(\langle Stat88 \rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q}$
$\langle fq \upharpoonright i_0, f' \upharpoonright i_0 \rangle \hookrightarrow T368(\langle Stat88 \rangle) \Rightarrow \quad fq \upharpoonright i_0 *_{\mathbb{Q}} f' \upharpoonright i_0 \in \mathbb{Q}$
$\langle fq \upharpoonright i_0, f' \upharpoonright j_0 \rangle \hookrightarrow T368(\langle Stat88 \rangle) \Rightarrow \quad fq \upharpoonright i_0 *_{\mathbb{Q}} f' \upharpoonright j_0 \in \mathbb{Q}$
$\langle \mathsf{Ra\_ABS}(fq \upharpoonright i_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0), \mathsf{Ra\_ABS}(f' \upharpoonright j_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \rangle \hookrightarrow T365 \Rightarrow$
$\qquad \mathsf{Ra\_ABS}(fq \upharpoonright i_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright i_0 -_{\mathbb{Q}} f' \upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f' \upharpoonright j_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq \upharpoonright i_0 -_{\mathbb{Q}} fq \upharpoonright j_0) \in \mathbb{Q}$
$\langle fq \upharpoonright i_0 *_{\mathbb{Q}} f' \upharpoonright j_0 \rangle \hookrightarrow T372(\langle Stat88 \rangle) \Rightarrow \quad Stat87:$
$\quad \mathsf{Rev}_{\mathbb{Q}}(fq \upharpoonright i_0 *_{\mathbb{Q}} f' \upharpoonright j_0) \in \mathbb{Q} \;\&\; fq \upharpoonright i_0 *_{\mathbb{Q}} f' \upharpoonright j_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(fq \upharpoonright i_0 *_{\mathbb{Q}} f' \upharpoonright j_0) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0, \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T370}(\langle \mathit{Stat88}\rangle) \Rightarrow$

$\quad \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \left(\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0\right) =$

$\qquad \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0$

$\langle \mathsf{fq}\upharpoonright \mathsf{j}_0, \mathsf{f}'\upharpoonright \mathsf{j}_0\rangle \hookrightarrow \mathit{T368}([\mathit{Stat89}, \mathit{Stat86}]) \Rightarrow \quad \mathit{Stat85}:$

$\quad \mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0 \in \mathbb{Q} \;\&\; \mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0 = \mathsf{f}'\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0$

$\langle \mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0\rangle \hookrightarrow \mathit{T372}(\langle \mathit{Stat85}\rangle) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T365}(\langle \mathit{Stat88}\rangle) \Rightarrow$

$\quad \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) \in \mathbb{Q}$

$\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0), \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0), \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0\rangle \hookrightarrow \mathit{T370}(\langle \mathit{Stat88}\rangle) \Rightarrow$

$\quad \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) =$

$\qquad \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \left(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0)\right)$

$\langle \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T365}(\langle \mathit{Stat86}\rangle) \Rightarrow$

$\quad \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) \in \mathbb{Q}$

-- By exploiting some of the above membership relations, we easily get

$$\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\,,$$

and hence

$$\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\,.$$

$\langle \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0), \mathsf{eps}_1\rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{eps}_1 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0)$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0), \mathsf{eps}_2\rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\langle \mathsf{eps}_1, \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0), \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T397} \Rightarrow$

$\quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

Suppose $\Rightarrow \quad \neg \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) \geqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0), \mathsf{m}, \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T376} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} \left(\mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\right) =$

$\quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\langle \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 = \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0) \vee$

$\quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

Suppose $\Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 = \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

EQUAL $\Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) = \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\langle \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0), \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2)\rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0$

$\langle \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0), \mathsf{m}\rangle \hookrightarrow \mathit{T393} \Rightarrow$

$\quad (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) *_{\mathbb{Q}} \mathsf{m} >_{\mathbb{Q}} \left(\mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\right) *_{\mathbb{Q}} \mathsf{m}$

$\langle \mathsf{m}, \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2\rangle \hookrightarrow \mathit{T368} \Rightarrow \quad \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) = (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) *_{\mathbb{Q}} \mathsf{m}$

$\langle \mathsf{m}, \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\rangle \hookrightarrow \mathit{T368} \Rightarrow \quad \left(\mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\right) *_{\mathbb{Q}} \mathsf{m} =$

$\mathsf{m} *_{\mathbb{Q}} \big(\mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\big)$

EQUAL $\Rightarrow$ $\mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) >_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2), \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\big\rangle \hookrightarrow \mathit{T384} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2) \geqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_AB}$

$\big\langle \mathsf{eps}_0, \mathsf{m} *_{\mathbb{Q}} (\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2), \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\big\rangle \hookrightarrow \mathit{T405} \Rightarrow$
$\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{eps}_0, \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\big\rangle \hookrightarrow \mathit{T384} \Rightarrow$
$\mathsf{eps}_0 \geqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

-- As a consequence, $\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2$ is greater than or equal to

$$\mathsf{Ra\_ABS}\left(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} (\mathsf{fq}\upharpoonright \mathsf{j}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0)\right) ,$$

because this quantity equals

$$\mathsf{Ra\_ABS}\left(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\right) ,$$

$\big\langle \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0), \mathsf{eps}_2\big\rangle \hookrightarrow \mathit{T384} \Rightarrow$ $\mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0), \mathsf{eps}_1\big\rangle \hookrightarrow \mathit{T384} \Rightarrow$ $\mathsf{eps}_1 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{eps}_1, \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0), \mathsf{eps}_2, \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)\big\rangle \hookrightarrow \mathit{T397} \Rightarrow$
$\mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

Suppose $\Rightarrow$ $\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 \neq$
$\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f'}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0\big\rangle \hookrightarrow \mathit{T371}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0 =$
$\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\big\langle \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0)\big\rangle \hookrightarrow \mathit{T365}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) =$
$\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0$

$\big\langle \mathsf{fq}\upharpoonright \mathsf{i}_0, \mathsf{f'}\upharpoonright \mathsf{j}_0\big\rangle \hookrightarrow \mathit{T391}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f'}\upharpoonright \mathsf{j}_0) =$
$\mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f'}\upharpoonright \mathsf{j}_0), \mathsf{fq}\upharpoonright \mathsf{i}_0, \mathsf{f'}\upharpoonright \mathsf{i}_0\big\rangle \hookrightarrow \mathit{T376}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f'}\upharpoonright \mathsf{j}_0) =$
$\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \big(\mathsf{f'}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f'}\upharpoonright \mathsf{j}_0)\big)$

$\big\langle \mathsf{fq}\upharpoonright \mathsf{i}_0, \mathsf{f'}\upharpoonright \mathsf{j}_0\big\rangle \hookrightarrow \mathit{T368}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 =$
$\mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0$

$\big\langle \mathsf{f'}\upharpoonright \mathsf{j}_0, \mathsf{fq}\upharpoonright \mathsf{j}_0\big\rangle \hookrightarrow \mathit{T391}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{j}_0) =$
$\mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0)$

$\big\langle \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0), \mathsf{f'}\upharpoonright \mathsf{j}_0, \mathsf{fq}\upharpoonright \mathsf{i}_0\big\rangle \hookrightarrow \mathit{T376}(\big\langle \mathit{Stat88}\big\rangle) \Rightarrow$ $\mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \big(\mathsf{fq}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0)\big) =$
$\mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{fq}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0)$

EQUAL $\big\langle \mathit{Stat87}\big\rangle \Rightarrow$
$\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0) =$
$\mathsf{fq}\upharpoonright \mathsf{i}_0 *_{\mathbb{Q}} \big(\mathsf{f'}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f'}\upharpoonright \mathsf{j}_0)\big) +_{\mathbb{Q}} \mathsf{f'}\upharpoonright \mathsf{j}_0 *_{\mathbb{Q}} \big(\mathsf{fq}\upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{fq}\upharpoonright \mathsf{j}_0)\big)$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow$ $\mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ $\mathit{Stat51}:\ \mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0 = \mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)$

$\mathsf{EQUAL}\ \langle \mathit{Stat51}\rangle \Rightarrow$
$\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) =$
$\mathsf{Ra\_ABS}\big(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\big)$

-- ..., in its turn

$$\mathsf{Ra\_ABS}\big(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\big)$$

is smaller than or equal to

$$\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{j}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\ ,$$

$\mathit{T10050} \Rightarrow$ $\mathit{Stat25}:\ \langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x}, \mathsf{y} \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(\mathsf{x} +_{\mathbb{Q}} \mathsf{y}) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{x}) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{y})\rangle$

$\langle \mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0), \mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\rangle \hookrightarrow \mathit{Stat25} \Rightarrow$
$\mathsf{Ra\_ABS}\big(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\big) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)\big) +_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(\mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\big)$

$\langle \mathsf{fq}\restriction \mathsf{i}_0, \mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0\rangle \hookrightarrow \mathit{T10046} \Rightarrow$ $\mathsf{Ra\_ABS}\big(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} (\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)\big) =$
$\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)$

$\langle \mathsf{f}'\restriction \mathsf{j}_0, \mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0\rangle \hookrightarrow \mathit{T10046} \Rightarrow$ $\mathsf{Ra\_ABS}\big(\mathsf{f}'\restriction \mathsf{j}_0 *_{\mathbb{Q}} (\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\big) =$
$\mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{j}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)$

$\mathsf{EQUAL}\ \langle \mathit{Stat51}\rangle \Rightarrow$
$\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{j}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0), \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{j}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\rangle \hookrightarrow \mathit{T384} \Rightarrow$
$\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{j}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0 *_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)$

-- ... which in its turn is smaller than or equal to

$$\mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) +_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{fq}\restriction \mathsf{j}_0)\ ,$$

$\langle \mathsf{m}\rangle \hookrightarrow \mathit{T394} \Rightarrow$ $\mathsf{m} *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Suppose} \Rightarrow$ $\neg \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)$

$\langle \mathsf{fq}\restriction \mathsf{i}_0\rangle \hookrightarrow \mathit{Stat55} \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) <_{\mathbb{Q}} \mathsf{m}$

$\langle \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0), \mathsf{m}\rangle \hookrightarrow \mathit{T384} \Rightarrow$ $\mathsf{m} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0)$

$\langle \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow \mathit{T384} \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee$
$\mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Suppose} \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}, \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0), \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)\rangle \hookrightarrow \mathit{T393} \Rightarrow$
$\mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0) >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{f}'\restriction \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{j}_0)$

$\langle m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0), \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0)\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Ra\_ABS}(fq\upharpoonright i_0)\rangle \hookrightarrow T394 \Rightarrow$ $\mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$

EQUAL $\Rightarrow$ $m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) = \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0)$

$\langle \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0), m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0)\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0)$

Suppose $\Rightarrow$ $\neg m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)$

$\langle f'\upharpoonright j_0\rangle \hookrightarrow Stat77 \Rightarrow$ $\mathsf{Ra\_ABS}(f'\upharpoonright j_0) <_{\mathbb{Q}} m$

$\langle \mathsf{Ra\_ABS}(f'\upharpoonright j_0), m\rangle \hookrightarrow T384 \Rightarrow$ $m >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright j_0)$

$\langle \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee$
$\mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) = \mathbf{0}_{\mathbb{Q}}$

Suppose $\Rightarrow$ $\mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle m, \mathsf{Ra\_ABS}(f'\upharpoonright j_0), \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)\rangle \hookrightarrow T393 \Rightarrow$
$m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) >_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)$

$\langle m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Ra\_ABS}(f'\upharpoonright j_0)\rangle \hookrightarrow T394 \Rightarrow$ $\mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$

EQUAL $\Rightarrow$ $m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) = \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)$

$\langle \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)$

$\langle m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0), \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0), m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)\rangle \hookrightarrow T397 \Rightarrow$
$m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0)$

-- …hence, by exploiting the transitivity laws which the ordering of rationals obeys, we get to the absurd conclusion that $eps_0 >_{\mathbb{Q}} eps_0$.

Use_def $(-_{\mathbb{Q}}) \Rightarrow$ $fq\upharpoonright i_0 \ast_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0 \ast_{\mathbb{Q}} f'\upharpoonright j_0 \in \mathbb{Q}$

$\langle fq\upharpoonright i_0 \ast_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0 \ast_{\mathbb{Q}} f'\upharpoonright j_0\rangle \hookrightarrow T10045(\langle Stat88\rangle) \Rightarrow$
$\mathsf{Ra\_ABS}(fq\upharpoonright i_0 \ast_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0 \ast_{\mathbb{Q}} f'\upharpoonright j_0) \in \mathbb{Q}$

$\langle m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), \mathsf{Ra\_ABS}(fq\upharpoonright i_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright j_0) \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), \mathsf{Ra\_ABS}(fq\upharpoonright i_0 \ast_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0 \ast_{\mathbb{Q}} f'\upharpoonright j_0)\rangle \hookrightarrow T40$

$m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 \ast_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0 \ast_{\mathbb{Q}} f'\upharpoonright j_0)$

$\langle m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), \mathsf{Ra\_ABS}(fq\upharpoonright i_0 \ast_{\mathbb{Q}} f'\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0 \ast_{\mathbb{Q}} f'\upharpoonright j_0), eps_0\rangle \hookrightarrow T406 \Rightarrow$
$m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0) >_{\mathbb{Q}} eps_0$

$\langle eps_0, m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright j_0) +_{\mathbb{Q}} m \ast_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} fq\upharpoonright j_0), eps_0\rangle \hookrightarrow T406 \Rightarrow$
$eps_0 >_{\mathbb{Q}} eps_0$

$\langle eps_0, eps_0\rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$
$\{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}((fq \ast_{\mathbb{Q}s} f')\upharpoonright i -_{\mathbb{Q}} (fq \ast_{\mathbb{Q}s} f')\upharpoonright j) >_{\mathbb{Q}} eps_0\} \subseteq \{i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(fq\upharpoonright i -_{\mathbb{Q}} fq\upharpoonright j) >_{\mathbb{Q}} eps_2 \vee \mathsf{Ra\_ABS}(f'\upharpoonright i -_{\mathbb{Q}} f'\upharpoonright j) >_{\mathbb{Q}} eps_1\}$

-- Since the inclusion just proves entails that the set on the left-hand side is finite, we have reached the desired contradiction, proving the statement of the present theorem.

$\langle \{ i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}(fq \restriction i -_{\mathbb{Q}} fq \restriction j) >_{\mathbb{Q}} eps_1 \vee \mathsf{Ra\_ABS}(f' \restriction i -_{\mathbb{Q}} f' \restriction j) >_{\mathbb{Q}} eps_2 \}, \{ i \cap j : i \in \mathbb{N}, j \in \mathbb{N} \mid \mathsf{Ra\_ABS}((fq *_{\mathbb{QS}} f') \restriction i -_{\mathbb{Q}} (fq *_{\mathbb{QS}} f') \restriction j) >_{\mathbb{Q}} eps_0 \} \rangle \hookrightarrow T162(\langle Stat17 \rangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 601 (414) $F, G \in \mathsf{RaCauchy} \rightarrow F /_{\mathbb{QS}} G \in \mathsf{RaCauchy}$. PROOF:

Suppose_not$(f, g) \Rightarrow$ $f, g \in \mathsf{RaCauchy} \;\&\; f /_{\mathbb{QS}} g \notin \mathsf{RaCauchy}$

TSomehow $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemma states that when f, f' and g, g' are rational sequences with f equivalent to g and f' equivalent to g' then the pointwise sum of f and f' is equivalent to the pointwise sum of g and g'.

Theorem 602 (10067) $\{F, G, Fp, Gp\} \subseteq \mathsf{RaSeq} \;\&\; \mathsf{Ra\_eqseq}(F, G) \;\&\; \mathsf{Ra\_eqseq}(Fp, Gp) \rightarrow \mathsf{Ra\_eqseq}(F +_{\mathbb{QS}} Fp, G +_{\mathbb{QS}} Gp)$. PROOF:

Suppose_not$(fq, gq, f', g') \Rightarrow$ $\{fq, gq, f', g'\} \subseteq \mathsf{RaSeq} \;\&\; \mathsf{Ra\_eqseq}(fq, gq) \;\&\; \mathsf{Ra\_eqseq}(f', g') \;\&\; \neg \mathsf{Ra\_eqseq}(fq +_{\mathbb{QS}} f', gq +_{\mathbb{QS}} g')$

-- For, assuming fq, gq, f', g' to be a counterexample to the statement of this lemma, we reach a contradiction by arguing as follows. The very definition of equivalence between rational sequences entails that for some positive rational number $\varepsilon = eps_0$, the set

$$\{ x : x \in \mathbf{domain}(fq +_{\mathbb{QS}} f') \mid \mathsf{Ra\_ABS}((fq +_{\mathbb{QS}} f') \restriction x -_{\mathbb{Q}} (gq +_{\mathbb{QS}} g') \restriction x) >_{\mathbb{Q}} \varepsilon \}$$

is infinite, whereas assuming that eps_1, eps_2 (also positive) aresuch that eps_0 is greater than the sum of eps_1, eps_2, the two analogous sets which have fq, gq, eps_1 and f', g', eps_2, respectively, in place of $fq +_{\mathbb{QS}} f'$, $gq +_{\mathbb{QS}} g'$, and eps_0, are finite.

Use_def(Ra_eqseq) $\Rightarrow$ *Stat1* : $\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{ x : x \in \mathbf{domain}(fq) \mid \mathsf{Ra\_ABS}(fq \restriction x -_{\mathbb{Q}} gq \restriction x) >_{\mathbb{Q}} \varepsilon \}) \rangle \;\&\;$ *Stat2* : $\langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{ x : x \in \mathbf{domain}(f') \mid \mathsf{Ra\_ABS}(f' \restriction x -_{\mathbb{Q}} g' \restriction x) >_{\mathbb{Q}} \varepsilon \}) \rangle \;\&\;$ *Stat3* : $\neg \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}\Big(\{ x : x \in \mathbf{domain}(fq +_{\mathbb{QS}} f') \mid \mathsf{Ra\_ABS}((fq +_{\mathbb{Q}}$

$\langle eps_0 \rangle \hookrightarrow Stat3 \Rightarrow$ $eps_0 \in \mathbb{Q} \;\&\; eps_0 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; \neg \mathsf{Finite}\Big(\{ x : x \in \mathbf{domain}(fq +_{\mathbb{QS}} f') \mid \mathsf{Ra\_ABS}((fq +_{\mathbb{QS}} f') \restriction x -_{\mathbb{Q}} (gq +_{\mathbb{QS}} g') \restriction x) >_{\mathbb{Q}} eps_0 \} \Big)$

$\langle eps_0 \rangle \hookrightarrow T10015 \Rightarrow$ *Stat20* : $\langle \exists e \in \mathbb{Q}, e' \in \mathbb{Q} \mid eps_0 >_{\mathbb{Q}} e \;\&\; e >_{\mathbb{Q}} e' \;\&\; e' >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; e >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; eps_0 >_{\mathbb{Q}} e +_{\mathbb{Q}} e' \rangle$

$\langle eps_1, eps_2 \rangle \hookrightarrow Stat20 \Rightarrow$ $eps_1, eps_2 \in \mathbb{Q} \;\&\; eps_2 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; eps_1 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \;\&\; eps_0 >_{\mathbb{Q}} eps_1 +_{\mathbb{Q}} eps_2$

$\langle eps_1 \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{Finite}(\{ x : x \in \mathbf{domain}(fq) \mid \mathsf{Ra\_ABS}(fq \restriction x -_{\mathbb{Q}} gq \restriction x) >_{\mathbb{Q}} eps_1 \})$

$\langle eps_2 \rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{Finite}(\{ x : x \in \mathbf{domain}(f') \mid \mathsf{Ra\_ABS}(f' \restriction x -_{\mathbb{Q}} g' \restriction x) >_{\mathbb{Q}} eps_2 \})$

-- However, reasoning by contradiction, we show that the above indicated infinite set is included in the union of the other two sets. Therefore, it cannot be infinite.

Suppose $\Rightarrow$ *Stat4* :

$\{x : x \in \mathbf{domain}(fq +_{QS} f') \mid \text{Ra\_ABS}((fq +_{QS} f') \upharpoonright x -_{Q} (gq +_{QS} g') \upharpoonright x) >_{Q} eps_0\} \not\subseteq \{x : x \in \mathbf{domain}(fq) \mid \text{Ra\_ABS}(fq \upharpoonright x -_{Q} gq \upharpoonright x) >_{Q} eps_1\} \cup \{x : x \in \mathbf{domain}(f') \mid \text{Ra\_A}$

-- The inclusion is proved by assuming it false and then deriving the absurdity that eps_0 is greater than itself. On the one hand, we prove that

$$eps_0 >_{Q} \text{Ra\_ABS}(fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0) +_{Q} \text{Ra\_ABS}(f' \upharpoonright i_0 -_{Q} g' \upharpoonright i_0)\,.$$

$\langle fq, f' \rangle \hookrightarrow T10062 \Rightarrow$ $fq +_{QS} f' \in \text{RaSeq} \;\&\; fq +_{QS} f' = \{[u, fq \upharpoonright u +_{Q} f' \upharpoonright u] : u \in \mathbb{N}\}$

$\langle gq, g' \rangle \hookrightarrow T10062 \Rightarrow$ $gq +_{QS} g' = \{[u, gq \upharpoonright u +_{Q} g' \upharpoonright u] : u \in \mathbb{N}\}$

$\langle fq \rangle \hookrightarrow T413a \Rightarrow$ *Stat11* : $\mathbf{domain}(fq) = \mathbb{N} \;\&\; \text{Svm}(fq) \;\&\; \mathbf{range}(fq) \subseteq \mathbb{Q}$

$\langle f' \rangle \hookrightarrow T413a \Rightarrow$ *Stat12* : $\mathbf{domain}(f') = \mathbb{N} \;\&\; \text{Svm}(f') \;\&\; \mathbf{range}(f') \subseteq \mathbb{Q}$

$\langle gq \rangle \hookrightarrow T413a \Rightarrow$ *Stat13* : $\mathbf{domain}(gq) = \mathbb{N} \;\&\; \text{Svm}(gq) \;\&\; \mathbf{range}(gq) \subseteq \mathbb{Q}$

$\langle g' \rangle \hookrightarrow T413a \Rightarrow$ *Stat14* : $\mathbf{domain}(g') = \mathbb{N} \;\&\; \text{Svm}(g') \;\&\; \mathbf{range}(g') \subseteq \mathbb{Q}$

$\langle fq +_{QS} f' \rangle \hookrightarrow T413a \Rightarrow$ $\mathbf{domain}(fq +_{QS} f') = \mathbb{N} \;\&\; \text{Svm}(fq +_{QS} f') \;\&\; \mathbf{range}(fq +_{QS} f') \subseteq \mathbb{Q}$

EQUAL $\langle Stat4 \rangle \Rightarrow$ *Stat4a* :

$\{x : x \in \mathbb{N} \mid \text{Ra\_ABS}((fq +_{QS} f') \upharpoonright x -_{Q} (gq +_{QS} g') \upharpoonright x) >_{Q} eps_0\} \not\subseteq \{x : x \in \mathbb{N} \mid \text{Ra\_ABS}(fq \upharpoonright x -_{Q} gq \upharpoonright x) >_{Q} eps_1\} \cup \{x : x \in \mathbb{N} \mid \text{Ra\_ABS}(f' \upharpoonright x -_{Q} g' \upharpoonright x) >_{Q} eps_2\}$

$\langle c \rangle \hookrightarrow Stat4a \Rightarrow$ *Stat5* :

$c \in \{x : x \in \mathbb{N} \mid \text{Ra\_ABS}((fq +_{QS} f') \upharpoonright x -_{Q} (gq +_{QS} g') \upharpoonright x) >_{Q} eps_0\} \;\&\; c \notin \{x : x \in \mathbb{N} \mid \text{Ra\_ABS}(fq \upharpoonright x -_{Q} gq \upharpoonright x) >_{Q} eps_1\} \;\&$
$c \notin \{x : x \in \mathbb{N} \mid \text{Ra\_ABS}(f' \upharpoonright x -_{Q} g' \upharpoonright x) >_{Q} eps_2\}$

$\langle i_0, i_0, i_0 \rangle \hookrightarrow Stat5 \Rightarrow$ *Stat10* :

$i_0 \in \mathbb{N} \;\&\; \text{Ra\_ABS}((fq +_{QS} f') \upharpoonright i_0 -_{Q} (gq +_{QS} g') \upharpoonright i_0) >_{Q} eps_0 \;\&\; \neg\text{Ra\_ABS}(fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0) >_{Q} eps_1 \;\&\; \neg\text{Ra\_ABS}(f' \upharpoonright i_0 -_{Q} g' \upharpoonright i_0) >_{Q} eps_2$

Suppose $\Rightarrow$ $fq \upharpoonright i_0 \notin \mathbb{Q}$

$\langle fq \rangle \hookrightarrow T66(\langle Stat11 \rangle) \Rightarrow$ *Stat24* : $fq \upharpoonright i_0 \notin \{fq \upharpoonright j : j \in \mathbf{domain}(fq)\}$

$\langle i_0 \rangle \hookrightarrow Stat24([Stat11, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat6* : $fq \upharpoonright i_0 \in \mathbb{Q}$

Suppose $\Rightarrow$ $gq \upharpoonright i_0 \notin \mathbb{Q}$

$\langle gq \rangle \hookrightarrow T66(\langle Stat13 \rangle) \Rightarrow$ *Stat25* : $gq \upharpoonright i_0 \notin \{gq \upharpoonright j : j \in \mathbf{domain}(gq)\}$

$\langle i_0 \rangle \hookrightarrow Stat25([Stat13, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ $gq \upharpoonright i_0 \in \mathbb{Q}$

$\langle gq \upharpoonright i_0 \rangle \hookrightarrow T372 \Rightarrow$ $\text{Rev}_{Q}(gq \upharpoonright i_0) \in \mathbb{Q}$

$\langle fq \upharpoonright i_0, \text{Rev}_{Q}(gq \upharpoonright i_0) \rangle \hookrightarrow T365(\langle Stat6 \rangle) \Rightarrow$ $fq \upharpoonright i_0 +_{Q} \text{Rev}_{Q}(gq \upharpoonright i_0) \in \mathbb{Q}$

Use_def $(-_{Q}) \Rightarrow$ $fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0 \in \mathbb{Q}$

$\langle fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0 \rangle \hookrightarrow T10045 \Rightarrow$ $\text{Ra\_ABS}(fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0) \in \mathbb{Q}$

$\langle \text{Ra\_ABS}(fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0), eps_1 \rangle \hookrightarrow T384(\langle Stat20 \rangle) \Rightarrow$ $eps_1 \geqslant_{Q} \text{Ra\_ABS}(fq \upharpoonright i_0 -_{Q} gq \upharpoonright i_0)$

Suppose $\Rightarrow$ $f' \upharpoonright i_0 \notin \mathbb{Q}$

$\langle f' \rangle \hookrightarrow T66(\langle Stat12 \rangle) \Rightarrow$ *Stat26* : $f' \upharpoonright i_0 \notin \{f' \upharpoonright j : j \in \mathbf{domain}(f')\}$

$\langle i_0 \rangle \hookrightarrow Stat26([Stat12, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ $f' \upharpoonright i_0 \in \mathbb{Q}$

Suppose $\Rightarrow$ $g' \upharpoonright i_0 \notin \mathbb{Q}$

$\langle g' \rangle \hookrightarrow T66(\langle Stat14 \rangle) \Rightarrow$ *Stat27* : $g' \upharpoonright i_0 \notin \{g' \upharpoonright j : j \in \mathbf{domain}(g')\}$

$\langle i_0 \rangle \hookrightarrow Stat27([Stat14, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ $g' \upharpoonright i_0 \in \mathbb{Q}$

$\langle g'\restriction i_0\rangle \hookrightarrow T372(\langle Stat6\rangle) \Rightarrow \quad \mathsf{Rev}_{\mathbb{Q}}(g'\restriction i_0) \in \mathbb{Q}$

$\langle f'\restriction i_0, \mathsf{Rev}_{\mathbb{Q}}(g'\restriction i_0)\rangle \hookrightarrow T365(\langle Stat6\rangle) \Rightarrow \quad f'\restriction i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(g'\restriction i_0) \in \mathbb{Q}$

$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow \quad f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0 \in \mathbb{Q}$

$\langle f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0\rangle \hookrightarrow T10045(\langle Stat6\rangle) \Rightarrow \quad \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0) \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0), \mathsf{eps}_2\rangle \hookrightarrow T384(\langle Stat20\rangle) \Rightarrow \quad \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)$

$\langle \mathsf{eps}_1, \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0), \mathsf{eps}_2, \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\rangle \hookrightarrow T397(\langle Stat20\rangle) \Rightarrow$
$\quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)$

$\langle \mathsf{eps}_1, \mathsf{eps}_2\rangle \hookrightarrow T365(\langle Stat20\rangle) \Rightarrow \quad \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2 \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0), \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\rangle \hookrightarrow T365(\langle Stat6\rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0) \in \mathbb{Q}$

$\langle \mathsf{eps}_0, \mathsf{eps}_1 +_{\mathbb{Q}} \mathsf{eps}_2, \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\rangle \hookrightarrow T405(\langle Stat1\rangle) \Rightarrow$
$\quad \mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)$

-- On the other hand, the sum

$$\mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)$$

of the absolute values of $fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0$, $f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0$ is greater than or equal to the absolute value of the sum of these two quantities; therefore it must be equal than the sum of $\mathsf{eps}_1, \mathsf{eps}_2$, and hence greater than or equal to eps_0.

$\mathsf{APPLY}\ \langle\,\rangle\ \mathsf{fcn\_symbol}\big(f(u) \mapsto fq\restriction u +_{\mathbb{Q}} f'\restriction u, g \mapsto fq +_{\mathbb{QS}} f', s \mapsto \mathbb{N}\big) \Rightarrow$
$\quad Stat22: \ \langle \forall x \mid (fq +_{\mathbb{QS}} f')\restriction x = \textbf{if } x \in \mathbb{N} \textbf{ then } fq\restriction x +_{\mathbb{Q}} f'\restriction x \textbf{ else } \emptyset \textbf{ fi}\rangle$

$\mathsf{APPLY}\ \langle\,\rangle\ \mathsf{fcn\_symbol}\big(f(u) \mapsto gq\restriction u +_{\mathbb{Q}} g'\restriction u, g \mapsto gq +_{\mathbb{QS}} g', s \mapsto \mathbb{N}\big) \Rightarrow$
$\quad Stat23: \ \langle \forall x \mid (gq +_{\mathbb{QS}} g')\restriction x = \textbf{if } x \in \mathbb{N} \textbf{ then } gq\restriction x +_{\mathbb{Q}} g'\restriction x \textbf{ else } \emptyset \textbf{ fi}\rangle$

$\langle i_0\rangle \hookrightarrow Stat22 \Rightarrow \quad (fq +_{\mathbb{QS}} f')\restriction i_0 = fq\restriction i_0 +_{\mathbb{Q}} f'\restriction i_0$

$\langle i_0\rangle \hookrightarrow Stat23 \Rightarrow \quad (gq +_{\mathbb{QS}} g')\restriction i_0 = gq\restriction i_0 +_{\mathbb{Q}} g'\restriction i_0$

$\mathsf{ALGEBRA} \Rightarrow \quad fq\restriction i_0 +_{\mathbb{Q}} f'\restriction i_0 -_{\mathbb{Q}} (gq\restriction i_0 +_{\mathbb{Q}} g'\restriction i_0) = fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)$

$\mathsf{EQUAL}\ \langle Stat10\rangle \Rightarrow \quad \mathsf{Ra\_ABS}\big(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\big) >_{\mathbb{Q}} \mathsf{eps}_0$

$T10050 \Rightarrow \quad Stat50: \ \langle \forall x, y \mid x, y \in \mathbb{Q} \rightarrow \mathsf{Ra\_ABS}(x +_{\mathbb{Q}} y) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(x) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(y)\rangle$

$\langle fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0, f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0\rangle \hookrightarrow Stat50 \Rightarrow$
$\quad \mathsf{Ra\_ABS}\big(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\big) \leqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)$

$\langle \mathsf{Ra\_ABS}\big(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\big), \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\rangle \hookrightarrow T384 \Rightarrow$
$\qquad \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\big)$

$\langle fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0, f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0\rangle \hookrightarrow T365(\langle Stat6\rangle) \Rightarrow$
$\quad fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0) \in \mathbb{Q}$

$\langle fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\rangle \hookrightarrow T10045(\langle Stat6\rangle) \Rightarrow$
$\quad \mathsf{Ra\_ABS}\big(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\big) \in \mathbb{Q}$

$\langle \mathsf{eps}_0, \mathsf{Ra\_ABS}(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0) +_{\mathbb{Q}} \mathsf{Ra\_ABS}(f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0), \mathsf{Ra\_ABS}\big(fq\restriction i_0 -_{\mathbb{Q}} gq\restriction i_0 +_{\mathbb{Q}} (f'\restriction i_0 -_{\mathbb{Q}} g'\restriction i_0)\big)\rangle \hookrightarrow T405(\langle Stat1\rangle) \Rightarrow$

$\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{gq}\upharpoonright\mathsf{i}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{i}_0)\right)$

$\langle \mathsf{eps}_0, \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{gq}\upharpoonright\mathsf{i}_0 +_{\mathbb{Q}} (\mathsf{f}'\upharpoonright\mathsf{i}_0 -_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{i}_0)), \mathsf{eps}_0 \rangle \hookrightarrow T10041a(\langle Stat1 \rangle) \Rightarrow$

$\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_0$

-- By transitivity of the ordering of rationals, we get the absurdity $\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_0$, whence the inclusion we were aiming at.

$\langle \mathsf{eps}_0, \mathsf{eps}_0 \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$

$\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{gq}\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\} \supseteq \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}') \mid \mathsf{Ra\_ABS}((\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}')\upharpoonright\mathsf{x} -_{\mathbb{Q}} (\mathsf{g}$

$\langle \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{gq}\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\}, \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\} \rangle \hookrightarrow T205 \Rightarrow$

$\mathsf{Finite}(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{gq}\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\})$

-- From the contradiction that the same set which, at the outset, we have assumed to be infinite turns out to be finite, the desired conclusion ensues immediately.

$\langle \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{gq}\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_1\} \cup \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{f}') \mid \mathsf{Ra\_ABS}(\mathsf{f}'\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_2\}, \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}') \mid \mathsf{Ra\_ABS}((\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}')\upharpoonright\mathsf{x} -_{\mathbb{Q}} (\mathsf{gq} +_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{ep}$

false; Discharge $\Rightarrow$ QED

-- A trivial corollary of the lemma stating that when f, f' and g, g' are rational sequences with f equivalent to g and f' equivalent to g' then the pointwise sum of f and f' is equivalent to the pointwise sum of g and g' is the analogous lemma stating the same for rational Cauchy sequences.

Theorem 603 (10067*a*) $\{\mathsf{F}, \mathsf{G}, \mathsf{Fp}, \mathsf{Gp}\} \subseteq \mathsf{RaCauchy}\ \&\ \mathsf{Ra\_eqseq}(\mathsf{F}, \mathsf{G})\ \&\ \mathsf{Ra\_eqseq}(\mathsf{Fp}, \mathsf{Gp}) \rightarrow \mathsf{Ra\_eqseq}(\mathsf{F} +_{\mathbb{QS}} \mathsf{Fp}, \mathsf{G} +_{\mathbb{QS}} \mathsf{Gp})$. PROOF:

Suppose_not$(\mathsf{fq}, \mathsf{gq}, \mathsf{f}', \mathsf{g}') \Rightarrow$ $\{\mathsf{fq}, \mathsf{gq}, \mathsf{f}', \mathsf{g}'\} \subseteq \mathsf{RaCauchy}\ \&\ \mathsf{Ra\_eqseq}(\mathsf{fq}, \mathsf{gq})\ \&\ \mathsf{Ra\_eqseq}(\mathsf{f}', \mathsf{g}')\ \&\ \neg\mathsf{Ra\_eqseq}(\mathsf{fq} +_{\mathbb{QS}} \mathsf{f}', \mathsf{gq} +_{\mathbb{QS}} \mathsf{g}')$

Use_def(RaCauchy) $\Rightarrow$ *Stat0* :

$\mathsf{fq} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\upharpoonright\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}\ \&\ Stat1:$

$\mathsf{f}' \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\upharpoonright\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}\ \&\ Stat2:\ \mathsf{gq} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}$

$\langle \rangle \hookrightarrow Stat0 \Rightarrow$ $\mathsf{fq} \in \mathsf{RaSeq}$

$\langle \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{f}' \in \mathsf{RaSeq}$

$\langle \rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{gq} \in \mathsf{RaSeq}$

$\langle \rangle \hookrightarrow Stat3 \Rightarrow$ $\mathsf{g}' \in \mathsf{RaSeq}$

$\langle \mathsf{fq}, \mathsf{gq}, \mathsf{f}', \mathsf{g}' \rangle \hookrightarrow T10067 \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Our next lemma states that when f, f' and g are rational Cauchy sequences with f equivalent to f' then the pointwise product of f and g is equivalent to the pointwise product of f' and g.

Theorem 604 (10068) $\{\mathsf{F}, \mathsf{Fp}, \mathsf{G}\} \subseteq \mathsf{RaCauchy}\ \&\ \mathsf{Ra\_eqseq}(\mathsf{F}, \mathsf{Fp}) \rightarrow \mathsf{Ra\_eqseq}(\mathsf{F} *_{\mathbb{QS}} \mathsf{G}, \mathsf{Fp} *_{\mathbb{QS}} \mathsf{G})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{fq},\mathsf{f}',\mathsf{g}') \Rightarrow \quad \{\mathsf{fq},\mathsf{f}',\mathsf{g}'\} \subseteq \mathsf{RaCauchy}\ \&\ \mathsf{Ra\_eqseq}(\mathsf{fq},\mathsf{f}')\ \&\ \neg\mathsf{Ra\_eqseq}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}', \mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')$

-- For, assuming $\mathsf{fq},\mathsf{gq},\mathsf{g}'$ to be a counterexample to the statement of this lemma, we reach a contradiction by arguing as follows. The very definition of equivalence between rational sequences entails that for some positive rational number $\varepsilon = \mathsf{eps}_0$, the set

$$\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') \,|\, \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\restriction\mathsf{x} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\restriction\mathsf{x}\big) >_{\mathbb{Q}} \varepsilon\}$$

is infinite, whereas the analogous set which has $\mathsf{fq},\mathsf{f}',\mathsf{eps}_0 /_{\mathbb{Q}} \mathsf{m}$ in place of $\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}'$, $\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}'$ is finite, where $\mathsf{m}\ \mathsf{eps}_1$ is a number exceeding the absolute values of all components of the sequence g'.

$\mathsf{Use\_def}(\mathsf{Ra\_eqseq}) \Rightarrow \quad \mathit{Stat1}:$
$\langle \forall \varepsilon \in \mathbb{Q} \,|\, \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}\big(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \,|\, \mathsf{Ra\_ABS}(\mathsf{fq}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\restriction\mathsf{x}) >_{\mathbb{Q}} \varepsilon\}\big)\rangle\ \&\ \mathit{Stat2}: \neg$
$\langle \forall \varepsilon \in \mathbb{Q} \,|\, \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}\Big(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') \,|\, \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\restriction\mathsf{x} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\restriction\mathsf{x}\big) >_{\mathbb{Q}} \varepsilon\}\Big)\rangle$

$\langle \mathsf{eps}_0 \rangle \hookrightarrow \mathit{Stat2} \Rightarrow \quad \mathsf{eps}_0 \in \mathbb{Q}\ \&\ \mathsf{eps}_0 >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}\ \&\ \neg\mathsf{Finite}\Big(\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') \,|\, \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\restriction\mathsf{x} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\restriction\mathsf{x}\big) >_{\mathbb{Q}} \mathsf{eps}_0\}\Big)$

$\langle \mathsf{g}' \rangle \hookrightarrow \mathit{T10061a} \Rightarrow \quad \mathit{Stat20}: \langle \exists \mathsf{x} \in \mathbb{Q}, \forall \mathsf{y} \in \mathbf{range}(\mathsf{g}') \,|\, \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{x}\rangle$

$\langle \mathsf{m} \rangle \hookrightarrow \mathit{Stat20}(\langle \mathit{Stat20} \rangle) \Rightarrow \quad \mathsf{m} \in \mathbb{Q}\ \&\ \mathit{Stat55}: \langle \forall \mathsf{y} \in \mathbf{range}(\mathsf{g}') \,|\, \mathsf{Ra\_ABS}(\mathsf{y}) <_{\mathbb{Q}} \mathsf{m}\rangle$

$\mathsf{Use\_def}(\mathsf{RaCauchy}) \Rightarrow \quad \mathit{Stat5}: \mathsf{g}' \in \{\mathsf{f} \in \mathsf{RaSeq} \,|\, \langle \forall \varepsilon \in \mathbb{Q} \,|\, \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}\big(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \,|\, \mathsf{Ra\_ABS}(\mathsf{f}\restriction\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction\mathsf{j}) >_{\mathbb{Q}} \varepsilon\}\big)\rangle\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat5} \Rightarrow \quad \mathsf{g}' \in \mathsf{RaSeq}$

$\langle \mathsf{g}' \rangle \hookrightarrow \mathit{T413a} \Rightarrow \quad \mathit{Stat13}: \mathbf{domain}(\mathsf{g}') = \mathbb{N}\ \&\ \mathsf{Svm}(\mathsf{g}')\ \&\ \mathbf{range}(\mathsf{g}') \subseteq \mathbb{Q}$

$\mathit{T371} \Rightarrow \quad \mathit{Stat30}: \mathbf{0}_{\mathbb{Q}} \in \mathbb{Q}$

$\mathsf{Suppose} \Rightarrow \quad \neg\mathsf{m} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\mathit{T182} \Rightarrow \quad \emptyset \in \mathbf{domain}(\mathsf{g}')$

$\langle \emptyset, \mathsf{g}' \rangle \hookrightarrow \mathit{T64} \Rightarrow \quad \mathsf{g}'\restriction\emptyset \in \mathbf{range}(\mathsf{g}')$

$\langle \mathsf{g}'\restriction\emptyset \rangle \hookrightarrow \mathit{Stat55} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{g}'\restriction\emptyset) <_{\mathbb{Q}} \mathsf{m}$

$\langle \mathsf{g}'\restriction\emptyset \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{g}'\restriction\emptyset) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Ra\_ABS}(\mathsf{g}'\restriction\emptyset), \mathsf{m} \rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{m} >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{g}'\restriction\emptyset)$

$\langle \mathsf{g}'\restriction\emptyset \rangle \hookrightarrow \mathit{T10045} \Rightarrow \quad \mathsf{Ra\_ABS}(\mathsf{g}'\restriction\emptyset) \in \mathbb{Q}$

$\langle \mathsf{m}, \mathsf{Ra\_ABS}(\mathsf{g}'\restriction\emptyset), \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T405} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{m} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow \mathit{T384} \Rightarrow \quad \mathsf{m} \neq \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m} \rangle \hookrightarrow \mathit{T380} \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q}\ \&\ \mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{m} \rangle \hookrightarrow \mathit{T395} \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\mathsf{Suppose} \Rightarrow \quad \neg\mathsf{Finite}\Big(\Big\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \,|\, \mathsf{Ra\_ABS}(\mathsf{fq}\restriction\mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\restriction\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\Big\}\Big)$

$\langle \mathsf{eps}_0, \mathbf{0}_{\mathbb{Q}}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow \mathit{T393} \Rightarrow \quad \mathit{Stat31}: \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})$

$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow \mathit{T368}(\langle \mathit{Stat30} \rangle) \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow \mathit{T394}(\langle \mathit{Stat30} \rangle) \Rightarrow \quad \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$

$\mathsf{EQUAL}\ \langle \mathit{Stat31}\rangle \Rightarrow\quad \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$
$\langle \mathsf{eps_0}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow \mathit{T368}(\langle \mathit{Stat1}\rangle) \Rightarrow\quad \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q}$
$\langle \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\rangle \hookrightarrow \mathit{Stat1} \Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge} \Rightarrow\quad \mathsf{Finite}\left(\left\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\right\}\right)$

-- We will reach the desired contradiction, thus proving the present lemma, by showing that the former of the two sets mentioned above is included in the latter. Hence, since the latter is finite, the former cannot be infinite. Temporarily assuming that the inclusion does not hold, we can find an unsigned integer $\mathsf{i_0}$ belonging to the former set but not belonging to the latter, so that

$$\mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{i_0} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{i_0}\big) >_{\mathbb{Q}} \mathsf{eps_0}\,,$$

$$\neg\mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i_0} -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{i_0}) >_{\mathbb{Q}} \mathsf{eps_0} /_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\,.$$

$\mathsf{Suppose} \Rightarrow\quad \mathit{Stat4}:\ \neg$
$\quad \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') \mid \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{x} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{x}\big) >_{\mathbb{Q}} \mathsf{eps_0}\} \subseteq \left\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\right\}$
$\mathsf{Use\_def}(\mathsf{RaCauchy}) \Rightarrow\quad \mathit{Stat3}:$
$\quad \mathsf{fq} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\upharpoonright\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}\ \&\ \mathit{Stat7}:$
$\qquad \mathsf{f}' \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\upharpoonright\mathsf{i} -_{\mathbb{Q}} \mathsf{f}\upharpoonright\mathsf{j}) >_{\mathbb{Q}} \varepsilon\})\rangle\}$
$\langle \mathsf{fq}, \mathsf{g}'\rangle \hookrightarrow \mathit{T10062} \Rightarrow\quad \mathsf{fq} *_{\mathbb{QS}} \mathsf{g}' \in \mathsf{RaSeq}\ \&\ \mathsf{fq} *_{\mathbb{QS}} \mathsf{g}' = \{[\mathsf{u}, \mathsf{fq}\upharpoonright\mathsf{u} *_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{f}', \mathsf{g}'\rangle \hookrightarrow \mathit{T10062} \Rightarrow\quad \mathsf{f}' *_{\mathbb{QS}} \mathsf{g}' = \{[\mathsf{u}, \mathsf{f}'\upharpoonright\mathsf{u} *_{\mathbb{Q}} \mathsf{g}'\upharpoonright\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle\ \rangle \hookrightarrow \mathit{Stat3} \Rightarrow\quad \mathsf{fq} \in \mathsf{RaSeq}$
$\langle\ \rangle \hookrightarrow \mathit{Stat7} \Rightarrow\quad \mathsf{f}' \in \mathsf{RaSeq}$
$\langle \mathsf{fq}\rangle \hookrightarrow \mathit{T413a} \Rightarrow\quad \mathit{Stat11}:\ \mathbf{domain}(\mathsf{fq}) = \mathbb{N}\ \&\ \mathsf{Svm}(\mathsf{fq})\ \&\ \mathbf{range}(\mathsf{fq}) \subseteq \mathbb{Q}$
$\langle \mathsf{f}'\rangle \hookrightarrow \mathit{T413a} \Rightarrow\quad \mathit{Stat12}:\ \mathbf{domain}(\mathsf{f}') = \mathbb{N}\ \&\ \mathsf{Svm}(\mathsf{f}')\ \&\ \mathbf{range}(\mathsf{f}') \subseteq \mathbb{Q}$
$\langle \mathsf{fq} *_{\mathbb{QS}} \mathsf{g}'\rangle \hookrightarrow \mathit{T413a} \Rightarrow\quad \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') = \mathbb{N}$
$\mathsf{EQUAL}\ \langle \mathit{Stat4}\rangle \Rightarrow\quad \mathit{Stat4a}:\ \neg$
$\quad \{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{x} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{x}\big) >_{\mathbb{Q}} \mathsf{eps_0}\} \subseteq \left\{\mathsf{x} : \mathsf{x} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{x}) >_{\mathbb{Q}} \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\right\}$
$\langle \mathsf{i_0}\rangle \hookrightarrow \mathit{Stat4a} \Rightarrow\quad \mathit{Stat10}:\ \mathsf{i_0} \in \mathbb{N}\ \&\ \mathsf{Ra\_ABS}\big((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{i_0} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\upharpoonright\mathsf{i_0}\big) >_{\mathbb{Q}} \mathsf{eps_0}\ \&\ \neg\mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i_0} -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{i_0}) >_{\mathbb{Q}} \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})$
$\mathsf{Use\_def}(-_{\mathbb{Q}}) \Rightarrow\quad \neg\mathsf{Ra\_ABS}\big(\mathsf{fq}\upharpoonright\mathsf{i_0} +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\upharpoonright\mathsf{i_0})\big) >_{\mathbb{Q}} \mathsf{eps_0} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})$

-- On the one hand, we observe that

$$\mathsf{eps_0} /_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\upharpoonright\mathsf{i_0} -_{\mathbb{Q}} \mathsf{f}'\upharpoonright\mathsf{i_0})\,,$$

$\mathsf{Suppose} \Rightarrow\quad \mathsf{fq}\upharpoonright\mathsf{i_0} \notin \mathbb{Q}$
$\langle \mathsf{fq}\rangle \hookrightarrow \mathit{T66}(\langle \mathit{Stat11}\rangle) \Rightarrow\quad \mathit{Stat24}:\ \mathsf{fq}\upharpoonright\mathsf{i_0} \notin \{\mathsf{fq}\upharpoonright\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$

$\langle \mathsf{i}_0 \rangle \hookrightarrow Stat24([Stat11, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat6:\ \mathsf{fq} \upharpoonright \mathsf{i}_0 \in \mathbb{Q}$

Suppose $\Rightarrow$ $\mathsf{f}' \upharpoonright \mathsf{i}_0 \notin \mathbb{Q}$

$\langle \mathsf{f}' \rangle \hookrightarrow T66(\langle Stat12 \rangle) \Rightarrow$ $Stat26:\ \mathsf{f}' \upharpoonright \mathsf{i}_0 \notin \{\mathsf{f}' \upharpoonright \mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{f}')\}$

$\langle \mathsf{i}_0 \rangle \hookrightarrow Stat26([Stat12, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{f}' \upharpoonright \mathsf{i}_0 \in \mathbb{Q}$

$\langle \mathsf{f}' \upharpoonright \mathsf{i}_0 \rangle \hookrightarrow T372(\langle Stat6 \rangle) \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0) \in \mathbb{Q}$

$\langle \mathsf{fq} \upharpoonright \mathsf{i}_0, \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0) \rangle \hookrightarrow T365(\langle Stat6 \rangle) \Rightarrow$ $\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0) \in \mathbb{Q}$

$\langle \mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0) \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \in \mathbb{Q}\ \&$

$\quad \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \geqslant_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

Suppose $\Rightarrow$ $\mathsf{g}' \upharpoonright \mathsf{i}_0 \notin \mathbf{range}(\mathsf{g}')$

$\langle \mathsf{g}' \rangle \hookrightarrow T66(\langle Stat13 \rangle) \Rightarrow$ $Stat27:\ \mathsf{g}' \upharpoonright \mathsf{i}_0 \notin \{\mathsf{g}' \upharpoonright \mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{g}')\}$

$\langle \mathsf{i}_0 \rangle \hookrightarrow Stat27([Stat13, Stat10]) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{g}' \upharpoonright \mathsf{i}_0 \in \mathbf{range}(\mathsf{g}')\ \&\ \mathsf{g}' \upharpoonright \mathsf{i}_0 \in \mathbb{Q}$

$\langle \mathsf{g}' \upharpoonright \mathsf{i}_0 \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{g}' \upharpoonright \mathsf{i}_0) \in \mathbb{Q}$

$\langle \mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0) \rangle \hookrightarrow T10045 \Rightarrow$ $\mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \in \mathbb{Q}$

$\langle \mathsf{eps}_0, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T368(\langle Stat1 \rangle) \Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \in \mathbb{Q}$

$\langle \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right), \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T384 \Rightarrow$

$\quad \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \geqslant_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

-- ...so that

$$\mathsf{eps}_0 \geqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq} \upharpoonright \mathsf{i}_0 -_{\mathbb{Q}} \mathsf{f}' \upharpoonright \mathsf{i}_0).$$

$\langle \mathsf{m}, \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \rangle \hookrightarrow T368 \Rightarrow$ $\mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \in \mathbb{Q}\ \&$

$\quad \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) = \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) *_{\mathbb{Q}} \mathsf{m}$

Suppose $\Rightarrow$ $\neg \mathsf{eps}_0 \geqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

$\langle \mathsf{eps}_0 \rangle \hookrightarrow T379 \Rightarrow$ $\mathsf{eps}_0 = \mathsf{eps}_0 *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}}$

$\langle \mathsf{m}, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T368 \Rightarrow$ $\mathsf{m} *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{m}$

$\langle \mathsf{m}, \mathsf{eps}_0, \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) \rangle \hookrightarrow T374 \Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} (\mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{m}) = (\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})) *_{\mathbb{Q}} \mathsf{m}$

EQUAL $\langle Stat30 \rangle \Rightarrow$ $\mathsf{eps}_0 = \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{m}$

$\langle \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}), \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \rangle \hookrightarrow T384 \Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \vee$

$\quad \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

Suppose $\Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) = \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

EQUAL $\Rightarrow$ $\mathsf{eps}_0 = \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

$\langle \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right), \mathsf{eps}_0 \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) >_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

$\langle \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}), \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right), \mathsf{m} \rangle \hookrightarrow T393(\langle Stat20 \rangle) \Rightarrow$

$\quad \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m}) *_{\mathbb{Q}} \mathsf{m} >_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) *_{\mathbb{Q}} \mathsf{m}$

EQUAL $\Rightarrow$ $\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

$\langle \mathsf{eps}_0, \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right) \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{eps}_0 \geqslant_{\mathbb{Q}} \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}\left(\mathsf{fq} \upharpoonright \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}' \upharpoonright \mathsf{i}_0)\right)$

-- On the other hand, we have that

$$\mathsf{Ra\_ABS}(g'\upharpoonright i_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright i_0) >_{\mathbb{Q}} \mathsf{eps}_0\,,$$

APPLY $\langle\,\rangle$ fcn_symbol $(f(u) \mapsto fq\upharpoonright u *_{\mathbb{Q}} g'\upharpoonright u, g \mapsto fq *_{\mathbb{Q}S} g', s \mapsto \mathbb{N}) \Rightarrow$

Stat22: $\langle \forall x \mid (fq *_{\mathbb{Q}S} g')\upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } fq\upharpoonright x *_{\mathbb{Q}} g'\upharpoonright x \textbf{ else } \emptyset \textbf{ fi}\rangle$

APPLY $\langle\,\rangle$ fcn_symbol $(f(u) \mapsto f'\upharpoonright u *_{\mathbb{Q}} g'\upharpoonright u, g \mapsto f' *_{\mathbb{Q}S} g', s \mapsto \mathbb{N}) \Rightarrow$

Stat23: $\langle \forall x \mid (f' *_{\mathbb{Q}S} g')\upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } f'\upharpoonright x *_{\mathbb{Q}} g'\upharpoonright x \textbf{ else } \emptyset \textbf{ fi}\rangle$

Use_def $(-_{\mathbb{Q}}) \Rightarrow$ *Stat77*: $\mathsf{Ra\_ABS}\Big((fq *_{\mathbb{Q}S} g')\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}\big((f' *_{\mathbb{Q}S} g')\upharpoonright i_0\big)\Big) >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle i_0\rangle \hookrightarrow$ *Stat22*([*Stat10*, *Stat10*]) $\Rightarrow$ $(fq *_{\mathbb{Q}S} g')\upharpoonright i_0 = fq\upharpoonright i_0 *_{\mathbb{Q}} g'\upharpoonright i_0$

$\langle i_0\rangle \hookrightarrow$ *Stat23*([*Stat10*, *Stat10*]) $\Rightarrow$ $(f' *_{\mathbb{Q}S} g')\upharpoonright i_0 = f'\upharpoonright i_0 *_{\mathbb{Q}} g'\upharpoonright i_0$

$\langle fq\upharpoonright i_0, g'\upharpoonright i_0\rangle \hookrightarrow$ *T368* $\Rightarrow$ $fq\upharpoonright i_0 *_{\mathbb{Q}} g'\upharpoonright i_0 \in \mathbb{Q}\ \&$
$fq\upharpoonright i_0 *_{\mathbb{Q}} g'\upharpoonright i_0 = g'\upharpoonright i_0 *_{\mathbb{Q}} fq\upharpoonright i_0$

$\langle f'\upharpoonright i_0, g'\upharpoonright i_0\rangle \hookrightarrow$ *T368* $\Rightarrow$ $f'\upharpoonright i_0 *_{\mathbb{Q}} g'\upharpoonright i_0 \in \mathbb{Q}\ \&$
$f'\upharpoonright i_0 *_{\mathbb{Q}} g'\upharpoonright i_0 = g'\upharpoonright i_0 *_{\mathbb{Q}} f'\upharpoonright i_0$

$\langle g'\upharpoonright i_0, f'\upharpoonright i_0\rangle \hookrightarrow$ *T391* $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(g'\upharpoonright i_0 *_{\mathbb{Q}} f'\upharpoonright i_0) = g'\upharpoonright i_0 *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)$

$\langle \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0), g'\upharpoonright i_0, fq\upharpoonright i_0\rangle \hookrightarrow$ *T376* $\Rightarrow$ $g'\upharpoonright i_0 *_{\mathbb{Q}} \big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big) =$
$g'\upharpoonright i_0 *_{\mathbb{Q}} fq\upharpoonright i_0 +_{\mathbb{Q}} g'\upharpoonright i_0 *_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)$

$\langle g'\upharpoonright i_0, fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\rangle \hookrightarrow$ *T10046* $\Rightarrow$ *Stat88*: $\mathsf{Ra\_ABS}\Big(g'\upharpoonright i_0 *_{\mathbb{Q}} \big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big)\Big) =$
$\mathsf{Ra\_ABS}(g'\upharpoonright i_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big)$

EQUAL $\langle$*Stat77*$\rangle \Rightarrow$ $\mathsf{Ra\_ABS}\Big(g'\upharpoonright i_0 *_{\mathbb{Q}} \big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big)\Big) >_{\mathbb{Q}} \mathsf{eps}_0$

EQUAL $\langle$*Stat88*$\rangle \Rightarrow$ $\mathsf{Ra\_ABS}(g'\upharpoonright i_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big) >_{\mathbb{Q}} \mathsf{eps}_0$

-- ...and therefore that

$$m *_{\mathbb{Q}} \mathsf{Ra\_ABS}(fq\upharpoonright i_0 -_{\mathbb{Q}} f'\upharpoonright i_0) >_{\mathbb{Q}} \mathsf{eps}_0\,.$$

Suppose $\Rightarrow$ $\neg m *_{\mathbb{Q}} \mathsf{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big) >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle g'\upharpoonright i_0\rangle \hookrightarrow$ *Stat55* $\Rightarrow$ $\mathsf{Ra\_ABS}(g'\upharpoonright i_0) <_{\mathbb{Q}} m$

$\langle \mathsf{Ra\_ABS}(g'\upharpoonright i_0), m\rangle \hookrightarrow$ *T384* $\Rightarrow$ $m >_{\mathbb{Q}} \mathsf{Ra\_ABS}(g'\upharpoonright i_0)$

Suppose $\Rightarrow$ $\neg \mathsf{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big), \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow$ *T384*($\langle$*Stat4*$\rangle$) $\Rightarrow$
$\mathsf{Ra\_ABS}\big(fq\upharpoonright i_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(f'\upharpoonright i_0)\big) = \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{Ra\_ABS}(g'\upharpoonright i_0)\rangle \hookrightarrow$ *T394* $\Rightarrow$ $\mathsf{Ra\_ABS}(g'\upharpoonright i_0) *_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = \mathbf{0}_{\mathbb{Q}}$

EQUAL $\langle$*Stat88*$\rangle \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathbf{0}_{\mathbb{Q}}, \mathsf{eps}_0, \mathbf{0}_{\mathbb{Q}}\rangle \hookrightarrow$ *T10041a* $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathbf{0}_{\mathbb{Q}}, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle \mathsf{m}, \mathsf{Ra\_ABS}(\mathsf{g}'\restriction \mathsf{i}_0), \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)) \rangle \hookrightarrow T393(\langle Stat20 \rangle) \Rightarrow$
$\mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)) >_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{g}'\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0))$

$\langle \mathsf{Ra\_ABS}(\mathsf{g}'\restriction \mathsf{i}_0), \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)) \rangle \hookrightarrow T368 \Rightarrow$
$\mathsf{Ra\_ABS}(\mathsf{g}'\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)) \in \mathbb{Q}$

$\langle \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)), \mathsf{Ra\_ABS}(\mathsf{g}'\restriction \mathsf{i}_0) *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)), \mathsf{eps}_0 \rangle \hookrightarrow T10041a(\langle Stat1 \rangle) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)) >_{\mathbb{Q}} \mathsf{eps}_0$

-- By exploiting one of the transitivity laws which the ordering of rational numbers obeys, we get that $\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_0$, an absurdity showing that the desired inclusion between sets actually holds.

$\langle \mathsf{eps}_0, \mathsf{m} *_{\mathbb{Q}} \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{i}_0 +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathsf{f}'\restriction \mathsf{i}_0)), \mathsf{eps}_0 \rangle \hookrightarrow T406 \Rightarrow$ $\mathsf{eps}_0 >_{\mathbb{Q}} \mathsf{eps}_0$

$\langle \mathsf{eps}_0, \mathsf{eps}_0 \rangle \hookrightarrow T384 \Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\} \supseteq \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') \mid \mathsf{Ra\_ABS}((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\restriction \mathsf{x} -_{\mathbb{Q}} (\mathsf{f}'$

-- The desired conclusion now follows immediately.

$\langle \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq}) \mid \mathsf{Ra\_ABS}(\mathsf{fq}\restriction \mathsf{x} -_{\mathbb{Q}} \mathsf{f}'\restriction \mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0 *_{\mathbb{Q}} \mathsf{Recip}_{\mathbb{Q}}(\mathsf{m})\}, \{\mathsf{x} : \mathsf{x} \in \mathbf{domain}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}') \mid \mathsf{Ra\_ABS}((\mathsf{fq} *_{\mathbb{QS}} \mathsf{g}')\restriction \mathsf{x} -_{\mathbb{Q}} (\mathsf{f}' *_{\mathbb{QS}} \mathsf{g}')\restriction \mathsf{x}) >_{\mathbb{Q}} \mathsf{eps}_0\} \rangle \hookrightarrow T162 \Rightarrow$ false; Discharge $\Rightarrow$

QED

-- An easy corollary of the preceding lemma is the following: When fq, f' and gq, g' are rational Cauchy sequences with fq equivalent to gq and f' equivalent to g', the pointwise product of fq and f' is equivalent to the pointwise product of f' and g'.

Theorem 605 (10069) $\{\mathsf{F}, \mathsf{G}, \mathsf{Fp}, \mathsf{Gp}\} \subseteq \mathsf{RaCauchy} \,\&\, \mathsf{Ra\_eqseq}(\mathsf{F}, \mathsf{G}) \,\&\, \mathsf{Ra\_eqseq}(\mathsf{Fp}, \mathsf{Gp}) \rightarrow \mathsf{Ra\_eqseq}(\mathsf{F} *_{\mathbb{QS}} \mathsf{Fp}, \mathsf{G} *_{\mathbb{QS}} \mathsf{Gp})$. PROOF:

Suppose_not$(\mathsf{fq}, \mathsf{gq}, \mathsf{f}', \mathsf{g}') \Rightarrow$ $\{\mathsf{fq}, \mathsf{gq}, \mathsf{f}', \mathsf{g}'\} \subseteq \mathsf{RaCauchy} \,\&\, \mathsf{Ra\_eqseq}(\mathsf{fq}, \mathsf{gq}) \,\&\, \mathsf{Ra\_eqseq}(\mathsf{f}', \mathsf{g}') \,\&\, \neg\mathsf{Ra\_eqseq}(\mathsf{fq} *_{\mathbb{QS}} \mathsf{f}', \mathsf{gq} *_{\mathbb{QS}} \mathsf{g}')$

-- For, assuming $\mathsf{fq}, \mathsf{gq}, \mathsf{f}', \mathsf{g}'$ to be a counterexample to the statement of this lemma, we reach a contradiction by arguing as follows. If follows from the preceding lemma that $\mathsf{gq} *_{\mathbb{Q}} \mathsf{g}'$ and $\mathsf{g}' *_{\mathbb{Q}} \mathsf{fq}$ are equivalent to $\mathsf{fq} *_{\mathbb{Q}} \mathsf{g}'$ and to $\mathsf{f}' *_{\mathbb{Q}} \mathsf{fq}$, respectively.

Use_def(RaCauchy) $\Rightarrow$ *Stat0* :
$\mathsf{fq} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction \mathsf{j}) >_{\mathbb{Q}} \varepsilon\}) \rangle\} \,\&\, Stat1 :$
$\mathsf{f}' \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}, \mathsf{j} \in \mathbb{N} \mid \mathsf{Ra\_ABS}(\mathsf{f}\restriction \mathsf{i} -_{\mathbb{Q}} \mathsf{f}\restriction \mathsf{j}) >_{\mathbb{Q}} \varepsilon\}) \rangle\} \,\&\, Stat2 :$ $\mathsf{gq} \in \{\mathsf{f} \in \mathsf{RaSeq} \mid \langle \forall \varepsilon \in \mathbb{Q} \mid \varepsilon >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \rightarrow \mathsf{Finite}(\{\mathsf{i} \cap \mathsf{j} : \mathsf{i} \in \mathbb{N}$

$\langle\,\rangle \hookrightarrow Stat0 \Rightarrow$ $\mathsf{fq} \in \mathsf{RaSeq}$

$\langle\,\rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{f}' \in \mathsf{RaSeq}$

$\langle\,\rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{gq} \in \mathsf{RaSeq}$

$\langle\,\rangle \hookrightarrow Stat3 \Rightarrow$ $\mathsf{g}' \in \mathsf{RaSeq}$

$\langle \mathsf{gq} \rangle \hookrightarrow T413a \Rightarrow$ $\mathsf{Ra\_eqseq}(\mathsf{gq}, \mathsf{gq})$

$\langle \mathsf{fq}, \mathsf{gq}, \mathsf{gq} \rangle \hookrightarrow T10063 \Rightarrow \quad \mathsf{Ra\_eqseq}(\mathsf{gq}, \mathsf{fq})$
$\langle \mathsf{g}' \rangle \hookrightarrow T413a \Rightarrow \quad \mathsf{Ra\_eqseq}(\mathsf{g}', \mathsf{g}')$
$\langle \mathsf{f}', \mathsf{g}', \mathsf{g}' \rangle \hookrightarrow T10063 \Rightarrow \quad \mathsf{Ra\_eqseq}(\mathsf{g}', \mathsf{f}')$
$\langle \mathsf{gq}, \mathsf{fq}, \mathsf{g}' \rangle \hookrightarrow T10068 \Rightarrow \quad \mathsf{Ra\_eqseq}(\mathsf{gq} *_{QS} \mathsf{g}', \mathsf{fq} *_{QS} \mathsf{g}')$
$\langle \mathsf{g}', \mathsf{f}', \mathsf{fq} \rangle \hookrightarrow T10068 \Rightarrow \quad \mathsf{Ra\_eqseq}(\mathsf{g}' *_{QS} \mathsf{fq}, \mathsf{f}' *_{QS} \mathsf{fq})$

-- The sequence $\mathsf{g}' *_{Q} \mathsf{fq}$ is easily shown to equal $\mathsf{fq} *_{Q} \mathsf{g}'$, and $\mathsf{f}' *_{Q} \mathsf{fq}$ is likewise shown to equal $\mathsf{fq} *_{Q} \mathsf{f}'$.

$\langle \mathsf{fq}, \mathsf{g}' \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{fq} *_{QS} \mathsf{g}' \in \mathsf{RaSeq} \; \& \; \mathsf{fq} *_{QS} \mathsf{g}' = \{[\mathsf{u}, \mathsf{fq}\restriction\mathsf{u} *_{Q} \mathsf{g}'\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{f}', \mathsf{fq} \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{f}' *_{QS} \mathsf{fq} \in \mathsf{RaSeq} \; \& \; \mathsf{f}' *_{QS} \mathsf{fq} = \{[\mathsf{u}, \mathsf{f}'\restriction\mathsf{u} *_{Q} \mathsf{fq}\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{gq}, \mathsf{g}' \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{gq} *_{QS} \mathsf{g}' \in \mathsf{RaSeq}$
$\langle \mathsf{g}', \mathsf{fq} \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{g}' *_{QS} \mathsf{fq} \in \mathsf{RaSeq} \; \& \; \mathsf{g}' *_{QS} \mathsf{fq} = \{[\mathsf{u}, \mathsf{g}'\restriction\mathsf{u} *_{Q} \mathsf{fq}\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{fq}, \mathsf{f}' \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{fq} *_{QS} \mathsf{f}' \in \mathsf{RaSeq} \; \& \; \mathsf{fq} *_{QS} \mathsf{f}' = \{[\mathsf{u}, \mathsf{fq}\restriction\mathsf{u} *_{Q} \mathsf{f}'\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{fq} \rangle \hookrightarrow T413a \Rightarrow \quad Stat11 : \; \mathbf{domain}(\mathsf{fq}) = \mathbb{N} \; \& \; \mathsf{Svm}(\mathsf{fq}) \; \& \; \mathbf{range}(\mathsf{fq}) \subseteq \mathbb{Q}$
$\langle \mathsf{f}' \rangle \hookrightarrow T413a \Rightarrow \quad Stat12 : \; \mathbf{domain}(\mathsf{f}') = \mathbb{N} \; \& \; \mathsf{Svm}(\mathsf{f}') \; \& \; \mathbf{range}(\mathsf{f}') \subseteq \mathbb{Q}$
$\langle \mathsf{g}' \rangle \hookrightarrow T413a \Rightarrow \quad Stat13 : \; \mathbf{domain}(\mathsf{g}') = \mathbb{N} \; \& \; \mathsf{Svm}(\mathsf{g}') \; \& \; \mathbf{range}(\mathsf{g}') \subseteq \mathbb{Q}$
Suppose $\Rightarrow \quad \mathsf{fq} *_{QS} \mathsf{g}' \neq \mathsf{g}' *_{QS} \mathsf{fq}$
EQUAL $\Rightarrow \quad Stat4 : \; \{[\mathsf{u}, \mathsf{fq}\restriction\mathsf{u} *_{Q} \mathsf{g}'\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\} \neq \{[\mathsf{u}, \mathsf{g}'\restriction\mathsf{u} *_{Q} \mathsf{fq}\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{u}_1 \rangle \hookrightarrow Stat4 \Rightarrow \quad Stat10 : \; \mathsf{u}_1 \in \mathbb{N} \; \& \; \mathsf{fq}\restriction\mathsf{u}_1 *_{Q} \mathsf{g}'\restriction\mathsf{u}_1 \neq \mathsf{g}'\restriction\mathsf{u}_1 *_{Q} \mathsf{fq}\restriction\mathsf{u}_1$
$\langle \mathsf{fq}\restriction\mathsf{u}_1, \mathsf{g}'\restriction\mathsf{u}_1 \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{fq}\restriction\mathsf{u}_1 \notin \mathbb{Q} \vee \mathsf{g}'\restriction\mathsf{u}_1 \notin \mathbb{Q}$
Suppose $\Rightarrow \quad \mathsf{fq}\restriction\mathsf{u}_1 \notin \mathbb{Q}$
$\langle \mathsf{fq} \rangle \hookrightarrow T66(\langle Stat11 \rangle) \Rightarrow \quad Stat24 : \; \mathsf{fq}\restriction\mathsf{u}_1 \notin \{\mathsf{fq}\restriction\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$
$\langle \mathsf{u}_1 \rangle \hookrightarrow Stat24([Stat11, Stat10]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad Stat6 : \; \mathsf{g}'\restriction\mathsf{u}_1 \notin \mathbb{Q}$
$\langle \mathsf{g}' \rangle \hookrightarrow T66(\langle Stat13 \rangle) \Rightarrow \quad Stat25 : \; \mathsf{g}'\restriction\mathsf{u}_1 \notin \{\mathsf{g}'\restriction\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{g}')\}$
$\langle \mathsf{u}_1 \rangle \hookrightarrow Stat25([Stat13, Stat10]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{fq} *_{QS} \mathsf{g}' = \mathsf{g}' *_{QS} \mathsf{fq}$
Suppose $\Rightarrow \quad \mathsf{f}' *_{QS} \mathsf{fq} \neq \mathsf{fq} *_{QS} \mathsf{f}'$
EQUAL $\Rightarrow \quad Stat5 : \; \{[\mathsf{u}, \mathsf{f}'\restriction\mathsf{u} *_{Q} \mathsf{fq}\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\} \neq \{[\mathsf{u}, \mathsf{fq}\restriction\mathsf{u} *_{Q} \mathsf{f}'\restriction\mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{u}_2 \rangle \hookrightarrow Stat5 \Rightarrow \quad Stat15 : \; \mathsf{u}_2 \in \mathbb{N} \; \& \; \mathsf{f}'\restriction\mathsf{u}_2 *_{Q} \mathsf{fq}\restriction\mathsf{u}_2 \neq \mathsf{fq}\restriction\mathsf{u}_2 *_{Q} \mathsf{f}'\restriction\mathsf{u}_2$
$\langle \mathsf{f}'\restriction\mathsf{u}_2, \mathsf{fq}\restriction\mathsf{u}_2 \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{f}'\restriction\mathsf{u}_2 \notin \mathbb{Q} \vee \mathsf{fq}\restriction\mathsf{u}_2 \notin \mathbb{Q}$
Suppose $\Rightarrow \quad \mathsf{f}'\restriction\mathsf{u}_2 \notin \mathbb{Q}$
$\langle \mathsf{f}' \rangle \hookrightarrow T66(\langle Stat12 \rangle) \Rightarrow \quad Stat26 : \; \mathsf{f}'\restriction\mathsf{u}_2 \notin \{\mathsf{f}'\restriction\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{f}')\}$
$\langle \mathsf{u}_2 \rangle \hookrightarrow Stat26([Stat12, Stat15]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad Stat7 : \; \mathsf{fq}\restriction\mathsf{u}_2 \notin \mathbb{Q}$
$\langle \mathsf{fq} \rangle \hookrightarrow T66(\langle Stat11 \rangle) \Rightarrow \quad Stat27 : \; \mathsf{fq}\restriction\mathsf{u}_2 \notin \{\mathsf{fq}\restriction\mathsf{j} : \mathsf{j} \in \mathbf{domain}(\mathsf{fq})\}$
$\langle \mathsf{u}_2 \rangle \hookrightarrow Stat27([Stat11, Stat15]) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow \quad \mathsf{f}' *_{QS} \mathsf{fq} = \mathsf{fq} *_{QS} \mathsf{f}'$

-- Therefore, by transitivity of the equivalence relation between rational sequences, we get that $\mathsf{gq} *_{Q} \mathsf{g}'$ and $\mathsf{fq} *_{Q} \mathsf{f}'$, a contradiction leading to the desired conclusion.

$\mathsf{EQUAL} \Rightarrow \quad \mathsf{Ra\_eqseq}(\mathsf{gq} *_{\mathsf{QS}} \mathsf{g}', \mathsf{g}' *_{\mathsf{QS}} \mathsf{fq})\ \&\ \mathsf{Ra\_eqseq}(\mathsf{g}' *_{\mathsf{QS}} \mathsf{fq}, \mathsf{fq} *_{\mathsf{QS}} \mathsf{f}')$
$\langle \mathsf{gq} *_{\mathsf{QS}} \mathsf{g}', \mathsf{g}' *_{\mathsf{QS}} \mathsf{fq}, \mathsf{fq} *_{\mathsf{QS}} \mathsf{f}' \rangle \hookrightarrow T10063 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- Proof of the algebraic rules for the operations on rational sequences introduced above will rest on the fact that two rational sequences are equal if and only if they have the same value for each integer, and on the rules for calculating the values at each integer of Next we prove that the algebraic operations on rational sequences introduced above obey the normal commutative, distributive, etc., algebraic laws. Our first two results state the commutative laws for addition and multiplication respectively. These results follow trivially from the pointwise definitions of the operations $+_{\mathsf{QS}}$ and $*_{\mathsf{QS}}$.

Theorem 606 (415) $\{\mathsf{F}, \mathsf{G}\} \subseteq \mathsf{RaSeq} \rightarrow \mathsf{F} +_{\mathsf{QS}} \mathsf{G} = \mathsf{G} +_{\mathsf{QS}} \mathsf{F}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{fq}, \mathsf{f}') \Rightarrow \quad \{\mathsf{fq}, \mathsf{f}'\} \subseteq \mathsf{RaSeq}\ \&\ \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' \neq \mathsf{f}' +_{\mathsf{QS}} \mathsf{fq}$
$\langle \mathsf{fq}, \mathsf{f}' \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{fq} +_{\mathsf{QS}} \mathsf{f}' = \{[\mathsf{u}, \mathsf{fq} \upharpoonright \mathsf{u} +_{\mathsf{Q}} \mathsf{f}' \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{f}', \mathsf{fq} \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{f}' +_{\mathsf{QS}} \mathsf{fq} = \{[\mathsf{u}, \mathsf{f}' \upharpoonright \mathsf{u} +_{\mathsf{Q}} \mathsf{fq} \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\mathsf{EQUAL} \Rightarrow \quad Stat19: \{[\mathsf{u}, \mathsf{fq} \upharpoonright \mathsf{u} +_{\mathsf{Q}} \mathsf{f}' \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\} \neq \{[\mathsf{u}, \mathsf{f}' \upharpoonright \mathsf{u} +_{\mathsf{Q}} \mathsf{fq} \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{i} \rangle \hookrightarrow Stat19 \Rightarrow \quad \mathsf{i} \in \mathbb{N}\ \&\ \mathsf{fq} \upharpoonright \mathsf{i} +_{\mathsf{Q}} \mathsf{f}' \upharpoonright \mathsf{i} \neq \mathsf{f}' \upharpoonright \mathsf{i} +_{\mathsf{Q}} \mathsf{fq} \upharpoonright \mathsf{i}$
$\langle \mathsf{fq} \rangle \hookrightarrow T10059 \Rightarrow \quad Stat21: \langle \forall \mathsf{h} \in \mathbb{N} \mid \mathsf{fq} \upharpoonright \mathsf{h} \in \mathbb{Q} \rangle$
$\langle \mathsf{f}' \rangle \hookrightarrow T10059 \Rightarrow \quad Stat22: \langle \forall \mathsf{h} \in \mathbb{N} \mid \mathsf{f}' \upharpoonright \mathsf{h} \in \mathbb{Q} \rangle$
$\langle \mathsf{i} \rangle \hookrightarrow Stat21 \Rightarrow \quad \mathsf{fq} \upharpoonright \mathsf{i} \in \mathbb{Q}$
$\langle \mathsf{i} \rangle \hookrightarrow Stat22 \Rightarrow \quad \mathsf{f}' \upharpoonright \mathsf{i} \in \mathbb{Q}$
$\langle \mathsf{fq} \upharpoonright \mathsf{i}, \mathsf{f}' \upharpoonright \mathsf{i} \rangle \hookrightarrow T365 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 607 (416) $\{\mathsf{F}, \mathsf{G}\} \subseteq \mathsf{RaSeq} \rightarrow \mathsf{F} *_{\mathsf{QS}} \mathsf{G} = \mathsf{G} *_{\mathsf{QS}} \mathsf{F}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{fq}, \mathsf{f}') \Rightarrow \quad \{\mathsf{fq}, \mathsf{f}'\} \subseteq \mathsf{RaSeq}\ \&\ \mathsf{fq} *_{\mathsf{QS}} \mathsf{f}' \neq \mathsf{f}' *_{\mathsf{QS}} \mathsf{fq}$
$\langle \mathsf{fq}, \mathsf{f}' \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{fq} *_{\mathsf{QS}} \mathsf{f}' = \{[\mathsf{u}, \mathsf{fq} \upharpoonright \mathsf{u} *_{\mathsf{Q}} \mathsf{f}' \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{f}', \mathsf{fq} \rangle \hookrightarrow T10062 \Rightarrow \quad \mathsf{f}' *_{\mathsf{QS}} \mathsf{fq} = \{[\mathsf{u}, \mathsf{f}' \upharpoonright \mathsf{u} *_{\mathsf{Q}} \mathsf{fq} \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\mathsf{EQUAL} \Rightarrow \quad Stat19: \{[\mathsf{u}, \mathsf{fq} \upharpoonright \mathsf{u} *_{\mathsf{Q}} \mathsf{f}' \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\} \neq \{[\mathsf{u}, \mathsf{f}' \upharpoonright \mathsf{u} *_{\mathsf{Q}} \mathsf{fq} \upharpoonright \mathsf{u}] : \mathsf{u} \in \mathbb{N}\}$
$\langle \mathsf{i} \rangle \hookrightarrow Stat19 \Rightarrow \quad \mathsf{i} \in \mathbb{N}\ \&\ \mathsf{fq} \upharpoonright \mathsf{i} *_{\mathsf{Q}} \mathsf{f}' \upharpoonright \mathsf{i} \neq \mathsf{f}' \upharpoonright \mathsf{i} *_{\mathsf{Q}} \mathsf{fq} \upharpoonright \mathsf{i}$
$\langle \mathsf{fq} \rangle \hookrightarrow T10059 \Rightarrow \quad Stat21: \langle \forall \mathsf{h} \in \mathbb{N} \mid \mathsf{fq} \upharpoonright \mathsf{h} \in \mathbb{Q} \rangle$
$\langle \mathsf{f}' \rangle \hookrightarrow T10059 \Rightarrow \quad Stat22: \langle \forall \mathsf{h} \in \mathbb{N} \mid \mathsf{f}' \upharpoonright \mathsf{h} \in \mathbb{Q} \rangle$
$\langle \mathsf{i} \rangle \hookrightarrow Stat21 \Rightarrow \quad \mathsf{fq} \upharpoonright \mathsf{i} \in \mathbb{Q}$
$\langle \mathsf{i} \rangle \hookrightarrow Stat22 \Rightarrow \quad \mathsf{f}' \upharpoonright \mathsf{i} \in \mathbb{Q}$
$\langle \mathsf{fq} \upharpoonright \mathsf{i}, \mathsf{f}' \upharpoonright \mathsf{i} \rangle \hookrightarrow T368 \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

-- It is equally easy to prove the associative laws for addition and multiplication of rational sequences. Once more, these results follow trivially from the pointwise definitions of the operations $+_{QS}$ and $*_{QS}$.

Theorem 608 (417) $\{F, G, H\} \subseteq \mathsf{RaSeq} \to (F +_{QS} G) +_{QS} H = F +_{QS} (G +_{QS} H)$. PROOF:

Suppose_not$(f', g', h') \Rightarrow \quad \{f', g', h'\} \subseteq \mathsf{RaSeq} \;\&\; f' +_{QS} g' +_{QS} h' \neq f' +_{QS} (g' +_{QS} h')$
$\langle f', g'\rangle \hookrightarrow T10062 \Rightarrow \quad f' +_{QS} g' \in \mathsf{RaSeq} \;\&\; f' +_{QS} g' = \{[u, f'\upharpoonright u +_Q g'\upharpoonright u] : u \in \mathbb{N}\}$
$\langle g', h'\rangle \hookrightarrow T10062 \Rightarrow \quad g' +_{QS} h' \in \mathsf{RaSeq} \;\&\; g' +_{QS} h' = \{[u, g'\upharpoonright u +_Q h'\upharpoonright u] : u \in \mathbb{N}\}$
$\langle f' +_{QS} g', h'\rangle \hookrightarrow T10062 \Rightarrow \quad f' +_{QS} g' +_{QS} h' = \{[u, (f' +_{QS} g')\upharpoonright u +_Q h'\upharpoonright u] : u \in \mathbb{N}\}$
$\langle f', g' +_{QS} h'\rangle \hookrightarrow T10062 \Rightarrow \quad f' +_{QS} (g' +_{QS} h') = \{[u, f'\upharpoonright u +_Q (g' +_{QS} h')\upharpoonright u] : u \in \mathbb{N}\}$
EQUAL $\Rightarrow \quad Stat19: \{[u, (f' +_{QS} g')\upharpoonright u +_Q h'\upharpoonright u] : u \in \mathbb{N}\} \neq \{[u, f'\upharpoonright u +_Q (g' +_{QS} h')\upharpoonright u] : u \in \mathbb{N}\}$
$\langle i\rangle \hookrightarrow Stat19 \Rightarrow \quad i \in \mathbb{N} \;\&\; (f' +_{QS} g')\upharpoonright i +_Q h'\upharpoonright i \neq f'\upharpoonright i +_Q (g' +_{QS} h')\upharpoonright i$
APPLY $\langle\,\rangle$ fcn_symbol$(f(u) \mapsto f'\upharpoonright u +_Q g'\upharpoonright u, g \mapsto f' +_{QS} g', s \mapsto \mathbb{N}) \Rightarrow$
$Stat23: \langle \forall x \mid (f' +_{QS} g')\upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } f'\upharpoonright x +_Q g'\upharpoonright x \textbf{ else } \emptyset \textbf{ fi}\rangle$
APPLY $\langle\,\rangle$ fcn_symbol$(f(u) \mapsto g'\upharpoonright u +_Q h'\upharpoonright u, g \mapsto g' +_{QS} h', s \mapsto \mathbb{N}) \Rightarrow$
$Stat24: \langle \forall x \mid (g' +_{QS} h')\upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } g'\upharpoonright x +_Q h'\upharpoonright x \textbf{ else } \emptyset \textbf{ fi}\rangle$
$\langle i\rangle \hookrightarrow Stat23 \Rightarrow \quad (f' +_{QS} g')\upharpoonright i = f'\upharpoonright i +_Q g'\upharpoonright i$
$\langle i\rangle \hookrightarrow Stat24 \Rightarrow \quad (g' +_{QS} h')\upharpoonright i = g'\upharpoonright i +_Q h'\upharpoonright i$
EQUAL $\Rightarrow \quad f'\upharpoonright i +_Q g'\upharpoonright i +_Q h'\upharpoonright i \neq f'\upharpoonright i +_Q (g'\upharpoonright i +_Q h'\upharpoonright i)$
$\langle f'\rangle \hookrightarrow T10059 \Rightarrow \quad Stat20: \langle \forall h \in \mathbb{N} \mid f'\upharpoonright h \in \mathbb{Q}\rangle$
$\langle i\rangle \hookrightarrow Stat20 \Rightarrow \quad f'\upharpoonright i \in \mathbb{Q}$
$\langle g'\rangle \hookrightarrow T10059 \Rightarrow \quad Stat21: \langle \forall h \in \mathbb{N} \mid g'\upharpoonright h \in \mathbb{Q}\rangle$
$\langle i\rangle \hookrightarrow Stat21 \Rightarrow \quad g'\upharpoonright i \in \mathbb{Q}$
$\langle h'\rangle \hookrightarrow T10059 \Rightarrow \quad Stat22: \langle \forall h \in \mathbb{N} \mid h'\upharpoonright h \in \mathbb{Q}\rangle$
$\langle i\rangle \hookrightarrow Stat22 \Rightarrow \quad h'\upharpoonright i \in \mathbb{Q}$
$\langle h'\upharpoonright i, f'\upharpoonright i, g'\upharpoonright i\rangle \hookrightarrow T370 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ QED

Theorem 609 (418) $\{F, G, H\} \subseteq \mathsf{RaSeq} \to (F *_{QS} G) *_{QS} H = F *_{QS} (G *_{QS} H)$. PROOF:

Suppose_not$(f', g', h') \Rightarrow \quad \{f', g', h'\} \subseteq \mathsf{RaSeq} \;\&\; f' *_{QS} g' *_{QS} h' \neq f' *_{QS} (g' *_{QS} h')$
$\langle f', g'\rangle \hookrightarrow T10062 \Rightarrow \quad f' *_{QS} g' \in \mathsf{RaSeq} \;\&\; f' *_{QS} g' = \{[u, f'\upharpoonright u *_Q g'\upharpoonright u] : u \in \mathbb{N}\}$
$\langle g', h'\rangle \hookrightarrow T10062 \Rightarrow \quad g' *_{QS} h' \in \mathsf{RaSeq} \;\&\; g' *_{QS} h' = \{[u, g'\upharpoonright u *_Q h'\upharpoonright u] : u \in \mathbb{N}\}$
$\langle f' *_{QS} g', h'\rangle \hookrightarrow T10062 \Rightarrow \quad f' *_{QS} g' *_{QS} h' = \{[u, (f' *_{QS} g')\upharpoonright u *_Q h'\upharpoonright u] : u \in \mathbb{N}\}$
$\langle f', g' *_{QS} h'\rangle \hookrightarrow T10062 \Rightarrow \quad f' *_{QS} (g' *_{QS} h') = \{[u, f'\upharpoonright u *_Q (g' *_{QS} h')\upharpoonright u] : u \in \mathbb{N}\}$
EQUAL $\Rightarrow \quad Stat19: \{[u, (f' *_{QS} g')\upharpoonright u *_Q h'\upharpoonright u] : u \in \mathbb{N}\} \neq \{[u, f'\upharpoonright u *_Q (g' *_{QS} h')\upharpoonright u] : u \in \mathbb{N}\}$
$\langle i\rangle \hookrightarrow Stat19 \Rightarrow \quad i \in \mathbb{N} \;\&\; (f' *_{QS} g')\upharpoonright i *_Q h'\upharpoonright i \neq f'\upharpoonright i *_Q (g' *_{QS} h')\upharpoonright i$

APPLY $\langle\,\rangle$ fcn_symbol$\big(\mathsf{f(u)} \mapsto \mathsf{f'}\restriction\mathsf{u} *_{\mathbb{Q}} \mathsf{g'}\restriction\mathsf{u}, \mathsf{g} \mapsto \mathsf{f'} *_{\mathbb{QS}} \mathsf{g'}, \mathsf{s} \mapsto \mathbb{N}\big) \Rightarrow$
Stat23 : $\langle\forall x \,|\, (f' *_{\mathbb{QS}} g')\restriction x = \textbf{if } x \in \mathbb{N} \textbf{ then } f'\restriction x *_{\mathbb{Q}} g'\restriction x \textbf{ else } \emptyset \textbf{ fi}\rangle$
APPLY $\langle\,\rangle$ fcn_symbol$\big(\mathsf{f(u)} \mapsto \mathsf{g'}\restriction\mathsf{u} *_{\mathbb{Q}} \mathsf{h'}\restriction\mathsf{u}, \mathsf{g} \mapsto \mathsf{g'} *_{\mathbb{QS}} \mathsf{h'}, \mathsf{s} \mapsto \mathbb{N}\big) \Rightarrow$
Stat24 : $\langle\forall x \,|\, (g' *_{\mathbb{QS}} h')\restriction x = \textbf{if } x \in \mathbb{N} \textbf{ then } g'\restriction x *_{\mathbb{Q}} h'\restriction x \textbf{ else } \emptyset \textbf{ fi}\rangle$
$\langle i\rangle \hookrightarrow$ *Stat23* $\Rightarrow$ $(f' *_{\mathbb{QS}} g')\restriction i = f'\restriction i *_{\mathbb{Q}} g'\restriction i$
$\langle i\rangle \hookrightarrow$ *Stat24* $\Rightarrow$ $(g' *_{\mathbb{QS}} h')\restriction i = g'\restriction i *_{\mathbb{Q}} h'\restriction i$
EQUAL $\Rightarrow$ $f'\restriction i *_{\mathbb{Q}} g'\restriction i *_{\mathbb{Q}} h'\restriction i \neq f'\restriction i *_{\mathbb{Q}} (g'\restriction i *_{\mathbb{Q}} h'\restriction i)$
$\langle f'\rangle \hookrightarrow$ *T10059* $\Rightarrow$ *Stat20* : $\langle\forall h \in \mathbb{N} \,|\, f'\restriction h \in \mathbb{Q}\rangle$
$\langle i\rangle \hookrightarrow$ *Stat20* $\Rightarrow$ $f'\restriction i \in \mathbb{Q}$
$\langle g'\rangle \hookrightarrow$ *T10059* $\Rightarrow$ *Stat21* : $\langle\forall h \in \mathbb{N} \,|\, g'\restriction h \in \mathbb{Q}\rangle$
$\langle i\rangle \hookrightarrow$ *Stat21* $\Rightarrow$ $g'\restriction i \in \mathbb{Q}$
$\langle h'\rangle \hookrightarrow$ *T10059* $\Rightarrow$ *Stat22* : $\langle\forall h \in \mathbb{N} \,|\, h'\restriction h \in \mathbb{Q}\rangle$
$\langle i\rangle \hookrightarrow$ *Stat22* $\Rightarrow$ $h'\restriction i \in \mathbb{Q}$
$\langle h'\restriction i, f'\restriction i, g'\restriction i\rangle \hookrightarrow$ *T374* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- It is easily seen that the zero and unit rational sequences play the proper algebraic roles.

Theorem 610 (419) $F \in \mathsf{RaSeq} \rightarrow F +_{\mathbb{QS}} \mathsf{RaSeq}_0 = F \,\&\, F *_{\mathbb{QS}} \mathsf{RaSeq}_1 = F$. PROOF:

Suppose_not(f) $\Rightarrow$ $f \in \mathsf{RaSeq} \,\&\, f +_{\mathbb{QS}} \mathsf{RaSeq}_0 \neq f \vee f *_{\mathbb{QS}} \mathsf{RaSeq}_1 \neq f$

-- Indeed, if we assume the contrary, either the zero rational sequence does not behave as additive unit element among rational sequences, or the unit rational does not behave as multiplicative unit element. Either alternative will lead to a contradiction, and the two proofs will closely resemble each other.

$\langle f\rangle \hookrightarrow$ *T10059* $\Rightarrow$ *Stat20* : $\langle\forall h \in \mathbb{N} \,|\, f\restriction h \in \mathbb{Q}\rangle$
Use_def(RaSeq) $\Rightarrow$ *Stat21* : $f \in \{f \subseteq \mathbb{N} \times \mathbb{Q} \,|\, \mathbf{domain}(f) = \mathbb{N} \,\&\, \mathsf{Svm}(f)\}$
$\langle\,\rangle \hookrightarrow$ *Stat21* $\Rightarrow$ $\mathbf{domain}(f) = \mathbb{N} \,\&\, \mathsf{Svm}(f)$
$\langle f\rangle \hookrightarrow$ *T65* $\Rightarrow$ $f = \{[i, f\restriction i] : i \in \mathbf{domain}(f)\}$

-- Assume first that RaSeq_0 does not behave as additive unit element.

Suppose $\Rightarrow$ *Stat1* : $f +_{\mathbb{QS}} \mathsf{RaSeq}_0 \neq f$

-- By the pointwise definition of $+_{\mathbb{QS}}$, the sequence $f +_{\mathbb{QS}} \mathsf{RaSeq}_0$ associates the rational value $f\restriction u +_{\mathbb{Q}} \mathsf{RaSeq}_0\restriction u$ with each unsigned integer u.

$\langle f, \mathsf{RaSeq}_0\rangle \hookrightarrow$ *T10062* $\Rightarrow$ $f +_{\mathbb{QS}} \mathsf{RaSeq}_0 = \big\{[u, f\restriction u +_{\mathbb{Q}} \mathsf{RaSeq}_0\restriction u] : u \in \mathbb{N}\big\}$
Use_def(RaSeq_0) $\Rightarrow$ $\mathsf{RaSeq}_0 = \mathbb{N} \times \{\mathbf{0}_{\mathbb{Q}}\}$

$\mathsf{Use\_def}(\times) \Rightarrow$ $\mathbb{N} \times \{\mathbf{0}_{\mathbb{Q}}\} = \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{0}_{\mathbb{Q}}\}\}$
Suppose $\Rightarrow$ *Stat2*: $\{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{0}_{\mathbb{Q}}\}\} \neq \{[x,\mathbf{0}_{\mathbb{Q}}] : x \in \mathbb{N}\}$
$\langle c \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $c \in \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{0}_{\mathbb{Q}}\}\} \leftrightarrow c \notin \{[x,\mathbf{0}_{\mathbb{Q}}] : x \in \mathbb{N}\}$
Suppose $\Rightarrow$ *Stat3*: $c \in \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{0}_{\mathbb{Q}}\}\}$ & $c \notin \{[x,\mathbf{0}_{\mathbb{Q}}] : x \in \mathbb{N}\}$
$\langle x_0, y_0, x_0 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat4*: $c \in \{[x,\mathbf{0}_{\mathbb{Q}}] : x \in \mathbb{N}\}$ & $c \notin \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{0}_{\mathbb{Q}}\}\}$
$\langle x_1, x_1, \mathbf{0}_{\mathbb{Q}} \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{RaSeq}_0 = \{[x,\mathbf{0}_{\mathbb{Q}}] : x \in \mathbb{N}\}$

-- On the other hand, $\mathsf{RaSeq}_0 \upharpoonright u$ has value $\mathbf{0}_{\mathbb{Q}}$ for all unsigned integers u; therefore, since $\mathbf{0}_{\mathbb{Q}}$ behaves as additive unit element among rational numbers, $f \upharpoonright u +_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright u$ always carries the value $f \upharpoonright u$. Accordingly the sequence $f +_{\mathbb{QS}} \mathsf{RaSeq}_0$, whose values coincide with those of f over the common domain $\mathbb{N}$, must coincide with f.

APPLY $\langle\,\rangle$ fcn_symbol$(f(u) \mapsto \mathbf{0}_{\mathbb{Q}}, g \mapsto \mathsf{RaSeq}_0, s \mapsto \mathbb{N}) \Rightarrow$
Stat23: $\langle \forall x \mid \mathsf{RaSeq}_0 \upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } \mathbf{0}_{\mathbb{Q}} \textbf{ else } \emptyset \textbf{ fi} \rangle$
$\langle d \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $d \in \{[u, f \upharpoonright u +_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright u] : u \in \mathbb{N}\} \leftrightarrow d \notin f$
Suppose $\Rightarrow$ *Stat6*: $d \notin \{[i, f \upharpoonright i] : i \in \mathbf{domain}(f)\}$ & *Stat5*: $d \in \{[u, f \upharpoonright u +_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright u] : u \in \mathbb{N}\}$
$\langle i_0 \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $i_0 \in \mathbb{N}$ & $d = [i_0, f \upharpoonright i_0 +_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright i_0]$
$\langle i_0 \rangle \hookrightarrow$ *Stat23* $\Rightarrow$ $\mathsf{RaSeq}_0 \upharpoonright i_0 = \mathbf{0}_{\mathbb{Q}}$
$\langle i_0 \rangle \hookrightarrow$ *Stat20* $\Rightarrow$ $f \upharpoonright i_0 \in \mathbb{Q}$
$\langle f \upharpoonright i_0 \rangle \hookrightarrow$ *T371* $\Rightarrow$ $f \upharpoonright i_0 +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = f \upharpoonright i_0$
EQUAL $\Rightarrow$ $d = [i_0, f \upharpoonright i_0]$
$\langle i_0 \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat7*: $d \in \{[i, f \upharpoonright i] : i \in \mathbf{domain}(f)\}$ & $d \notin \{[u, f \upharpoonright u +_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright u] : u \in \mathbb{N}\}$
$\langle i_1, i_1 \rangle \hookrightarrow$ *Stat7* $\Rightarrow$ $d = [i_1, f \upharpoonright i_1]$ & $i_1 \in \mathbb{N}$ & $d \neq [i_1, f \upharpoonright i_1 +_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright i_1]$
$\langle i_1 \rangle \hookrightarrow$ *Stat23* $\Rightarrow$ $\mathsf{RaSeq}_0 \upharpoonright i_1 = \mathbf{0}_{\mathbb{Q}}$
$\langle i_1 \rangle \hookrightarrow$ *Stat20* $\Rightarrow$ $f \upharpoonright i_1 \in \mathbb{Q}$
$\langle f \upharpoonright i_1 \rangle \hookrightarrow$ *T371* $\Rightarrow$ $f \upharpoonright i_1 +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} = f \upharpoonright i_1$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat11*: $f *_{\mathbb{QS}} \mathsf{RaSeq}_1 \neq f$

-- We have drawn a contradiction from the assumption that RaSeq_0 does not behave as additive unit element; hence we must now reason under the assumption that RaSeq_1 does not behave as additive unit element. By the pointwise definition of $*_{\mathbb{QS}}$, the sequence $f *_{\mathbb{QS}} \mathsf{RaSeq}_1$ associates the rational value $f \upharpoonright u *_{\mathbb{Q}} \mathsf{RaSeq}_0 \upharpoonright u$ with each unsigned integer u.

$\langle f, \mathsf{RaSeq}_1 \rangle \hookrightarrow$ *T10062* $\Rightarrow$ $f *_{\mathbb{QS}} \mathsf{RaSeq}_1 = \{[u, f \upharpoonright u *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright u] : u \in \mathbb{N}\}$
$\mathsf{Use\_def}(\mathsf{RaSeq}_1) \Rightarrow$ $\mathsf{RaSeq}_1 = \mathbb{N} \times \{\mathbf{1}_{\mathbb{Q}}\}$
$\mathsf{Use\_def}(\times) \Rightarrow$ $\mathsf{RaSeq}_1 = \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{1}_{\mathbb{Q}}\}\}$
Suppose $\Rightarrow$ *Stat12*: $\{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{1}_{\mathbb{Q}}\}\} \neq \{[x,\mathbf{1}_{\mathbb{Q}}] : x \in \mathbb{N}\}$
$\langle c' \rangle \hookrightarrow$ *Stat12* $\Rightarrow$ $c' \in \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{1}_{\mathbb{Q}}\}\} \leftrightarrow c' \notin \{[x,\mathbf{1}_{\mathbb{Q}}] : x \in \mathbb{N}\}$
Suppose $\Rightarrow$ *Stat13*: $c' \in \{[x,y] : x \in \mathbb{N}, y \in \{\mathbf{1}_{\mathbb{Q}}\}\}$ & $c' \notin \{[x,\mathbf{1}_{\mathbb{Q}}] : x \in \mathbb{N}\}$

$\langle x_2, y_2, x_2 \rangle \hookrightarrow Stat13 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat14: \ c' \in \{[x, \mathbf{1}_{\mathbb{Q}}] : x \in \mathbb{N}\} \ \& \ c' \notin \{[x, y] : x \in \mathbb{N}, y \in \{\mathbf{1}_{\mathbb{Q}}\}\}$

$\langle x_3, x_3, \mathbf{1}_{\mathbb{Q}} \rangle \hookrightarrow Stat14 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{RaSeq}_1 = \{[x, \mathbf{1}_{\mathbb{Q}}] : x \in \mathbb{N}\}$

-- On the other hand, $\mathsf{RaSeq}_1 \upharpoonright u$ has value $\mathbf{1}_{\mathbb{Q}}$ for all unsigned integers u; therefore, since $\mathbf{1}_{\mathbb{Q}}$ behaves as multiplicative unit element among rational numbers, $f \upharpoonright u *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright u$ always carries the value $f \upharpoonright u$. Accordingly the sequence $f *_{\mathbb{QS}} \mathsf{RaSeq}_1$, whose values coincide with those of f over the common domain $\mathbb{N}$, must coincide with f.

APPLY $\langle\ \rangle$ fcn_symbol$(f(u) \mapsto \mathbf{1}_{\mathbb{Q}}, g \mapsto \mathsf{RaSeq}_1, s \mapsto \mathbb{N}) \Rightarrow$

$Stat33: \ \langle \forall x \mid \mathsf{RaSeq}_1 \upharpoonright x = \textbf{if } x \in \mathbb{N} \textbf{ then } \mathbf{1}_{\mathbb{Q}} \textbf{ else } \emptyset \textbf{ fi} \rangle$

$\langle d' \rangle \hookrightarrow Stat11 \Rightarrow$ $d' \in \{[u, f \upharpoonright u *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright u] : u \in \mathbb{N}\} \leftrightarrow d' \notin f$

Suppose $\Rightarrow$ $Stat16: \ d' \notin \{[i, f \upharpoonright i] : i \in \mathbf{domain}(f)\} \ \& \ Stat15: \ d' \in \{[u, f \upharpoonright u *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright u] : u \in \mathbb{N}\}$

$\langle i_2 \rangle \hookrightarrow Stat15 \Rightarrow$ $i_2 \in \mathbb{N} \ \& \ d' = [i_2, f \upharpoonright i_2 *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright i_2]$

$\langle i_2 \rangle \hookrightarrow Stat33 \Rightarrow$ $\mathsf{RaSeq}_1 \upharpoonright i_2 = \mathbf{1}_{\mathbb{Q}}$

$\langle i_2 \rangle \hookrightarrow Stat20 \Rightarrow$ $f \upharpoonright i_2 \in \mathbb{Q}$

$\langle f \upharpoonright i_2 \rangle \hookrightarrow T379 \Rightarrow$ $f \upharpoonright i_2 *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = f \upharpoonright i_2$

EQUAL $\Rightarrow$ $d' = [i_2, f \upharpoonright i_2]$

$\langle i_2 \rangle \hookrightarrow Stat16 \Rightarrow$ false; Discharge $\Rightarrow$ $Stat17: \ d' \in \{[i, f \upharpoonright i] : i \in \mathbf{domain}(f)\} \ \& \ d' \notin \{[u, f \upharpoonright u *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright u] : u \in \mathbb{N}\}$

$\langle i_3, i_3 \rangle \hookrightarrow Stat17 \Rightarrow$ $d' = [i_3, f \upharpoonright i_3] \ \& \ i_3 \in \mathbb{N} \ \& \ d' \neq [i_3, f \upharpoonright i_3 *_{\mathbb{Q}} \mathsf{RaSeq}_1 \upharpoonright i_3]$

$\langle i_3 \rangle \hookrightarrow Stat33 \Rightarrow$ $\mathsf{RaSeq}_1 \upharpoonright i_3 = \mathbf{1}_{\mathbb{Q}}$

$\langle i_3 \rangle \hookrightarrow Stat20 \Rightarrow$ $f \upharpoonright i_3 \in \mathbb{Q}$

$\langle f \upharpoonright i_3 \rangle \hookrightarrow T379 \Rightarrow$ $f \upharpoonright i_3 *_{\mathbb{Q}} \mathbf{1}_{\mathbb{Q}} = f \upharpoonright i_3$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 611 ( ) $N, M \in \mathbb{Z} \ \& \ M \neq [\emptyset, \emptyset] \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}(M) \rightarrow \langle \exists k \in \mathbb{Z} \mid \mathsf{is\_nonneg}_{\mathbb{N}}(N -_{\mathbb{Z}} k *_{\mathbb{Z}} M) \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}((k +_{\mathbb{Z}} [1, \emptyset]) *_{\mathbb{Z}} M) -_{\mathbb{Z}} N \rangle$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $n, m \in \mathbb{Z} \ \& \ m \neq [\emptyset, \emptyset] \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}(m) \ \& \ \neg \langle \exists k \in \mathbb{Z} \mid \mathsf{is\_nonneg}_{\mathbb{N}}(n -_{\mathbb{Z}} k *_{\mathbb{Z}} m) \ \& \ \mathsf{is\_nonneg}_{\mathbb{N}}((k +_{\mathbb{Z}} [1, \emptyset]) *_{\mathbb{Z}} m) -_{\mathbb{Z}} n \rangle$

THUS $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 612 ( ) $N \in \mathbb{R} \rightarrow N +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(N) = \mathbf{0}_{\mathbb{R}}$. PROOF:

Suppose_not$(n) \Rightarrow$ $n \in \mathbb{R} \ \& \ n +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(n) \neq \mathbf{0}_{\mathbb{R}}$

$\langle n \rangle \hookrightarrow T99999 \Rightarrow$ $n \subseteq \mathbb{Q}$

Use_def$(\mathsf{Rev}_{\mathbb{R}}) \Rightarrow$ $n +_{\mathbb{R}} \{\mathsf{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v : u \in \mathbb{Q} \backslash n, v \in \mathbf{0}_{\mathbb{R}}\} \neq \mathbf{0}_{\mathbb{R}}$

Use_def$(+_{\mathbb{R}}) \Rightarrow$ $\{x +_{\mathbb{Q}} y : x \in n \ \& \ y \in \{\mathsf{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v : u \in \mathbb{Q} \backslash n, v \in \mathbf{0}_{\mathbb{R}}\}\} \neq \mathbf{0}_{\mathbb{R}}$

SIMPLF $\Rightarrow$ $\{x +_{\mathbb{Q}} (\mathsf{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v) : x \in n, u \in \mathbb{Q} \backslash n, v \in \mathbf{0}_{\mathbb{R}}\} \neq \mathbf{0}_{\mathbb{R}}$

Suppose $\Rightarrow$ $\{x +_{\mathbb{Q}} (\mathrm{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v) : x \in n, u \in \mathbb{Q}\backslash n, v \in \mathbf{0}_{\mathbb{R}}\} \not\subseteq \mathbf{0}_{\mathbb{R}}$

$\langle \mathrm{Memb}(c) \rangle \Rightarrow$ *Stat1* : $c \in \{x +_{\mathbb{Q}} (\mathrm{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v) : x \in n, u \in \mathbb{Q}\backslash n, v \in \mathbf{0}_{\mathbb{R}}\}$ & $c \notin \mathbf{0}_{\mathbb{R}}$

$\langle a_1, b_1, c_1 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $c = a_1 +_{\mathbb{Q}} (\mathrm{Rev}_{\mathbb{Q}}(b_1) +_{\mathbb{Q}} c_1)$ & $a_1 \in n$ & $b_1 \in \mathbb{Q}\backslash n$ & $c_1 \in \mathbf{0}_{\mathbb{R}}$

THUS $\Rightarrow$ $a_1, b_1, c_1 \in \mathbb{Q}$

ALGEBRA $\Rightarrow$ $\mathrm{Rev}_{\mathbb{Q}}(b_1) \in \mathbb{Q}$

THUS $\Rightarrow$ false; Discharge $\Rightarrow$ $\{x +_{\mathbb{Q}} (\mathrm{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v) : x \in n, u \in \mathbb{Q}\backslash n, v \in \mathbf{0}_{\mathbb{R}}\} \not\supseteq \mathbf{0}_{\mathbb{R}}$

$\langle \mathrm{Memb}(d) \rangle \Rightarrow$ $d \in \mathbf{0}_{\mathbb{R}}$ & $d \notin \{x +_{\mathbb{Q}} (\mathrm{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v) : x \in n, u \in \mathbb{Q}\backslash n, v \in \mathbf{0}_{\mathbb{R}}\}$

THUS $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 613 ( ) $N, M \in \mathbb{R} \to N \subseteq M \vee M \subseteq N$. PROOF:

Suppose_not(n) $\Rightarrow$ $n, m \in \mathbb{R}$ & $n \not\subseteq m$ & $m \not\subseteq n$

Use_def($\mathbb{R}$) $\Rightarrow$ *Stat1* :

$n \in \{s : s \subseteq \mathbb{Q} \mid s \neq \emptyset$ & $s \neq \mathbb{Q}$ & $\langle \forall x \in s, \exists y \in s \mid y >_{\mathbb{Q}} x \rangle$ & $\langle \forall x \in s, y \in \mathbb{Q} \mid x >_{\mathbb{Q}} y \to y \in s \rangle\}$ &

$m \in \{s : s \subseteq \mathbb{Q} \mid s \neq \emptyset$ & $s \neq \mathbb{Q}$ & $\langle \forall x \in s, \exists y \in s \mid y >_{\mathbb{Q}} x \rangle$ & $\langle \forall x \in s, y \in \mathbb{Q} \mid x >_{\mathbb{Q}} y \to y \in s \rangle\}$

$\langle a, b \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $n \subseteq \mathbb{Q}$ & *Stat2* :

$\langle \forall x \in n, n \in \mathbb{Q} \mid x >_{\mathbb{Q}} y \to y \in n \rangle$ & $m \subseteq \mathbb{Q}$ & $m \neq \emptyset$ & *Stat3* : $\langle \forall x \in m, y \in \mathbb{Q} \mid x >_{\mathbb{Q}} y \to y \in m \rangle$

$\langle \mathrm{Memb}(c) \rangle \Rightarrow$ $c \in n$ & $c \notin m$

$\langle \mathrm{Memb}(d) \rangle \Rightarrow$ $d \in m$ & $d \notin n$

ELEM $\Rightarrow$ $c \neq d$ & $c, d \in \mathbb{Q}$

Suppose $\Rightarrow$ $c >_{\mathbb{Q}} d$

$\langle c, d \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg c >_{\mathbb{Q}} d$

Suppose $\Rightarrow$ $d >_{\mathbb{Q}} c$

$\langle d, c \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg d >_{\mathbb{Q}} c$

T384 $\Rightarrow$ $\neg\mathrm{is\_nonneg}_{\mathbb{Q}}(c -_{\mathbb{Q}} d)$ & $\neg\mathrm{is\_nonneg}_{\mathbb{Q}}(d -_{\mathbb{Q}} c)$

ALGEBRA $\Rightarrow$ $\neg\mathrm{is\_nonneg}_{\mathbb{Q}}(c -_{\mathbb{Q}} d)$ & $\neg\mathrm{is\_nonneg}_{\mathbb{Q}}(\mathrm{Rev}_{\mathbb{Q}}(c -_{\mathbb{Q}} d))$

$\langle c, d \rangle \hookrightarrow$ *T99999* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 614 ( ) $N, M \in \mathbb{R} \to N \cup M \in \mathbb{R}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, m \in \mathbb{R}$ & $n \cup m \notin \mathbb{R}$

$\langle n, m \rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \subseteq m \vee m \subseteq n$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 615 ( ) $N \in \mathbb{R} \to \#N_{\mathbb{R}} \in \mathbb{R}$ & $N \subseteq \#N_{\mathbb{R}}$. PROOF:

Suppose_not(n) $\Rightarrow$ $n \in \mathbb{R} \,\&\, \#n_{\mathbb{R}} \notin \mathbb{R} \vee n \not\subseteq \#n_{\mathbb{R}}$
Use_def$(\#)_{\mathbb{R}}$ $\Rightarrow$ $\#n_{\mathbb{R}} = n \cup \text{Rev}_{\mathbb{R}}(n)$
ELEM $\Rightarrow$ $n \cup \text{Rev}_{\mathbb{R}}(n) \notin R$
$\langle n \rangle \hookrightarrow$ *T99999* $\Rightarrow$ $\text{Rev}_{\mathbb{R}}(n) \in R$
$\langle n, \text{Rev}_{\mathbb{R}}(n) \rangle \hookrightarrow$ *T99999* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 616 ( ) $N, M \in \mathbb{R} \rightarrow N = M +_{\mathbb{R}} (N -_{\mathbb{R}} M)$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, m \in \mathbb{R} \,\&\, n \neq m +_{\mathbb{R}} (n -_{\mathbb{R}} m)$
Use_def$(-_{\mathbb{R}})$ $\Rightarrow$ $n \neq m +_{\mathbb{R}} \left(n +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(m)\right)$
ALGEBRA $\Rightarrow$ $n \neq n +_{\mathbb{R}} \left(m +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(m)\right)$
$\langle m \rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \neq n +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle n \rangle \hookrightarrow$ *T99999* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 617 ( ) $N, M \in \mathbb{R} \rightarrow N \,|\!*\!|_{\mathbb{R}}\, M = M \,|\!*\!|_{\mathbb{R}}\, N$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, m \in \mathbb{R} \,\&\, n \,|\!*\!|_{\mathbb{R}}\, m = m \,|\!*\!|_{\mathbb{R}}\, n$
Use_def$(|\!*\!|_{\mathbb{R}})$ $\Rightarrow$ $\{u *_{\mathbb{Q}} v : u \in \#n_{\mathbb{R}}, v \in \#m_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v)\} \cup \mathbf{0}_{\mathbb{R}} \neq \{u *_{\mathbb{Q}} v : u \in \#m_{\mathbb{R}}, v \in \#n_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v)\} \cup \mathbf{0}_{\mathbb{R}}$
ELEM $\Rightarrow$ $\{u *_{\mathbb{Q}} v : u \in \#n_{\mathbb{R}}, v \in \#m_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v)\} \neq \{v *_{\mathbb{Q}} u : u \in \#n_{\mathbb{R}}, v \in \#m_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u)\}$
ELEM $\Rightarrow$ $\neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} V \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} U) \leftrightarrow \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} U \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} V)$
EQUAL $\Rightarrow$ $\{u *_{\mathbb{Q}} v : u \in \#n_{\mathbb{R}}, v \in \#m_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v)\} \neq \{v *_{\mathbb{Q}} u : u \in \#n_{\mathbb{R}}, v \in \#m_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v)\}$
$\langle n \rangle \hookrightarrow$ *T99999* $\Rightarrow$ $\#n_{\mathbb{R}} \in R$
$\langle m \rangle \hookrightarrow$ *T99999* $\Rightarrow$ $\#m_{\mathbb{R}} \in R$
THUS $\Rightarrow$ $\#n_{\mathbb{R}} \subseteq \mathbb{Q} \,\&\, \#m_{\mathbb{R}} \subseteq \mathbb{Q}$
Suppose $\Rightarrow$ *Stat1* : $\neg\langle \forall u \in \#n_{\mathbb{R}}, v \in \#n_{\mathbb{R}} \mid u *_{\mathbb{Q}} v = v *_{\mathbb{Q}} u \rangle$
Pred_monot $\Rightarrow$ $\neg\langle \forall u \in \mathbb{Q}, v \in \mathbb{Q} \mid u *_{\mathbb{Q}} v = v *_{\mathbb{Q}} u \rangle$
$\langle a, b \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $a, b \in \mathbb{Q} \,\&\, a *_{\mathbb{Q}} b \neq b *_{\mathbb{Q}} a$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $\langle \forall u \in \#n_{\mathbb{R}}, v \in \#n_{\mathbb{R}} \mid u *_{\mathbb{Q}} v = v *_{\mathbb{Q}} u \rangle$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 618 ( ) $N, M \in \mathbb{R} \rightarrow N *_{\mathbb{R}} M = M *_{\mathbb{R}} N$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, m \in \mathbb{R} \,\&\, n *_{\mathbb{R}} m \neq m *_{\mathbb{R}} n$
$\langle n, m \rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \,|\!*\!|_{\mathbb{R}}\, n = m \,|\!*\!|_{\mathbb{R}}\, n$
Use_def$(*_{\mathbb{R}})$ $\Rightarrow$ $\neg(n \supseteq \mathbf{0}_{\mathbb{R}} \leftrightarrow m \supseteq \mathbf{0}_{\mathbb{R}} \leftrightarrow m \supseteq \mathbf{0}_{\mathbb{R}} \leftrightarrow n \supseteq \mathbf{0}_{\mathbb{R}})$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 619 ( ) $\mathsf{N} \in \mathbb{R} \rightarrow \#\mathsf{N}_{\mathbb{R}} = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{N}) \textbf{ then } \mathsf{N} \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(\mathsf{N}) \textbf{ fi}$. PROOF:

Suppose_not(n) $\Rightarrow$ $n \in \mathbb{R} \,\&\, \#n_{\mathbb{R}} \neq \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(n) \textbf{ then } n \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(n) \textbf{ fi}$

Use_def$(\#)_{\mathbb{R}}$ $\Rightarrow$ $\#n_{\mathbb{R}} = n \cup \mathsf{Rev}_{\mathbb{R}}(n)$

Use_def$(\mathsf{is_nonneg}_{\mathbb{R}})$ $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(n) \leftrightarrow \mathbf{0}_{\mathbb{R}} \subseteq n$

Suppose $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(n)$

ELEM $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} \subseteq n \,\&\, \mathsf{Rev}_{\mathbb{R}}(n) \nsubseteq n$

$\langle \mathsf{Memb}(c) \rangle$ $\Rightarrow$ $c \in \mathsf{Rev}_{\mathbb{R}}(n) \,\&\, c \notin n$

Use_def$(\mathsf{Rev}_{\mathbb{R}})$ $\Rightarrow$ $Stat1:\ c \in \{\mathsf{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v :\ u \in \mathbb{Q} \backslash n, v \in \mathbf{0}_{\mathbb{R}}\}$

$\langle a, b \rangle \hookrightarrow Stat1 \Rightarrow$ $c = \mathsf{Rev}_{\mathbb{Q}}(a) +_{\mathbb{Q}} b \,\&\, a \in \mathbb{Q} \,\&\, a \notin n \,\&\, b \in \mathbf{0}_{\mathbb{R}}$

Use_def$(\mathbf{0}_{\mathbb{R}})$ $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} = \{x \in \mathbb{Q} \mid \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x\} \,\&\, Stat2:\ b \in \{x \in \mathbb{Q} \mid \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x\}$

$\langle z \rangle \hookrightarrow Stat2 \Rightarrow$ $b \in \mathbb{Q} \,\&\, \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} b$

Suppose $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} a$

Suppose $\Rightarrow$ $Stat3:\ a \notin \{x \in \mathbb{Q} \mid \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x\}$

$\langle z_2 \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $a \in \mathbf{0}_{\mathbb{R}}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} a$

$\langle a \rangle \hookrightarrow T99999 \Rightarrow$ $a >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} \vee a = \mathbf{0}_{\mathbb{Q}}$

$\langle b \rangle \hookrightarrow T99999 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(b) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle a, \mathbf{0}_{\mathbb{Q}}, b, \mathbf{0}_{\mathbb{Q}} \rangle$ $\Rightarrow$ $a +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(b) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}} +_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

ALGEBRA $\Rightarrow$ $a +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(b) >_{\mathbb{Q}} \mathbf{0}_{\mathbb{Q}}$

$\langle a, b \rangle \hookrightarrow T99999 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(a +_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(b))$

ALGEBRA $\Rightarrow$ $\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(a) +_{\mathbb{Q}} b$

Suppose $\Rightarrow$ $Stat4:\ \mathsf{Rev}_{\mathbb{Q}}(a) +_{\mathbb{Q}} b \notin \{x \in \mathbb{Q} \mid \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x\}$

$\langle z_3 \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(a) +_{\mathbb{Q}} b \in \mathbf{0}_{\mathbb{R}}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \mathsf{is\_nonneg}_{\mathbb{R}}(n)$

ELEM $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} \nsubseteq n \,\&\, n \nsubseteq \mathsf{Rev}_{\mathbb{R}}(n)$

$\langle \mathsf{Memb}(d) \rangle$ $\Rightarrow$ $d \in \mathbf{0}_{\mathbb{R}} \,\&\, d \notin n$

Use_def$(\mathbf{0}_{\mathbb{R}})$ $\Rightarrow$ $Stat5:\ d \in \{x \in \mathbb{Q} \mid \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x\}$

$\langle z_5 \rangle \hookrightarrow Stat6 \Rightarrow$ $\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} d$

$\langle \mathbf{0}_{\mathbb{Q}}, d \rangle \hookrightarrow T99999 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(d) >_{\mathbb{Q}} \mathsf{Rev}_{\mathbb{Q}}(\mathbf{0}_{\mathbb{Q}})$

$\langle \mathsf{Rev}_{\mathbb{Q}}(d), \mathbf{0}_{\mathbb{Q}}, d \rangle \hookrightarrow T99999 \Rightarrow$ $\mathsf{Rev}_{\mathbb{Q}}(d) >_{\mathbb{Q}} d$

$\langle \mathsf{Memb}(e) \rangle$ $\Rightarrow$ $e \in n \,\&\, e \notin \mathsf{Rev}_{\mathbb{R}}(n)$

Use_def$(\mathsf{Rev}_{\mathbb{R}})$ $\Rightarrow$ $Stat7:\ e \notin \{\mathsf{Rev}_{\mathbb{Q}}(u) +_{\mathbb{Q}} v :\ u \in \mathbb{Q} \backslash n, v \in \mathbf{0}_{\mathbb{R}}\}$

Use_def$(\mathbb{R})$ $\Rightarrow$ $Stat8:\ n \in \{s :\ s \subseteq \mathbb{Q} \mid s \neq \emptyset \,\&\, s \neq \mathbb{Q} \,\&\, \langle \forall x \in s, \exists y \in s \mid y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in s, y \in \mathbb{Q} \mid x >_{\mathbb{Q}} y \rightarrow y \in s \rangle\}$

$\langle z_4 \rangle \hookrightarrow Stat8 \Rightarrow$ $Stat9:\ \langle \forall x \in n, y \in \mathbb{Q} \mid x >_{\mathbb{Q}} y \rightarrow y \in n \rangle$

$\langle e, d \rangle \hookrightarrow Stat9 \Rightarrow \quad \neg e >_{\mathbb{Q}} d$
$\langle Rev_{\mathbb{Q}}(d), d \rangle \hookrightarrow Stat9 \Rightarrow \quad Rev_{\mathbb{Q}}(d) \notin n$
$THUS \Rightarrow \quad d, e, Rev_{\mathbb{Q}}(d), Rev_{\mathbb{Q}}(e) \in \mathbb{Q}$
$ELEM \Rightarrow \quad e \neq d$
$\langle e, d \rangle \hookrightarrow T99999 \Rightarrow \quad d >_{\mathbb{Q}} e$
$\langle d, Rev_{\mathbb{R}}(e), e, Rev_{\mathbb{R}}(e) \rangle \hookrightarrow T99999 \Rightarrow \quad d +_{\mathbb{Q}} Rev_{\mathbb{Q}}(d) >_{\mathbb{Q}} e +_{\mathbb{Q}} Rev_{\mathbb{Q}}(d)$
$ALGEBRA \Rightarrow \quad \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d)$
$Suppose \Rightarrow \quad e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d) \notin \mathbf{0}_{\mathbb{R}}$
$Use\_def(\mathbf{0}_{\mathbb{R}}) \Rightarrow \quad Stat10: \ e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d) \notin \{x \in \mathbb{Q} \mid \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} x\}$
$\langle e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d) \rangle \hookrightarrow Stat10 \Rightarrow \quad false; \qquad Discharge \Rightarrow \quad e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d) \in \mathbf{0}_{\mathbb{R}}$
$\langle Rev_{\mathbb{Q}}(d), e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d) \rangle \hookrightarrow Stat7 \Rightarrow \quad e \neq Rev_{\mathbb{Q}}(Rev_{\mathbb{Q}}(d)) +_{\mathbb{Q}} (e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d)) \vee Rev_{\mathbb{Q}}(d) \notin \mathbb{Q} \backslash n \vee e +_{\mathbb{R}} Rev_{\mathbb{Q}}(d) \notin \mathbf{0}_{\mathbb{R}}$
$ALGEBRA \Rightarrow \quad false; \qquad Discharge \Rightarrow \quad$ QED

Theorem 620 ( ) $N \in \mathbb{R} \rightarrow \#N_{\mathbb{R}} \in \mathbb{R} \,\&\, \#N_{\mathbb{R}} >_{\mathbb{R}} N \vee \#N_{\mathbb{R}} = N \,\&\, \#N_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \vee \#N_{\mathbb{R}} = \mathbf{0}_{\mathbb{R}} \,\&\, is\_nonneg_{\mathbb{R}}(\#N_{\mathbb{R}})$. PROOF:

$Suppose\_not(n) \Rightarrow \quad n \in \mathbb{R} \,\&\, \neg(\#n_{\mathbb{R}} \in \mathbb{R} \,\&\, \#n_{\mathbb{R}} >_{\mathbb{R}} n \vee \#n_{\mathbb{R}} = n \,\&\, \#n_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \vee \#n_{\mathbb{R}} = \mathbf{0}_{\mathbb{R}} \,\&\, is\_nonneg_{\mathbb{R}}(\#n_{\mathbb{R}}))$
$Suppose \Rightarrow \quad is\_nonneg_{\mathbb{R}}(n)$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \#n_{\mathbb{R}} = n$
$ELEM \Rightarrow \quad false; \qquad Discharge \Rightarrow \quad \neg is\_nonneg_{\mathbb{R}}(n)$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \#n_{\mathbb{R}} = Rev_{\mathbb{R}}(n)$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad is\_nonneg_{\mathbb{R}}(Rev_{\mathbb{R}}(n))$
$ELEM \Rightarrow \quad \neg Rev_{\mathbb{R}}(n) >_{\mathbb{R}} n \,\&\, Rev_{\mathbb{R}}(n) \neq n$
$Use\_def(>_{\mathbb{R}}) \Rightarrow \quad \neg is\_nonneg_{\mathbb{R}}(Rev_{\mathbb{R}}(n) -_{\mathbb{R}} n)$
$ALGEBRA \Rightarrow \quad \neg is\_nonneg_{\mathbb{R}}(Rev_{\mathbb{R}}(n) -_{\mathbb{R}} Rev_{\mathbb{R}}(n))$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad is\_nonneg_{\mathbb{R}}(Rev_{\mathbb{R}}(n))$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad false; \qquad Discharge \Rightarrow \quad$ QED

Theorem 621 ( ) $N \in \mathbb{R} \rightarrow \#N_{\mathbb{R}} = \#Rev_{\mathbb{R}_{\mathbb{R}}}(N)$. PROOF:

$Suppose\_not(n) \Rightarrow \quad n \in \mathbb{R} \,\&\, \#n_{\mathbb{R}} \neq \#Rev_{\mathbb{R}_{\mathbb{R}}}(n)$
$ALGEBRA \Rightarrow \quad \mathbf{0}_{\mathbb{R}} = Rev_{\mathbb{R}}(\mathbf{0}_{\mathbb{R}})$
$T99999 \Rightarrow \quad is\_nonneg_{\mathbb{R}}(\mathbf{0}_{\mathbb{R}}) \,\&\, is\_nonneg_{\mathbb{R}}(Rev_{\mathbb{R}}(\mathbf{0}_{\mathbb{R}}))$
$Suppose \Rightarrow \quad n = \mathbf{0}_{\mathbb{R}}$
$\langle \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow T99999 \Rightarrow \quad \#n_{\mathbb{R}} = n$
$\langle Rev_{\mathbb{R}}(\mathbf{0}_{\mathbb{R}}) \rangle \hookrightarrow T99999 \Rightarrow \quad false; \qquad Discharge \Rightarrow \quad n \neq \mathbf{0}_{\mathbb{R}}$
$Suppose \Rightarrow \quad is\_nonneg_{\mathbb{R}}(n)$

$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \neg \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n))$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \#n_{\mathbb{R}} = n$
$\langle \text{Rev}_{\mathbb{R}}(n) \rangle \hookrightarrow T99999 \Rightarrow \quad \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n))$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \text{is\_nonneg}_{\mathbb{R}}(n)$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n))$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \#n_{\mathbb{R}} = \text{Rev}_{\mathbb{R}}(n)$
$\langle \text{Rev}_{\mathbb{R}}(n) \rangle \hookrightarrow T99999 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 622 ( ) $N, M \in \mathbb{R} \;\&\; \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(M)) \to N >_{\mathbb{R}} N +_{\mathbb{R}} M \lor N = N +_{\mathbb{R}} M$. PROOF:

Suppose_not$(n, m) \Rightarrow \quad n, m \in \mathbb{R} \;\&\; \text{is_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(m)) \;\&\; \neg n >_{\mathbb{R}} n +_{\mathbb{R}} m \;\&\; n \neq n +_{\mathbb{R}} m$
Use_def$(>_{\mathbb{R}}) \Rightarrow \quad \text{Rev}_{\mathbb{R}}(n) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \lor \text{Rev}_{\mathbb{R}}(m) = \mathbf{0}_{\mathbb{R}}$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \mathbf{0}_{\mathbb{R}} >_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(m)) \lor \text{Rev}_{\mathbb{R}}(m) = \mathbf{0}_{\mathbb{R}}$
ALGEBRA $\Rightarrow \quad \mathbf{0}_{\mathbb{R}} >_{\mathbb{R}} m \lor m = \mathbf{0}_{\mathbb{R}}$
$\langle n, \mathbf{0}_{\mathbb{R}}, n, m \rangle \hookrightarrow T99999 \Rightarrow \quad n +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} >_{\mathbb{R}} n +_{\mathbb{R}} m \lor n +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} = n +_{\mathbb{R}} m$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 623 ( ) $N, M \in \mathbb{R} \;\&\; \text{is\_nonneg}_{\mathbb{R}}(N) \;\&\; \neg \text{is\_nonneg}_{\mathbb{R}}(M) \to N >_{\mathbb{R}} \#N +_{\mathbb{R}} M_{\mathbb{R}} \lor N = \#N +_{\mathbb{R}} M_{\mathbb{R}} \lor \text{Rev}_{\mathbb{R}}(M) >_{\mathbb{R}} \#N +_{\mathbb{R}} M_{\mathbb{R}} \lor \text{Rev}_{\mathbb{R}}(M) = \#N +_{\mathbb{R}} M_{\mathbb{R}}$. PROOF:

Suppose_not$(n, m) \Rightarrow \quad n, m \in \mathbb{R} \;\&\; \text{is_nonneg}_{\mathbb{R}}(n) \;\&\; \neg \text{is_nonneg}_{\mathbb{R}}(m) \;\&\; \neg(n >_{\mathbb{R}} \#n +_{\mathbb{R}} m_{\mathbb{R}} \lor n = \#n +_{\mathbb{R}} m_{\mathbb{R}} \lor \text{Rev}_{\mathbb{R}}(m) >_{\mathbb{R}} \#n +_{\mathbb{R}} m_{\mathbb{R}} \lor \text{Rev}_{\mathbb{R}}(m) = \#n +_{\mathbb{R}} m_{\mathbb{R}})$
$\langle m \rangle \hookrightarrow T99999 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(m))$
$\langle n, m \rangle \hookrightarrow T99999 \Rightarrow \quad n >_{\mathbb{R}} n +_{\mathbb{R}} m \lor n = n +_{\mathbb{R}} m$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \neg \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) \lor n = \mathbf{0}_{\mathbb{R}}$
$\langle \text{Rev}_{\mathbb{R}}(n), \text{Rev}_{\mathbb{R}}(m) \rangle \hookrightarrow T99999 \Rightarrow \quad \text{Rev}_{\mathbb{R}}(m) >_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(m) +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(n) \lor \text{Rev}_{\mathbb{R}}(m) = \text{Rev}_{\mathbb{R}}(m) +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(n)$
Suppose $\Rightarrow \quad \text{is\_nonneg}_{\mathbb{R}}(n +_{\mathbb{R}} m)$
$\langle n +_{\mathbb{R}} m \rangle \hookrightarrow T99999 \Rightarrow \quad \#n +_{\mathbb{R}} m_{\mathbb{R}} = n +_{\mathbb{R}} m$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \text{is\_nonneg}_{\mathbb{R}}(n +_{\mathbb{R}} m)$
$\langle n, m \rangle \hookrightarrow T99999 \Rightarrow \quad \#n +_{\mathbb{R}} m_{\mathbb{R}} = \text{Rev}_{\mathbb{R}}(n +_{\mathbb{R}} m)$
ALGEBRA $\Rightarrow \quad \#n +_{\mathbb{R}} m_{\mathbb{R}} = \text{Rev}_{\mathbb{R}}(m) +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(n)$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 624 ( ) $N, M \in \mathbb{R} \to N +_{\mathbb{R}} \#M_{\mathbb{R}} >_{\mathbb{R}} n \lor n +_{\mathbb{R}} \#M_{\mathbb{R}} = n$. PROOF:

Suppose_not$(n, m) \Rightarrow \quad n, m \in \mathbb{R} \;\&\; \neg(n +_{\mathbb{R}} \#m_{\mathbb{R}} >_{\mathbb{R}} n \lor n +_{\mathbb{R}} \#m_{\mathbb{R}} = n)$
$\langle m \rangle \hookrightarrow T99999 \Rightarrow \quad \text{is\_nonneg}_{\mathbb{R}}(\#m_{\mathbb{R}})$

$\mathsf{Use\_def}(>_{\mathbb{R}}) \Rightarrow \quad \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \vee \#\mathsf{m}_{\mathbb{R}} = \mathbf{0}_{\mathbb{R}}$

$\langle \mathsf{n}, \mathsf{n}, \#\mathsf{m}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow T99999 \Rightarrow \quad \mathsf{n} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \mathsf{n} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \vee \mathsf{n} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{n} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

Theorem 625 ( ) $\mathsf{N}, \mathsf{M} \in \mathbb{R} \rightarrow \#\mathsf{N}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{M}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{N} +_{\mathbb{R}} \mathsf{M}_{\mathbb{R}} \vee \#\mathsf{N}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{M}_{\mathbb{R}} = \#\mathsf{N} +_{\mathbb{R}} \mathsf{M}_{\mathbb{R}}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}, \mathsf{m}) \Rightarrow \quad \mathsf{n}, \mathsf{m} \in \mathbb{R} \,\&\, \neg(\#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}})$

$\langle \mathsf{n} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \textbf{ then } \mathsf{n} \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) \textbf{ fi}$

$\langle \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{m}_{\mathbb{R}} = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m}) \textbf{ then } \mathsf{n} \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}) \textbf{ fi}$

$\langle \mathsf{n} +_{\mathbb{R}} \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m}) \textbf{ then } \mathsf{n} +_{\mathbb{R}} \mathsf{m} \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m}) \textbf{ fi}$

$\mathsf{Suppose} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$

$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m})$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \neg\big(\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})\big)$

$\mathsf{Suppose} \Rightarrow \quad \neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$

$\langle \mathsf{n} \rangle \hookrightarrow T99999 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}\big(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n})\big)$

$\langle \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}\big(\mathsf{Rev}_{\mathbb{R}}(\mathsf{m})\big)$

$\langle \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}) \rangle \hookrightarrow T99999 \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}\big(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})\big)$

$\mathsf{ALGEBRA} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}\big(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m})\big)$

$\langle \mathsf{Rev}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m}) \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{Rev}_{\mathbb{R}_{\mathbb{R}}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m}) = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m})$

$\langle \mathsf{n} +_{\mathbb{R}} \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n} +_{\mathbb{R}} \mathsf{m})$

$\mathsf{ALGEBRA} \Rightarrow \quad \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})$

$\langle \mathsf{n} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n})$

$\langle \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})$

$\mathsf{ELEM} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \vee \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$

$\langle \#\mathsf{n}_{\mathbb{R}}, \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n}_{\mathbb{R}}$

$\langle \#\mathsf{m}_{\mathbb{R}}, \mathsf{n} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{m}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{n}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} \vee \#\mathsf{m}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{n}_{\mathbb{R}} = \#\mathsf{n}_{\mathbb{R}}$

$\mathsf{ALGEBRA} \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{m}_{\mathbb{R}}$

$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat1}: \ \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$

$\langle \mathsf{n} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} = \mathsf{n}$

$\langle \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})$

$\langle \mathsf{n}, \mathsf{m} \rangle \hookrightarrow T99999 \Rightarrow \quad \mathsf{n} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \mathsf{n} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}) >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}) = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}}$

$\mathsf{ELEM} \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}}$

$\langle \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}}, \#\mathsf{n}_{\mathbb{R}} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee$
$\quad \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}}$

$\langle \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}}, \#\mathsf{n}_{\mathbb{R}} \rangle \hookrightarrow T99999 \Rightarrow \quad \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee$
$\quad \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} +_{\mathbb{R}} \mathsf{m}_{\mathbb{R}}$

$\mathsf{ELEM} \Rightarrow$ false; Discharge $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$
$\langle \mathsf{LIKEWISE}(\mathit{Stat1}\backslash\mathit{Stat2}, \mathsf{n} \mapsto \mathsf{m}, \mathsf{m} \mapsto \mathsf{n})\rangle \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 626 ( ) $\mathsf{N}, \mathsf{M} \in \mathbb{R} \rightarrow \#\mathsf{N}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{M}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{N} -_{\mathbb{R}} \mathsf{M}_{\mathbb{R}} \vee \#\mathsf{N}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{M}_{\mathbb{R}} = \#\mathsf{N} -_{\mathbb{R}} \mathsf{M}_{\mathbb{R}}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}, \mathsf{m}) \Rightarrow$ $\mathsf{n}, \mathsf{m} \in \mathbb{R} \,\&\, \neg(\#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} -_{\mathbb{R}} \mathsf{m}_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} -_{\mathbb{R}} \mathsf{m}_{\mathbb{R}})$
$\mathsf{Use\_def}(-_{\mathbb{R}}) \Rightarrow$ $\neg(\#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} >_{\mathbb{R}} \#\mathsf{n} -_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} -_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})_{\mathbb{R}})$
$\langle \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\neg(\#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{Rev}_{\mathbb{R}_{\mathbb{R}}}(\mathsf{m}) >_{\mathbb{R}} \#\mathsf{n} -_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})_{\mathbb{R}} \vee \#\mathsf{n}_{\mathbb{R}} +_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \#\mathsf{n} -_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})_{\mathbb{R}})$
$\langle \mathsf{n}, \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 627 ( ) $\mathsf{N}, \mathsf{M} \in \mathbb{R} \rightarrow \#\mathsf{N}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{M}_{\mathbb{R}} = \#\mathsf{N} *_{\mathbb{R}} \mathsf{M}_{\mathbb{R}}$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{n}, \mathsf{m}) \Rightarrow$ $\mathsf{n}, \mathsf{m} \in \mathbb{R} \,\&\, \#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} \neq \#\mathsf{n} *_{\mathbb{R}} \mathsf{m}_{\mathbb{R}}$
$\langle \mathsf{n}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\#\mathsf{n}_{\mathbb{R}} = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \textbf{ then } \mathsf{n} \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) \textbf{ fi}$
$\langle \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\#\mathsf{m}_{\mathbb{R}} = \textbf{if } \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m}) \textbf{ then } \mathsf{n} \textbf{ else } \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}) \textbf{ fi}$
$\mathsf{Suppose} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$
$\mathsf{ELEM} \Rightarrow$ $\#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{n} *_{\mathbb{R}} \mathsf{m}$
$\langle \mathsf{n}, \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n} *_{\mathbb{R}} \mathsf{m})$
$\langle \mathsf{n} *_{\mathbb{R}} \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ false; Discharge $\Rightarrow$ $\neg(\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m}))$
$\mathsf{Suppose} \Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$
$\mathsf{ELEM} \Rightarrow$ $\#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})$
$\langle \mathsf{n}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}))$
$\langle \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{m}))$
$\langle \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}))$
$\mathsf{ALGEBRA} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n} *_{\mathbb{R}} \mathsf{m}) \,\&\, \#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{n} *_{\mathbb{R}} \mathsf{m}$
$\langle \mathsf{n} *_{\mathbb{R}} \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ false; Discharge $\Rightarrow$ $\neg(\neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m}))$
$\mathsf{Suppose} \Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$
$\mathsf{ELEM} \Rightarrow$ $\#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) *_{\mathbb{R}} \mathsf{m}$
$\langle \mathsf{n}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}))$
$\langle \mathsf{n}, \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}) *_{\mathbb{R}} \mathsf{m})$
$\mathsf{ALGEBRA} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n} *_{\mathbb{R}} \mathsf{m})) \,\&\, \#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{Rev}_{\mathbb{R}}(\mathsf{n} *_{\mathbb{R}} \mathsf{m})$
$\langle \mathsf{Rev}_{\mathbb{R}}(\mathsf{n} *_{\mathbb{R}} \mathsf{m})\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n}) \,\&\, \neg\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m})$
$\mathsf{ELEM} \Rightarrow$ $\#\mathsf{n}_{\mathbb{R}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{R}} = \mathsf{n} *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m})$
$\langle \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(\mathsf{m}))$
$\langle \mathsf{n}, \mathsf{m}\rangle \hookrightarrow \mathit{T99999} \Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{n} *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{m}))$

ALGEBRA $\Rightarrow$ $\quad \mathsf{is\_nonneg}_{\mathbb{R}}(n *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(m))\ \&\ \#n_{\mathbb{R}} *_{\mathbb{R}} \#m_{\mathbb{R}} = n *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(m)$
$\langle n *_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(m)\rangle \hookrightarrow T99999 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 628 ( ) $N, M \in \mathbb{R}\ \&\ M \neq \mathbf{0}_{\mathbb{R}} \rightarrow \#N_{\mathbb{R}} /_{\mathbb{R}} \#M_{\mathbb{R}} = \#N /_{\mathbb{R}} M_{\mathbb{R}}$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $\quad n, m \in \mathbb{R}\ \&\ \mathit{TO\_BE\_CONTINUED}$
$\mathit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 629 ( ) $N, M \in \mathbb{R} \rightarrow N\ |{*}|_{\mathbb{R}}\ M \in \mathbb{R}$. PROOF:

Suppose_not$(n, m) \Rightarrow$ $\quad n, m \in \mathbb{R}\ \&\ n\ |{*}|_{\mathbb{R}}\ m \notin \mathbb{R}$
Use_def$(\ |{*}|_{\mathbb{R}}\ ) \Rightarrow$ $\quad \{u *_{\mathbb{Q}} v : u \in \#n_{\mathbb{R}}, v \in \#m_{\mathbb{R}} \mid \neg(\mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} u \vee \mathbf{0}_{\mathbb{Q}} >_{\mathbb{Q}} v)\} \cup \mathbf{0}_{\mathbb{R}} \notin \mathbb{R}$
$\mathit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 630 ( ) $N, M \in \mathbb{R} \rightarrow N *_{\mathbb{R}} M \in \mathbb{R}$. PROOF:

$\mathit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 631 ( ) $K, n, m \in \mathbb{R} \rightarrow n +_{\mathbb{R}} (m +_{\mathbb{R}} K) = (n +_{\mathbb{R}} m) +_{\mathbb{R}} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow$ $\quad k, n, m \in \mathbb{R}\ \&\ n +_{\mathbb{R}} (m +_{\mathbb{R}} k) = (n +_{\mathbb{R}} m) +_{\mathbb{R}} k$
$\mathit{THUS} \Rightarrow$ $\quad k \subseteq \mathbb{Q}\ \&\ n \subseteq \mathbb{Q}\ \&\ m \subseteq \mathbb{Q}$
Use_def$(+_{\mathbb{R}}) \Rightarrow$ $\quad \{u +_{\mathbb{Q}} v : u \in n, v \in \{u +_{\mathbb{Q}} v : u \in m, v \in k\}\} \neq \{u +_{\mathbb{Q}} v : u \in \{u +_{\mathbb{Q}} v : u \in n, v \in m\}, v \in k\}$
SIMPLF $\Rightarrow$ $\quad \{u +_{\mathbb{Q}} (v +_{\mathbb{Q}} w) : u \in n, v \in m, w \in k\} \neq \{u +_{\mathbb{Q}} v +_{\mathbb{Q}} w : u \in n, v \in m, w \in k\}$
Suppose $\Rightarrow$ $\quad \neg\langle \forall u \in m, v \in n, w \in k \mid u +_{\mathbb{Q}} (v +_{\mathbb{Q}} w) = (u +_{\mathbb{Q}} v) +_{\mathbb{Q}} w\rangle$
Pred_monot $\Rightarrow$ $\quad \mathit{Stat1} : \neg\langle \forall u \in \mathbb{Q}, v \in \mathbb{Q}, w \in \mathbb{Q} \mid u +_{\mathbb{Q}} (v +_{\mathbb{Q}} w) = (u +_{\mathbb{Q}} v) +_{\mathbb{Q}} w\rangle$
$\langle u, v, w\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\quad u, v, w \in \mathbb{Q}\ \&\ u +_{\mathbb{Q}} (v +_{\mathbb{Q}} w) \neq u +_{\mathbb{Q}} v +_{\mathbb{Q}} w$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $\quad \langle \forall u \in m, v \in n, w \in k \mid u +_{\mathbb{Q}} (v +_{\mathbb{Q}} w) = (u +_{\mathbb{Q}} v) +_{\mathbb{Q}} w\rangle$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 632 ( ) $N \in \mathbb{R} \rightarrow \mathsf{Rev}_{\mathbb{R}}(\mathsf{Rev}_{\mathbb{R}}(N)) = N$. PROOF:

$\text{Suppose\_not}(n) \Rightarrow \quad n \in \mathbb{R} \,\&\, \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) \neq n$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \text{Rev}_{\mathbb{R}}(n) \in \mathbb{R}$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \text{Rev}_{\mathbb{R}}(n) +_{\mathbb{R}} n = \mathbf{0}_{\mathbb{R}}$
$\langle \text{Rev}_{\mathbb{R}}(n) \rangle \hookrightarrow T99999 \Rightarrow \quad \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(n) = \mathbf{0}_{\mathbb{R}}$
$\text{ELEM} \Rightarrow \quad \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(n) +_{\mathbb{R}} n = \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} n$
$\langle \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)), \text{Rev}_{\mathbb{R}}(n), n \rangle \hookrightarrow T99999 \Rightarrow \quad (\text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) +_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(n)) +_{\mathbb{R}} n = \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) +_{\mathbb{R}} (\text{Rev}_{\mathbb{R}}(n) +_{\mathbb{R}} n)$
$\text{ELEM} \Rightarrow \quad \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} n = \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n)) \rangle \hookrightarrow T99999 \Rightarrow \quad \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} n = \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n))$
$\langle \mathbf{0}_{\mathbb{R}}, n \rangle \hookrightarrow T99999 \Rightarrow \quad n +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} = \text{Rev}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(n))$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow$ QED

Theorem 633 ( ) $K, N, M \in \mathbb{R} \rightarrow N *_{\mathbb{R}} (M *_{\mathbb{R}} K) = (N *_{\mathbb{R}} M) *_{\mathbb{R}} K$. PROOF:

$\text{Suppose\_not}(k, n, m) \Rightarrow \quad k, n, m \in \mathbb{R} \,\&\, \textit{TO\_BE\_CONTINUED}$
$\textit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 634 ( ) $K, N, M \in \mathbb{R} \rightarrow N *_{\mathbb{R}} (M +_{\mathbb{R}} K) = N *_{\mathbb{R}} M +_{\mathbb{R}} N *_{\mathbb{R}} K$. PROOF:

$\text{Suppose\_not}(k, n, m) \Rightarrow \quad k, n, m \in \mathbb{R} \,\&\, \textit{TO\_BE\_CONTINUED}$
$\textit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 635 ( ) $X, Y \in \mathbb{R} \,\&\, \text{is\_nonneg}_{\mathbb{R}}(x) \,\&\, \text{is\_nonneg}_{\mathbb{R}}(y) \rightarrow \text{is\_nonneg}_{\mathbb{R}}(x +_{\mathbb{R}} y) \,\&\, \text{is\_nonneg}_{\mathbb{R}}(x *_{\mathbb{R}} y)$. PROOF:

$\text{Suppose\_not}(n, m) \Rightarrow \quad n, m \in \mathbb{R} \,\&\, \text{is\_nonneg}_{\mathbb{R}}(n) \,\&\, \text{is\_nonneg}_{\mathbb{R}}(m) \,\&\, \neg(\text{is\_nonneg}_{\mathbb{R}}(m +_{\mathbb{R}} n) \,\&\, \text{is\_nonneg}_{\mathbb{R}}(m *_{\mathbb{R}} n))$
$\textit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 636 ( ) $M \in \mathbb{R} \rightarrow M = M *_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$. PROOF:

$\textit{TO\_BE\_CONTINUED} \Rightarrow$ QED

Theorem 637 ( ) $M \in \mathbb{R} \,\&\, M \neq \mathbf{0}_{\mathbb{R}} \rightarrow \text{Recip}_{\mathbb{Q}}(M) \in \mathbb{R} \,\&\, M *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(M) = \mathbf{1}_{\mathbb{R}}$. PROOF:

TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 638 ( ) $N, M \in \mathbb{R} \,\&\, M \neq \mathbf{0}_{\mathbb{R}} \rightarrow N = M *_{\mathbb{R}} (N /_{\mathbb{R}} M)$. PROOF:

TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 639 ( ) $X \in \mathbb{R} \rightarrow \text{is\_nonneg}_{\mathbb{R}}(X) \vee \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(X)) \,\&\, \left(\text{is\_nonneg}_{\mathbb{R}}(X) \,\&\, \text{is\_nonneg}_{\mathbb{R}}(\text{Rev}_{\mathbb{R}}(X)) \rightarrow X = \mathbf{0}_{\mathbb{R}}\right)$. PROOF:

TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 640 ( ) $X \in \mathbb{R} \rightarrow X = X *_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$. PROOF:

TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 641 ( ) $X, Y \in \mathbb{R} \,\&\, \text{is\_nonneg}_{\mathbb{R}}(X) \,\&\, \text{is\_nonneg}_{\mathbb{R}}(Y) \,\&\, X +_{\mathbb{R}} Y = \mathbf{0}_{\mathbb{R}} \rightarrow X = \mathbf{0}_{\mathbb{R}} \,\&\, Y = \mathbf{0}_{\mathbb{R}}$. PROOF:

Suppose_not$(m, n) \Rightarrow \quad m, n \in \mathbb{R} \,\&\, \text{is_nonneg}_{\mathbb{R}}(m) \,\&\, \text{is_nonneg}_{\mathbb{R}}(n) \,\&\, m +_{\mathbb{R}} n = \mathbf{0}_{\mathbb{R}} \,\&\, \neg(m = \mathbf{0}_{\mathbb{R}} \,\&\, n = \mathbf{0}_{\mathbb{R}})$
ALGEBRA $\Rightarrow \quad m = \text{Rev}_{\mathbb{R}}(n) \,\&\, n = \text{Rev}_{\mathbb{R}}(m)$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow \quad n = \mathbf{0}_{\mathbb{R}}$
$\langle m \rangle \hookrightarrow T99999 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

Theorem 642 ( ) $X, Y, X_1 \in \mathbb{R} \,\&\, X >_{\mathbb{R}} Y \,\&\, X_1 >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \rightarrow X *_{\mathbb{R}} X_1 >_{\mathbb{R}} Y *_{\mathbb{R}} X_1$. PROOF:

Suppose_not$(m, n, k) \Rightarrow \quad m, n, k \in \mathbb{R} \,\&\, m >_{\mathbb{R}} n \,\&\, k >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \,\&\, \neg m *_{\mathbb{R}} k >_{\mathbb{R}} n *_{\mathbb{R}} k$
ALGEBRA $\Rightarrow \quad m *_{\mathbb{R}} k -_{\mathbb{R}} n *_{\mathbb{R}} k = (m -_{\mathbb{R}} n) *_{\mathbb{R}} k$
Use_def$(>_{\mathbb{R}}) \Rightarrow \quad \text{is_nonneg}_{\mathbb{R}}(m -_{\mathbb{R}} n) \,\&\, m \neq n \,\&\, \neg\text{is_nonneg}_{\mathbb{R}}\left((m -_{\mathbb{R}} n) *_{\mathbb{R}} k\right) \vee m *_{\mathbb{R}} k \neq n *_{\mathbb{R}} k$
$\langle m -_{\mathbb{R}} n, k \rangle \hookrightarrow T99999 \Rightarrow \quad m *_{\mathbb{R}} k \neq n *_{\mathbb{R}} k$
ALGEBRA $\Rightarrow \quad m -_{\mathbb{R}} n = \emptyset \,\&\, (m -_{\mathbb{R}} n) *_{\mathbb{R}} k = \emptyset$
$\langle m -_{\mathbb{R}} n, k \rangle \hookrightarrow T99999 \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

Theorem 643 ( ) $X \in \mathbb{R} \,\&\, X >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \rightarrow \text{Recip}_{\mathbb{Q}}(X) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$. PROOF:

Suppose_not(m) $\Rightarrow$ $m \in \mathbb{R} \;\&\; m >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \;\&\; \neg \text{Recip}_{\mathbb{Q}}(m) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$\langle \text{Recip}_{\mathbb{Q}}(m) \rangle \hookrightarrow T99999 \Rightarrow$ $\text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \vee \text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)) = \mathbf{0}_{\mathbb{R}}$

Suppose $\Rightarrow$ $\text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)) = \mathbf{0}_{\mathbb{R}}$

ELEM $\Rightarrow$ $\text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)) *_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(m) = \mathbf{0}_{\mathbb{R}} *_{\mathbb{R}} \text{Rev}_{\mathbb{R}}(m)$

ALGEBRA $\Rightarrow$ $\mathbf{1}_{\mathbb{R}} = \mathbf{0}_{\mathbb{R}}$

ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$\langle \text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)), m \rangle \hookrightarrow T99999 \Rightarrow$ $\text{Rev}_{\mathbb{R}}(\text{Recip}_{\mathbb{Q}}(m)) *_{\mathbb{R}} m >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

ALGEBRA $\Rightarrow$ $\text{Rev}_{\mathbb{R}}(\mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$T99999 \Rightarrow$ $\mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$\langle \text{Rev}_{\mathbb{R}}(\mathbf{1}_{\mathbb{R}}), \mathbf{0}_{\mathbb{R}}, \mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow T99999 \Rightarrow$ $\text{Rev}_{\mathbb{R}}(\mathbf{1}_{\mathbb{R}}) +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

ALGEBRA $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

Use_def($>_{\mathbb{R}}$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 644 ( ) $X, Y \in \mathbb{R} \;\&\; X >_{\mathbb{R}} Y \rightarrow X >_{\mathbb{R}} (X +_{\mathbb{R}} Y) /_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} \cup \mathbf{1}_{\mathbb{R}}) \;\&\; (X +_{\mathbb{R}} Y) /_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} \cup \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} Y$. PROOF:

Suppose_not(m, n) $\Rightarrow$ $m, n \in \mathbb{R} \;\&\; m >_{\mathbb{R}} n \;\&\; \neg\big(m >_{\mathbb{R}} (m +_{\mathbb{R}} n) /_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} \cup \mathbf{1}_{\mathbb{R}}) \;\&\; (m +_{\mathbb{R}} n) /_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} \cup \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} n\big)$

$\langle m, n, n, n \rangle \hookrightarrow T99999 \Rightarrow$ $m +_{\mathbb{R}} n >_{\mathbb{R}} n +_{\mathbb{R}} n$

$\langle m, n, m, m \rangle \hookrightarrow T99999 \Rightarrow$ $m +_{\mathbb{R}} m >_{\mathbb{R}} n +_{\mathbb{R}} m$

ALGEBRA $\Rightarrow$ $m +_{\mathbb{R}} n >_{\mathbb{R}} n *_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}})$

ALGEBRA $\Rightarrow$ $m *_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} m +_{\mathbb{R}} n$

ALGEBRA $\Rightarrow$ $\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}, \mathbf{1}_{\mathbb{R}} \cup \mathbf{1}_{\mathbb{R}} \in \mathbb{Q}$

$\langle \mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}, \mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

ALGEBRA $\Rightarrow$ $\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$\langle \mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} \rangle \hookrightarrow T99999 \Rightarrow$ $\text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$

$\langle m +_{\mathbb{R}} n, n *_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}), \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) \rangle \hookrightarrow T99999 \Rightarrow$ $(m +_{\mathbb{R}} n) *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} n *_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}})$

ALGEBRA $\Rightarrow$ $(m +_{\mathbb{R}} n) *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} n$

$\langle m *_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}), m +_{\mathbb{R}} n, \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) \rangle \hookrightarrow T99999 \Rightarrow$ $n *_{\mathbb{R}} (\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) >_{\mathbb{R}} m +_{\mathbb{R}} n *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}})$

ALGEBRA $\Rightarrow$ $n >_{\mathbb{R}} m +_{\mathbb{R}} n *_{\mathbb{R}} \text{Recip}_{\mathbb{Q}}(\mathbf{1}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}})$

Use_def($/_{\mathbb{R}}$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Least Upper Bound

Theorem 645 ( ) $S \neq \emptyset \;\&\; S \subseteq \mathbb{R} \rightarrow \bigcup S \in \mathbb{R} \vee \bigcup S = \mathbb{Q}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $s \neq \emptyset \;\&\; s \subseteq \mathbb{R} \;\&\; \bigcup s \notin \mathbb{R} \;\&\; \bigcup s \neq \mathbb{Q}$

Use_def($\bigcup$) $\Rightarrow$ $\bigcup s = \{y : x \in s, y \in x\}$

$\langle$Memb(r)$\rangle$ $\Rightarrow$ $r \in s \;\&\; r \in \mathbb{R}$

Use_def$(\mathbb{R}) \Rightarrow$ $\mathit{Stat1}: \; r \in \{s : s \subseteq \mathbb{Q} \,|\, s \neq \emptyset \,\&\, s \neq \mathbb{Q} \,\&\, \langle \forall x \in s, \exists y \in s \,|\, y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in s, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in s \rangle \}$
$\langle a \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $r \subseteq \mathbb{Q} \,\&\, r \neq \emptyset \,\&\, r \neq \mathbb{Q} \,\&\, \langle \forall x \in r, \exists y \in r \,|\, y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in r, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in r \rangle$
$\langle \mathrm{Memb}(q) \rangle \Rightarrow$ $q \in r$
Suppose $\Rightarrow$ $\mathit{Stat2}: \; \{y : x \in s, y \in x\} = \emptyset$
$\langle r, q \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ $\bigcup s \neq \emptyset$
Suppose $\Rightarrow$ $\{y : x \in s, y \in x\} \nsubseteq \mathbb{Q}$
$\langle \mathrm{Memb}(c) \rangle \Rightarrow$ $\mathit{Stat3}: \; c \in \{y : x \in s, y \in x\} \,\&\, c \notin \mathbb{Q}$
$\langle a_1, b_1 \rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $c \in s \,\&\, y \in c$
ELEM $\Rightarrow$ $\mathit{Stat4}: \; c \in \{s : s \subseteq \mathbb{Q} \,|\, s \neq \emptyset \,\&\, s \neq \mathbb{Q} \,\&\, \langle \forall x \in s, \exists y \in s \,|\, y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in s, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in s \rangle \}$
$\langle a_2 \rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ $c \subseteq \mathbb{Q} \,\&\, y \in \mathbb{Q}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\bigcup s \subseteq \mathbb{Q}$
ELEM $\Rightarrow$ $\mathit{Stat5}: \; \bigcup s \notin \{s : s \subseteq \mathbb{Q} \,|\, s \neq \emptyset \,\&\, s \neq \mathbb{Q} \,\&\, \langle \forall x \in s, \exists y \in s \,|\, y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in s, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in s \rangle \}$
$\langle \{y : x \in s, y \in x\} \rangle \hookrightarrow \mathit{Stat5} \Rightarrow$
$\neg \langle \forall x \in \{y : x \in s, y \in x\}, \exists y \in \{y : x \in s, y \in x\} \,|\, y >_{\mathbb{Q}} x \rangle \vee$
$\neg \langle \forall x \in \{y : x \in s, y \in x\}, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in \{y : x \in s, y \in x\} \rangle$
SIMPLF $\Rightarrow$ $\neg \langle \forall x \in s, u \in x, \exists v \in s, w \in v \,|\, w >_{\mathbb{Q}} u \rangle \vee \neg \langle \forall x \in s, y \in x, u \in \mathbb{Q} \,|\, y >_{\mathbb{Q}} u \rightarrow u \in \{y : x \in s, y \in x\} \rangle$
Suppose $\Rightarrow$ $\mathit{Stat6}: \; \neg \langle \forall x \in s, y \in x, u \in \mathbb{Q} \,|\, y >_{\mathbb{Q}} u \rightarrow u \in \{y : x \in s, y \in x\} \rangle$
$\langle a_3, b_3, c_3 \rangle \hookrightarrow \mathit{Stat6} \Rightarrow$ $a_3 \in s \,\&\, b_3 \in a_3 \,\&\, c_3 \in \mathbb{Q} \,\&\, b_3 >_{\mathbb{Q}} c_3 \,\&\, \mathit{Stat7}: \; c_3 \notin \{y : x \in s, y \in x\}$
$\langle a_3, c_3 \rangle \hookrightarrow \mathit{Stat7} \Rightarrow$ $c_3 \notin a_3$
ELEM $\Rightarrow$ $a_3 \in \mathbb{R}$
Use_def$(\mathbb{R}) \Rightarrow$ $\mathit{Stat8}: \; a_3 \in \{s : s \subseteq \mathbb{Q} \,|\, s \neq \emptyset \,\&\, s \neq \mathbb{Q} \,\&\, \langle \forall x \in s, \exists y \in s \,|\, y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in s, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in s \rangle \}$
$\langle aa \rangle \hookrightarrow \mathit{Stat8} \Rightarrow$ $\langle \forall x \in a_3, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in a_3 \rangle$
$\langle a_3, c_3 \rangle \hookrightarrow \mathit{Stat8} \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat9}: \; \neg \langle \forall x \in s, u \in x, \exists v \in s, w \in v \,|\, w >_{\mathbb{Q}} u \rangle$
$\langle a_4, c_4, a_4 \rangle \hookrightarrow \mathit{Stat9} \Rightarrow$ $a_4 \in s \,\&\, c_4 \in a_4 \,\&\, \mathit{Stat10}: \; \neg \langle \exists w \in a_4 \,|\, w >_{\mathbb{Q}} c_4 \rangle$
ELEM $\Rightarrow$ $a_4 \in \mathbb{R}$
Use_def$(\mathbb{R}) \Rightarrow$ $\mathit{Stat11}: \; a_4 \in \{s : s \subseteq \mathbb{Q} \,|\, s \neq \emptyset \,\&\, s \neq \mathbb{Q} \,\&\, \langle \forall x \in s, \exists y \in s \,|\, y >_{\mathbb{Q}} x \rangle \,\&\, \langle \forall x \in s, y \in \mathbb{Q} \,|\, x >_{\mathbb{Q}} y \rightarrow y \in s \rangle \}$
$\langle ab \rangle \hookrightarrow \mathit{Stat8} \Rightarrow$ $\mathit{Stat12}: \; \langle \forall x \in a_4, \exists y \in a_4 \,|\, y >_{\mathbb{Q}} x \rangle$
$\langle ab \rangle \hookrightarrow \mathit{Stat12} \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 646 ( ) $X \in \mathbb{R} \,\&\, \mathrm{is\_nonneg}_{\mathbb{R}}(X) \rightarrow \sqrt{X} \in \mathbb{R} \,\&\, \mathrm{is\_nonneg}_{\mathbb{R}}(\sqrt{X}) \,\&\, \sqrt{X} *_{\mathbb{R}} \sqrt{X} = X$. PROOF:

Suppose_not$(n) \Rightarrow$ $n \in \mathbb{R} \,\&\, \mathrm{is\_nonneg}_{\mathbb{R}}(n) \,\&\, \neg(\sqrt{n} \in \mathbb{R} \,\&\, \mathrm{is\_nonneg}_{\mathbb{R}}(\sqrt{n}) \,\&\, \sqrt{n} *_{\mathbb{R}} \sqrt{n} = n)$
Use_def$(\sqrt{\ }) \Rightarrow$ $\sqrt{n} = \bigcup \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq x\}$
Suppose $\Rightarrow$ $\mathit{Stat1}: \; \mathbf{0}_{\mathbb{R}} \notin \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\}$
$\langle \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathbf{0}_{\mathbb{R}} \notin \mathbb{R} \vee \mathbf{0}_{\mathbb{R}} *_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} \nsubseteq n$
ALGEBRA $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} \nsubseteq n$
Use_def$(\mathrm{is_nonneg}_{\mathbb{R}}) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} \in \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\}$

ELEM $\Rightarrow$ $\{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\} \neq \emptyset$
Use_def($\bigcup$) $\Rightarrow$ $\bigcup\{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\} = \{u : v \in \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\}, u \in v\}$
SIMPLF $\Rightarrow$ $\sqrt{n} = \{u : y \in \mathbb{R}, u \in y \,|\, y *_{\mathbb{R}} y \subseteq n\}$
T99999 $\Rightarrow$ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathbf{1}_{\mathbb{R}})$
ALGEBRA $\Rightarrow$ $\mathbf{1}_{\mathbb{R}} \neq \mathbf{0}_{\mathbb{R}}$
Use_def($>_{\mathbb{R}}$) $\Rightarrow$ $\mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle n, \mathbf{1}_{\mathbb{R}}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle n, \mathbf{1}_{\mathbb{R}}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} >_{\mathbb{R}} n$
Suppose $\Rightarrow$ *Stat2* : $n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} \subseteq \{u : y \in \mathbb{R}, u \in y \,|\, y *_{\mathbb{R}} y \subseteq n\}$
$\langle a, b\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} \subseteq a \;\&\; a *_{\mathbb{R}} a \subseteq n$
$\langle n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}, a\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $a \geqslant_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$
$\langle a *_{\mathbb{R}} a, n\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \geqslant_{\mathbb{R}} a *_{\mathbb{R}} a$
$\langle n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}, a, n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}, a\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \geqslant_{\mathbb{R}} (n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) *_{\mathbb{R}} (n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}})$
ALGEBRA $\Rightarrow$ $n \geqslant_{\mathbb{R}} n +_{\mathbb{R}} (n +_{\mathbb{R}} n *_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}})$
$\langle n\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \geqslant_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle n, n\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n *_{\mathbb{R}} n \geqslant_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} *_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
ALGEBRA $\Rightarrow$ $n *_{\mathbb{R}} n \geqslant_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle n, n *_{\mathbb{R}} n\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n +_{\mathbb{R}} n *_{\mathbb{R}} n \geqslant_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
ALGEBRA $\Rightarrow$ $n +_{\mathbb{R}} n *_{\mathbb{R}} n \geqslant_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}$
$\langle n +_{\mathbb{R}} n *_{\mathbb{R}} n, \mathbf{1}_{\mathbb{R}}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n +_{\mathbb{R}} n *_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} \geqslant_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$
ALGEBRA $\Rightarrow$ $n +_{\mathbb{R}} n *_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} \geqslant_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$
$\langle n, n +_{\mathbb{R}} n *_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n +_{\mathbb{R}} (n +_{\mathbb{R}} n *_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}) \geqslant_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$
$\langle n, n +_{\mathbb{R}} (n +_{\mathbb{R}} n *_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}), n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $n \geqslant_{\mathbb{R}} n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}$
$\langle n, n +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ false; Discharge $\Rightarrow$ $\bigcup\{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq x\} \neq \mathbb{R}$
$\langle \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\}\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $\sqrt{n} \in \mathbb{R}$
Suppose $\Rightarrow$ $\neg\mathsf{is\_nonneg}_{\mathbb{R}}(\sqrt{n})$
Use_def($\mathsf{is\_nonneg}_{\mathbb{R}}$) $\Rightarrow$ $\mathbf{0}_{\mathbb{R}} \nsubseteq \sqrt{n}$
$\langle \mathsf{Memb}(c)\rangle \Rightarrow$ $c \in \mathbf{0}_{\mathbb{R}} \;\&\;$ *Stat3* : $c \notin \{u : v \in \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\}, u \in v\}$
$\langle \mathbf{0}_{\mathbb{R}}, c\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\neg(\mathbf{0}_{\mathbb{R}} \in \{y : y \in \mathbb{R} \,|\, y *_{\mathbb{R}} y \subseteq n\} \;\&\; c \in \mathbf{0}_{\mathbb{R}})$
ELEM $\Rightarrow$ $\sqrt{n} *_{\mathbb{R}} \sqrt{n} \neq n$
Suppose $\Rightarrow$ $\sqrt{n} *_{\mathbb{R}} \sqrt{n} \nsubseteq n$
TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 647 ( ) $X, Y \in \mathbb{R} \;\&\; Y *_{\mathbb{R}} Y = X \;\&\; \mathsf{is\_nonneg}_{\mathbb{R}}(Y) \rightarrow Y = \sqrt{X}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, m \in \mathbb{R} \;\&\; m *_{\mathbb{R}} m = n \;\&\; \mathsf{is\_nonneg}_{\mathbb{R}}(m) \;\&\; m \neq \sqrt{n}$
$\langle n\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $\sqrt{n} \in \mathbb{R} \;\&\; \sqrt{n} *_{\mathbb{R}} \sqrt{n} = m *_{\mathbb{R}} m \;\&\; \mathsf{is\_nonneg}_{\mathbb{R}}(\sqrt{n})$
ALGEBRA $\Rightarrow$ $(\sqrt{n} -_{\mathbb{R}} m) *_{\mathbb{R}} (\sqrt{n} +_{\mathbb{R}} m) = \mathbf{0}_{\mathbb{R}}$
$\langle \sqrt{n} -_{\mathbb{R}} m, \sqrt{n} +_{\mathbb{R}} m\rangle \hookrightarrow$ *T99999* $\Rightarrow$ $\sqrt{n} -_{\mathbb{R}} m = \mathbf{0}_{\mathbb{R}} \vee \sqrt{n} +_{\mathbb{R}} m = \mathbf{0}_{\mathbb{R}}$

Suppose $\Rightarrow$ $\sqrt{n} -_{\mathbb{R}} m = \mathbf{0}_{\mathbb{R}}$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ $\sqrt{n} +_{\mathbb{R}} m = \mathbf{0}_{\mathbb{R}}$
ALGEBRA $\Rightarrow$ $\sqrt{n} = \mathrm{Rev}_{\mathbb{R}}(m)$
$\langle m \rangle \hookrightarrow T99999 \Rightarrow$ $m = \mathbf{0}_{\mathbb{R}}$
ALGEBRA $\Rightarrow$ $\mathrm{Rev}_{\mathbb{R}}(\sqrt{n}) = m$
$\langle \sqrt{n} \rangle \hookrightarrow T99999 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 648 ( ) $X \in \mathbb{R}$ & $\mathrm{is\_nonneg}_{\mathbb{R}}(X)$ & $Y \in \mathbb{R}$ & $\mathrm{is\_nonneg}_{\mathbb{R}}(Y) \rightarrow \sqrt{X *_{\mathbb{R}} Y} = \sqrt{X} *_{\mathbb{R}} \sqrt{Y}$. PROOF:

Suppose_not$(n) \Rightarrow$ $n \in \mathbb{R}$ & $\mathrm{is\_nonneg}_{\mathbb{R}}(n)$ & $m \in \mathbb{R}$ & $\mathrm{is\_nonneg}_{\mathbb{R}}(m)$ & $\sqrt{n *_{\mathbb{R}} m} \neq \sqrt{n} *_{\mathbb{R}} \sqrt{m}$
ALGEBRA $\Rightarrow$ $n *_{\mathbb{R}} m \in \mathbb{R}$
$\langle n, m \rangle \hookrightarrow T99999 \Rightarrow$ $\mathrm{is\_nonneg}_{\mathbb{R}}(n *_{\mathbb{R}} m)$
$\langle n \rangle \hookrightarrow T99999 \Rightarrow$ $\sqrt{n} \in \mathbb{R}$ & $\sqrt{n} *_{\mathbb{R}} \sqrt{n} = n$ & $\mathrm{is\_nonneg}_{\mathbb{R}}(\sqrt{n})$
$\langle m \rangle \hookrightarrow T99999 \Rightarrow$ $\sqrt{m} \in \mathbb{R}$ & $\sqrt{m} *_{\mathbb{R}} \sqrt{m} = m$ & $\mathrm{is\_nonneg}_{\mathbb{R}}(\sqrt{m})$
ELEM $\Rightarrow$ $\sqrt{n} *_{\mathbb{R}} \sqrt{n} *_{\mathbb{R}} (\sqrt{m} *_{\mathbb{R}} \sqrt{m}) = n *_{\mathbb{R}} m$
$\langle n, m \rangle \hookrightarrow T99999 \Rightarrow$ $\mathrm{is\_nonneg}_{\mathbb{R}}(\sqrt{m} *_{\mathbb{R}} \sqrt{m})$
$\langle n *_{\mathbb{R}} m, \sqrt{m} *_{\mathbb{R}} \sqrt{m} \rangle \hookrightarrow T99999 \Rightarrow$ false; Discharge $\Rightarrow$ QED

17 Complex numbers

-- Complex Numbers
DEF 58. $\mathbb{C} =_{\mathrm{Def}} \mathbb{R} \times \mathbb{R}$
-- Complex Sum
DEF 59. $X +_{\mathbb{C}} Y =_{\mathrm{Def}} \left[X^{[1]} +_{\mathbb{R}} Y^{[1]}, X^{[2]} +_{\mathbb{R}} Y^{[2]}\right]$
-- Complex Product
DEF 60. $X *_{\mathbb{C}} Y =_{\mathrm{Def}} \left[X^{[1]} *_{\mathbb{R}} Y^{[1]} -_{\mathbb{R}} X^{[2]} *_{\mathbb{R}} Y^{[2]}, X^{[1]} *_{\mathbb{R}} Y^{[2]} +_{\mathbb{R}} X^{[2]} *_{\mathbb{R}} Y^{[1]}\right]$
-- Complex Norm
DEF 61. $\#X_{\mathbb{C}} =_{\mathrm{Def}} \sqrt{X^{[1]} *_{\mathbb{R}} X^{[1]} +_{\mathbb{R}} X^{[2]} *_{\mathbb{R}} X^{[2]}}$
-- Complex reciprocal
DEF 62. $\mathrm{Recip}_{\mathbb{C}}(X) =_{\mathrm{Def}} \left[X^{[1]} /_{\mathbb{R}} (\#X_{\mathbb{C}} *_{\mathbb{R}} \#X_{\mathbb{C}}), \mathrm{Rev}_{\mathbb{R}}(X^{[2]} /_{\mathbb{R}} (\#X_{\mathbb{C}} *_{\mathbb{R}} \#X_{\mathbb{C}}))\right]$

-- Complex Quotient
DEF 63. $X /_{\mathbb{C}} Y =_{\mathrm{Def}} X *_{\mathbb{C}} \mathrm{Recip}_{\mathbb{C}}(Y)$

DEF 63*a*. $\mathsf{Rev}_{\mathbb{C}}(\mathsf{X}) =_{\mathrm{Def}} \left[\mathsf{Rev}_{\mathbb{R}}(\mathsf{X}^{[1]}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{X}^{[2]})\right]$

DEF 63*b*. $\mathsf{X} -_{\mathbb{C}} \mathsf{Y} =_{\mathrm{Def}} \mathsf{X} +_{\mathbb{C}} \mathsf{Rev}_{\mathbb{C}}(\mathsf{Y})$

DEF 63*x*. $\mathbf{0}_{\mathbb{C}} =_{\mathrm{Def}} [\mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]$

DEF 63*y*. $\mathbf{1}_{\mathbb{C}} =_{\mathrm{Def}} [\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]$

Theorem 649 ( ) $(\mathsf{X},\mathsf{Y} \in \mathbb{R} \rightarrow [\mathsf{X},\mathsf{Y}] \in \mathbb{C})$ & $(\mathsf{m} \in \mathbb{C} \rightarrow \mathsf{m} = \left[\mathsf{m}^{[1]}, \mathsf{m}^{[2]}\right]$ & $\mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{R})$. PROOF:

Suppose_not$(\mathsf{x},\mathsf{y}) \Rightarrow$ $(\mathsf{x},\mathsf{y} \in \mathbb{R}$ & $[\mathsf{x},\mathsf{y}] \notin \mathbb{C}) \vee \left(\mathsf{m} \in \mathbb{C}\right.$ & $\neg(\mathsf{m} = \left[\mathsf{m}^{[1]}, \mathsf{m}^{[2]}\right]$ & $\left.\mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{R})\right)$
Use_def$(\mathbb{C}) \Rightarrow$ $[\mathsf{x},\mathsf{y}] \notin \mathbb{R} \times \mathbb{R}$ & $\mathsf{m} \in \mathbb{R} \times \mathbb{R}$
Use_def$(\times) \Rightarrow$ $[\mathsf{x},\mathsf{y}] \notin \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{R}, \mathsf{y} \in \mathbb{R}\}$ & *Stat1* : $\mathsf{m} \in \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{R}, \mathsf{y} \in \mathbb{R}\}$
Suppose $\Rightarrow$ *Stat2* : $[\mathsf{x},\mathsf{y}] \notin \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{R}, \mathsf{y} \in \mathbb{R}\}$
$\langle \mathsf{x},\mathsf{y} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3* : $\mathsf{m} \in \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathbb{R}, \mathsf{y} \in \mathbb{R}\}$
$\langle \mathsf{a},\mathsf{b} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\mathsf{m} = [\mathsf{a},\mathsf{b}]$ & $\mathsf{a},\mathsf{b} \in \mathbb{R}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 650 ( ) $\mathsf{N},\mathsf{M} \in \mathbb{C} \rightarrow \mathsf{N} +_{\mathbb{C}} \mathsf{M} \in \mathbb{C}$. PROOF:

Suppose_not$(\mathsf{n},\mathsf{m}) \Rightarrow$ $\mathsf{n},\mathsf{m} \in \mathbb{C}$ & $\mathsf{n} +_{\mathbb{C}} \mathsf{m} \notin \mathbb{C}$
THUS $\Rightarrow$ $\mathsf{n}^{[1]}, \mathsf{n}^{[2]}, \mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{R}$
Use_def(C_PLUS) $\Rightarrow$ $\left[\mathsf{n}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[1]}, \mathsf{n}^{[2]} +_{\mathbb{R}} \mathsf{m}^{[2]}\right] \notin \mathbb{C}$
ALGEBRA $\Rightarrow$ $\mathsf{n}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[1]}, \mathsf{n}^{[2]} +_{\mathbb{R}} \mathsf{m}^{[2]} \in \mathbb{R}$
$\langle \mathsf{n}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[1]}, \mathsf{n}^{[2]} +_{\mathbb{R}} \mathsf{m}^{[2]} \rangle \hookrightarrow$ *T99999* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 651 ( ) $\mathsf{N},\mathsf{M} \in \mathbb{C} \rightarrow \mathsf{N} +_{\mathbb{C}} \mathsf{M} = \mathsf{M} +_{\mathbb{C}} \mathsf{N}$. PROOF:

Suppose_not$(\mathsf{n},\mathsf{m}) \Rightarrow$ $\mathsf{n},\mathsf{m} \in \mathbb{C}$ & $\mathsf{n} +_{\mathbb{C}} \mathsf{m} = \mathsf{m} +_{\mathbb{C}} \mathsf{n}$
THUS $\Rightarrow$ $\mathsf{n}^{[1]}, \mathsf{n}^{[2]}, \mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{R}$
Use_def(C_PLUS) $\Rightarrow$ $\left[\mathsf{n}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[1]}, \mathsf{n}^{[2]} +_{\mathbb{R}} \mathsf{m}^{[2]}\right] \neq \left[\mathsf{m}^{[1]} +_{\mathbb{R}} \mathsf{n}^{[1]}, \mathsf{m}^{[2]} +_{\mathbb{R}} \mathsf{n}^{[2]}\right]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 652 ( ) $\mathsf{N} \in \mathbb{C} \rightarrow \mathsf{N} = \mathsf{N} +_{\mathbb{C}} \mathbf{0}_{\mathbb{C}}$. PROOF:

Suppose_not$(\mathsf{n}) \Rightarrow$ $\mathsf{n} \in \mathbb{C}$ & $\mathsf{n} \neq \mathsf{n} +_{\mathbb{C}} \mathbf{0}_{\mathbb{C}}$
THUS $\Rightarrow$ $\mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{R}$ & $\mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right]$

Use_def(C_PLUS) $\Rightarrow$ $\mathsf{n} \neq \left[\mathsf{n}^{[1]} +_{\mathbb{R}} \mathbf{0}_{\mathbb{C}}{}^{[1]}, \mathsf{n}^{[2]} +_{\mathbb{R}} \mathbf{0}_{\mathbb{C}}{}^{[2]}\right]$
Use_def($\mathbf{0}_{\mathbb{C}}$) $\Rightarrow$ $\mathsf{n} \neq \left[\mathsf{n}^{[1]} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}, \mathsf{n}^{[2]} +_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}\right]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 653 ( ) $\mathsf{N} \in \mathbb{C} \to \mathsf{Rev}_{\mathbb{C}}(\mathsf{N}) \in \mathbb{C} \;\&\; \mathsf{Rev}_{\mathbb{C}}\left(\mathsf{Rev}_{\mathbb{C}}(\mathsf{N})\right) = \mathsf{N}$. PROOF:

Suppose_not(n) $\Rightarrow$ $\mathsf{n} \in \mathbb{C} \;\&\; \mathsf{Rev}_{\mathbb{C}}(\mathsf{n}) \notin \mathbb{C} \vee \mathsf{Rev}_{\mathbb{C}}\left(\mathsf{Rev}_{\mathbb{C}}(\mathsf{n})\right) \neq \mathsf{n}$
THUS $\Rightarrow$ $\mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{R} \;\&\; \mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right]$
Use_def($\mathsf{Rev}_{\mathbb{C}}$) $\Rightarrow$ $\mathsf{Rev}_{\mathbb{C}}(\mathsf{n}) = \left[\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]})\right]$
ALGEBRA $\Rightarrow$ $\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]}) \in \mathbb{R}$
$\left\langle \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]}) \right\rangle \hookrightarrow T99999$ $\Rightarrow$ $\mathsf{Rev}_{\mathbb{C}}(\mathsf{n}) \in \mathbb{C}$
Use_def($\mathsf{Rev}_{\mathbb{C}}$) $\Rightarrow$ $\mathsf{Rev}_{\mathbb{C}}\left(\mathsf{Rev}_{\mathbb{C}}(\mathsf{n})\right) = \left[\mathsf{Rev}_{\mathbb{R}}\left(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]})\right), \mathsf{Rev}_{\mathbb{R}}\left(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]})\right)\right]$
ALGEBRA $\Rightarrow$ $\mathsf{Rev}_{\mathbb{R}}\left(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]})\right) = \mathsf{n}^{[1]} \;\&\; \mathsf{Rev}_{\mathbb{R}}\left(\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]})\right) = \mathsf{n}^{[2]}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 654 ( ) $\mathsf{N} \in \mathbb{C} \to \mathsf{N} +_{\mathbb{C}} \mathsf{Rev}_{\mathbb{C}}(\mathsf{N}) = \mathbf{0}_{\mathbb{C}}$. PROOF:

Suppose_not(n) $\Rightarrow$ $\mathsf{n} \in \mathbb{C} \;\&\; \mathsf{n} +_{\mathbb{C}} \mathsf{Rev}_{\mathbb{C}}(\mathsf{n}) \neq \mathbf{0}_{\mathbb{C}}$
THUS $\Rightarrow$ $\mathsf{n}^{[1]}, \mathsf{n}^{[2]} \in \mathbb{R} \;\&\; \mathsf{n} = \left[\mathsf{n}^{[1]}, \mathsf{n}^{[2]}\right]$
Use_def(C_PLUS) $\Rightarrow$ $\mathbf{0}_{\mathbb{C}} \neq \left[\mathsf{n}^{[1]} +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{C}}{}^{[1]}(\mathsf{n}), \mathsf{n}^{[2]} +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{C}}{}^{[2]}(\mathsf{n})\right]$
Use_def($\mathsf{Rev}_{\mathbb{C}}$) $\Rightarrow$ $\mathsf{Rev}_{\mathbb{C}}(\mathsf{n}) = \left[\mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]}), \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]})\right]$
ELEM $\Rightarrow$ $\mathbf{0}_{\mathbb{C}} \neq \left[\mathsf{n}^{[1]} +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[1]}), \mathsf{n}^{[2]} +_{\mathbb{R}} \mathsf{Rev}_{\mathbb{R}}(\mathsf{n}^{[2]})\right]$
ALGEBRA $\Rightarrow$ $\mathbf{0}_{\mathbb{C}} \neq [\mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]$
Use_def($\mathbf{0}_{\mathbb{C}}$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 655 ( ) $\mathsf{N}, \mathsf{M} \in \mathbb{C} \to \mathsf{N} = \mathsf{M} +_{\mathbb{C}} (\mathsf{N} -_{\mathbb{C}} \mathsf{M})$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $\mathsf{n}, \mathsf{m} \in \mathbb{C} \;\&\; \mathsf{n} \neq \mathsf{m} +_{\mathbb{C}} (\mathsf{n} -_{\mathbb{C}} \mathsf{m})$
Use_def(C_MINUS) $\Rightarrow$ $\mathsf{n} \neq \mathsf{m} +_{\mathbb{C}} \left(\mathsf{n} +_{\mathbb{C}} \mathsf{Rev}_{\mathbb{C}}(\mathsf{m})\right)$
ALGEBRA $\Rightarrow$ $\mathsf{n} \neq \mathsf{n} +_{\mathbb{C}} \left(\mathsf{m} +_{\mathbb{C}} \mathsf{Rev}_{\mathbb{C}}(\mathsf{m})\right)$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 656 ( ) $\mathsf{N}, \mathsf{M} \in \mathbb{C} \to \mathsf{N} *_{\mathbb{C}} \mathsf{M} = \mathsf{M} *_{\mathbb{C}} \mathsf{N}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, m \in \mathbb{C} \,\&\, n *_{\mathbb{C}} m = m *_{\mathbb{C}} n$
THUS $\Rightarrow$ $n^{[1]}, n^{[2]} \in \mathbb{R} \,\&\, n = \left[n^{[1]}, n^{[2]}\right] \,\&\, m^{[1]}, m^{[2]} \in \mathbb{R} \,\&\, m = \left[m^{[1]}, m^{[2]}\right]$
Use_def(C_TIMES) $\Rightarrow$ $\left[n^{[1]} *_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} m^{[2]}, n^{[1]} *_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} m^{[1]} *_{\mathbb{R}} n^{[2]}\right] \neq \left[m^{[1]} *_{\mathbb{R}} n^{[1]} -_{\mathbb{R}} m^{[2]} *_{\mathbb{R}} n^{[2]}, m^{[1]} *_{\mathbb{R}} n^{[2]} +_{\mathbb{R}} n^{[1]} *_{\mathbb{R}} m^{[2]}\right]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 657 ( ) $N \in \mathbb{C} \rightarrow \#N_{\mathbb{C}} \in \mathbb{R} \,\&\, \text{is\_nonneg}_{\mathbb{R}}(\#N_{\mathbb{C}})$. PROOF:

Suppose_not(n) $\Rightarrow$ $n \in \mathbb{C} \,\&\, \#N_{\mathbb{C}} \notin \mathbb{R} \vee \neg\text{is\_nonneg}_{\mathbb{R}}(\#n_{\mathbb{C}})$
THUS $\Rightarrow$ $n^{[1]}, n^{[2]} \in \mathbb{R} \,\&\, n = \left[n^{[1]}, n^{[2]}\right] \,\&\, m^{[1]}, m^{[2]} \in \mathbb{R} \,\&\, m = \left[m^{[1]}, m^{[2]}\right]$
$\langle n^{[1]} \rangle \hookrightarrow T99999 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{R}}(n^{[1]} *_{\mathbb{R}} n^{[1]})$
$\langle n^{[2]} \rangle \hookrightarrow T99999 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{R}}(n^{[2]} *_{\mathbb{R}} n^{[2]})$
$\langle n^{[1]} *_{\mathbb{R}} n^{[1]}, n^{[2]} *_{\mathbb{R}} n^{[2]} \rangle \hookrightarrow T99999 \Rightarrow$ $\text{is\_nonneg}_{\mathbb{R}}(n^{[1]} *_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} n^{[2]})$
Use_def$(\#)_{\mathbb{C}}$ $\Rightarrow$ $\#n_{\mathbb{C}} = \sqrt{\text{is\_nonneg}_{\mathbb{R}}}(n^{[1]} *_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} n^{[2]})$
$\langle n^{[1]} *_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} n^{[2]} \rangle \hookrightarrow T99999 \Rightarrow$ $\#n_{\mathbb{C}} \in \mathbb{R} \,\&\, \text{is\_nonneg}_{\mathbb{R}}(\#n_{\mathbb{C}})$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 658 ( ) $N \in \mathbb{C} \rightarrow \#N_{\mathbb{C}} = \#\text{Rev}_{\mathbb{C}_{\mathbb{C}}}(N)$. PROOF:

Suppose_not(n) $\Rightarrow$ $n \in \mathbb{C} \,\&\, \#n_{\mathbb{C}} = \#\text{Rev}_{\mathbb{C}_{\mathbb{C}}}(n)$
THUS $\Rightarrow$ $n^{[1]}, n^{[2]} \in \mathbb{R} \,\&\, n = \left[n^{[1]}, n^{[2]}\right] \,\&\, m^{[1]}, m^{[2]} \in \mathbb{R} \,\&\, m = \left[m^{[1]}, m^{[2]}\right]$
Use_def$(\text{Rev}_{\mathbb{C}})$ $\Rightarrow$ $\text{Rev}_{\mathbb{C}}(n) = \left[\text{Rev}_{\mathbb{R}}(n^{[1]}), \text{Rev}_{\mathbb{R}}(n^{[1]})\right]$
Use_def$(\#)_{\mathbb{C}}$ $\Rightarrow$ $\#n_{\mathbb{C}} = \sqrt{n^{[1]} *_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} n^{[2]}} \,\&\, \#\text{Rev}_{\mathbb{C}_{\mathbb{C}}}(n) = \sqrt{\text{Rev}_{\mathbb{C}}^{[1]}(n) *_{\mathbb{R}} \text{Rev}_{\mathbb{C}}^{[1]}(n) +_{\mathbb{R}} \text{Rev}_{\mathbb{C}}^{[2]}(n) *_{\mathbb{R}} \text{Rev}_{\mathbb{C}}^{[2]}(n)}$
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 659 ( ) $N, M \in \mathbb{C} \rightarrow \#N_{\mathbb{C}} +_{\mathbb{C}} \#M_{\mathbb{C}} >_{\mathbb{R}} \#N +_{\mathbb{C}} M_{\mathbb{C}} \vee \#N_{\mathbb{C}} +_{\mathbb{C}} \#M_{\mathbb{C}} = \#N +_{\mathbb{C}} M_{\mathbb{C}}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, n \in \mathbb{C} \,\&\, \neg(\#n_{\mathbb{C}} +_{\mathbb{C}} \#m_{\mathbb{C}} >_{\mathbb{R}} \#n +_{\mathbb{C}} m_{\mathbb{C}} \vee \#n_{\mathbb{C}} +_{\mathbb{C}} \#m_{\mathbb{C}} = \#n +_{\mathbb{C}} m_{\mathbb{C}})$
THUS $\Rightarrow$ $n^{[1]}, n^{[2]} \in \mathbb{R} \,\&\, n = \left[n^{[1]}, n^{[2]}\right] \,\&\, m^{[1]}, m^{[2]} \in \mathbb{R} \,\&\, m = \left[m^{[1]}, m^{[2]}\right]$
TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 660 ( ) $N, M \in \mathbb{C} \rightarrow \#N_{\mathbb{C}} +_{\mathbb{C}} \#M_{\mathbb{C}} >_{\mathbb{R}} \#N +_{\mathbb{C}} M_{\mathbb{C}} \vee \#N_{\mathbb{C}} +_{\mathbb{C}} \#M_{\mathbb{C}} = \#N -_{\mathbb{C}} M_{\mathbb{C}}$. PROOF:

Suppose_not(n, m) $\Rightarrow$ $n, n \in \mathbb{C} \,\&\, \neg(\#n_{\mathbb{C}} +_{\mathbb{C}} \#m_{\mathbb{C}} >_{\mathbb{R}} \#n +_{\mathbb{C}} m_{\mathbb{C}} \vee \#n_{\mathbb{C}} +_{\mathbb{C}} \#m_{\mathbb{C}} = \#n -_{\mathbb{C}} m_{\mathbb{C}})$
THUS $\Rightarrow$ $n^{[1]}, n^{[2]} \in \mathbb{R} \,\&\, n = \left[n^{[1]}, n^{[2]}\right] \,\&\, m^{[1]}, m^{[2]} \in \mathbb{R} \,\&\, m = \left[m^{[1]}, m^{[2]}\right]$

TO_BE_CONTINUED ⇒ QED

Theorem 661 ( ) $N, M \in \mathbb{C} \to \#N_{\mathbb{C}} \ast_{\mathbb{C}} \#M_{\mathbb{C}} = \#N \ast_{\mathbb{C}} M_{\mathbb{C}}$. PROOF:

Suppose_not(n, m) ⇒ $n, n \in \mathbb{C} \ \&\ \#n_{\mathbb{C}} \ast_{\mathbb{C}} \#m_{\mathbb{C}} \neq \#n \ast_{\mathbb{C}} m_{\mathbb{C}}$
THUS ⇒ $n^{[1]}, n^{[2]} \in \mathbb{R} \ \&\ n = \left[n^{[1]}, n^{[2]}\right] \ \&\ m^{[1]}, m^{[2]} \in \mathbb{R} \ \&\ m = \left[m^{[1]}, m^{[2]}\right]$
Use_def $(\#)_{\mathbb{C}}$ ⇒ $\#n_{\mathbb{C}} = \sqrt{n^{[1]} \ast_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} n^{[2]}} \ \&\ \#m_{\mathbb{C}} = \sqrt{m^{[1]} \ast_{\mathbb{R}} m^{[1]} +_{\mathbb{R}} m^{[2]} \ast_{\mathbb{R}} m^{[2]}} \ \&$
$\#n \ast_{\mathbb{C}} m_{\mathbb{C}} = \sqrt{n \ast_{\mathbb{C}} m^{[1]} \ast_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[1]} +_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[2]} \ast_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[2]}}$
$\left\langle n^{[1]} \ast_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} n^{[2]} \right\rangle \hookrightarrow$ *T99999* ⇒ $(\#n_{\mathbb{C}} \ast_{\mathbb{R}} \#n_{\mathbb{C}} = n^{[1]} \ast_{\mathbb{R}} n^{[1]}) +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} n^{[2]}$
$\left\langle m^{[1]} \ast_{\mathbb{R}} m^{[1]} +_{\mathbb{R}} m^{[2]} \ast_{\mathbb{R}} m^{[2]} \right\rangle \hookrightarrow$ *T99999* ⇒ $\#m_{\mathbb{C}} \ast_{\mathbb{R}} \#m_{\mathbb{C}} = m^{[1]} \ast_{\mathbb{R}} m^{[1]} +_{\mathbb{R}} m^{[2]} \ast_{\mathbb{R}} m^{[2]}$
$\left\langle n \ast_{\mathbb{C}} m^{[1]} \ast_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[1]} +_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[2]} \ast_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[2]} \right\rangle \hookrightarrow$ *T99999* ⇒ $\#n \ast_{\mathbb{C}} m_{\mathbb{C}} \ast_{\mathbb{R}} \#n \ast_{\mathbb{C}} m_{\mathbb{C}} =$
$n \ast_{\mathbb{C}} m^{[1]} \ast_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[1]} +_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[2]} \ast_{\mathbb{R}} n \ast_{\mathbb{C}} m^{[2]}$
Use_def(C_TIMES) ⇒ $n \ast_{\mathbb{C}} m = \left[n^{[1]} \ast_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[2]}, n^{[1]} \ast_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[1]}\right]$
ELEM ⇒ $\#n \ast_{\mathbb{C}} m_{\mathbb{C}} \ast_{\mathbb{R}} \#n \ast_{\mathbb{C}} m_{\mathbb{C}} =$
$(n^{[1]} \ast_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[2]}) \ast_{\mathbb{R}} (n^{[1]} \ast_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[2]}) +_{\mathbb{R}} (n^{[1]} \ast_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[1]}) \ast_{\mathbb{R}} (n^{[1]} \ast_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[1]})$
ELEM ⇒ $\#n_{\mathbb{C}} \ast_{\mathbb{R}} \#n_{\mathbb{C}} \ast_{\mathbb{R}} (\#m_{\mathbb{C}} \ast_{\mathbb{R}} \#m_{\mathbb{C}}) = (n^{[1]} \ast_{\mathbb{R}} n^{[1]} +_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} n^{[2]}) \ast_{\mathbb{R}} (m^{[1]} \ast_{\mathbb{R}} m^{[1]} +_{\mathbb{R}} m^{[2]} \ast_{\mathbb{R}} m^{[2]})$
ALGEBRA ⇒ $\#n_{\mathbb{C}} \ast_{\mathbb{R}} \#m_{\mathbb{C}} \ast_{\mathbb{R}} (\#n_{\mathbb{C}} \ast_{\mathbb{R}} \#m_{\mathbb{C}}) = \#n \ast_{\mathbb{C}} m_{\mathbb{C}} \ast_{\mathbb{R}} \#n \ast_{\mathbb{C}} m_{\mathbb{C}}$
ELEM ⇒ $\sqrt{\#n_{\mathbb{C}} \ast_{\mathbb{R}} \#m_{\mathbb{C}} \ast_{\mathbb{R}} (\#n_{\mathbb{C}} \ast_{\mathbb{R}} \#m_{\mathbb{C}})} = \sqrt{\#n \ast_{\mathbb{C}} m_{\mathbb{C}} \ast_{\mathbb{R}} \#n \ast_{\mathbb{C}} m_{\mathbb{C}}}$
ALGEBRA ⇒ $n \ast_{\mathbb{C}} m \in \mathbb{C}$
TO_BE_CONTINUED ⇒ QED

Theorem 662 ( ) $N, M \in \mathbb{C} \ \&\ M \neq \mathbf{0}_{\mathbb{C}} \to \#N_{\mathbb{C}} /_{\mathbb{R}} \#M_{\mathbb{C}} = \#N /_{\mathbb{C}} M_{\mathbb{C}}$. PROOF:

Suppose_not(n, m) ⇒ $n, m \in \mathbb{C} \ \&\ m \neq \mathbf{0}_{\mathbb{C}} \ \&\ \#n_{\mathbb{C}} /_{\mathbb{R}} \#m_{\mathbb{C}} = \#n /_{\mathbb{C}} m_{\mathbb{C}}$
THUS ⇒ $n^{[1]}, n^{[2]} \in \mathbb{R} \ \&\ n = \left[n^{[1]}, n^{[2]}\right] \ \&\ m^{[1]}, m^{[2]} \in \mathbb{R} \ \&\ m = \left[m^{[1]}, m^{[2]}\right]$
TO_BE_CONTINUED ⇒ QED

Theorem 663 ( ) $N, M \in \mathbb{C} \to N \ast_{\mathbb{C}} M \in \mathbb{C}$. PROOF:

Suppose_not(n, m) ⇒ $n, m \in \mathbb{C} \ \&\ n \ast_{\mathbb{C}} m \notin \mathbb{C}$
Use_def(C_TIMES) ⇒ $\left[n^{[1]} \ast_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[2]}, n^{[1]} \ast_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} m^{[1]} \ast_{\mathbb{R}} n^{[2]}\right] \notin \mathbb{C}$
ALGEBRA ⇒ $n^{[1]} \ast_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[2]}, n^{[1]} \ast_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} m^{[1]} \ast_{\mathbb{R}} n^{[2]} \in \mathbb{R}$
$\left\langle n^{[1]} \ast_{\mathbb{R}} m^{[1]} -_{\mathbb{R}} n^{[2]} \ast_{\mathbb{R}} m^{[2]}, n^{[1]} \ast_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} m^{[1]} \ast_{\mathbb{R}} n^{[2]} \right\rangle \hookrightarrow$ *T99999* ⇒ false; Discharge ⇒ QED

Theorem 664 ( ) $K, N, M \in \mathbb{C} \rightarrow N +_{c} (M +_{c} K) = (N +_{c} M) +_{c} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow$ $n, n, k \in \mathbb{C}$ & $n +_{c} (m +_{c} k) \neq n +_{c} m +_{c} k$
THUS $\Rightarrow$ $n^{[1]}, n^{[2]}, m^{[1]}, m^{[2]}, k^{[1]}, k^{[2]} \in \mathbb{R}$
Use_def(C_PLUS) $\Rightarrow$ $\left[n^{[1]} +_{\mathbb{R}} (m^{[1]} +_{\mathbb{R}} k^{[1]}), n^{[2]} +_{\mathbb{R}} (m^{[2]} +_{\mathbb{R}} k^{[2]})\right] \neq \left[n^{[1]} +_{\mathbb{R}} m^{[1]} +_{\mathbb{R}} k^{[1]}, n^{[2]} +_{\mathbb{R}} m^{[2]} +_{\mathbb{R}} k^{[2]}\right]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 665 ( ) $K, N, M \in \mathbb{C} \rightarrow N *_{c} (M *_{c} K) = (N *_{c} M) *_{c} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow$ $n, n, k \in \mathbb{C}$ & $n *_{c} (m *_{c} k) \neq n *_{c} m *_{c} k$
Loc_def $\Rightarrow$ $an = n^{[1]}$
Loc_def $\Rightarrow$ $dn = n^{[2]}$
Loc_def $\Rightarrow$ $am = m^{[1]}$
Loc_def $\Rightarrow$ $dm = m^{[2]}$
Loc_def $\Rightarrow$ $ak = k^{[1]}$
Loc_def $\Rightarrow$ $dk = kn^{[2]}$
Use_def(C_TIMES) $\Rightarrow$
$[an *_{\mathbb{R}} (am *_{\mathbb{R}} ak -_{\mathbb{R}} dm *_{\mathbb{R}} dk) -_{\mathbb{R}} dn *_{\mathbb{R}} (am *_{\mathbb{R}} dk +_{\mathbb{R}} ak *_{\mathbb{R}} dm), an *_{\mathbb{R}} (am *_{\mathbb{R}} dk +_{\mathbb{R}} ak *_{\mathbb{R}} dm) +_{\mathbb{R}} (am *_{\mathbb{R}} ak -_{\mathbb{R}} dm *_{\mathbb{R}} dk) *_{\mathbb{R}} dn] \neq$
$[(an *_{\mathbb{R}} am -_{\mathbb{R}} dn *_{\mathbb{R}} dm) *_{\mathbb{R}} ak -_{\mathbb{R}} (an *_{\mathbb{R}} dm +_{\mathbb{R}} am *_{\mathbb{R}} dn) *_{\mathbb{R}} dk, (an *_{\mathbb{R}} am -_{\mathbb{R}} dn *_{\mathbb{R}} dm) *_{\mathbb{R}} dk +_{\mathbb{R}} ak *_{\mathbb{R}} (an *_{\mathbb{R}} dm +_{\mathbb{R}} am *_{\mathbb{R}} dn)]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 666 ( ) $K, N, M \in \mathbb{C} \rightarrow N *_{c} (M +_{c} K) = N *_{c} M +_{c} N *_{c} K$. PROOF:

Suppose_not$(k, n, m) \Rightarrow$ $n, n, k \in \mathbb{C}$ & $n *_{c} (m +_{c} k) \neq n *_{c} m +_{c} n *_{c} k$
TO_BE_CONTINUED $\Rightarrow$ QED

Theorem 667 ( ) $M \in \mathbb{C} \rightarrow M = M *_{c} \mathbf{1}_{c}$. PROOF:

Suppose_not$(m) \Rightarrow$ $m \in \mathbb{C}$ & $m \neq m *_{c} \mathbf{1}_{c}$
THUS $\Rightarrow$ $m^{[1]}, m^{[2]} \in \mathbb{R}$ & $m = \left[m^{[1]}, m^{[2]}\right]$
Use_def(C_TIMES) $\Rightarrow$ $m \neq \left[n^{[1]} *_{\mathbb{R}} \mathbf{1}_{c}^{[1]} -_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} \mathbf{1}_{c}^{[2]}, n^{[1]} *_{\mathbb{R}} \mathbf{1}_{c}^{[2]} +_{\mathbb{R}} \mathbf{1}_{c}^{[1]} *_{\mathbb{R}} n^{[2]}\right]$
Use_def$(\mathbf{1}_{c}) \Rightarrow$ $m \neq \left[n^{[1]} *_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} -_{\mathbb{R}} n^{[2]} *_{\mathbb{R}} \mathbf{0}_{\mathbb{R}}, n^{[1]} *_{\mathbb{R}} \mathbf{0}_{\mathbb{R}} +_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} *_{\mathbb{R}} n^{[2]}\right]$
ALGEBRA $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Theorem 668 ( ) $\mathsf{M} \in \mathbb{C} \,\&\, \mathsf{M} \neq \mathbf{0}_{\mathbb{C}} \rightarrow \mathsf{Recip}_{\mathbb{C}}(\mathsf{M}) \in \mathbb{C} \,\&\, \mathsf{M} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{M}) = \mathbf{1}_{\mathbb{C}}$. PROOF:

Suppose_not(m) ⇒ $\mathsf{m} \in \mathbb{C} \,\&\, \mathsf{m} \neq \mathbf{0}_{\mathbb{C}} \,\&\, \neg\big(\mathsf{Recip}_{\mathbb{C}}(\mathsf{m}) \in \mathbb{C} \,\&\, \mathsf{m} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{m}) = \mathbf{1}_{\mathbb{C}}\big)$
THUS ⇒ $\mathsf{m}^{[1]}, \mathsf{m}^{[2]} \in \mathbb{R} \,\&\, \mathsf{m} = \big[\mathsf{m}^{[1]}, \mathsf{m}^{[2]}\big]$
Use_def$(\mathbf{0}_{\mathbb{C}})$ ⇒ $\mathsf{m} \neq [\mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]$
ELEM ⇒ $\neg(\mathsf{m}^{[1]} = \mathbf{0}_{\mathbb{R}} \,\&\, \mathsf{m}^{[2]} = \mathbf{0}_{\mathbb{R}})$
Use_def$(\#)_{\mathbb{C}}$ ⇒ $\#\mathsf{m}_{\mathbb{C}} = \sqrt{\mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]}}$
Loc_def ⇒ $\mathsf{nrm} = \mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]}$
$\langle \mathsf{m}^{[1]} \rangle \hookrightarrow$ *T99999* ⇒ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]})$
$\langle \mathsf{m}^{[1]}, \mathsf{m}^{[1]} \rangle \hookrightarrow$ *T99999* ⇒ $\mathsf{m}^{[1]} \neq \mathsf{R}_0 \rightarrow \mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]} \neq \mathsf{R}_0$
$\langle \mathsf{m}^{[2]} \rangle \hookrightarrow$ *T99999* ⇒ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]})$
$\langle \mathsf{m}^{[2]}, \mathsf{m}^{[2]} \rangle \hookrightarrow$ *T99999* ⇒ $\mathsf{m}^{[2]} \neq \mathsf{R}_0 \rightarrow \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]} \neq \mathsf{R}_0$
ELEM ⇒ $\neg(\mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]} = \mathbf{0}_{\mathbb{R}} \,\&\, \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]} = \mathbf{0}_{\mathbb{R}})$
$\langle \mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]}, \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]} \rangle \hookrightarrow$ *T99999* ⇒ $\mathsf{is\_nonneg}_{\mathbb{R}}(\mathsf{nrm})$
$\langle \mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]}, \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]} \rangle \hookrightarrow$ *T99999* ⇒ $\mathsf{nrm} \neq \emptyset$
$\langle \mathsf{nrm} \rangle \hookrightarrow$ *T99999* ⇒ $\#\mathsf{m}_{\mathbb{C}} *_{\mathbb{R}} \#\mathsf{m}_{\mathbb{C}} = \mathsf{nrm}$
Use_def$(\mathsf{Recip}_{\mathbb{C}})$ ⇒ $\mathsf{Recip}_{\mathbb{C}}(\mathsf{m}) = \big[\mathsf{m}^{[1]} /_{\mathbb{R}} \mathsf{nrm}, \mathsf{Rev}_{\mathbb{R}}(\mathsf{x}^{[2]} /_{\mathbb{R}} \mathsf{nrm})\big]$
Use_def(C_TIMES) ⇒ $\mathsf{M} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{M}) = \big[\mathsf{m}^{[1]} *_{\mathbb{R}} (\mathsf{m}^{[1]} /_{\mathbb{R}} \mathsf{nrm}) -_{\mathbb{R}} \mathsf{m}^{[2]} *_{\mathbb{R}} \big(\mathsf{Rev}_{\mathbb{R}}(\mathsf{m}^{[2]}) /_{\mathbb{R}} \mathsf{nrm}\big), \mathsf{m}^{[1]} *_{\mathbb{R}} \big(\mathsf{Rev}_{\mathbb{R}}(\mathsf{m}^{[2]}) /_{\mathbb{R}} \mathsf{nrm}\big) +_{\mathbb{R}} \mathsf{m}^{[2]} *_{\mathbb{R}} (\mathsf{m}^{[1]} /_{\mathbb{R}} \mathsf{nrm})\big]$
ALGEBRA ⇒ $\mathsf{M} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{M}) = \big[(\mathsf{m}^{[1]} *_{\mathbb{R}} \mathsf{m}^{[1]} +_{\mathbb{R}} \mathsf{m}^{[2]} *_{\mathbb{R}} \mathsf{m}^{[2]}) /_{\mathbb{R}} \mathsf{nrm}, \mathbf{0}_{\mathbb{R}}\big]$
ELEM ⇒ $\mathsf{M} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{M}) = [\mathsf{nrm} /_{\mathbb{R}} \mathsf{nrm}, \mathbf{0}_{\mathbb{R}}]$
Use_def$(/_{\mathbb{R}})$ ⇒ $\mathsf{M} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{M}) = \big[\mathsf{nrm} *_{\mathbb{R}} \mathsf{Recip}_{\mathbb{R}}(\mathsf{nrm}), \mathbf{0}_{\mathbb{R}}\big]$
$\langle \mathsf{nrm} \rangle \hookrightarrow$ *T99999* ⇒ false; Discharge ⇒ QED

Theorem 669 ( ) $\mathsf{N}, \mathsf{M} \in \mathbb{C} \,\&\, \mathsf{M} \neq \mathbf{0}_{\mathbb{C}} \rightarrow \mathsf{N} = \mathsf{M} *_{\mathbb{C}} (\mathsf{N} /_{\mathbb{C}} \mathsf{M})$. PROOF:

Suppose_not(m) ⇒ $\mathsf{n}, \mathsf{m} \in \mathbb{C} \,\&\, \mathsf{m} \neq \mathbf{0}_{\mathbb{C}} \,\&\, \mathsf{n} \neq \mathsf{m} *_{\mathbb{C}} (\mathsf{n} /_{\mathbb{C}} \mathsf{m})$
Use_def(C_OVER) ⇒ $\mathsf{n} \neq \mathsf{m} *_{\mathbb{C}} \big(\mathsf{n} *_{\mathbb{C}} \mathsf{Recip}_{\mathbb{C}}(\mathsf{m})\big)$
ALGEBRA ⇒ false; Discharge ⇒ QED

Theorem 670 ( ) $\mathbf{0}_{\mathbb{C}}, \mathbf{1}_{\mathbb{C}} \in \mathbb{C}$. PROOF:

Use_def$(\mathbf{0}_{\mathbb{C}})$ ⇒ $\mathbf{0}_{\mathbb{C}} = [\mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]$
Use_def$(\mathbf{1}_{\mathbb{C}})$ ⇒ $\mathbf{1}_{\mathbb{C}} = [\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]$
ALGEBRA ⇒ $\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}} \in \mathbb{R}$
$\langle \mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow$ *T99999* ⇒ $\mathbf{0}_{\mathbb{C}} \in \mathbb{C}$
$\langle \mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}} \rangle \hookrightarrow$ *T99999* ⇒ QED

-- % Sums for Real Maps with finite domains

APPLY $\langle \Sigma_\Theta : \sum_{\mathbb{R}} \rangle$ sigma_theory$(s \mapsto \mathbb{R}, \oplus \mapsto +_{\mathbb{R}}, e \mapsto \mathbf{0}_{\mathbb{R}}) \Rightarrow$

Theorem 671 (real_sigma) $\mathsf{Svm}(f) \,\&\, \mathbf{range}(f) \subseteq \mathbb{R} \,\&\, \mathsf{Finite}(f) \rightarrow \sum_{\mathbb{R}} f \in \mathbb{R} \,\&\, p \in f \rightarrow \sum_{\mathbb{R}} \{p\} = f(p^{[2]}) \,\&\, \langle \forall a \mid \sum_{\mathbb{R}} f = \sum_{\mathbb{R}} f_{|\mathbf{domain}(f) \cap a} +_{\mathbb{R}} \sum_{\mathbb{R}} f_{|\mathbf{domain}(f) \setminus a} \rangle$.

-- Sums of absolutely convergent infinite series

Def 64. $\sum_{\mathbb{R}}^{\infty} X \;=_{\mathrm{Def}}\; \bigcup \{ \sum_{\mathbb{R}} X_{|s} : s \subseteq \mathbf{domain}(X) \mid \mathsf{Finite}(s) \}$

-- Real functions of a real variable

Def 65. $\mathbb{F} \;=_{\mathrm{Def}}\; \{ f \subseteq \mathbb{R} \times \mathbb{R} \mid \mathsf{Svm}(f) \,\&\, \mathbf{domain}(f) = \mathbb{R} \}$

-- Sum of Real Functions

Def 66. $X +_{\mathbb{F}} Y \;=_{\mathrm{Def}}\; \{ [x, X \upharpoonright x +_{\mathbb{R}} Y \upharpoonright x] : x \in \mathbb{R} \}$

-- Product of Real Functions

Def 67. $X *_{\mathbb{F}} Y \;=_{\mathrm{Def}}\; \{ [x, X \upharpoonright x *_{\mathbb{R}} Y \upharpoonright x] : x \in \mathbb{R} \}$

-- LUB of a set of Real Functions

Def 68. $\mathsf{LUB}(X) \;=_{\mathrm{Def}}\; \{ [x, \bigcup \{ f \upharpoonright x : f \in X \}] : x \in \mathbb{R} \}$

-- Constant zero function

Def 69. $\mathbf{0}_{\mathbb{F}} \;=_{\mathrm{Def}}\; \{ [x, \mathbf{0}_{\mathbb{R}}] : x \in \mathbb{R} \}$

Theorem 672 ( ) $N, M \in \mathbb{F} \rightarrow N +_{\mathbb{F}} M = M +_{\mathbb{F}} N$. Proof:

Qed

Theorem 673 ( ) $N, M \in \mathbb{F} \rightarrow N +_{\mathbb{F}} M = M +_{\mathbb{F}} N$. Proof:

Qed

Theorem 674 ( ) $N, M \in \mathbb{F} \rightarrow N *_{\mathbb{F}} M = M *_{\mathbb{F}} N$. Proof:

Qed

Theorem 675 ( ) $K, N, M \in \mathbb{F} \rightarrow N +_{\mathbb{F}} (M +_{\mathbb{F}} K) = (N +_{\mathbb{F}} M) +_{\mathbb{F}} K$. Proof:

Qed

Theorem 676 ( ) $K, N, M \in \mathbb{F} \rightarrow N *_{\mathbb{F}} (M *_{\mathbb{F}} K) = (N *_{\mathbb{F}} M) *_{\mathbb{F}} K$. Proof:

Qed

Theorem 677 ( ) $K, N, M \in \mathbb{F} \rightarrow N *_{\mathbb{F}} (M +_{\mathbb{F}} K) = N *_{\mathbb{F}} M +_{\mathbb{F}} N *_{\mathbb{F}} K$. Proof:

Qed

-- % Sums for series of real functions

APPLY $\langle \Sigma_{\Theta} : \sum_{\mathbb{F}} \rangle$ sigma_theory$(\mathbb{F}, +_{\mathbb{F}}, \mathbf{0}_{\mathbb{F}}) \Rightarrow$

Theorem 678 (real_function_sigma) $\mathsf{Svm}(\mathsf{ser})\ \&\ \mathbf{range}(\mathsf{ser}) \subseteq \mathbb{F}\ \&\ \mathsf{Finite}(\mathsf{ser}) \rightarrow \big(\sum_{\mathbb{F}} \mathsf{ser} \in \mathbb{F}\ \&\ p \in \mathsf{ser} \longrightarrow \sum_{\mathbb{R}} \{p\} = \mathsf{ser}(p^{[2]})\big)\ \&\ \big\langle \forall a \mid \sum_{\mathbb{R}} \mathsf{ser} = \sum_{\mathbb{R}} \mathsf{ser}_{|\mathbf{domain}(\mathsf{ser}) \cap a} +_{\mathbb{R}} \sum_{\mathbb{R}} \mathsf{ser}_{|\mathbf{c}}$

-- Sums of absolutely convergent infinite series of real functions

Def 71. $\sum_{\mathbb{F}}^{\infty} X \quad =_{\mathrm{Def}} \quad \mathsf{LUB}\big(\{\sum_{\mathbb{R}} X_{|s} : s \subseteq \mathbf{domain}(X) \mid \mathsf{Finite}(s)\}\big)$

-- % Product of a nonempty family of sets ; Note - this is also the real greatest lower bound

Def 72. $\mathsf{GLB}(X) \quad =_{\mathrm{Def}} \quad \{x \in \mathbf{arb}(X) \mid \langle \forall y \in X \mid x \in y \rangle\}$

-- Block function

Def 73. $\mathsf{Bl\_f}(X, Y, U) \quad =_{\mathrm{Def}} \quad \{[x, \textbf{if } X \subseteq x\ \&\ x \subseteq Y \textbf{ then } U \textbf{ else } \mathbf{0}_{\mathbb{R}} \textbf{ fi}] : x \in \mathbb{R}\}$

-- Block function integral

Def 74. $\mathsf{BFInt}(X) \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{c *_{\mathbb{R}} (b -_{\mathbb{R}} a) : a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R} \mid \mathsf{Bl\_f}(a, b, c) = X\})$

-- Block functions

Def 75. $\mathsf{RBF} \quad =_{\mathrm{Def}} \quad \{\mathsf{Bl\_f}(a, b, c) : a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}\}$

-- Comparison of real functions

Def 76. $X >_{\mathbb{F}} Y \quad \leftrightarrow_{\mathrm{Def}} \quad X \neq Y\ \&\ \langle \forall x \in \mathbb{R} \mid X \upharpoonright x \supseteq Y \upharpoonright x \rangle$

-- Lebesgue Upper Integral of a Positive Function

Def 77. $\int^{+} X \quad =_{\mathrm{Def}} \quad \mathsf{GLB}\big(\{\sum_{\mathbb{R}}^{\infty} \{[n, \mathsf{BFInt}(\mathsf{ser} \upharpoonright n)] : n \in \mathbb{N}\} : \mathsf{ser} \subseteq \mathbb{N} \times \mathsf{RBF} \mid \mathsf{Svm}(\mathsf{ser})\ \&\ \sum_{\mathbb{F}}^{\infty} \mathsf{ser} >_{\mathbb{F}} X\}\big)$

-- Positive Part of real function

Def 78. $\mathsf{Pos\_part}(X) \quad =_{\mathrm{Def}} \quad \{[x, \textbf{if } X \upharpoonright x \supseteq \mathbf{0}_{\mathbb{R}} \textbf{ then } X \upharpoonright x \textbf{ else } \mathbf{0}_{\mathbb{R}} \textbf{ fi}] : x \in \mathbb{R}\}$

-- Reverse of a real function

Def 79. $\mathsf{Rev}_{\mathbb{F}}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \{[\mathsf{x}, \mathsf{Rev}_{\mathbb{R}}(\mathsf{X}\restriction\mathsf{x})] : \mathsf{x} \in \mathbb{R}\}$

-- Lebesgue Integral

Def 80. $\int \mathsf{X} \quad =_{\mathrm{Def}} \quad \int^{+} \mathsf{Pos\_part}(\mathsf{X}) -_{\mathbb{R}} \int^{+} \mathsf{Pos\_part}\big(\mathsf{Rev}_{\mathbb{F}}(\mathsf{X})\big)$

-- Continuous function of a real variable

Def 81. $\mathsf{is\_continuous}_{\mathbb{F}}(\mathsf{X}) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{X} \subseteq \mathbb{R} \times \mathbb{R} \;\&\; \mathsf{Svm}(\mathsf{X}) \;\&$
$\big\langle \forall \mathsf{x} \in \mathbf{domain}(\mathsf{X}), \varepsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, \mathsf{y} \in \mathbf{domain}(\mathsf{X}) \mid \delta >_{\mathbb{R}} \emptyset \;\&\; \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \;\&\; \varepsilon \neq \mathbf{0}_{\mathbb{R}} \;\&\; \delta \supseteq \#\mathsf{x} -_{\mathbb{R}} \mathsf{y}_{\mathbb{R}} \rightarrow \varepsilon \supseteq \#\mathsf{X}\restriction\mathsf{x} -_{\mathbb{R}} \mathsf{X}\restriction\mathsf{y}_{\mathbb{R}} \big\rangle$

-- Euclidean n - space

Def 82. $\mathsf{E}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \{\mathsf{f} \subseteq \mathsf{X} \times \mathbb{R} \mid \mathsf{Svm}(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathsf{X}\}$

-- Euclidean norm

Def 83. $\|\mathsf{X}\|_{\mathbb{R}} \quad =_{\mathrm{Def}} \quad \sqrt{\sum_{\mathbb{F}} \mathsf{X}}$

-- Difference of Real Functions

Def 84. $\mathsf{X} -_{\mathbb{F}} \mathsf{Y} \quad =_{\mathrm{Def}} \quad \{[\mathsf{x}, \mathsf{X}\restriction\mathsf{x} -_{\mathbb{R}} \mathsf{Y}\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{domain}(\mathsf{X})\}$

-- Continuous function on Euclidean n - space

Def 85. $\mathsf{is\_continuous\_REnF}(\mathsf{X}, \mathsf{Y}) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{X} \subseteq \mathsf{E}(\mathsf{Y}) \times \mathsf{E}(\mathsf{Y}) \;\&\; \mathsf{Svm}(\mathsf{X}) \;\&$
$\big\langle \forall \mathsf{x} \in \mathbf{domain}(\mathsf{X}), \varepsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, \mathsf{y} \in \mathbf{domain}(\mathsf{X}) \mid \delta \supseteq \mathbf{0}_{\mathbb{R}} \;\&\; \delta \neq \mathbf{0}_{\mathbb{R}} \;\&\; \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \;\&\; \varepsilon \neq \mathbf{0}_{\mathbb{R}} \;\&\; \delta \supseteq \|\mathsf{x} -_{\mathbb{F}} \mathsf{y}\|_{\mathbb{R}} \rightarrow \varepsilon \supseteq \#\mathsf{X}\restriction\mathsf{x} -_{\mathbb{R}} \mathsf{X}\restriction\mathsf{y}_{\mathbb{R}} \big\rangle$

-- Difference - and - diagonal trick

Def 86. $\mathsf{DD}(\mathsf{X}, \mathsf{Y}) \quad =_{\mathrm{Def}} \quad \big\{\textbf{if } \mathsf{x}\restriction\emptyset \neq \mathsf{x}\restriction 1 \textbf{ then } \big(\mathsf{X}\restriction(\mathsf{x}\restriction\emptyset) -_{\mathbb{R}} \mathsf{X}\restriction(\mathsf{x}\restriction 1)\big) /_{\mathbb{R}} (\mathsf{x}\restriction\emptyset -_{\mathbb{R}} \mathsf{x}\restriction 1) \textbf{ else } \mathsf{Y}\restriction(\mathsf{x}\restriction\emptyset) \textbf{ fi} : \mathsf{x} \in \mathsf{E}(2)\big\}$

-- Derivative of function of a real variable

Def 87. $\mathsf{Der}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \mathbf{arb}\Big(\Big\{\mathsf{df} \in \mathbb{F} \mid \mathbf{domain}(\mathsf{X}) = \mathbf{domain}(\mathsf{df}) \;\&\; \mathsf{is\_continuous\_REnF}\big(\mathsf{DD}(\mathsf{X}, \mathsf{df})_{|\mathbf{domain}(\mathsf{X}) \times \mathbf{domain}(\mathsf{X})}, 2\big)\Big\}\Big)$

-- Complex functions of a complex variable

Def 88. $\mathbb{CF} \quad =_{\mathrm{Def}} \quad \{\mathsf{f} \subseteq \mathbb{C} \times \mathbb{C} \mid \mathsf{Svm}(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathbb{C}\}$

-- Complex Euclidean n - space

Def 91. $\mathsf{CE}(\mathsf{X}) \quad =_{\mathrm{Def}} \quad \{\mathsf{f} \subseteq \mathsf{X} \times \mathbb{C} \mid \mathsf{Svm}(\mathsf{f}) \;\&\; \mathbf{domain}(\mathsf{f}) = \mathsf{X}\}$

-- Difference - and - diagonal trick , complex case

Def 89. $\mathsf{CDD}(\mathsf{X}, \mathsf{Y}) \quad =_{\mathrm{Def}} \quad \big\{\textbf{if } \mathsf{x}\restriction\emptyset \neq \mathsf{x}\restriction 1 \textbf{ then } \big(\mathsf{X}\restriction(\mathsf{x}\restriction\emptyset) -_{\mathbb{C}} \mathsf{X}\restriction(\mathsf{x}\restriction 1)\big) /_{\mathbb{C}} (\mathsf{x}\restriction\emptyset -_{\mathbb{C}} \mathsf{x}\restriction 1) \textbf{ else } \mathsf{Y}\restriction(\mathsf{x}\restriction\emptyset) \textbf{ fi} : \mathsf{x} \in \mathsf{CE}(2)\big\}$

-- Continuous function of a complex variable

Def 90. $\mathsf{is\_continuous}_{\mathbb{CF}}(\mathsf{X}) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{X} \subseteq \mathbb{C} \times \mathbb{C} \;\&\; \mathsf{Svm}(\mathsf{X}) \;\&$
$\big\langle \forall \mathsf{x} \in \mathbf{domain}(\mathsf{X}), \varepsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, \mathsf{y} \in \mathbf{domain}(\mathsf{X}) \mid \delta \supseteq \mathbf{0}_{\mathbb{R}} \;\&\; \delta \neq \mathbf{0}_{\mathbb{R}} \;\&\; \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \;\&\; \varepsilon \neq \mathbf{0}_{\mathbb{R}} \;\&\; \delta \supseteq \|\mathsf{x} -_{\mathbb{C}} \mathsf{y}\|_{\mathbb{R}} \rightarrow \varepsilon \supseteq \|\mathsf{X}\restriction\mathsf{x} -_{\mathbb{C}} \mathsf{X}\restriction\mathsf{y}\|_{\mathbb{R}} \big\rangle$

-- Complex Euclidean norm

DEF 92. $\|X\|_{\mathbb{C}} =_{\text{Def}} \sqrt{\sum_{\mathbb{F}}\{[m, \#X\restriction m_{\mathbb{C}} *_{\mathbb{R}} \#X\restriction m_{\mathbb{C}}] : m \in \mathbf{domain}(X)\}}$

-- Difference of Complex Functions

DEF 93. $X -_{\mathbb{CF}} Y =_{\text{Def}} \{[x, X\restriction x -_{\mathbb{C}} Y\restriction x] : x \in \mathbb{C}\}$

-- Continuous function on Complex Euclidean n - space

DEF 94. $\text{is\_continuous\_CEnF}(X, Y) \leftrightarrow_{\text{Def}} X \subseteq CE(Y) \times CE(Y) \,\&\, \text{Svm}(X) \,\&$
$\langle \forall x \in \mathbf{domain}(X), \varepsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, y \in \mathbf{domain}(X) \mid \delta \supseteq \mathbf{0}_{\mathbb{R}} \,\&\, \delta \neq \mathbf{0}_{\mathbb{R}} \,\&\, \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \,\&\, \varepsilon \neq \mathbf{0}_{\mathbb{R}} \,\&\, \delta \supseteq \|x -_{\mathbb{CF}} y\|_{\mathbb{C}} \rightarrow \varepsilon \supseteq \|X\restriction x -_{\mathbb{CF}} X\restriction y\|_{\mathbb{C}} \rangle$

-- Derivative of function of a complex variable

DEF 95. $\text{CDer}(X) =_{\text{Def}} \mathbf{arb}\Big(\Big\{df \in \mathbb{CF} \mid \mathbf{domain}(X) = \mathbf{domain}(df) \,\&\, \text{is\_continuous\_CEnF}\big(\text{CDD}(X, df)_{|\mathbf{domain}(X) \times \mathbf{domain}(X)}, 2\big)\Big\}\Big)$

-- Open set in the complex plane

DEF 97. $\text{is\_open\_C\_set}(X) \leftrightarrow_{\text{Def}} X \subseteq \mathbb{C} \,\&\, \text{is\_continuous}_{\mathbb{CF}}(\{[z, \textbf{if } z \in X \textbf{ then } [\mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}] \textbf{ else } [\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}] \textbf{ fi}] : z \in \mathbb{C}\})$

-- Analytic function of a complex variable

DEF 98. $\text{is\_analytic}_{\mathbb{CF}}(X) \leftrightarrow_{\text{Def}} \text{is\_continuous}_{\mathbb{CF}}(X) \,\&\, \text{is\_open\_C\_set}(\mathbf{domain}(X)) \,\&\, \text{CDer}(X) \neq \emptyset$

-- Complex exponential function

DEF 99. $\text{C\_exp\_fcn} =_{\text{Def}} \mathbf{arb}(\{f : f \subseteq \mathbb{C} \times \mathbb{C} \mid \text{is\_analytic}_{\mathbb{CF}}(f) \,\&\, \text{CDer}(f) = f \,\&\, f\restriction[\mathbf{0}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}] = [\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}]\})$

-- The constant pi

DEF 100. $\pi =_{\text{Def}} \mathbf{arb}(\{x \in \mathbb{R} \mid x \supseteq \mathbf{0}_{\mathbb{R}} \,\&\, x \neq \mathbf{0}_{\mathbb{R}} \,\&\, \text{C\_exp\_fcn}([\mathbf{0}_{\mathbb{R}}, x]) = [\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}] \,\&\, \langle \forall y \in \mathbb{R} \mid \text{C\_exp\_fcn}([\mathbf{0}_{\mathbb{R}}, y]) = [\mathbf{1}_{\mathbb{R}}, \mathbf{0}_{\mathbb{R}}] \rightarrow y = x \vee \mathbf{0}_{\mathbb{R}} \supseteq y \rangle\})$

-- Continuous complex function on the reals

DEF 101. $\text{is\_continuous\_CoRF}(X) \leftrightarrow_{\text{Def}} X \subseteq \mathbb{R} \times \mathbb{C} \,\&\, \text{Svm}(X) \,\&$
$\langle \forall x \in \mathbf{domain}(X), \varepsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, y \in \mathbf{domain}(X) \mid \delta >_{\mathbb{R}} \emptyset \,\&\, \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \,\&\, \varepsilon \neq \mathbf{0}_{\mathbb{R}} \,\&\, \delta \supseteq \#x -_{\mathbb{R}} y_{\mathbb{R}} \rightarrow \varepsilon \supseteq \|X\restriction x -_{\mathbb{C}} X\restriction y\|_{\mathbb{R}} \rangle$

-- Difference - and - diagonal trick , real - to - complex case

DEF 102. $\text{CRDD}(X, Y) =_{\text{Def}} \{\textbf{if } x\restriction\emptyset \neq x\restriction 1 \textbf{ then } (X\restriction(x\restriction\emptyset) -_{\mathbb{C}} X\restriction(x\restriction 1)) /_{\mathbb{C}} (x\restriction\emptyset -_{\mathbb{C}} x\restriction 1) \textbf{ else } Y\restriction(x\restriction\emptyset) \textbf{ fi} : x \in E(2)\}$

-- Continuous complex function on E (n)

DEF 103. $\text{is\_continuous\_CREnF}(X, Y) \leftrightarrow_{\text{Def}} X \subseteq E(Y) \times \mathbb{C} \,\&\, \text{Svm}(X) \,\&$
$\langle \forall x \in \mathbf{domain}(X), \varepsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, y \in \mathbf{domain}(X) \mid \delta >_{\mathbb{R}} \emptyset \,\&\, \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \,\&\, \varepsilon \neq \mathbf{0}_{\mathbb{R}} \,\&\, \delta \supseteq \|x -_{\mathbb{F}} y\|_{\mathbb{R}} \rightarrow \varepsilon \supseteq \#X\restriction x -_{\mathbb{CF}} X\restriction y_{\mathbb{C}} \rangle$

-- Derivative of complex function of a real variable

DEF 104. $\text{CRDer}(X) =_{\text{Def}} \mathbf{arb}\Big(\Big\{df \in \mathbb{CF} \mid \mathbf{domain}(X) = \mathbf{domain}(df) \,\&\, \text{is\_continuous\_CREnF}\big(\text{CRDD}(X, df)_{|\mathbf{domain}(X) \times \mathbf{domain}(X)}, 2\big)\Big\}\Big)$

-- Real Interval

DEF 105. $\text{Interval}(X, Y) =_{\text{Def}} \{x \in \mathbb{R} \mid X \subseteq x \,\&\, x \subseteq Y\}$

-- Continuously differentiable curve in the complex plane

DEF 106. $\mathsf{is\_CD\_curv}(X, Y, U) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{is\_continuous\_CoRF}(X) \,\&\, \mathbf{domain}(X) = \mathsf{Interval}(Y, U) \,\&\, \emptyset \neq \mathsf{CRDer}(X) \,\&\, \mathsf{is\_continuous\_CoRF}\big(\mathsf{CRDer}(X)\big)$

-- Complex line integral

DEF 107. $\oint_{U}^{V}(X, Y) \quad =_{\mathrm{Def}}$

$$\left[\int \left\{\left[x, \textbf{if } x \notin \mathsf{Interval}(U, V) \textbf{ then } \mathbf{0}_{\mathbb{R}} \textbf{ else } X \restriction (Y \restriction x) *_{\mathbb{C}} \mathsf{CRDer}(Y) \restriction x^{[1]} \textbf{ fi}\right] : x \in \mathbb{R}\right\}, \int \left\{\left[x, \textbf{if } x \notin \mathsf{Interval}(U, V) \textbf{ then } \mathbf{0}_{\mathbb{R}} \textbf{ else } X \restriction (Y \restriction x) *_{\mathbb{C}} \mathsf{CRDer}(Y) \restriction x^{[2]} \textbf{ fi}\right] : x \in \mathbb{R}\right\}\right]$$

-- Cauchy integral theorem

Theorem 679 ( ) $\mathsf{is\_analytic}_{\mathbb{CF}}(f) \rightarrow$

$\big\langle \exists \varepsilon \in \mathbb{R} \mid \varepsilon \supseteq \mathbf{0}_{\mathbb{R}} \,\&\, \varepsilon \neq \mathbf{0}_{\mathbb{R}} \,\&\, \big\langle \forall crv_1, crv_2 \mid \mathsf{is\_CD\_curv}(crv_1, \mathbf{0}_{\mathbb{R}}, \mathbf{1}_{\mathbb{R}}) \,\&\, \mathsf{is\_CD\_curv}(crv_1, \mathbf{0}_{\mathbb{R}}, \mathbf{1}_{\mathbb{R}}) \,\&\, crv_1 \restriction \mathbf{0}_{\mathbb{R}} = crv_1 \restriction \mathbf{1}_{\mathbb{R}} \,\&\, crv_2 \restriction \mathbf{0}_{\mathbb{R}} = crv_2 \restriction \mathbf{1}_{\mathbb{R}} \,\&\, \big\langle \forall x \in \mathsf{Interval}(\mathbf{0}_{\mathbb{R}}, \mathbf{1}_{\mathbb{R}}) \mid \varepsilon \supseteq \# crv_1 \restriction x -_{\mathbb{C}}$

QED

-- Cauchy integral formula

Theorem 680 ( ) $\mathsf{is\_analytic}_{\mathbb{CF}}(f) \,\&\, \mathbf{domain}(f) \supseteq \{z \in \mathbb{C} : \# z_{\mathbb{C}} \subseteq \mathbf{1}_{\mathbb{R}}\} \rightarrow$

$\big\langle \forall z \in \mathbb{C} \mid \# z_{\mathbb{C}} \subseteq \mathbf{1}_{\mathbb{R}} \,\&\, \# z_{\mathbb{C}} \neq \mathbf{1}_{\mathbb{R}} \rightarrow f \restriction z = \oint_{\mathbf{0}_{\mathbb{R}}}^{\pi +_{\mathbb{R}} \pi}(\{[x, f \restriction x /_{\mathbb{C}} (x -_{\mathbb{C}} z)] : x \in \mathbb{C} \setminus \{z\}\}, \{[x, \mathsf{C\_exp\_fcn}([\mathbf{0}_{\mathbb{R}}, x])] : x \in \mathbb{R}\}) /_{\mathbb{C}} [\mathbf{0}_{\mathbb{R}}, \pi +_{\mathbb{R}} \pi] \big\rangle$. PROOF:

QED

-- END HERE — — — — Beyond this point, the number of steps of definition needed to reach any concept, say, of classical functional analysis can be estimated by counting the number of definitions needed to reach the corresponding point in any standard reference on this subject, e. g. Dunford-Schwartz.