Inductive sets

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Inductive sets occur often in mathematics and in computer science. The most classical of them is the set $\mathbb{N}$ of *natural numbers*, generated from the singleton $\{0\}$ by the unitary increment operation. Another classical example is the set of all *finite lists* with components drawn from a fixed, though generic, base set A. A third example is the set of all *terms over a signature*. Inductive sets can be conceived of dynamically: namely, in each specific instance we can identify a set of *seeds* and a *generating map* which—we may think—repeatedly puts new elements into a set whose initial value is the set of seeds and whose ending value, the inductive set, is reached when a certain 'plateau' (that is, the fulfillment of a certain closure property) has been achieved, perhaps by a transfinite number of iterations. Below, in specifying the notion of inductive set formally, we will insist that seeds should not be reachable again as the construction proceeds. More generally, the construction of an inductive set should be carried out under some guarantee that no element of the inductive set can admit more than one construction. This will make reasoning about the elements of an inductive set particularly plain and effective.

Notice that the generating map is not necessarily single-valued: while in the paradigmatic case of natural numbers it is such, to best treat the other two cases mentioned above we will figure out two suitable multi-valued generating maps. (For uniformity, as regards $\mathbb{N}$, we will proceed from the definitions

$$0 =_{\text{Def}} \emptyset \quad \text{and} \quad \mathsf{hasNext}(X, Y) \leftrightarrow_{\text{Def}} Y = X \;\mathsf{with}\; X,$$

even though the definition $\mathsf{next}(X) =_{\text{Def}} X \;\mathsf{with}\; X$ might seem better tailored to the case than the one of $\mathsf{hasNext}$.) Another crucial design choice is whether the generating map should be a 'small' relation (representable as a set of pairs) or a global relation (representable by a formula $\varphi \equiv \varphi(X, Y)$). We opt here for a *global map*.

A more than adequate technical surrogate for the (intuitively more appealing) dynamic construction of an inductive set will consist of two steps:

1. Identify a superset of the desired inductive set. This must contain the seeds and must be closed w.r.t. the generating map.

2. Extract the inductive set from the superset determined at Step 1.

Since inductive sets are normally infinite, the *infinity axiom*

$$\mathbf{s}_\infty \neq \emptyset \;\&\; (\forall x \in \mathbf{s}_\infty)(\{x\} \in \mathbf{s}_\infty)$$

will be needed to effect Step 1. To make Step 2 a plain routine matter, we will design a THEORY offering adequate support to it.

This revolution of replacing potential infinity by actual infinity in mathematical reasoning was made by Cantor and Dedekind in the late nineteenth century. Thanks to the availability of the recursive definition scheme, a single actual infinite set, even one which is as indistinct as $\mathbf{s}_\infty$ is, suffices to get started.

1 How to frame an inductive set

To convey an intuitive grasp of the notion of inductive set, we state beforehand that the set $\mathbf{s}_\infty$ may fail to be inductive *relative* to the singleton set $\{\mathbf{arb}\,(\mathbf{s}_\infty)\}$ of seeds and to the generating map $\mathsf{Sngl}(X,Y) \leftrightarrow_{\text{Def}} Y = \{X\}$. Momentarily assuming for definiteness that $\mathbf{arb}\,(\mathbf{s}_\infty) = \emptyset$, we should not regard $\mathbf{s}_\infty$ as being inductive relative to $\{\emptyset\}, \mathsf{Sngl}$ if it had such 'superfluous' elements as $\{\emptyset, \{\emptyset\}\}$ or $\{\emptyset, \{\emptyset, \{\{\emptyset\}\}\}\}$ instead of consisting of *only* the 'mandatory' elements in the following infinite list:

$$\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\{\{\emptyset\}\}\}, \ldots$$

We will develop in Sec.2.2 the machinery needed to form the subset of $\mathbf{s}_\infty$ consisting *precisely* of the sets $\mathbf{arb}\,(\mathbf{s}_\infty)$, $\{\mathbf{arb}\,(\mathbf{s}_\infty)\}$, and $\{\cdots\{\mathbf{arb}\,(\mathbf{s}_\infty)\}\cdots\}$. The best we can say now is that $\mathbf{s}_\infty$ FRAMES the desired inductive set. Below (Sec.1.1–Sec.1.4) we will exploit $\mathbf{s}_\infty$ to construct sets which frame other important inductive sets such as the set $\mathbb{N}$ of all natural numbers, the family $\mathbb{H}$ of all hereditarily finite sets, and the set $\mathsf{tuples}(A)$ of all finite lists over A.

Before continuing, we must provide the formal definition of INDUCTIVELY CLOSED SET, which presupposes a couple of notions regarding a global map R (in a sense *into-ness* and *injectivity*):

$$\begin{array}{rcl}
\mathsf{Maps}(R,S,T) & \leftrightarrow_{\text{Def}} & (\forall x \in S)(\forall y)\big(R(x,y) \rightarrow y \in T\big), \\
\mathsf{Disj}(R,S) & \leftrightarrow_{\text{Def}} & (\forall u,v \in S)(\forall y)\big(R(u,y)\ \&\ R(v,y) \rightarrow u = v\big), \\
\mathsf{IndClosed}(N,R,A) & \leftrightarrow_{\text{Def}} & A \subseteq N\ \&\ \mathsf{Maps}(R,N,N \setminus A)\ \&\ \mathsf{Disj}(R,N) \\
 & & \&\ (\forall t)\big(A \subseteq t\ \&\ \mathsf{Maps}(R,t,t) \rightarrow N \subseteq t\big).
\end{array}$$

A set S which candidates to frame a set N such that $\mathsf{IndClosed}(N,R,A)$ will be required to meet a *less stringent* condition than being inductively closed, namely the following:

$$\mathsf{Frames}(S,R,A) \quad \leftrightarrow_{\text{Def}} \quad A \subseteq S\ \&\ \mathsf{Maps}(R,S,S \setminus A)\ \&\ \mathsf{Disj}(R,S).$$

1.1 How to frame the inductive set $\mathbb{N}$ of natural numbers

After recalling that

$$0 =_{\text{Def}} \emptyset, \quad \mathsf{next}(X) =_{\text{Def}} X\ \mathsf{with}\ X, \quad \text{and} \quad \mathsf{hasNext}(X,Y) \leftrightarrow_{\text{Def}} Y = \mathsf{next}(X),$$

let us recursively define for all X:

$$f(X) \quad =_{\text{Def}} \quad \{0\} \cup \{\, \mathsf{next}(v) : u \in X,\ v \in f(u)\,\}\,.$$

It is obvious that $\mathsf{SngVal}(\mathsf{hasNext})$ holds, where *single-valuedness* is defined as follows:

$$\mathsf{SngVal}(R) \quad \leftrightarrow_{\mathrm{Def}} \quad (\forall x, u, v)(R(x,u) \,\&\, R(x,v) \;\rightarrow\; u = v).$$

Moreover, we have the following:

Lemma 1 $\mathsf{Frames}(f(\mathbf{s}_\infty), \mathsf{hasNext}, \{0\})$.

Proof. Obviously $0 \in f(\mathbf{s}_\infty)$ and $\mathbf{arb}\,(\mathbf{s}_\infty) \in \mathbf{s}_\infty \,\&\, 0 \in f(\mathbf{arb}\,(\mathbf{s}_\infty))$ hold, and therefore $\mathsf{next}(0) \in f(\mathbf{s}_\infty)$. It can also be proved that $Y \in f(\mathbf{s}_\infty) \;\rightarrow\; \mathsf{next}(Y) \in f(\mathbf{s}_\infty)$: If $Y = 0$ this has just been seen; When $0 \neq Y \in f(\mathbf{s}_\infty)$, pick u, v such that $u \in \mathbf{s}_\infty \,\&\, v \in f(u) \,\&\, Y = \mathsf{next}(v)$, hence $Y \in f(\{u\})$, where $\{u\} \in \mathbf{s}_\infty$, so that $\mathsf{next}(Y) \in f(\mathbf{s}_\infty)$. We readily get $\mathsf{Maps}(\mathsf{hasNext}, f(\mathbf{s}_\infty), f(\mathbf{s}_\infty) \setminus \{0\})$ from the above argument. Then we get $\mathsf{next}(X) = \mathsf{next}(Y) \;\rightarrow\; X = Y$ since, if by absurd hypothesis $X \neq Y$ held along with $X \cup \{X\} = Y \cup \{Y\}$, then $X \in Y \,\&\, Y \in X$ would hold, and hence the set $\{X, Y\}$ would violate the regularity axiom. Hence $\mathsf{Disj}(\mathsf{hasNext}, f(\mathbf{s}_\infty))$ holds, and the thesis follows.

■

1.2 How to frame the family of hereditarily finite sets

The construction of a set H satisfying both $\emptyset \in H$ and the implication

$$X \in H \;\rightarrow\; \mathcal{P}(X) \subseteq H$$

for all X, parallels very closely the construction just seen, the main change being that we define

$$f(X) \quad =_{\mathrm{Def}} \quad \{\emptyset\} \cup \{\, v \cup \mathcal{P}(v) : u \in X,\; v \in f(u) \,\},$$

and replace $\mathsf{hasNext}$ by the single-valued map $\mathsf{hasPow}(X, Y) \leftrightarrow_{\mathrm{Def}} Y = X \cup \mathcal{P}(X)$.

The only detail where the proof of $\mathsf{Frames}(f(\mathbf{s}_\infty), \mathsf{hasPow}, \{\emptyset\})$ differs from the proof of Lemma 1 is the way we get $X \cup \mathcal{P}(X) = Y \cup \mathcal{P}(Y) \;\rightarrow\; X = Y$: If by absurd hypothesis $X \cup \mathcal{P}(X) = Y \cup \mathcal{P}(Y)$ and $X \neq Y$ held together, then we would have $(X \subseteq Y \,\vee\, X \in Y) \,\&\, (Y \subseteq X \,\&\, Y \in X)$; but we must discard $X \subseteq Y \subseteq X$ (which would contradict $X \neq Y$), as well as $X \in Y \in X$ (which would conflict with the regularity axiom), as well as $X \subseteq Y \in X$ (which would imply $Y \in Y$, conflicting with regularity), as well as $Y \subseteq X \in Y$.

1.3 How to frame the inductive sets of based tuples

Constructing a set which frames the inductive set which we will elect as the domain of all flat tuples over a base set A will be easier if we begin with a THEORY, devoid of assumptions, on ordered pairs:[1]

[1] Inside the THEORY $\mathsf{orderedPair}$, if we adopt the pair construction proposed by Jack, then we can recursively define len as follows:

$$\mathsf{len}(T) =_{\mathrm{Def}} \mathbf{arb}\left(\{\mathsf{next}(\mathsf{len}(r)) : x \in T,\; y \in x,\; r \in y \mid (\exists l)(T = [l, r])\}\right) .$$

THEORY orderedPair()
==> (cons, car, cdr, nl, len)

$\mathsf{car}\big(\mathsf{cons}(X,Y)\big) = X$

$\mathsf{cdr}\big(\mathsf{cons}(X,Y)\big) = Y$

$\mathsf{cons}(X,Y) = \mathsf{cons}(U,V) \rightarrow X = U \;\&\; Y = V$

$\mathsf{nl} \neq \mathsf{cons}(X,Y)$ -- *plausibly, nl is an alias of* $\emptyset$

$\mathsf{len}(\mathsf{nl}) = 0$

$\mathsf{len}(\mathsf{cons}(X,Y)) = \mathsf{next}(\mathsf{len}(Y))$

END orderedPair

We make the invocation

APPLY $([-\,,\,-], \mathsf{hd}, \mathsf{tl}, [\,], \mathsf{lth})$ orderedPair()

in sight of exploiting the operations thus introduced to build the desired theory

THEORY tuples()
==> (tups, len)

$[\,] \in \mathsf{tups}(A)$

$V \in \mathsf{tups}(A) \rightarrow A \times \{V\} \subseteq \mathsf{tups}(A) \setminus \{[\,]\}$

$[\,] \in T \;\&\; (\forall v \in T)(A \times \{v\} \subseteq T) \rightarrow \mathsf{tups}(A) \subseteq T$

$\mathsf{len}([\,]) = 0$

$\mathsf{len}([-\,,\,T]) = \mathsf{next}(\mathsf{len}(T))$

END tuples

During the construction of this theory, before we can indicate how to build the inductive set $\mathsf{tups}(A)$ for each fixed A, we will need a set that frames $\mathsf{tups}(A)$. We concentrate on this problem for the time being, postponing to the end of Sec.2.2 the discussion on how to use the framing set to achieve what is desired.

We recursively define an auxiliary function tups in two parameters:

$$\mathsf{tups}(A,V) =_{\mathrm{Def}} \{[\,]\} \cup \bigcup\{A \times \mathsf{tups}(A,w) : w \in V\}\,.$$

E.g., it should be intuitively clear that when V is a finite ordinal (intended *à la* von Neumann), $\mathsf{tups}(A,V)$ will consist of those tuples over the base set A whose length does not exceed V.

For any fixed A, we then put

$$\begin{array}{rcl} \mathsf{Pads}(V,W) & \leftrightarrow_{\mathrm{Def}} & W \in A \times \{V\}, \\ \mathsf{tupp} & \leftrightarrow_{\mathrm{Def}} & \mathsf{tups}(A, \mathsf{s}_\infty), \end{array}$$

after which we can easily prove that

$$\begin{array}{l} [\,] \in \mathsf{tupp}, \\ V \in \mathsf{tupp} \;\&\; \mathsf{Pads}(V,W) \rightarrow W \in \mathsf{tupp}, \\ \mathsf{Maps}(\mathsf{Pads}, \mathsf{tupp}, \mathsf{tupp} \setminus \{[\,]\}), \\ U \neq V \;\&\; \mathsf{Pads}(U,W) \rightarrow \neg\, \mathsf{Pads}(V,W), \end{array}$$

to wit,

$$\mathsf{Frames}(\mathsf{tupp},\ \mathsf{Pads},\ \{[\,]\})\,.$$

1.4 How to frame the inductive sets of terms

Once we will own the theory tuples specified above, in order to frame the desired inductive set terms of all terms over a signature S, we can proceed as follows.

We will begin with an auxiliary function terms in two parameters:

$$\mathsf{terms}(S,X) =_{\mathrm{Def}} \bigcup\{[\mathsf{car}(p), args] : \; p \in S, y \in X, args \in \mathsf{tups}(\mathsf{terms}(S,y)) \mid \mathsf{len}(args) = \mathsf{cdr}(p)\}\,,$$

where tups and len result from an invocation

$$\mathbf{APPLY}\ (\mathsf{tups}, \mathsf{len})\ \ \mathsf{tuples}()\ ,$$

and where each p in S is interpreted as a "symbol" whose two components are the "lexeme" and the degree (often called "arity"), respectively.

For any fixed signature σ, after (locally) putting

$$\begin{array}{rcl}
\mathsf{termm} & =_{\mathrm{Def}} & \mathsf{terms}(\sigma, \mathsf{s}_\infty)\ , \\
\mathsf{Pads}(V,W) & =_{\mathrm{Def}} & \big(\exists p \in \sigma\big)\big(\exists a \in \mathsf{tups}(\mathsf{termm}) \setminus \{[\,]\}\big)\big(W = [\mathsf{car}(p), a]\ \&\ \mathsf{len}(a) = \mathsf{cdr}(p) \\
& & \qquad \&\ \mathsf{car}(a) = V\big), \\
\mathsf{consts} & \leftrightarrow_{\mathrm{Def}} & \big\{\,\big[\mathsf{car}(p), [\,]\big] : p \in \sigma \mid \mathsf{cdr}(p) = 0\,\big\},
\end{array}$$

one can prove that

$$\mathsf{Frames}(\mathsf{termm},\ \mathsf{Pads},\ \mathsf{consts})\ .$$

2 Theories related to inductive closure

[$\cdots$TO BE COMPLETED$\cdots$]

2.1 Weak induction

What confers an inductive set n its appeal is the following THEORY,

THEORY $\mathsf{weakInduction}(\mathsf{n}, \mathsf{r}, \mathsf{a}, \mathsf{p})$
  $\mathsf{IndClosed}(\mathsf{n}, \mathsf{r}, \mathsf{a})$
  $X \in \mathsf{a} \rightarrow \mathsf{p}(X)$
  $X \in \mathsf{n}\ \&\ \mathsf{p}(X)\ \&\ \mathsf{r}(X,Y) \rightarrow \mathsf{p}(Y)$
==>
  $\mathsf{a} = \emptyset \rightarrow \mathsf{n} = \emptyset$
  $\mathsf{Exhs}(\mathsf{r}, \mathsf{n} \setminus \mathsf{a}, \mathsf{n})$
  -- *Hint: if any* $w \in \{v \in \mathsf{n} \setminus \mathsf{a} \mid (\forall x \in \mathsf{n})(\neg \mathsf{r}(x,v))\}$ *existed, removing*
  --   *it from* n *would lead to a set contradicting the minimality of* n
  $X \in \mathsf{n} \rightarrow \mathsf{p}(X)$
  -- *Hint:* $\mathsf{a} \subseteq \{x \in \mathsf{n} \mid \mathsf{p}(x)\}\ \&\ \mathsf{Maps}(\mathsf{r}, \{x \in \mathsf{n} \mid \mathsf{p}(x)\}, \{x \in \mathsf{n} \mid \mathsf{p}(x)\})$
END $\mathsf{weakInduction}$

where the *surjectivity* notion

$$\mathsf{Exhs}(R,T,S) \quad \leftrightarrow_{\mathrm{Def}} \quad (\forall y \in T)(\exists x \in S)(R(x,y))$$

is involved.

Weak induction constitutes a familiar reasoning template which every reader knows from experience (if only with arithmetic induction) to be extremely versatile; moreover, this scheme can be generalized into *strong* induction, as we will see in Sec.2.3.

2.2 Getting an inductive set from a framing set

The following THEORY circumscribes a given set a of seeds with an inductive set n, while also associating an inductive subtree with each element of n:

THEORY indClosure(s, r, a)
  Frames(s, r, a)
==> (n, indCl)
-- *Hint:* $\mathsf{indCl}(B) =_{\text{Def}} \bigcap\{t \subseteq \mathsf{s} \mid (B \subseteq s \rightarrow B \subseteq t) \;\&\; \mathsf{Maps}(\mathsf{r}, t, t)\}$, *i.e.,*
-- $\mathsf{indCl}(B) =_{\text{Def}} \{x \in \mathsf{s} \mid (\forall t \subseteq \mathsf{s})((B \subseteq \mathsf{s} \rightarrow B \subseteq t) \;\&\; \mathsf{Maps}(\mathsf{r}, t, t) \rightarrow x \in t)\}$
  $\mathsf{n} = \mathsf{indCl}(\mathsf{a})$
  $\mathsf{a} \subseteq \mathsf{n} \;\&\; \mathsf{n} \subseteq \mathsf{s}$
  $B \subseteq \mathsf{n} \;\&\; (\forall x \in \mathsf{indCl}(B))(\forall y)\big(\mathsf{r}(x, y) \rightarrow y \notin B\big) \rightarrow \mathsf{IndClosed}(\mathsf{indCl}(B), \mathsf{r}, B)$
  $B \subseteq \mathsf{a} \rightarrow \mathsf{IndClosed}(\mathsf{indCl}(B), \mathsf{r}, B)$
  $\mathsf{IndClosed}(\mathsf{n}, \mathsf{r}, \mathsf{a})$
  $\mathsf{Exhs}(\mathsf{r}, \mathsf{indCl}(B) \setminus B, \mathsf{indCl}(B))$
  $B \subseteq \mathsf{n} \rightarrow B \subseteq \mathsf{indCl}(B) \;\&\; \mathsf{indCl}(B) \subseteq \mathsf{n}$
  $Y \in \mathsf{n} \;\&\; X \neq Y \;\&\; X \in \mathsf{indCl}(\{Y\}) \rightarrow Y \notin \mathsf{indCl}(\{X\})$
  $X \in \mathsf{n} \rightarrow \mathsf{IndClosed}(\mathsf{indCl}(\{X\}), \mathsf{r}, \{X\})$
END indClosure

A well-founded relation is naturally associated with any inductive set n endowed with subtrees:

THEORY subTree(n, r, a, tree)
  $\mathsf{IndClosed}(\mathsf{n}, \mathsf{r}, \mathsf{a})$
  $X \in \mathsf{n} \rightarrow \mathsf{IndClosed}(\mathsf{tree}(X), \mathsf{r}, \{X\})$
==>
  $X \in \mathsf{n} \rightarrow \mathsf{tree}(X) \subseteq \mathsf{n}$
  --*Hint:* **APPLY** (t, indCl) indClosure(n, r, a) *provides* $\mathsf{indCl}(\{X\})$
  -- *s.t.* $\mathsf{Maps}(\mathsf{r}, \mathsf{indCl}(\{X\}), \mathsf{indCl}(\{X\})) \;\&\; \mathsf{indCl}(\{X\}) \subseteq \mathsf{t} \subseteq \mathsf{n}$
  $X \in \mathsf{n} \rightarrow \neg\, \mathsf{r}(X, X)$
  -- *Hint: if* $X \in \mathsf{n} \;\&\; \mathsf{r}(X, X)$*, then removal of* X *from* n *would lead*
  -- *to a set contradicting the minimality of* n
  $T \neq \emptyset \;\&\; T \subseteq \mathsf{n} \rightarrow (\exists m \in T)(\forall u \in T)\big(m \notin \mathsf{tree}(u) \setminus \{u\}\big)$
  -- *N.B.: this paves the way to recursive constructions over* n
  $X \in \mathsf{n} \;\&\; \mathsf{r}(X, Y) \rightarrow \mathsf{tree}(Y) \subsetneq \mathsf{tree}(X)$
  $X \in \mathsf{n} \;\&\; Y \in \mathsf{tree}(X) \;\&\; X \in \mathsf{tree}(Y) \rightarrow X = Y$
  $\mathsf{a} \neq \emptyset \;\&\; (\forall x \in \mathsf{n})(\exists y)\big(\, R(x, y)\,\big) \rightarrow \mathsf{Infinite}(n)$
  -- *here the following notion of* Infinite *is being referred to:*
  -- $\mathsf{Infinite}(I) =_{Def} (\exists c)\big((\exists k \in c)(k \subseteq I) \;\&\; (\forall k \in c)(\exists h \in c)(h \subsetneq k)\big)$,
  -- *and a clue on how to get a witness* c *of the infiniteness of* n *is:*

-- *pick* $c =_{Def} \{\mathsf{tree}(X) : X \in n\}$

END subTree

[$\cdots$TO BE COMPLETED$\cdots$]

2.3 Strong induction

THEORY strongInduction(n, r, a, tree, p)

$\mathsf{IndClosed}(\mathsf{n}, \mathsf{r}, \mathsf{a})$

$X \in n \rightarrow \mathsf{IndClosed}(\mathsf{tree}(X), \mathsf{r}, \{X\})$

$Y \in n \;\&\; (\forall x \in n)\big(Y \in \mathsf{tree}(x) \setminus \{x\} \rightarrow \mathsf{p}(x)\big) \rightarrow \mathsf{p}(Y)$

==>

$X \in n \rightarrow p(X)$

-- *Hint: Assuming the contrary, we could fix (exploiting* subTree*)*

-- *an* m *in* n *s.t.* $\neg\, \mathsf{p}(m) \;\&\; (\forall u \in n)\big(m \in \mathsf{tree}(u) \setminus \{u\} \rightarrow \mathsf{p}(u)\big)$

END strongInduction

2.4 Uniqueness of natural numbers

Let us now momentarily restrict our study to the special case when the seeds (whose set is passed as third parameter to the THEORYes weakInduction, indClosure, and subTree seen above) form a singleton set $\mathsf{a} = \{\,\mathsf{e}\,\}$, and moreover the generating map (which is passed as second parameter to the said THEORYes) is single-valued.

To ease the subsequent discussion, we introduce notions which combine single-valuedness with the notions Maps, Exhs, and Disj (*into-ness*, *surjectivity*, and *injectivity*) introduced earlier. Along with them, we introduce the new notion of *bijectivity*, which like the others refers to a global function G instead of to a more generic global dyadic relation. Moreover, we supply a restricted notion of inductively closed set:

$$\begin{array}{rcl}
\mathsf{Sends}(G,S,T) & \leftrightarrow_{\text{Def}} & (\forall x \in S)\big(G(x) \in T\big), \\
\mathsf{Surj}(G,T,S) & \leftrightarrow_{\text{Def}} & (\forall y \in T)(\exists x \in S)\big(G(x) = y\big), \\
\mathsf{Inj}(G,S,T) & \leftrightarrow_{\text{Def}} & (\forall u, v \in S)(\forall y \in T)\big(G(u) = y \;\&\; G(v) = y \rightarrow u = v\big), \\
\mathsf{Bij}(G,S,T) & \leftrightarrow_{\text{Def}} & \mathsf{Surj}(G,T,S) \;\&\; \mathsf{Inj}(G,S,T), \\
\mathsf{SuccClosed}(N,G,A) & \leftrightarrow_{\text{Def}} & A \subseteq N \;\&\; \mathsf{Sends}(G,N,N \setminus A) \;\&\; \mathsf{Inj}(G,N,N) \\
& & \quad \& \; (\forall t)\big(A \subseteq t \;\&\; \mathsf{Sends}(G,t,t) \rightarrow N \subseteq t\big)\,.
\end{array}$$

As one should expect, the following THEORY, where segments take the place of trees, easily ensues from subTree:

THEORY subSegm(n, succ, e, segm)

$\mathsf{n} = \mathsf{segm}(\mathsf{e})$

$\mathsf{SngVal}(\mathsf{succ})$

$\mathsf{e} \in \mathsf{n}$

$X \in \mathsf{n} \rightarrow \mathsf{IndClosed}(\mathsf{segm}(X), \mathsf{succ}, \{X\})$

==>

$X \in \mathsf{n}\ \&\ \mathsf{succ}(X,Y)\ \rightarrow\ \mathsf{segm}(Y) = \mathsf{segm}(X) \setminus \{X\}$
-- *i.e.,* $U \in \mathsf{n}\ \&\ \mathsf{succ}(U,V)\ \&\ W \in \mathsf{segm}(U) \setminus \{U\}\ \rightarrow\ W \in \mathsf{segm}(V)$
$T \neq \emptyset\ \&\ T \subseteq \mathsf{n}\ \rightarrow\ (\exists m \in T)(\forall u \in T)\big(u \in \mathsf{segm}(m)\big)$
$X, Y \in \mathsf{n}\ \rightarrow\ X \in \mathsf{segm}(Y) \vee Y \in \mathsf{segm}(X)$
$U, V, W \in \mathsf{n}\ \&\ V \in \mathsf{segm}(U)\ \&\ W \in \mathsf{segm}(V)\ \rightarrow\ W \in \mathsf{segm}(U)$
$(\forall x \in \mathsf{n})(\exists y)\big(\ R(x,y)\ \big)\ \rightarrow\ \mathsf{Infinite}(\mathsf{n})$
END subSegm

It should be clear that such n, e, and g are meant to represent the natural numbers, their first element, and their successor function, respectively. The fact that this representation is essentially unique should emerge from a theory with the following traits:

THEORY uniqNat(n, g, e, nn, gg, ee)
SuccClosed(n, g, { e })
SuccClosed(nn, gg, { ee })
==> (p, q)
p(e) = ee
q(ee) = e
$X \in \mathsf{n}\ \rightarrow\ \mathsf{p}\big(\mathsf{f}(X)\big) = \mathsf{ff}\big(\mathsf{p}(X)\big)$
$Y \in \mathsf{nn}\ \rightarrow\ \mathsf{q}\big(\mathsf{ff}(Y)\big) = \mathsf{f}\big(\mathsf{q}(Y)\big)$
Bij(p, n, nn)
Bij(q, nn, n)
$X \in \mathsf{n}\ \rightarrow\ q\big(\mathsf{p}(X)\big) = X$
$Y \in \mathsf{nn}\ \rightarrow\ p\big(\mathsf{q}(Y)\big) = Y$
END uniqNat

2.5 Free closure relative to given constructors

[$\cdots$TO BE COMPLETED$\cdots$]

Inductive sets can also be generated by a set ff of constructors:

THEORY freeClosure(s, ff, a)
$\mathsf{a} \subseteq \mathsf{s}$
$F \in \mathsf{ff}\ \rightarrow\ \mathsf{Maps}(F, \mathsf{s}, \mathsf{s} \setminus \mathsf{a})$
$F \in \mathsf{ff}\ \rightarrow\ \mathsf{Disj}(F, \mathsf{s})$
$F, G \in \mathsf{ff}\ \&\ F \neq G\ \&\ F(U,Y)\ \rightarrow\ \neg\, G(V,Y)$
-- *Accordingly,* $\mathsf{Maps}(\bigcup \mathsf{ff}, \mathsf{s}, \mathsf{s} \setminus \mathsf{a})\ \&\ \mathsf{Disj}(\bigcup \mathsf{ff}, \mathsf{s})$
==> (n, tree)
$\mathsf{IndClosed}(\mathsf{n}, \bigcup \mathsf{ff}, \mathsf{a})$
$X \in \mathsf{n}\ \rightarrow\ \mathsf{IndClosed}(\mathsf{tree}(X), \bigcup \mathsf{ff}, \{X\})$
END freeClosure

However, the constructors have, in many applications, a degree (or *"arity"*) which should be taken into account. We will see below how to deal with this complication.

[$\cdots$TO BE COMPLETED$\cdots$]